STABILITY AND SPECTRA OF BLOW-UP IN PROBLEMS WITH QUASI-LINEAR GRADIENT DIFFUSIVITY

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1 Introduction and Statement of Main Results

A significant feature characterising many nonlinear evolution partial differential equations is that their solutions form singularities. The study of such singularities, in particular their spatial and temporal structure, location and stability is of much interest both for an understanding of the physical problems modelled by such equations (such as combustion or the possible onset of turbulence) and also for the deep analytic problems that they pose. It is a feature of such problems, especially those coming from the mechanics of combustion, that the singularities display a strong structural stability. This is despite being singular and unstable in time in the strict Lyapunov sense.
A common method of understanding such structures is to rescale the problem so that the singularity becomes regular or even a steady state in the rescaled variables and then to linearise about this state. Usually, the linearisation of the nonlinear evolution equation about such a state reduces it to a linear eigenvalue problem for a differential operator and spectral theory can be applied to determine its stability. In the first approximation, the spectral properties and eigenspaces can describe the corresponding stable, unstable and the centre manifold which play a fundamental role in the understanding of the stability of the steady states and of the general evolution of the nonlinear equation.

In this paper we present a series of results and methods concerning the study of a quasi-linear parabolic problem, which does not fall into the class of equation amenable to the above analysis and is, possibly, the simplest one exhibiting the following non-standard properties:
(i) The evolution behaviour is highly non-stationary exhibiting blow-up in a *finite* time T. In a suitable rescaling the singularity becomes a steady state which is **absolutely unstable** in the usual Lyapunov sense. However, it is established that the maximally monotone shape is stable in a structural (rescaled) sense.
(i) Linearization about a homogeneous steady state does not lead necessarily to a linear problem. This provides us with a kind of a **nonlinear eigenvalue problem** for the quasilinear "linearized" equation. Nevertheless, we show how different nonlinear spectra (countable or finite) can be constructed which affect the asymptotic behaviour in the original non-linearised evolution equation.

We choose our basic problem as a natural generalization of a classical combustion equation which takes the following form

$$u_t = (|u_x|^\sigma u_x)_x + e^u \tag{1}$$

which formally can be treated as a combustion model in a non-linear medium. We consider for (1) the initial boundary-value problem for $x \in (-1,1)$, $t > 0$, with the Dirichlet zero boundary condition $u = 0$ for $x = \pm 1$, $t > 0$, and given bounded initial data $u(0,x) = u_0(x) \geq 0$ in $[-1,1]$.

The gradient diffusivity operator or the p-Laplacian operator $\mathbf{A}(u) = (|u_x|^\sigma u_x)_x$, as well as the more general operator

$$\mathbf{A}(u) = (|(u^m)_x|^\sigma (u^m)_x)_x, \quad m > 0,$$

occurs in the models of turbulent diffusion in porous media flows according to Leibenson, [Le1] and [Le2], see also Barenblatt [B]. Such quasilinear parabolic equations, $u_t = \mathbf{A}(u)$, are known as equations of non-Newtonian elastic ($m = 1$) or polytropic ($m \neq 1$) filtration with the local velocity of propagation satisfying a nonlinear Darcy law, cf. Khristianovich [Kh], Muskat [M] and also [AS], [MP]. Further references on applications and the main properties of the solutions can be found in the survey [K] and in [EV], [GKe].
Observe also that different types of gradient-dependent nonlinearities are widely used in mean curvature flow equations related to differential geometry and to problems of motion of a closed hypersurface in $\mathbf{R}^N$ by its mean curvature, [Br] and [CGG]. They occur in phase transition phenomena, [Gu]. A typical example is the equation

$$u_t = \frac{u_{xx}}{1+(u_x)^2} - \frac{1}{u}, \quad x \in \mathbf{R}, \ t > 0,$$

which describes (after a suitable surface parametrization) the evolution of cylindrically symmetric hypersurfaces moving by mean curvature in $\mathbf{R}^3$, [ES] and [SS].

The quasi-linear problem (1) reduces to a semi-linear one when $\sigma = 0$:

$$u_t = u_{xx} + e^u, \tag{2}$$

which is the famous Frank-Kamenetskii equation in the theory of adiabatic explosion and flame propagation, see [ZBLM]. Both the solutions of (1) and of (2) become infinite at a single point in a finite time if the initial data is suitably large. Furthermore, both (1) and (2) are invariant under a group of scaling transformations.

We demonstrate however, that the exponent $\sigma = 0$ separates two different forms of behaviour such that if $\sigma = 0$ all asymptotic profiles evolve in an approximately self-similar manner such that the solution profile is not invariant to the set of transformations that leaves the original equation invariant. In contrast, if $\sigma > 0$, then monotone initial data leads to monotone self-similar solutions which are invariant under these transformations. We also demonstrate the existence of a critical set of exponents σ_k (which includes $\sigma = 0$), at which similar transitions occur between approximately and true self-similar behaviour although this time for non-monotone initial data. Furthermore we give arguments to support the conjecture that if $\sigma > \phi \equiv (\sqrt{5}-1)/2$ (the Golden Mean (!)) then all initial data leads to self-similar behaviour.

Roughly speaking, we discuss the following conclusions (here and later on by a *linear* spectrum we mean that which is constructed via reduction to a standard linear eigenvalue problem):

(i) If $\sigma > \sigma_\infty \in (0, \phi)$ then we prove that there exists a *countable* spectrum of self-similar solutions to (1), which play a role of nonlinear eigenfunctions (completeness remains an open problem).

(ii) If $0 < \sigma < \sigma_\infty$ then we establish that the nonlinear spectrum is finite. The rest of the approximate ("linear") eigenfunctions to (1) are constructed via a special perturbation technique and come from a somewhat standard linear spectral analysis.

(iii) If $\sigma \leq 0$ then no nonlinear spectrum occurs and the above perturbation technique produces all the "linear" eigenfunctions which describe approximate self-similar solutions.

In fact, we describe a **spectral transition phenomenon**: when σ varies from 0 to σ_∞, the first eigenfunctions of the linear spectrum (which is known to be "complete" for $\sigma = 0$) are partially transformed into the corresponding nonlinear eigenfunctions. Finally, for $\sigma = \sigma_\infty$ this process is finished and for $\sigma > \sigma_\infty$ the spectrum becomes fully nonlinear.

A further survey of some of the remarkable ideas concerning the role of nonlinear eigenfunctions in a general theory of self-organization of nonlinear media can be found in a survey paper by Kurdyumov [Ku] together with a list of related references.

This paper reviews the results obtained by the authors together with Bill Dold in [BDG] and [BG1], [BG2], [BG3] which are concerned with the blow-up of the solutions of (1). The proofs of all the results will be sketched only and more details can be found in the above papers. We also include a series of new results on the asymptotic behaviour of blow-up described by a singularly perturbed Hamilton-Jacobi equation. This occurs when the nonlinear spectra are not complete and an extra evolution perturbation analysis is needed.

1.1. Main results on self-similar solutions

We firstly consider the case $\sigma > 0$ and shall assume that the initial data $u_0(x)$ is large enough so that finite-time blow-up occurs at a time T and at a single point x^*. The existence of such a time and such a point follows from standard comparison arguments and is described in [BE1], [SGKM] and the references therein. (The contrasting, but related examples of steady solutions are discussed in [GPP].) The main result proved in [BDG] completely characterises the form that the blow-up solution takes provided that the initial data has a single maximum in the interval $[-1, 1]$. In particular, the blow-up is *self-similar* when x is close to x^* and t is close to T. To see this we observe that the equation (1) is invariant under the transformation

$$u = \log(\lambda) + u, \quad x = \lambda^{1/(\sigma+2)}x, \quad t = \lambda t, \tag{3}$$

and posseses (formally) self-similar solutions of the form

$$u(x,t) = -\log(T-t) + \theta\left(\frac{|x-x^*|}{(T-t)^{1/(\sigma+2)}}\right). \tag{4}$$

Here $\theta(y)$ satisfies the nonlinear *ordinary* differential equation

$$B(\theta) \equiv (|\theta_y|^\sigma \theta_y)_y - \frac{y}{\sigma+2}\theta_y + e^\theta - 1 = 0, \quad \theta_y(0) = 0 \ \text{(symmetry)}. \tag{5}$$

Our principle result is as follows

Theorem 1.1. *Suppose that $\sigma > 0$ and $u_0 \equiv u_0(|x|)$ has a unique maximum on the interval $[-1,1]$. Then the asymptotic behaviour as $t \to T$ is self-similar in the following sense: the $\omega-$limit set of the rescaled orbit*

$$\theta(y,t) = \log(T-t) + u(y(T-t)^{1/(\sigma+2)}, t) \tag{6}$$

(where we consider convergence on compact subsets in y), comprises non-zero, monotone, self-similar profiles $\theta_1(y)$ satisfying (5). Moreover, if $g(y)$ is in the ω-limit set then there exists a constant C_+ such that for $y > 0$

$$g(y) \le -(\sigma+2)\log(y) + C_+, \quad \frac{1}{\sigma+2} y g_y + 1 \ge 0. \tag{7}$$

We establish this result in Section 3 by using a mixture of comparison results and energy estimates. The function $\theta_1(y)$ in the above limit has the further property that

$$\theta(y) + (\sigma+2)\log(y) \to C \quad \text{as} \quad y \to \infty, \tag{8}$$

where C is a (bounded) constant. This condition is forced by the boundary conditions satisfied by the function $u(x,t)$.

A similar description of the $\omega-$limit set was originally proposed by Kapila [Kap] and Kassoy & Poland [KP] for the case of linear diffusion when $\sigma = 0$ and was supported by some numerical evidence. However, the argument for the existence of a self-similar limit of the function $u(x,t)$ breaks down as there is then no suitable solution of (5) which also satisfies the growth condition (8) and indeed all non-zero solutions grow *exponentially* as $y \to \infty$. This result was conjectured by Dold [D] and proved by Eberly [E] (see also the extensions of this result in [BQ]). (It is interesting that a similar result applies in two dimensions but not in three [BE2], [ET]). The corresponding behaviour of $u(x,t)$ when $\sigma = 0$ is more subtle and is *Approximately Self-Similar* in the sense that the rescaled function $\theta(y,t)$ converges to zero on compact sets in y and does not give much insight into the structure of $u(x,t)$ close to blow-up. (We observe that even in the case $N = 3$ when a nontrivial self-similar solution exists, that this solution is unstable and the asymptotic behaviour of 'almost all' the solutions is still approximately self-similar.) More details of the semi-linear problem are given in Section 2.

We thus have a most interesting change in the behaviour of the solutions of (1) as σ increases from zero. This behaviour is governed to a great extent by the behaviour of the solutions of the ordinary differential equation problem (5) which also satisfy the growth condition (8). As a straightforward consequence, we derive the existence of the simplest non-trivial self-similar solution (the first nonlinear eigenfunction).

Corollary 1.2. *For all $\sigma > 0$ the system (5),(8) has a non-trivial solution $\theta_1(y)$ with a unique (positive) maximum at $y = 0$.*

Theorem 1.1 and Corollary 1.2 are proved in Section 3.
We expect that such a *monotone* solution is unique and that in Theorem 1.1 the $\omega-$limit set is exactly $\theta_1(y)$. This question of uniqueness is, however, still open.

Let us now describe some properties of the above solution $\theta_1(y)$ for small $\sigma > 0$. In view of possible nonuniqueness we fix a profile θ_1 with the minimal value at the origin $\theta_1(0) > 0$. The following conjecture is still an open problem.

Conjecture 1.3. *For small $\sigma > 0$*
(i) $\theta_1(0) \sim \sigma/2$,
(ii) $\theta_1(y) \sim -(\sigma+2)\log(y) + \log\left(\frac{4}{\sigma}\right) \quad as \quad y \to \infty$.

The above formal result is derived in Section 4 and gives some insight into the different behaviour of the solutions in the limit of $\sigma = 0^+$. It is clear from this formal estimate that the formal limit of the solutions of (5), (8) is the null solution. We shall make this comparison more precise by studying the difference in the behaviour of the function $u(x,t)$ as a function of t when $\sigma = 0$ and σ is close to zero.

As we remarked earlier, the value $\sigma = 0$ represents an important point at which the form of the solutions of the equation (1) changes. In fact, a combination of analytic, numerical and formal arguments implies there are an infinite number of values of $\sigma \equiv \sigma_k > 0$ at which similar changes in the behaviour of the non-monotone solutions of (1) occur. We term these points *transition points*, and they are determined by the existence of different types of (self-similar) solutions of the equation (5), (8). We describe these as follows.

Conjecture 1.4. *There is an infinite sequence $\sigma_k \geq 0$ of values of σ such that $0 = \sigma_1 < \sigma_2 < ... < \sigma_k < ...$ for which*
(i) There exists a finite limit, $\sigma_k \to \sigma_\infty \approx 0.60$ as $k \to \infty$,
(ii) a solution $\theta_k(y)$ of (5), (8) with precisely k maxima/minima exists iff $\sigma > \sigma_k$,
(iii) as $\sigma \to \sigma_k^+$ there is a constant $A_k > 0$ such that

$$\theta_k(0) \sim (-1)^{k-1} A_k (\sigma - \sigma_k)_+.$$

Numerical experiments indicate that $\sigma_2 = 0.49...$ and $\sigma_3 = 0.57...$.

This result which is formally derived in [BG2] is strongly implied by the following result proved in [BG1].

Theorem 1.5. *Let* $\phi \equiv \frac{\sqrt{5}-1}{2} = 0.61803...$ *be the positive root of the quadratic equation* $\phi^2 + \phi = 1$ *(so that* ϕ *is the golden ratio). There is a* $\sigma_\infty < \phi$ *such that*
(i) if $\sigma > \sigma_\infty$ *then the problem (5),(8) has a countably infinite number of distinct solutions* θ_k *described above so that* $\theta_k(0) \neq 0$ *and each* $\theta_k(y)$ *has exactly* k *maxima and minima for* $y \geq 0$.
(ii) if $\sigma > \phi$ *then the problem (5),(8) has a further infinite set of solutions* $\psi_k(y)$ *such that* $\psi_k(y) \equiv 0$ *on a non-empty interval* $[0, a_k]$ *(where* $\{a_k > 0\}$ *is a strictly monotone sequence) and* $\psi_k(y) = 0$ *at precisely* $k-1$ *points in the interval* (a_k, ∞).

The conclusions of the above section are summarised in Figure 1 for which the value of $\theta_k(0)$ is plotted as a function of σ. This graph was obtained numerically and agrees excellently with the formal asymptotics given in Conjecture 1.4. In Figure 2 we present the corresponding graphs of the solutions $\theta_k(y)$ and $\psi_k(y)$ of (5), (8).

In Section 5 we sketch the proof of Theorem 1.5 and give arguments to support Conjecture 1.4. These arguments make extensive use of phase-plane techniques and the result in Theorem 1.5 corresponds to a homoclinic bifurcation point when $\sigma = \phi$.

Thus, according to Theorem 1.5 for $\sigma > \sigma_\infty$ there is a countable set of self-similar solutions of (1) with the corresponding functions $\{\theta_k(y)\}$ having an arbitrary given number of maxima, and for $\sigma > \phi$ there is also a set of self-similar solutions with *flat* behaviour in a neighbourhood of the origin.

We conjecture that if $\sigma > \phi$ then *all* initial data for (1) leads to self-similar growth taking one of these forms.
Let us note that countable spectra of self-similar solutions were constructed in [GKPS] for the quasilinear equation with a power source

$$u_t = (|u_x|^\sigma u_x)_x + u^p,$$

with $\sigma > 1$, $p > \sigma + 1$ (some aspects of the N-dimensional equation was also studied). The existence of critical exponents σ_∞ and ϕ for this equation remains an open problem.

1.2. Some results on approximately self-similar solutions

Following the conjecture in Conjecture 1.4, it appears that in the case $\sigma \in (0, \sigma_\infty)$, say $\sigma \in (\sigma_k, \sigma_{k+1})$, there exist k different exact self-similar solutions, with $\theta_k(y)$ having exactly k maxima and minima for $y \geq 0$. The behaviour of solutions of (1) which eventually in time have *more* than k maxima/minima is *approximately self-similar.* Such approximately self-similar solutions are formally constructed in Section 6 by using a matching argument combining a spectral analysis in an *inner* region with an outer solution where the solution is governed by the equation without diffusion:

$$u_t = e^u,$$

the so-called *Hamilton-Jacobi* region.

This observation extends the well known result (see Section 2) that if $\sigma = 0$ then no initial data $u_0(x)$ for $u(x,t)$ leads to solutions which evolve in a non-trivial self-similar fashion. Indeed if the function $u(x,t)$ eventually has a unique maximum then on compact sets in the variable y the rescaled solution converges to zero such that

$$|\log(T-t)\theta(y,t)| \le C \quad \text{is uniformly bounded away from zero as} \quad t \to T. \tag{9}$$

Herrero and Velazquez [HV3] have shown that if $\sigma = 0$ then there exist solutions which evolve in a particular non-monoton manner up to $t = T$ (with two maxima, see Section 2.2 below), also grow in an approximately self-similar evolution but that the rescaled function converges to zero more rapidly (exponentially) than in (9). The most general result on existence of all such unstable blow-up patterns for the semi-linear equation $u_t = \Delta u + u^p$, $p > 1$, was proved by Amadori [A] by means of a topological construction similar to that in [B]. The discussion in Section 6 extends this result to show that the behaviour of the form given in (9) occurs for solutions of (1) which evolve with k maxima/minima precisely when σ takes one of the values σ_k.

Thus the transition points σ_k play the same role as $\sigma = 0$ in differentiating between solution-types which evolve in a self-similar or approximately self-similar manner. Indeed, solution profiles which lead to approximately self-similar behaviour if $\sigma < \sigma_k$ can instead give self-similar solutions if $\sigma > \sigma_k$.

In Section 7 we show briefly that the case of negative σ is similar to that of $\sigma = 0$ in that no self-similar solutions appear and the behaviour of all the solutions is governed by the corresponding approximate self-similar solutions.

1.3. Discussion

There are many questions unresolved in the previous discussion. In particular, establishing the uniqueness of the monotone solutions in the $\omega-$limit set described in Theorem 1.1 (as well as the uniqueness of prifiles with a fixed number of maxima/minima), and the closely related question of proving the results quoted in the Conjecture 1.4. However, it now seems as though we have the beginnings of a complete picture which explains the evolution of the solutions of (1) and it is expected that a similar picture will be found for many other related problems.

2 General Results

We now review some of the elementary theory of the *initial value* problem for (1) and will discuss the existing results for the *semi-linear* case of $\sigma = 0$.

2.1. *Local existence.* The quasi-linear equation (1) will, in general admit weak solutions for times $t < T$. These solutions will also be classical at any point for which $u_x \neq 0$. We define such a weak solution $u(x,t)$ to be a continuous function satisfying the boundary contitions which has a weak derivative $u_x \in L^{2(\sigma+1)}(\Omega_{T'})$ where $\Omega_{T'} = [-1,1] \times (0,T')$, which also satisfies the identity

$$\int_{\Omega_{T'}} (|u_x|^\sigma u_x \phi_x - e^u \phi - u\phi_t) dx dt - \int_{-1}^{1} u_0(x)\phi(x,0) dx = 0 \tag{10}$$

where ϕ is any compactly supported test function $\phi \in C_0^\infty(\Omega_{T'})$.

The local existence and uniqueness of such a solution can be found in Lions [L] and Kalashnikov [K] and the references therein. Furthermore it is shown in [AE] and [AR] that such solutions have a continuous first derivative u_x. Indeed, the weak solution can be constructed as a limit of the sequence $u_\epsilon(x,t)$ of solutions to a *regularised* form of (1) where the derivative term is replaced by

$$((|u_x|^2 + \epsilon^2)^{\sigma/2} u_x)_x,$$

so that as $\epsilon \to 0$

$$u_\epsilon \to u \quad \text{and} \quad (u_\epsilon)_x \to u_x$$

on any compact subset of $\Omega_{T'}$. This form of the equation is the most convenient for numerical calculations where ϵ is taken to be suitably small.

It is possible to use standard techniques described in the review article by Levine [Lev] to prove that if u_0 is suitably large then the weak solutions of (1) must blow-up in a finite time and, moreover, at a single point (see Section 3).

2.2. The semilinear case. When $\sigma = 0$ then problem (1) reduces to the semi-linear equation

$$u_t = u_{xx} + e^u. \tag{11}$$

The question of the blow-up for equations like (10) was first rigorously addressed by Khudiaev, Kaplan, Friedman, Fujita and others in the 60-th, see lists of references in the books [BE1], [SGKM, Chapter 4] and in [BK], [Lev]. Since then many papers have been published on estimating the blow-up time T and the profile close to the blow-up point x^*. Friedman & McLeod [FM] showed that there was a constant δ such that if T was the blow-up time then

$$-\log(T-t) \le u(0,t) \quad \text{and} \quad u(x,t) \le -\log(\delta(T-t)), \quad \delta \in (0,1). \tag{12}$$

Furthermore the results in [BE1] show that

$$\log(T-t) + u(x,t) \to 0 \tag{13}$$

uniformly on sets for which $|x - x^*| \le C(T-t)^{1/2}$, so that as $t \to T$

$$u(0,t) = -\log(T-t) + o(1). \tag{14}$$

(Similar estimates were obtained by Galaktionov & Posashkov [GP1] and by Giga & Kohn [GK2] for problems with polynomial source terms.) These results motivate a search for solutions $u(x,t)$ close to the function $-\log(T-t)$ on the above subsets in x. As was stated in the introduction, attempts to find self-similar solutions of the blow-up profile close to the blow-up point which were of the form given in (4), failed because there is no solution to the problem (5) which also satisfies (8).

To overcome this problem the self-similar description of blow-up given in (4) must be extended to consider functions $\theta(y,s)$ which depend upon the rescaled time variable

$$s = -\log(T-t).$$

The function $\theta(y,s)$ then satisfies the following semi-linear parabolic equation

$$\theta_s = \theta_{yy} - \frac{1}{2}y\theta_y + e^{\theta} - 1, \quad \theta_y(0,s) = 0 \tag{15}$$

together with initial data and a condition on the boundary which now goes to infinity.

The time-dependent equation (15) clearly has the steady solution $\theta(y,s) \equiv 0$ which from (13) we see to be an attractor over compact sets in y. It is then natural to linearise (15) about this zero solution and to determine the eigenvalues $\{\lambda_k\}$ of the corresponding linear operator $A_1\theta = \theta_{yy} - y\theta_y/2 + \theta$. This procedure can be found in [BB], [FK], [GHV], [HV2]. In the symmetric case where the resulting eigenvalues λ_k are

$$\lambda_k = k - 1, \quad k = 0, 1, 2, ...,$$

with corresponding eigenfunctions being the Hermite polynomials $H_{2k}(y/2)$. The instability implied by the negativity of λ_0 is related to perturbations of the zero solution which blow-up at a time different from T and if we only consider perturbations with the same blow-up time then the most significant eigenvalue from a point of view of stability analysis is λ_1. This satisfies the condition

$$\lambda_1 = 0. \tag{16}$$

Significantly, we show in Section 6 that for the corresponding system with non-zero σ we have

$$\lambda_1 = -\sigma.$$

We observe that the null solution is non-hyperbolic when $\sigma = 0$ (and we show in Section 6 that this is also the case when $\sigma = \sigma_k$). In contrast the null solution *is* hyperbolic when σ is small. The result (16) means that a Centre-Manifold reduction can be applied to help analyse the problem (15). Details of this reduction are given in e.g. [FK] and [BB]. From this reduction it follows that on compact sets in y a solution which is monotonic evolves in the manner: for $s \gg 1$

$$\theta(y,s) \sim \frac{1}{2s}(1 - \frac{y^2}{2}) \tag{17}$$

a result proved by several authors. This result clearly shows that such a rescaled solution $\theta(y,s)$ converges *slowly* to the zero solution on such compact sets. It was established in [HV3] that there exists non-monotone inital data which gives rise to rescaled solutions $\theta(y,s)$ which corresponds to the more oscillatory eigenfunction H_4 (i.e. $k = 2$) with positive eigenvalue $\lambda_2 = 1$. Existence of such patterns for all k's is proved in [A]. Such non-monotone rescaled solutions then converge rapidly (indeed exponentially fast) to zero.

A more general description of the form of the blow-up can be obtained by recasting the equation (15) in terms of the *ignition* variable

$$\eta = (x - x^*)(T-t)^{-1/2}|\log(T-t)|^{-1/2}. \tag{18}$$

This variable was first introduced by Hocking, Stuart & Stewartson [HSS] in their studies of semilinear equations like $u_t = u_{xx} + Au^3 + Bu$ with a cubic nonlinearity, and by Dold [D] for the exponential nonlinearity. (Subsequent numerical and asymptotic studies have made much use of this variable.) When recast in terms of this variable the equation (15) essentially reduces in order to a first-order Hamilton-Jacobi equation with the effect of the diffusion operator u_{xx} dissapearing in the limit as $t \to T$. This limit was studied formally in [D] and rigorously in [BB], [B], [HV1], [BrKu] (see also [GHV]). In particular the following results is true: as $t \to T$ uniformly on compact sets in η

$$u(\eta(T-t)^{1/2}|\log(T-t)|^{1/2}, t) + \log(T-t) \to -\log(1 + \frac{\eta^2}{4}). \tag{19}$$

This result has also been confirmed numerically by Budd, Huang and Russell [BHR]. It is shown further in [BHR] that an adaptive mesh method based on equidistributing the function e^u automatically places mesh points along curves such that η is constant. (Similar estimates for polynomial source terms were studied numerically in [GP3] and in [BerK].)

Away from the blow-up point the function $u(x,t)$ tends toward a well defined function $u(x,T)$ as $t \to T$. This function is finite except at the point x^* and for t sufficiently close to T, $u(x,t)$ increases monotonically toward $u(x,T)$ [GP1]. The profile of $u(x,T)$ for x close to x^* was also determined formally by Dold [D] and proven by Bressan [B] to be of the form

$$u(x,T) = -2\log|x| + \log|\log|x|| + \log(8) + o(1) \quad \text{as} \quad x \to x^*. \tag{20}$$

The formulae given in (17), (19) and (20) essentially give a complete description of the stable and generic *approximately self-similar blow-up* of the semi-linear problem.

3 The ω-limit set for the quasi-linear equation

The main objective of this section is to sketch the proof of Theorem 1.1.

As a first estimate we have the following

Theorem 3.1. *Suppose that $\sigma > 0$ and u_0 and $|u_0'|^{\sigma+1} \in C^1$ are both bounded over the interval $[-1,1]$, $u_0(x) \equiv u_0(|x|)$ is decreasing, and that*

$$(|u_0'|^\sigma u_0')'(0) < 0. \tag{21}$$

Then there exist strictly positive constants C_+, M_- and M_+ which depend upon σ such that for small x

$$u(x,T) = \lim_{t \to T} u(x,t) \le -(\sigma+2)\log(x) + C_+, \tag{22}$$

and at $x = 0$

$$M_- \equiv \log\left(\frac{2+3\sigma}{2(\sigma+1)}\right) \le u(0,t) + \log(T-t) \le M_+. \tag{23}$$

We contrast this expression with the result of (14) for which the equivalent term for M_- is zero. We also note that as $\sigma \to 0$ then $M_- \to \sigma/2$ (and it can be shown that $C_+ \to \log(4/\sigma)$). This bound also implies that $M_- < \theta(0)$ and the asymptotic result quoted in Formal Proposition 1.3 indicates that these results are sharp in the limit of small σ. Numerical calculations show that this estimate ceases to be sharp for larger values of σ.

Sketch of proof. The proof for both the estimate (22), and the lower bound in (23) is based upon the technique introduced by Friedman & McLeod [FM] and relies upon the construction of a function $J(x,t)$ for which estimates are made using the Maximum Principle. In particular we define

$$J(x,t) = |u_x|^\sigma u_x + xF(u),$$

where

$$F(u) = \frac{\sigma}{2(1+\sigma)} e^u(1 - Ae^{-\alpha u}), \quad \alpha > 0, A > 0,$$

is chosen such that

$$F, F_u, F_{uu} \ge 0, \quad F(0) < 1,$$

and

$$2(1+\sigma)FF_u + e^u(F_u - (\sigma+1)F) \le 0 \quad \text{if} \quad u > 0.$$

Then after some manipulation it follows that $J(x,t)$ almost everywhere satisfies the following parabolic differential inequality:

$$J_t \le (\sigma+1)|u_x|^\sigma J_{xx} + (\sigma+1)(e^u - 2F_u)J. \tag{24}$$

Thus, provided that $J(x,0) \le 0$ (which is true provided that A and α are chosen suitably) it follows from the Maximum Principle that

$$J(x,t) \le 0$$

for all x and t.
We conclude that the function $u(x,t)$ satisfies the inequality

$$w(x,t) \equiv |u_x|^\sigma u_x + x\frac{\sigma}{2(1+\sigma)} e^u(1 - Ae^{-\alpha u}) \le 0, \tag{25}$$

for all $x \ge 0$ where $A \in (0,1)$ and $\alpha > 0$ are constants. Now, as $w(0,t) \equiv 0$ it follows from (25) that $w_x(0,t) \le 0$ and hence

$$(|u_x|^\sigma u_x)_x + \mu e^u + \mu x e^u u_x \le 0, \tag{26}$$

where μ is any real number which can be taken arbitrarily close to $\sigma/2(1+\sigma)$. Integrating (25) over the interval $(0,x)$ gives (22) for small x. Furthermore, substituting (26) into the differential equation (1) gives

$$u_t \leq (1-\mu)e^u$$

which gives the lower bound in (23) with $M_- = -\log(1-\mu)$.

To obtain the second estimate in (23) we compare $u(x,t)$ with a similarity solution of (1) which blows up in the same time T. This takes the form

$$u(x,t) = -\log(T-t) + \theta(y;\mu)$$

where $\theta(y;\mu)$ is a solution of the ordinary differential equation (5) with $\theta(0;\mu) = \mu$. In the next sections we will investigate the solutions of this equation in some detail and in particular will look at the slowly-growing solutions satisfying (8). To obtain the result in the theorem (and indeed to prove the existence of the slowly-growing solutions) we need the following estimate on the general solutions of (5).

Lemma 3.2. *If $\mu > 0, \sigma \geq 0$ then*
(i) there is a global weak solution of (5) such that near the origin

$$\theta(y;\mu) = \mu - \frac{\sigma+1}{\sigma+2}(e^{\mu}-1)^{1/(\sigma+1)}y^{(\sigma+2)/(\sigma+1)}(1+o(1)), \tag{27}$$

(ii) if μ is sufficiently large then $\theta(y;\mu)$ is monotone decreasing in y.

Proof of Lemma 3.2. The result *(i)* follows from fairly standard estimates on the differential equation with details given in [BDG].
To prove *(ii)* we construct a family $g_\alpha(y)$ of strict lower solutions to the equation (5). A simple calculation shows that the function

$$g_\alpha(y) = A - \alpha y$$

satisfies the differential inequality

$$B(g_\alpha) > 0$$

where B is the operator in (5), provided that A (independent of α) is sufficiently large, $A > \sigma + 1 - \log(\sigma+2)$. From this we deduce using the Maximum Principle that if $\theta(y)$ intersects the function g_{α_0} at a point $y_1 > 0$ with $\theta(y_1) = g_{\alpha_0}(y_1)$ and $\theta_y < (g_{\alpha_0})_y$ at this point then

$$\theta(y) < g_\alpha(y), \quad \theta_y < 0 \quad \text{if} \quad y > y_1. \tag{28}$$

Indeed $\theta_y \leq (g_\alpha)_y$ for any $\alpha > \alpha_0$.
To describe the function $\theta(y)$ on the compact interval $[0, y_1]$ we observe that if θ is sufficiently large, then the equation (5) is uniformly approximated by the equation

$$(|\theta_y|^\sigma \theta_y)_y + e^\theta = 0 \tag{29}$$

(which is the equation satisfied by the steady state solution of (1)). This equation has a strictly decreasing solution. By rescaling (5) and applying standard arguments of continuous dependence, we deduce that θ is also monotone in this range and is uniformly approximated by the solution of (29).
By combining these two estimates we deduce that if μ is sufficiently large that $\theta(y)$ decreases monotonically to $-\infty$ on the interval $[0, \infty)$.

End of proof of Theorem 3.1. To conclude the proof of Theorem 3.1 we compare $u(x,t)$ with the self-similar solution u_* of (1) such that

$$u_*(x,t) = -\log(T-t) + \theta(y;\mu)$$

where μ is taken sufficiently large so that $\theta(y;\mu)$ is monotone decreasing. Let $I(t;\mu)$ be the number of intersections of these two solutions on the interval $[0,1]$. By taking μ sufficiently large we can assume that

$$u_0(0) < u_*(0,0) \quad \text{and} \quad I(0,\mu) = 1.$$

As $u_*(1,t) < 0 \equiv u(1,t)$ it follows by the Maximum Principle (the Sturmian argument) that $I(t;\mu) \leq 1$ for $t > 0$. It now follows that

$$\sup u(x,t) \equiv u(0,t) \leq u_*(0,t),$$

as a contradiction of this fact would lead to them having different blow-up times. From this result we can immediately deduce the result of the lemma by taking

$$M_+ = u_*(0,t) + \log(T-t) \equiv \mu.$$

End of proof of Theorem 1.1. By using the estimates in Theorem 3.1 we can now establish the result given in Theorem 1.1. The proof relies upon the construction of a suitable (approximate) Lyapunov function for the parabolic equation satisfied by the rescaled function $\theta(y,s)$. See also [G]. There are several technical difficulties with this construction owing to the lack of regularity of the solution when $\sigma \neq 0$. We give here only the briefest outline of the proof.

The function $\theta(y,s)$ satisfies the parabolic equation

$$\theta_s = B(\theta) \equiv a(\theta_y)\theta_{yy} + b(y,\theta,\theta_y), \quad \theta(l(s),s) = -s, \tag{30}$$

where

$$a(w) = (\sigma+1)|w|^\sigma, \quad b(y,\theta,w) = -\frac{1}{\sigma+2}yw + e^\theta - 1, \quad l(s) = e^{s/(\sigma+2)}, \tag{31}$$

where $\theta(y,0) = \theta_0(y)$ is monotone decreasing in y. We have already established bounds for θ and related bounds can be determined for θ_s, indeed there is a constant C such that

$$\theta_y < 0, \quad |\theta_s| \leq C(1+y) \quad \text{if} \quad y < l. \tag{32}$$

To prove Theorem 1.1 it is now necessary to show that θ converges to a non-zero stationary solution of problem (30) which is also a solution of (5), (8).

Suppose now that for a given y_0, θ, w we consider the following initial value problem for $\phi(y_0, y, \theta, w)$

$$a(\phi_y)\phi_{yy} + b(y, \phi, \phi_y) = 0, \quad \phi(y_0) = \theta, \quad \phi_y(y_0) = w. \tag{33}$$

It is shown in [BDG] that if we define the following functions:

$$\rho(y, \theta, w) = \exp\left\{-\frac{1}{(\sigma+1)(\sigma+2)}\int_0^y z|\phi_z(y, z, \theta, w)|^{-\sigma} dz\right\}, \tag{34}$$

and

$$\Phi(y, \theta, w) = (\sigma+1)\int_0^w (w-\eta)|\eta|^\sigma \rho(y, \theta, \eta) d\eta + \int_v^{-1} \rho(y, \mu, 0)(e^\mu - 1) d\mu, \tag{35}$$

then the function

$$L[v](s) \equiv \int_0^{l(s)} \Phi(y, \theta, \theta_y) dy \tag{36}$$

is an approximate Lyapounov function for the system (30) which is almost non-increasing along the solution trajectory. (In fact L is constructed as a limit of a sequence of Lyapounov functions for the regularised problem. The technicalities behind this construction are covered in detail in [BDG].) By differentiating the function L with respect to s it can then be shown that if $s^* \gg 1$ is fixed and $S > s^*$ then

$$\int_{s^*}^S ds \int_0^{l(s)} \rho(y, \theta, \theta_y)(\theta_s)^2 dy \le D \tag{37}$$

where the constant D does not depend upon S.

If we replace $l(s)$ by L in (37), it is shown in [BDG] that for the solution $\theta(y, s)$ close to a non-trivial stationary one, $\rho(y, \theta, \theta_y)$ is bounded below by a constant $\rho_0 > 0$ so that

$$\int_{s^*}^S ds \int_0^L (\theta_s)^2 dy \le D/\rho_0. \tag{38}$$

As a consequence of this estimate, the equation (30) has an ω-limit set which is a subset of those functions satisfying (5) on the interval $[0, L]$. By making more careful estimates we can then extend this result to apply for all $L < \infty$.

Thus, the ω-limit set of (30) contains non-trivial solutions of the problem (5). The first of the estimates for the profile of $g(y)$ given in (7) follows immediately from (22). The second of the estimates follows from a rescaling of the inequality that $u_t > 0$ for t sufficiently close to T in the domain $\{u \gg 1\}$. The proof of this result is very similar to that given in [GP1], [GP2].

4 Asymptotic estimates for small σ

The previous section gives a proof of the existence of a monotone self-similar profile $\theta_1(y)$ but does not give much information abouts its spatial form. We will now derive an approximation to this for small (positive) σ by using a formal matching argument (with full details given in [BG2]). Surprisingly, the estimates we obtain for small σ seem (from numerical evidence) to work well for quite large values of σ. From the formal estimates we can then make interesting deductions about the behaviour of the corresponding function $u(x,t)$. To do this analysis we first establish the following property of the solution $\theta_1(y)$, which is strongly implied by careful numerical calculations. Given $\sigma > 0$, we denote $\theta_1(y;\sigma)$ a monotone solution with the minimal value $\theta_1(0;\sigma)$ at the origin.

Proposition 4.1. *As $\sigma \to 0^+$ then $\mu(\sigma) \equiv \theta_1(0;\sigma) \to 0$.*

Proof. We argue by contradiction. Assume that there exists a sequence $\{\sigma_k > 0\} \to 0$ such that $\mu(\sigma_k) \to \mu_* > 0$. Let $\mu_* < \infty$. Then the solution $\theta_1(y;0)$, with $\theta_1(0;0) = \mu_*$, is strictly monotone and is rapidly growing as $y \to \infty$, $\theta_1(y;0) \sim -e^{y^2/4}$ for $y \gg 1$. This is a straightforward consequence of the nonexistence result from [E]. By the continuous dependence of $\theta_1(y;\sigma)$ upon $\sigma \approx 0$ on compact subsets in y (which follows from Lemma 3.2 and monotonicity of $\theta_1(y;0)$, see also comments below) we then conclude, arguing as in the proof of Lemma 3.2 by comparison with the family $\{g_\alpha\}$ of lower solutions, that all the solutions $\theta_1(y;\sigma_k)$ with $\mu(\sigma_k) \approx \mu_*$ are rapidly growing (in the sense of Theorem 5.2 below). This contradicts the assumption.
If $\mu_\infty = \infty$ then we first rescale the equation setting $\bar\theta_k(\bar y) = \theta_1(y;\sigma_k) - \mu_k$, $\bar y = y e^{\mu_k/(\sigma_k+2)}$, and passing to the limit $\mu_k \to \infty$ we arrive at a similar conclusion for the stationary equation for the limit function $\bar\theta_\infty$.

It follows from this proposition and the continuous dependence upon the boundary values that on compact intervals in y the solution $\theta_1(y)$ will be small. Hence one can approximate the nonlinear function $e^{\theta_1} - 1$ by $\theta_1 + \theta_1^2/2 + O(\theta_1^3)$. It is this approximation which is the basis for our formal calculations.

We start by making the rescaling

$$\theta(y) = \mu v(z), \quad z = y\,\mu^{-\sigma/(\sigma+2)}. \tag{39}$$

With this rescaling, equation (5) becomes

$$(|v_z|^\sigma v_z)_z - \frac{z}{\sigma+2} v_z + \frac{e^{\mu v} - 1}{\mu} = 0, \quad v(0) = 1, \quad v_z(0) = 0. \tag{40}$$

The rescaled equation (40) can then be studied over two ranges: an *inner region* which comprises a compact interval over which $\mu v \ll 1$ and an *outer region* which includes the limit of $z \to \infty$. In this latter limit the function $\theta(y)$ must grow slowly to match the condition (8) and this forces the rescaled function $v(z)$ to have the form

$$\mu v(z) \sim D - (\sigma+2)\log(z), \tag{41}$$

where D is a suitable constant. The method of matched asymptotic expansions can be applied to study (40) as there is a *matching region* for which both of the conditions $\mu v \ll 1$ and $z \gg 1$ are valid.

The inner region. In this region (which includes the origin) we assume that μv is small. As σ is also small, it follows from Lemma 3.2 that the nonlinear equation (40) can be approximated to leading order by the linear equation

$$v_{zz} - \frac{z}{2}v_z + v = 0, \quad v(0) = 1, \quad v_z(0) = 0. \tag{42}$$

Observe that due to Lemma 3.2 there holds $|\theta_y|^\sigma = y^{\sigma/(\sigma+1)}(1 + o(1)) \to 1$ as $\sigma \to 0$ uniformly on subsets $[\delta, \epsilon_0]$, $\delta > 0$ is arbitrarily small. Since the derivative θ_y is strictly negative and uniformly (in $\sigma \geq 0$ small) bounded below, from (39) we derive the approximating linear operator given in (41).
The differential equation in (42) has two linearly independent solutions $\phi(z)$ and $\psi(z)$ such that $\phi(z)$ has the closed form

$$\phi(z) = 1 - \frac{z^2}{2} \tag{43}$$

and if z is large $\psi(z)$ can be approximated by:

$$\psi(z) \sim 4e^{z^2/4}\left(\frac{1}{z^3} + \frac{12}{z^5} + ...\right). \tag{44}$$

The function $\phi(z)$ matches both the boundary conditions for (40), (42) and grows slowly as $z \to \infty$ (in contrast to $\psi(z)$ which grows very rapidly in this limit). Thus $\phi(z)$ is an appropriate inner approximation to the solution $v(z)$ of (40). Hence, provided μz^2 is small

$$v(z) \approx 1 - \frac{z^2}{2}.$$

Now, if σ and μz^2 are both small then (40) can be regarded as a perturbation of (42). In particular, if we set $v(z) = \phi(z) + w(z)$ then to leading order:

$$w_{zz} - \frac{z}{2}w_z + w = \sigma f(z) + \mu g(z) + \mathcal{O}(\mu^2 + \sigma^2), w(0) = 0, \quad w_z(0) = 0, \tag{45}$$

where

$$f(z) = -(\log|\phi_z|\phi_z)_z - \frac{1}{4}z\phi_z, \quad g(z) = -\frac{\phi^2}{2}. \tag{46}$$

If we ignore the terms of order μ^2, σ^2 in (45) then the function $w(z)$ can be determined by solving a simple linear differential equation. The linearly independent solutions of (42), the function $w(z)$, can be determined by using the method of Variation of Constants. Indeed, a simple calculation shows that the Wronskian $W(z)$ of $\phi(z), \psi(z)$ is given by

$$W(z) = -e^{z^2/4},$$

and hence, $w(z)$ can be expressed in the form

$$w(z) = -A(z)\psi(z) + B(z)\phi(z). \tag{47}$$

Here

$$A(z) = \int_0^z e^{-s^2/4}[\sigma f(s) + \mu g(s)]\phi(s) ds$$

and

$$B(z) = \int_0^z e^{-s^2/4}[\sigma f(s) + \mu g(s)]\psi(s) ds.$$

An inspection of (44), (47) reveals that as z increases, then the contribution of $\psi(z)$ to the function $w(z)$ will (in general) force exponential growth. Indeed, if (40) is considered as an initial value problem in z then most of the solutions for almost all small $\mu > 0$ have rapid growth in the limit of large z. As we seek a solution of (40) with a special value of $\mu = \mu(\sigma) > 0$, which ultimately grows slowly, it is essential that the function $w(z)$ should also grow slowly as $z \to \infty$. This forces the condition

$$\lim_{z \to \infty} A(z) = 0. \tag{48}$$

The condition (48) is implied (to leading order) iff

$$\sigma \int_0^\infty e^{-s^2/4} f(s)\phi(s) ds + \mu \int_0^\infty e^{-s^2/4} g(s)\phi(s) ds = 0.$$

Both of these integrals can (remarkably) be expressed in terms of the error function and we deduce from them that the function $w(z)$ grows slowly for large z if to leading order

$$\mu = \frac{\sigma}{2}(1 + o(1)). \tag{49}$$

The result (49) gives a simple relation between μ and σ which agrees well with numerical calculations for values of σ as large as 1 and from these calculations appears to be an upper bound for larger values of σ. We observe, that if $\sigma = 0$ then the condition (49) forces $\mu = 0$ and that all solutions of (40) with non-zero μ will grow exponentially for large z. Provided that (49) is satisfied we may calculate the asymptotic form of $w(z)$ for large z. The details of this calculation are given in [BG2].

We conclude from this inner calculation that to leading order

$$v(z) = 1 - \frac{z^2}{2} + \mathcal{O}(\sigma z^2) \quad \text{if} \quad \mu = \frac{\sigma}{2} \quad \text{and} \quad z \ll \sigma^{-1/2}. \tag{50}$$

The outer region. In the range of $z \to \infty$ the functional form (41) imposes the condition that $(|v_z|^\sigma v_z)_z$ is small compared with the other terms in (40). If we consider this to be true for an outer region of values for z then (40) may be approximated to leading order by the first-order ordinary differential equation

$$-\frac{z v_z}{\sigma + 2} + \frac{e^{\mu v} - 1}{\mu} = 0. \tag{51}$$

This problem has the exact solution

$$v(z) = -\frac{1}{\mu} \log(1 + E z^{\sigma+2}), \tag{52}$$

where E is an arbitrary positive constant. For very large z, (52) gives

$$\mu v(z) \sim -(\sigma+2)\log(z) - \log(E),$$

so that from (41)

$$D = -\log(E).$$

If, in contrast, $Ez^{\sigma+2}$ is small then from (52) we deduce that to leading order

$$\mu v(z) = -Ez^{\sigma+2} = -Ez^2 + \mathcal{O}(\sigma E z^2 \log(z)). \tag{53}$$

The expression (53) is only a valid expression for $v(z)$ if, for this functional form of $v(z)$, $(|v_z|^\sigma v_z)_z$ is still small compared with the other terms in (40) so that (51) remains a good approximation to the original differential equation. For small σ, there holds $(|v_z|^\sigma v_z)_z \approx v_{zz}$ which in expression (53) is approximated by $-2E/\mu$. In comparison, the terms zv_z and $(e^{\mu v}-1)/\mu$ are both of order Ez^2/μ which dominates $-2E/\mu$ for large z^2. So the condition for (50) and (53) for (53) to be valid is simply that

$$z^2 \gg 1,$$

which is the appropriate outer region for z.

Matching. The expressions (50) and (53) are both valid descriptions of $v(z)$ in a *matching region* given by

$$1 \ll z \ll \sigma^{-1/2},$$

which is non-empty provided that σ is small. Comparing (53) with the inner expansion for $v(z)$ given in (50) we see that if σ is small and z is large then the leading order terms of the form z^2 match provided that

$$E = \frac{\mu}{2} = \frac{\sigma}{4}. \tag{54}$$

We note that as E is positive, then this matching is only possible if both μ and σ are positive. This strongly implies that self-similar solutions are only found if $\sigma > 0$.

More detailed arguments presented in [BG2] show that this matching can be made precise to higher orders than those given here.

Thus, if $\mu = \sigma/2$ and σ is small, then the function $\theta_1(y)$ can be described by rescaling $v(z)$. To simplify this rescaling we observe that for small σ

$$\sigma^{\sigma/(\sigma+2)} = 1 + \mathcal{O}(\sigma\log(\sigma))$$

and hence to leading order in σ we have

$$z \sim y.$$

Using this result we have:

$$\theta_1(y) \sim \frac{\sigma}{2}\left(1 - \frac{y^2}{2}\right) \quad \text{for} \quad y \ll \sigma^{-1/2}, \tag{55}$$

$$\theta_1(y) \sim -\log(1+\frac{\sigma}{4}y^{\sigma+2}) \quad \text{for} \quad 1 \ll y, \tag{56}$$

$$\theta_1(y) \sim -(\sigma+2)\log(y) + \log\left(\frac{4}{\sigma}\right) \quad \text{for} \quad y \gg \sigma^{-1/2}. \tag{57}$$

This gives the results quoted in Conjecture 1.3.

Comparison with the case $\sigma = 0$. The formal results above afford a comparison of the self-similar behaviour of the function $u(x,t)$ for $\sigma > 0$ with the approximately self-similar behaviour given when $\sigma = 0$. In particular we may compare the respective functions $\theta_1(y)$ and $\theta(y,s)$. The results corresponding to (55) and (56) are (17) and (19) respectively. We see from these results that the two results are comparable if in the limit of $\sigma \to 0$ we replace the constant σ by the time dependent term $1/|\log(T-t)|$, which tends to zero as $t \to T$.

If we now consider the limit function $u(x,T)$ then, in the case of self-similar evolution when $\sigma > 0$, we have for small x

$$u(x,T) \approx -(\sigma+2)\log|x| + \log\left(\frac{4}{\sigma}\right). \tag{58}$$

This expression can be compared directly with (20) for which the term $\log(4/\sigma)$ is replaced by $\log|\log|x||$.

5 A countable set of self-similar solutions

The exact and formal analysis of the last two sections has given important information about the monotone self-similar function described by $\theta_1(y)$. In fact, the problem (5), (8) has many other solutions provided that σ is sufficiently large. We now prove the existence of these, establish Theorem 1.5 and give supporting arguments for Conjecture 1.4. Our proof will be based upon comparison results and will make extensive use of phase-plane arguments.

We showed in Section 3 that if μ is sufficiently large then the solution $\theta(y)$ of (5) is monotone decreasing in y and in fact grows rapidly as $y \to \infty$. The plan of attack in this section is to look at solutions of (5) for which $\theta(0) \equiv \mu$ is small. If σ is sufficiently large, these solutions will be shown to have an arbitrary given number of oscillations. Thus there are critical values of μ for which the number of oscillations of the solutions of (5) changes and these are the *slowly growing* solutions which satisfy the growth condition (8). Indeed, numerical calculations of the self-similar solutions are made by slowly changing the value of μ and finding those values for which the number of oscillations (and hence behaviour at infinity) changes.

We start by making some general statements about the solutions of the initial value problem (5).

Theorem 5.1. *The problem (5) has two one-parameter forms of solution.* **P-solutions** *(parametrised by μ) which have the local form given by (27) and* **Q-solutions** *for which there is a constant $a > 0$ such that $\theta(y) \equiv 0$ if $y < a$ and as $y \to a^+$*

$$\theta_{\pm}(y) = \pm(\sigma+2)^{-1/\sigma}\left(\frac{\sigma}{\sigma+1}\right)^{(\sigma+1)/\sigma} a^{1/\sigma}(y-a)_+^{(\sigma+1)/\sigma}(1+o(1)). \tag{59}$$

Theorem 5.2. *As $y \to \infty$ the solutions of (5) behave in one of two ways:*
(i) **rapidly growing solutions** *such that*

$$\theta(y) \sim \left[\frac{1}{2(\sigma+1)}\left(\frac{1}{\sigma+1}\right)^{\sigma+1}\right]^{1/\sigma} y^{(\sigma+2)/\sigma}, \tag{60}$$

(ii) **slowly growing solutions** *such that there is a constant C for which*

$$\theta(y) \sim -(\sigma+2)\log(y) + C + o(1). \tag{61}$$

It is, of course, the slowly growing solutions which correspond to similarity solutions of the original PDE problem (1).

The proof of Theorem 5.1 makes use of simple local estimates and is given in [BG1]. (As a consequence of the proof it can also be shown that if $y_1 < y_2$ are points of successive maxima then $\theta(y_2) > \theta(y_1)$.) The proof of Theorem 5.2 is rather harder and (as in Section 3) is based on a comparison of $\theta(y)$ with the one parameter family g_α described in the proof of Lemma 3.2. The significant result here is (28) which implies that if $\theta(y)$ intersects the function g_{α_0} with some α_0 at a point y_1 then

$$\theta(y) \to -\infty \quad \text{as} \quad y \to \infty.$$

In [BG1] it is proved that such an intersection does indeed occur for all of the solutions described in Lemma 5.1 implying that all such solutions tend to $-\infty$. In this case, the term e^θ in (5) becomes a small perturbation to the simpler equation

$$(|\theta_y|^\sigma \theta_y)_y - \frac{y}{\sigma+2}\theta_y - 1 = 0.$$

This equation has the two solutions (60) and (61). The result in Lemma 5.2 then follows from the standard theory of perturbed dynamical systems.

Small Solutions. We now turn our attention to the study of those solutions of (5) which are *small* on a compact interval which includes the origin. Such solutions approximately satisfy the *linearised* equation

$$(|g_y|^\sigma g_y)_y - \frac{y}{\sigma+2}g_y + g = 0, \quad g_y(0) = 0. \tag{62}$$

A study of this equation is important for two reasons, firstly it gives important information about the oscillation properties of the solutions of (5) on compact intervals in y and secondly it is the equation satisfied by a leading order asymptotic approximation to $\theta(y)$ and we use its properties to generalise the asymptotic calculation in Section 4.

Although (62) is a nonlinear differential equation and does not seem much simpler than (5), it is most significant that its solutions are invariant under the following rescaling

$$g(y) \to \mu g(y\mu^{(\sigma+2)/\sigma}). \tag{63}$$

We can exploit this invariance by introducing new coordinates φ, p and s defined by

$$g(y) = y^{(\sigma+2)/\sigma}\varphi(s), \quad s = \log(y), \quad p = \varphi_s. \tag{64}$$

Substituting into (62) gives the following first-order differential equation

$$\frac{dp}{d\varphi} = -\frac{1}{(\sigma+1)p}\left[\frac{a\varphi - mp}{\left|\frac{(\sigma+2)}{\sigma}\varphi + p\right|^{\sigma}} + b\varphi + cp\right], \tag{65}$$

where

$$a = \frac{\sigma-1}{\sigma}, \quad m = \frac{1}{\sigma+2}, \quad b = \frac{2(\sigma+1)(\sigma+2)}{\sigma^2}, \quad c = \frac{(\sigma+1)(\sigma+4)}{\sigma}. \tag{66}$$

The equation (65) defines a set of trajectories in the (φ, p) phase-plane. Any two solutions of (62) related by the rescaling (63) will be mapped onto the same trajectory of (65).
Like its fully nonlinear counterpart, the system (62) admits P-solutions and Q-solutions which can be both slowly growing or rapidly growing as $y \to \infty$. These define related trajectories for (65). We state these as follows

Theorem 5.3. (Local behaviour.) *Up to rescaling, the system (62), (65) admits P-solutions of the form*

$$g(y) \approx 1 - \frac{\sigma+1}{\sigma+2}y^{(\sigma+2)/(\sigma+1)} \quad as \quad y \to 0 \tag{67}$$

or

$$p \approx -\frac{\sigma+2}{\sigma}\varphi - \varphi^{1/(\sigma+1)} \quad as \quad \varphi \to \infty, \quad s, p \to -\infty. \tag{68}$$

It also admits Q-solutions such that

$$g(y) \approx \pm(\sigma+2)^{-1/\sigma}(\sigma/(\sigma+1))^{(\sigma+1)/\sigma}(y-1)_+^{(\sigma+1)/\sigma} \quad as \quad y \to 1 \tag{69}$$

or

$$p \approx \pm m^{1/(\sigma+1)}\varphi^{1/(\sigma+1)} \quad as \quad p, \varphi \to 0, \quad s \to -\infty. \tag{70}$$

The solutions in (70) in fact describe the unstable manifold W_u of the origin for the equation (65).

Some simple local analysis of the phase plane for (65) identifies two possible asymptotic behaviours as $y, s \to \infty$. In particular, if $\sigma < 1$ the system (65) has precisely three singular points: the origin, π_+ and π_- where

$$\pi_\pm = (\pm\varphi_0, 0), \quad \varphi_0 = \frac{\sigma}{\sigma+2}(-a/b)^{1/\sigma}. \tag{71}$$

The points $\pi_\pm$ are stable spiral attractors if $\sigma < \frac{2}{3}$ and are unstable if $\sigma > \frac{2}{3}$, coalescing with the origin if $\sigma \geq 1$. They correspond directly to *rapidly growing* solutions of (62) of the form

$$g(y) = \pm\varphi_0 y^{(\sigma+2)/\sigma}. \tag{72}$$

If $\sigma < 1$ then the singular point at the origin has a stable manifold W_s which has the local form

$$W_s \equiv \{p \sim \frac{a}{m}\varphi + \alpha\varphi^{\sigma+1}\}. \tag{73}$$

This manifold corresponds directly to *slowly growing* solutions of (62) such that

$$g(y) \sim Ay^{\sigma+2} \quad \text{as} \quad y \to \infty, \tag{74}$$

where A can be an arbitrary constant. These slowly growing solutions have precisely the same form in y as the asymptotic expression for the solutions of (5) given in (53). This observation forms the basis of a further asymptotic calculation of the solutions of (5) when $\theta(y)$ is small on compact intervals and σ is close to one of the transition points σ_k.

For two special values of σ we may construct exact slowly growing solutions of (62) which are P-solutions and Q-solutions respectively

Lemma 5.4. *(a) If $\sigma = 0$ then (62) has a slowly growing P-solution of the form*

$$g(y) = 1 - \frac{y^2}{2}, \tag{75}$$

(b) if $\sigma = \phi \equiv (\sqrt{5}-1)/2$ then (62) has a slowly growing Q-solution of the form

$$g(y) = \pm B(y-1)_+^{\phi+2}, \quad B = (\phi+1)^{-2/\phi}(\phi+2)^{-(\phi+1)/\phi}, \tag{76}$$

where the latter solution corresponds precisely to a homoclinic orbit of (65) connecting the origin to itself.

These results follow immediately by substituting (75), (76) into the differential equation (62). Although easy to prove they in fact lie at the heart of the analysis of the system (5) and are the key to the special importance of the two values of $\sigma = 0$ and $\sigma = \phi$.

From numerical calculations, it is evident that there are a series of values of σ at which there is a slowly-growing P-solution of (62) with precisely k zeros. These values are precisely the transition points σ_k so that $\sigma_1 = 0$.

We now state some basic qualitative features of the geometry of the phase plane of the equation (65).

1. There is a positive σ_∞ such that if $0 < \sigma < \sigma_\infty < \phi$ then equation (65) has no limit cycles. All trajectories tend either to the origin or to one of the singular points $\pi_\pm$.

2. At precisely the points $\sigma = \sigma_k < \sigma_\infty$ the P-solutions with the local form given in (68) also form the stable manifold of the origin.

3. At $\sigma = \sigma_\infty$ a stable and an unstable limit cycle are created in a saddle-node bifurcation. When σ is slightly greater than σ_∞ then both limit cycles enclose the origin and the unstable limit cycle lies inside the stable limit cycle.

4. The stable limit cycle persists for all $\sigma > \sigma_\infty$ and continues to enclose the origin. Any P-solution is attracted onto this trajectory.

5. As σ increases, the unstable limit cycle shrinks and becomes the homoclinic orbit (76) when $\sigma = \phi$. For $\sigma < \phi$ the unstable manifolds of the origin are attracted toward the points $\pi_\pm$, and for $\sigma > \phi$ onto the limit-cycle.

6. For $\phi < \sigma < \frac{2}{3}$ the unstable limit cycle divides into two separate unstable limit cycles enclosing the points $\pi_\pm$ and for $\sigma \to \frac{2}{3}$ the limit cycles contract onto the singular points.

The existence of a limit cycle. The proof of Theorem 1.5 follows (in part) from a proof of the results 3. and 4. above describing the existence of a stable limit cycle. This follows from an application of the Poincaré-Bendixson Theorem and relies on the construction of an invariant region in the phase-space given by the following lemma.

Lemma 5.5. *Let $\alpha = 1 + 1/\sigma$ and $B^\sigma = [1 - (\sigma + 2 - \alpha)]/[\alpha^{(\sigma+1)}(\sigma+1)(\alpha-1)]$. Further, let Γ be the curve in phase-space given by rescaling the function*

$$G(y) = \pm B(y-1)_+^\alpha \tag{77}$$

in terms of p and φ, so that Γ is a closed curve which includes the origin. Then, if $\mathcal{R}$ is the interior of Γ

- *if $\sigma < \phi$ then any trajectory starting on Γ enters $\mathcal{R}$,*
- *if $\sigma = \phi$ then Γ is precisely the homoclinic orbit (76),*
- *if $\sigma > \phi$ then any trajectory starting on Γ leaves $\mathcal{R}$.*

The region $\mathcal{R}$ also determines the local behaviour of the stable and unstable manifolds of the origin.

Lemma 5.6. *If $\sigma > \phi$ $\quad(\sigma < \phi)$ then the stable manifold of the origin lies inside (outside) $\mathcal{R}$ and the unstable manifold of the origin lies outside (inside) $\mathcal{R}$.*

The proof of these results uses the Maximum Principle and local estimates of the orbits close to the origin. It is given in detail in [BG1].

We may now prove the following

Theorem 5.7. *(i) There is a unique value of $\sigma > 0$, namely $\sigma = \phi$, for which the origin lies on a homoclinic orbit.*
(ii) There is a positive $\sigma_\infty < \phi$ such that for $\sigma > \sigma_\infty$ a stable limit cycle exists which encloses the origin.

Proof. (i) This result follows from Lemma 5.6 and the invariance of the region $\mathcal{R}$. To prove (ii) on existence of the limit cycle we observe that if $\sigma > \phi$ then all trajectories leave $\mathcal{R}$ and do not enter the origin. Some simple estimates (also given in [BG1]) demonstrate that there is a (large) region $\mathcal{V}$ enclosing the origin into which all trajectories enter. Thus, by the Poincaré-Bendixson Theorem, a stable limit cycle exists for $\sigma > \phi$ and by continuity arguments for σ smaller than, but close to, ϕ. Indeed, it is destroyed at σ_∞ when it coalesces with the unstable limit cycle.

We can now deduce some important properties of the solutions of equation (62).

Corollary 5.8. *(a) If $\sigma > \sigma_\infty$ then any P-solution of (62) must oscillate about zero an infinite number of times (with oscillations of increasing size) as $y \to \infty$.*
(b) If $\sigma > \phi$ then any Q-solution of (62) must also oscillate an infinite number of times.
(c) If $\sigma < \phi$ then no Q-solution has a zero for $y > a$.

Note. We observe from simple continuity arguments that by taking σ sufficiently close to (but smaller than) σ_∞ that we can obtain a corresponding P-solution which will oscillate an arbitrarily large number of times.

Proof. (a) For $\sigma > \sigma_\infty$ at least one stable limit cycle exists which encloses all the singular points. Thus any trajectory which is unbounded as $s \to \infty$ (such as a P-solution) must oscillate an infinite number of times about the origin.
(b) If $\sigma > \phi$ then the unstable manifold of the origin (a Q-solution) lies outside the region $\mathcal{R}$ and is hence attracted onto the stable limit cycle and oscillates about the origin.
(c) If $\sigma < \phi$ then the unstable manifold of the origin lies wholly inside $\mathcal{R}$ and is eventually attracted onto one of the singular points $\pi_\pm$ without ever crossing the line $\varphi = 0$.

The behaviour of solutions for $\sigma < \sigma_\infty$.

The behaviour of the solutions in this case is less clear. For $\sigma > 0$ it appears from numerical experiments, that the number of zeros $N(\sigma)$ of the P-solutions of (62) is an *increasing* function of σ which changes by one as σ passes through each of the transition points σ_k so that

$$N(\sigma) = k + 1 \quad \text{if} \quad \sigma_k < \sigma < \sigma_{k+1}. \tag{78}$$

A complete proof of (78) is still open. However the following result is proved in [BG1]

Lemma 5.9. *The following hold:*
(i) $N(0) = 1$,
(ii) $N(\sigma) > 1$ *if* $\sigma > 0$,
(iii) $N(\sigma) = 2$ *if* σ *is close to* 0.

The proof of this lemma is similar in spirit to that of the existence of the limit cycle and involves the construction of a further invariant region in phase-space.

Estimates of the values of the transition points σ_k can be made numerically and we find that

$$\sigma_1 = 0, \quad \sigma_2 = 0.49..., \quad \sigma_3 = 0.57..., \quad ..., \sigma_\infty = 0.60... .$$

The existence of P-solutions. We now prove Theorem 1.5 (i).

If $\sigma > \sigma_\infty$ then the P-solution of (62) oscillates an infinite number of times. Thus, if $[0, Y]$ is a compact interval and μ is sufficiently small it follows by continuity that the solution $\theta(y)$ of (5) can have arbitrarily many zeros. In contrast, if μ is sufficiently large, then $\theta(y)$ is monotone and has only one zero on $[0, \infty)$. The values of μ at which the number of zeros of $\theta(y)$ on $[0, \infty)$ changes are precisely those for which (5) has *slowly growing* solutions.

To make this more precise we introduce two non-intersecting sets of P-solutions of (5):

$$B_* = \{\mu \neq 0 : \theta(y) \quad \text{satisfies} \quad (60)\}, \tag{79}$$

$$G_* = \{\mu \neq 0 : \theta(y) \quad \text{satisfies} \quad (61)\}. \tag{80}$$

We have already shown that for any $\sigma > 0$ there exists a constant $\mu_+ > 0$ such that $(\mu_+, \infty) \subset B_*$. In [BG1] this result is extended (by making use of similar comparison arguments) to show that there is a second constant $\mu_- < 0$ such that $(-\infty, \mu_-) \subset B_*$ with $\theta(y)$ having exactly one maxima if $\mu < \mu_-$.

We now introduce two further sets of values for μ:

$$W_{2m-1} = \{\mu > 0 : \theta(y) \quad \text{has at least} \quad 2m+1 \quad \text{maxima and minima for} \quad y \geq 0\}, \tag{81}$$

$$W_{2m} = \{\mu < 0 : \theta(y) \quad \text{has at least} \quad 2m+2 \quad \text{maxima and minima for} \quad y \geq 0\}. \tag{82}$$

Our previous results have shown that if $\sigma > \sigma_\infty$ then

$$W_k \neq \emptyset \quad \text{for any} \quad k = 1, 2, \ldots . \tag{83}$$

Moreover, we can also conclude from Lemma 5.9 that W_1 is non-empty for any $\sigma > 0$, and by continuity that W_k will be non-empty for arbitrarily large k as $\sigma \to \sigma_\infty$ from below. We conjecture that W_k is non-empty iff $\sigma > \sigma_k$.

The results on the behaviour of $\theta(y)$ if $|\mu|$ is large demonstrate the existence of values μ_k such that

$$\mu_{2m-1} = \sup W_{2m-1} \quad \text{and} \quad \mu_{2m} = \inf W_{2m}. \tag{84}$$

The following result now follows by simple continuity arguments.

Lemma 5.10. *For any $k = 1, 2, \ldots$*

$$\theta(y; \mu_k) \quad \textit{has exactly} \quad k \quad \textit{maxima and minima for} \quad y \geq 0. \tag{85}$$

To complete the proof we apply the following lemma

Lemma 5.11. *(i) The set B_* is open and if $\mu_* \in B_*$ then the function $\theta(y;\mu)$ has a constant number of maxima and minima in a neighbourhood of μ_*.*
(ii) The set G_* is closed and $\mu_k \in G_*$ for each k.

Proof. (i) If $\mu_* \in B_*$ then $\theta(y;\mu_*)$ satisfies (60) and hence there exists an α such that (28) is true for all y greater than some y_1. For μ in a neighbourhood of μ_* (28) will remain valid for slightly different values of y_1 and α. In this neighbourhood $\theta(y;\mu)$ cannot satisfy (61) and hence satisfies (60) so that $\mu \in B_*$. Hence B_* is open. Now, for $y > y_1, \theta(y;\mu_*)$ is strictly monotone with no maxima or minima, and this property will be shared by $\theta(y;\mu)$. By continuity the number of maxima/minima of $\theta(y;\mu)$ must remain constant in the interval $[0, y_1]$ for μ in a neighbourhood of μ_* and hence the number of maxima/minima is constant over the whole real line.

(ii) The set G_* is the complement of B_* and hence is closed. As μ_k is the sup (inf) of the set W_k it follows from (i) and Lemma 5.9 that μ_k cannot lie in the open set B_* (as the number of maxima/minima of $\theta(y;\mu)$ changes in a neighbourhood of μ_k). Thus $\mu_k \in G_*$.

From (ii) we immediately deduce that $\theta(y;\mu_k)$ satisfies (61) and hence is the desired similarity solution. We have thus established the following

Lemma 5.12.
(a) If $\sigma > \sigma_\infty$ then for all k there is a slowly growing solution of (5) with k zeros for $y > 0$.
(b) If $\sigma > 0$ then there is a slowly growing solution with one zero.
(c) As $\sigma \to \sigma_\infty$ from below then there will exist slowly growing solutions with an increasing number of zeros.

The proof of this result is now immediate using the result (83) and the comments following. The formal results (i) and (ii) in Conjecture 1.4 are then related to the conjecture (77) about the behaviour of $N(\sigma)$ and the consequent implications on the sets W_k.

Existence of the Q-solutions. The proof of Theorem 1.5 (ii) in this case is very similar to that for the P-solutions and relies on the observation in Corollary 5.8 that the Q-solutions of the linearised problem (62) oscillate about the origin iff $\sigma > \phi$. The proof breaks down if $\sigma < \phi$ and the numerical evidence strongly indicates that Q-solutions do not exist in this range.

The Q-solutions of (5) are characterised by the interval $[0, a]$ over which each is identically zero. The value of a determines the behaviour of the solutions and (roughly) a large (small) value of a corresponds to a large (small) value of μ for the P-solutions.

There are a number of results about the Q-solutions which mimic those for the P-solutions and are also proved by using comparison arguments. We present these without proof - with further details given in [BG1]. The solutions $\theta_\pm(y; a)$ were defined in Theorem 5.1.

Lemma 5.13. *Let*

$$B_*^\pm = \{a : \theta_\pm(y; a) \quad \textit{is a Q-solution satisfying} \quad (60)\},$$
$$G_*^\pm = \{a : \theta_\pm(y; a) \quad \textit{is a Q-solution satisfying} \quad (61)\}.$$

Then

- *For any $\sigma > 0$ there exists a large constant a_* such that $(a_*, \infty) \subset B_*^\pm$; $\theta_-(y; a)$ is strictly decreasing and $\theta_+(y; a)$ has exactly one maxima, if $a > a_*$.*
- *If $\sigma > \phi$ then for all k there exists a sufficiently small a for which $\theta_\pm(y; a)$ has at least k maxima/minima.*
- *The sets $B_*^\pm$ are open.*

We can now define sets $W_k^\pm$ similar to (81), (82) with μ "replaced" by a, and calculate

$$a_{2m-1} = \sup W_{2m-1}^+, \quad a_{2m} = \sup W_{2m}^-.$$

If $\sigma > \phi$ then these values of a_k exist and by the same reasoning as for the P-solutions the functions

$$\theta_+(y; a_{2m-1}) \quad \text{and} \quad \theta_-(y; a_{2m})$$

are slowly-growing as in (61) and hence are similarity solutions.

This concludes the proof of Theorem 1.5.

Asymptotics for σ close to σ_k and σ close to ϕ. The asymptotics for the solutions for σ close to σ_k is very similar to that described in Section 4 for σ small. In this case the inner form of the solutions takes the form

$$v(z) = (-1)^{k-1} g(z),$$

where $g(z)$ is the slowly growing P-solution of (62) such that $g(0) = 0$, $g(z)$ has k zeros and there is a positive constant E such that $g(z) \sim (-1)^k E z^{\sigma+2}$ as $z \to \infty$. Such a solution $g(z)$

of (62) exists only when $\sigma = \sigma_k$ and corresponds to a P-solution in the phase-space which is also the stable manifold of the origin. The sign of $v(z)$ ensures that $v(z) \sim -Ez^{\sigma+2}$ for large z which then matches with the expression (53). The value of μ is proportional to $\sigma - \sigma_k$, and can be calculated by using very similar integral estimates to those in Section 4.

If σ is close to ϕ then the solution of (5) is a rescaling of the exact solution given in (76). In particular, if we consider a Q-solution $\theta(y)$ which is zero on the interval $[0, a]$ then setting

$$\mu^{\sigma/(\sigma+2)} = a,$$

the rescaled function $v(z)$ satisfies (40) with $v(z) = 0$ on the interval $[0, 1]$. In the inner region, $v(z)$ is then given by (76) to leading order. By similar arguments to those given before, we have μ proportional to $\sigma - \phi$ and deduce that as $\sigma \to \phi$ there is a constant A such that

$$a = A(\sigma - \phi)^{\phi/(\phi+2)}. \tag{86}$$

This asymptotic estimate agrees very closely with numerical calculations of the Q-solutions.

6 Approximately self-similar solutions for $0 < \sigma < \sigma_\infty$

We have now established that if $\sigma > \sigma_\infty$ then there is a countable set P of self-similar solutions with an arbitrary given number of maxima and minima (and zeros). We can then expect that the solutions from P exhaust all possible forms of asymptotic blow-up behaviour under the assumption that in the rescaled self-similar variable no flat part near the origin exists. (We also know that if $\sigma > \phi$ then there exists another set Q of solutions which are flat at the origin and that combining different solutions for $x > 0$ and $x < 0$ can give solutions with a non-symmetric behaviour as $t \to T$.)

We now fix an arbitrary $\sigma \in (0, \sigma_\infty)$ and using formal arguments, consider the possible forms of the evolution of the solution.

The nonlinear spectral problem in the inner region. We begin by rewriting the original equation (1) in the self-similar variables (5) to give

$$\theta_s = B(\theta) \equiv (|\theta_y|^\sigma \theta_y)_y - \frac{y}{\sigma+2}\theta_y + e^\theta - 1, \quad s = -\log(T-t). \tag{87}$$

Here the operator B is as in (5) and we look for solutions satisfying the same conditions as (8). The self-similar solutions constructed earlier are (stationary) non-trivial fixed points of (87) as $s \to \infty$. In contrast, we now study solutions of (87) which decay so that

$$\theta(y, s) \to 0 \quad \text{as} \quad s \to \infty \quad \text{on compact subsets in } y. \tag{88}$$

The main problem now is to determine all possible rates of decay in (88). In doing so we linearise equation (87) so that if $s \gg 1$ then on compact sets in y we have

$$\theta_s = D(\theta) + \frac{1}{2}\theta^2 + \mathcal{O}(\theta^3), \tag{89}$$

where $D(\theta)$ is the *linearised* (but still nonlinear) operator already defined in (61)

$$D(\theta) = (|\theta_y|^\sigma \theta_y)_y - \frac{y}{\sigma+2}\theta_y + \theta. \tag{90}$$

If we only consider the leading order terms in (89) we arrive at the non-stationary linearised equation

$$g_s = D(g) \quad \text{for} \quad s \gg 1. \tag{91}$$

The behaviour of the form (88) is expected to be given by a set of self-similar solutions of equation (91) of the form

$$g(y,s) = e^{-\lambda s} w(\eta), \quad \eta = y\, e^{\sigma \lambda s/(\sigma+2)}, \tag{92}$$

where $\lambda \geq -1$ is a constant (the eigenvalue) and $w(\eta)$ (the corresponding eigenfunction) solves the equation

$$D(w) = -\lambda(w + \frac{\sigma}{\sigma+2}\eta w_\eta) \quad \text{for} \quad \eta > 0, \tag{93}$$

which is derived by substituting (92) into (91). In order to make it possible to match the behaviour of (92) with the corresponding one in the *outer region* (see below) we impose the restriction

$$|w(\eta)| \quad \text{has slow growth as} \quad \eta \to \infty. \tag{94}$$

In this case (93),(94) form a natural eigenvalue problem when augmented with the condition that $w_\eta = 0$.

Expanding the eigenvalue equation (93) we obtain

$$(|w_\eta|^\sigma w_\eta)_\eta - \frac{1+\lambda\sigma}{\sigma+2}\eta w_\eta + (1+\lambda)w = 0. \tag{95}$$

If we now set

$$h(\eta) = (1+\lambda)^{-1/\sigma} w(\eta) \tag{96}$$

then this reduces to the standard form

$$(|h_\eta|^\sigma h_\eta)_\eta - m\eta h_\eta + h = 0 \tag{97}$$

where

$$m \equiv m(\sigma,\lambda) = \frac{1+\lambda\sigma}{(\sigma+2)(1+\lambda)}. \tag{98}$$

If $\lambda = 0$ then this is precisely the linearised equation (61) studied in Section 5. This equation has *slowly growing* solutions for which

$$|w(\eta)| = C\eta^{1/m}(1+o(1)) \quad \text{as} \quad \eta \to \infty. \tag{99}$$

Similarly it has *rapidly growing* solutions such that

$$|w(\eta)| \sim D\eta^{(\sigma+2)/\sigma}. \tag{100}$$

Using the same phase-plane analysis as in Section 5 we arrive at a similar result on the existence of a countable set of eigenfunctions $\{w_k, \lambda = \lambda_k\}$. Before stating this it is significant that there exist explicit expressions for the first two eigenfunctions. Namely, we have

$$\lambda_0 = -1, \quad w_0(\eta) = 1, \tag{101}$$

and the more complicated second eigenfunction

$$\lambda_1 = -\sigma, \quad w_1(\eta) = (1-\sigma)^{1/\sigma}\left(1 - \frac{\sigma+1}{\sigma+2}\eta^{\frac{\sigma+2}{\sigma+1}}\right). \tag{102}$$

The value of λ_1 corresponds to

$$m \equiv m(\sigma) = \frac{\sigma+1}{\sigma+2}.$$

We observe that when $\sigma > 0$ both eigenvalues are negative, and correspond to a subset of the unstable manifold of the trivial solution of (91).

We now state our main result.

Proposition 6.1. *For each fixed $\sigma \in (0, \sigma_\infty)$ there exists a monotone increasing sequence $\{\lambda_k = \lambda_k(\sigma)\}$ such that*

$$\lambda_k(\sigma) \to \lambda_\infty(\sigma) < \frac{1-\sigma-\sigma^2}{\sigma} \quad \textit{as} \quad j \to \infty, \tag{103}$$

such that the corresponding eigenfunctions $\{w_k(\eta)\}$ satisfying (93),(99) have exactly k maxima and minima for $\eta \geq 0$. There is a corresponding decreasing sequence

$$m_k(\sigma) \to m_\infty(\sigma) > \sigma/(\sigma+1).$$

Remark. The phase-plane analysis shows that slowly growing Q-solutions do not exist so that such approximate self-similar solutions cannot be constructed.

Proof. A full proof of this result is given in [BG3] however we can motivate exactly as in Section 4. In the $(\sigma, m)-$plane the curve $m = 1/(\sigma+2)$ corresponds to $\lambda = 0$. When

$$m \equiv m_{hom} = \sigma/(\sigma+1), \quad \lambda \equiv \lambda_{hom} = \frac{1-\sigma-\sigma^2}{\sigma}$$

the equation (97) has a slowly growing *homoclinic* solution (in the appropriate phase-plane) exactly as in Section 5 which takes the form

$$w(\eta) = C(\eta - 1)_+^{1/m}.$$

A limit cycle is then created at some $m_\infty > m_{hom}$ corresponding to λ_∞. From the existence of this limit cycle we infer the existence of the infinity of values λ_k.

For those values $m_k(\sigma) < 1/(\sigma+2)$ the corresponding λ_k are positive and describe the *stability* set of the trivial solution of (91). The modes corresponding to $m_k > 1/(\sigma+2)$ are unstable. If $\sigma \in (\sigma_k, \sigma_{k+1})$ the problem (86) has the k different steady solutions θ_k which form attractors for the resulting unstable modes. The curves $m_k(\sigma)$ intersect the curve $1/(\sigma+2)$ precisely when $\sigma = \sigma_k$ marking again that these are transition points between self-similar and approximately self-similar forms of behaviour.

A comparison with the spectrum of the Hermite operator for $\sigma = 0$. If we consider the curves $m_k(\sigma)$ as functions of σ then when $\sigma = 0$ they may be computed in terms of a standard eigenvalue problem for the linear second order equation

$$A_1 w \equiv w_{\eta\eta} - \frac{1}{2}\eta w_\eta + w = -\lambda w, \tag{104}$$

which can be written in divergence form using the weight function $\rho(\eta) = e^{-\eta^2/4}$:

$$\frac{1}{\rho}(\rho w_\eta)_\eta + w = -\lambda w. \tag{105}$$

The operator A_1 is self-adjoint in the space H^1_ρ weighted by ρ and plays an important role in the analysis in [FK], [GHV], [HV1], [HV2]. It has eigenvalues $\lambda_k = k - 1,\ k = 0, 1, ..,$ so that (since (95) makes no sense for $\sigma = 0$, we rescale the space variable by setting $\sqrt{1+\lambda}\eta \to \eta$)

$$m_k(0) = \frac{1}{2(1+\lambda_k)} = \frac{1}{2k}, \quad k = 1, 2, \dots. \tag{106}$$

The corresponding eigenfunctions are the even Hermite polynomials H_{2k}.

Power law decay in the resonance cases $\sigma = \sigma_k$. If $\lambda_k(\sigma) \neq 0$ for all j then the spectrum is bounded away from zero (the fixed point is hyperbolic) and all possible types of stable behaviour are governed by the approximately self-similar solutions (92).

We now consider, formally, the resonance case of $\lambda_k(\sigma) = 0$, i.e., $\sigma = \sigma_k$. In the simplest case we have $k = 1$ so that $\sigma = \sigma_1 = 0$. This leads immediately to the case considered in Section 2. Let us now presume that $\sigma_k > 0$. Then using the natural similarity invariance of the operator D in (89) we perform the following transformation (recall that $\eta = y$ if $\lambda = 0$ in (91))

$$f(\eta, s) = a(s) w(\xi, s), \quad \xi = (a(s))^{-\sigma/(\sigma+2)}\,\eta, \tag{107}$$

where

$$a(s) = f(0, s) \to 0 \quad \text{as} \quad s \to \infty. \tag{108}$$

Then the function w solves the equation (where from hereon we omit higher order terms)

$$w_s = D(w) + \frac{a_s}{a}\left(\frac{\sigma}{\sigma+2}\xi w_\xi - w\right) + \frac{1}{2}aw^2 + ..., \tag{109}$$

which looks like an asymptotically small perturbation of the autonomous (in time s) equation (91). Since after the transformation (107) the problem is reduced to the study of the stabilisation

$$w(\xi, s) \to w_k(\xi) \quad \text{as} \quad s \to \infty \tag{110}$$

we perform a standard linearisation procedure setting $w(s) = w_k + Y(s)$. Recall that w_k solves the stationary equation $D(w_k) = 0$. This yields the equation for the perturbation

$$Y_s = A_k Y + \frac{a_s}{a}\left(\frac{\sigma}{\sigma+2}\xi(w_k)_\xi - w_k\right) + \frac{1}{2}aw_k^2 + \dots, \tag{111}$$

where A_k is the linear operator

$$A_k Y = p(\xi)Y_{\xi\xi} + q(\xi)Y_\xi + Y, \quad Y_\xi(0) = 0, \tag{112}$$

with the coefficients

$$p(\xi) = (\sigma+1)|w_k'|^\sigma, \tag{113}$$

$$q(\xi) = \sigma(\sigma+1)\frac{|w_k'|^\sigma}{w_k'}w_k'' - \frac{\xi}{\sigma+2} \equiv \frac{\sigma-1}{\sigma+2}\xi - \sigma\frac{w_k}{w_k'}. \tag{114}$$

One can see that if $k = 1$ then (112) yields the Hermite operator in (104).

The operator A_k reduces to divergence form and can be treated as a symmetric operator in a weighted space $L^2_\rho(\mathbf{R}_+)$, with a weighting function $\rho(\xi) \geq 0$ which is uniquely determined by the coefficients $p(\xi)$ and $q(\xi)$. We now prove the most important property of the operator A_k.

Proposition 6.2. *For any $k \geq 1$ the singular point $\xi = \infty$ is in the limit-point case for the operator A_k in $L^2_\rho(\mathbf{R}_+)$.*

Proof. For $k = 1$ when A_1 takes the Hermite form (103) the result is classical. If $k \geq 2$ then one can calculate from (112), (113) using the slow growth condition

$$w_k(\xi) = -c\,\xi^{\sigma+2}(1 + o(1)), \quad \xi \to \infty,$$

where c denotes different positive constants, that for $\xi \gg 1$

$$p(\xi) = c\,\xi^{\sigma(\sigma+1)}(1 + o(1)), \quad q(\xi) = -\frac{\xi}{\sigma+2}(1 + o(1)).$$

Therefore the approximating operator takes the form (after rescaling in ξ)

$$\tilde{A}_k Y = \xi^{\sigma(\sigma+1)}Y'' - \frac{\xi}{\sigma+2}Y' + Y.$$

The corresponding rapidly (exponentially fast) growing solution

$$Y \sim O(\exp\{c\xi^{2-\sigma(\sigma+1)}\}), \quad \xi \gg 1,$$

(recall that $2 - \sigma(\sigma+1) > 1$ if $\sigma < \sigma_\infty < \phi$) does not belong to the corresponding space $L^2_{\tilde{\rho}}$. Thus $\xi = \infty$ is in the limit-point case of a singular endpoint [LS]. Since this conclusion is true after small perturbations of the coefficients, the result holds for the operator A_k.

The operator A_k admits other singular points where $p(\xi) = 0$, i.e., those degeneracy points where $w_k'(\xi) = 0$. In a small neighbourhood of such points, say $\xi = \xi_0$, there hold $p(\xi) = (\sigma+1)|\xi - \xi_0|^{\sigma/(\sigma+1)}(1 + o(1))$ and $q(\xi) = \sigma|\xi - \xi_0|^{-1/(\sigma+1)}\text{sign}(\xi - \xi_0)(1 + o(1))$.

We first consider in more detail the most delicate case $\xi_0 = 0$ where the approximating operator takes the form

$$\bar{A}_k Y = (\sigma+1)\xi^{\sigma/(\sigma+1)} Y'' + \sigma \xi^{-1/(\sigma+1)} Y' + Y.$$

with the symmetry condition $Y'(0) = 0$. The corresponding givergence form

$$\bar{A}_k Y = \frac{1}{\bar{\rho}}\left(Y'' + \frac{N-1}{\xi} Y'\right) + Y \equiv \frac{1}{\bar{\rho}}\frac{1}{\xi^{N-1}}(\xi^{N-1}Y')' + Y,$$

where $\bar{\rho}(\xi) = (\sigma+1)^{-1}\xi^{-\sigma/(\sigma+1)}$, contains the radial symmetric Laplace-like operator Δ_N in $L^2_{\bar{\rho}}(\mathbf{R}^N)$ with the non-integer dimension $N = (2\sigma+1)/(\sigma+1) \in (1,2)$. The equation $\Delta_N Y = 0$ admits a unique linearly independent regular nontrivial solution $Y_1 = 1$. The second solution $Y_2 = \xi^{2-N} = \xi^{1/(\sigma+1)} \in H^1_{\bar{\rho}}$ does not satisfy the symmetry condition. Thus, the origin $\xi = 0$ (as well as $\xi = \infty$) is in the limit-point case.
Consider a finite $\xi_0 > 0$. Then replacing $\xi - \xi_0 > 0$ by ξ (the case $\xi - \xi_0 < 0$ is similar) we again arrive at the approximating operator $\bar{A}_k$. Since no symmetry condition is now imposed at $\xi = 0$, this singular point is in the limit-circle case and it can be treated as a regular one. The uniqueness of the continuation through this point is guaranteed by the condition $(\bar{\rho})^{-1}\Delta_N Y \in L^2_{\bar{\rho}}$, where $L^2_{\bar{\rho}}$ means the weighted space of functions defined for both $\xi > 0$ ($\xi - \xi_0 > 0$) and $\xi < 0$ ($\xi - \xi_0 < 0$). See [LS].
The above conclusions make it possible to deduce that A_k admits a self-adjoint extension in the weighted space $L^2_\rho(\mathbf{R}_+)$ with a *discrete* spectrum $\{\nu_j\}$ and with the corresponding orthonormal basis of eigenfunctions $\left\{\widetilde{H}_{2j}(\xi)\right\}$. Then using a standard approach for the asymptotic expansion we set

$$Y(\xi,s) = \sum_{j=1}^{\infty} b_{2j}(s)\widetilde{H}_{2j}(\xi) \quad \text{for} \quad s \gg 1. \tag{115}$$

Substituting (114) into (110) and multiplying by $\widetilde{H}_2(\xi)$ in L^2_ρ and using the symmetry of A_k we find that the first coefficient $b_2(s)$ satisfies

$$b_2' = \frac{a_s}{a}\gamma_1 + a\gamma_2 + \dots, \tag{116}$$

where

$$\gamma_1 = \frac{\sigma}{\sigma+2}\left(\xi w_k', \widetilde{H}_2\right) - (w_k, \widetilde{H}_2), \quad \gamma_2 = \frac{1}{2}\left(w_k^2, \widetilde{H}_2\right), \tag{117}$$

and the brackets denote the inner product in L^2_ρ. By our assumptions, $b_2(s) \to 0$ as $s \to \infty$ and thus $a(s)$ must satisfy the equation

$$a_s = -\frac{\gamma_2}{\gamma_1}a^2 + \dots \quad \text{for} \quad s \gg 1, \tag{118}$$

which formally yields

$$a(s) = \frac{\gamma_1}{\gamma_2}\frac{1}{s}(1 + o(1)) \quad \text{as} \quad s \to \infty. \tag{119}$$

Thus in the stable set of the rescaled profile $w_k(\xi)$, the asymptotics for large s is given by (107), (119), i.e., as $s \to \infty$

$$f(\eta, s) = \frac{\delta}{s} w_k \left(\delta^{-\sigma/(\sigma+2)} s^{\sigma/(\sigma+2)} \eta \right) + \dots , \tag{120}$$

where

$$\delta = \frac{\gamma_1}{\gamma_2}.$$

Finally we conclude that in the *inner region* the behaviour of the solution as $t \to T$ is given by:

(i) In the nonresonance case, $\lambda_k(\sigma) \neq 0$,

$$u(x,t) = -\log(T-t) + (T-t)^{\lambda_k} w_k(\eta) + \dots , \tag{121}$$

with

$$\eta = y\, e^{\sigma \lambda_k s/(\sigma+2)} \equiv x(T-t)^{-(1+\sigma\lambda_k)/(\sigma+2)}. \tag{122}$$

(ii) In the resonance case $\lambda_k = 0$ $(\sigma = \sigma_k)$, (119) gives

$$u(x,t) = -\log(T-t) + \frac{\delta}{|\log(T-t)|} w_k \left(\delta^{-\sigma/(\sigma+2)} |\log(T-t)|^{\sigma/(\sigma+2)} y \right) + \dots , \tag{123}$$

with

$$y = \frac{x}{(T-t)^{1/(\sigma+2)}}.$$

The behaviour in the outer domain (and matching).

The asymptotic behaviour in the outer region is typical of combustion problems in that it is governed by the first-order (diffusion free) equation

$$u_t = e^u. \tag{124}$$

The non-resonance case. Introducing in (1) the rescaled variables

$$u(x,t) = -\log(T-t) + g(\zeta, s), \quad \zeta = \frac{x}{(T-t)^{\alpha}}, \tag{125}$$

where $\alpha > 0$ is a free constant (to be determined) satisfying

$$\alpha < \frac{1}{\sigma+2}, \tag{126}$$

we arrive at the following equation

$$g_s = -\alpha \zeta g_\zeta + e^g - 1 + e^{-\gamma s} (|g_\zeta|^\sigma g_\zeta)_\zeta, \tag{127}$$

where, in view of (126), $\gamma = 1 - \alpha(\sigma+2) > 0$. The rescaled spatial variable in (125) satisfies: if $\zeta = O(1)$ then $y \gg 1$ and the outer region is defined as $\{\zeta = O(1)\}$.

Equation (127) is an exponentially small, singular perturbation of the Hamilton-Jacobi equation

$$g_s = -\alpha\zeta g_\zeta + e^g - 1. \tag{128}$$

A general approach to study such a singularly perturbed dynamical systems (stability of the ω-limits) is given in [GV]. Using certain interior regularity results similar to that in [GV] we can expect that $g(\zeta, s)$ approaches a stationary solution of the autonomous equation (128) as $s \to \infty$. Explicitly solving equation (128) in this case we have

$$g(\zeta, s) = -\log(1 + A\zeta^{1/\alpha}) + \dots \quad \text{as} \quad s \to \infty, \tag{129}$$

i.e., in the outer region, as $t \to T$

$$u(x,t) = -\log(T-t) - \log\left(1 + A\frac{x^{1/\alpha}}{(T-t)}\right) + \dots. \tag{130}$$

We now compare the outer expansion (130) with the inner one (121) in the non-resonance case. In view of the slow-growth behaviour (99) we have from (121) with $\eta \gg 1$ that

$$u(x,t) = -\log(T-t) - C\frac{x^{1/m_k}}{(T-t)} + \dots. \tag{131}$$

On the other hand, the outer expansion (130) yields for $\eta \gg 1$ that

$$u(x,t) = -\log(T-t) - A\frac{x^{1/\alpha}}{(T-t)} + \dots \tag{132}$$

A formal matching procedure then implies that $A = C$ and that the exponent α is given by

$$\alpha = m_k = \frac{1 + \lambda_k\sigma}{(\sigma+2)(1+\lambda_k)}. \tag{133}$$

The resonance case. In this case we introduce other spatial recaled variables:

$$u(x,t) = -\log(T-t) + g(\zeta, s), \quad \zeta = \frac{y}{|\log(T-t)|^\beta}, \tag{134}$$

with the exponent β to be determined. Substituting into (1) we obtain

$$g_s = -\frac{1}{\sigma+2}\zeta g_\zeta + e^g - 1 + \frac{1}{s^{\beta(\sigma+2)}}(|g_\zeta|^\sigma g_\zeta)_\zeta + \frac{1}{s}\beta\zeta g_\zeta, \tag{135}$$

which is a perturbation of the first-order equation (128) with $\alpha = 1/(\sigma+2)$. Assuming convergence to the corresponding stationary profile (129) and substituting (134) we obtain that in the outer region

$$u(x,t) = -\log(T-t) - \log\left[1 + A\frac{x^{\sigma+2}}{(T-t)|\log(T-t)|^{\beta(\sigma+2)}}\right] + \dots. \tag{136}$$

On the other hand, using (99) with $1/m = (\sigma + 2)$ we conclude from the inner expansion (123), that if $y^{\sigma+2}s^{-\sigma} \gg 1$ then

$$u(x,t) = -\log(T-t) - C\delta^{1-\sigma}\frac{x^{\sigma+2}}{(T-t)|\log(T-t)|^{1-\sigma}}. \tag{137}$$

It follows from the outer expansion (136) that if $y^{\sigma+2}s^{-\beta} \ll 1$ then

$$u(x,t) = -\log(T-t) - A\frac{x^{\sigma+2}}{(T-t)|\log(T-t)|^{\beta(\sigma+2)}.} \tag{138}$$

Finally we get that

$$\beta = \frac{\sigma+1}{\sigma+2}, \tag{139}$$

which reduces to the case $\beta = \frac{1}{2}$ when $\sigma = 0$, which agrees with the results in Section 2.

7 The behaviour for $\sigma < 0$

The main distinctive feature of negative $\sigma \in (-1, 0)$ is that there is no non-trivial self-similar solution. Therefore all possible types of asymptotic behaviour as $t \to T$ are given by approximate self-similar solutions (as in $\sigma = 0$). These are constructed using the same matching techniques as in Section 6.

In the construction of the first approximately self-similar solution which is monotone in $\eta > 0$ we first take the explicit representation of the inner behaviour via (102) so that

$$u(x,t) = -\log(T-t) + (T-t)^{|\sigma|}w_1(\eta) + ..., \quad \eta = x(T-t)^{(1-\sigma^2)(\sigma+2)}. \tag{140}$$

Therefore, as in Section 6, we conclude that in the outer region the solution behaves like (130) with the exponent (see (133) with $\lambda_1 = -\sigma$)

$$\alpha = m_1 = \frac{\sigma+1}{\sigma+2}. \tag{141}$$

Finally, we observe that no resonant behaviour exists if $\sigma \in (-1, 0)$.

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January 1997