Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain *

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Abstract: Closing the loop around an exponentially stable single-input single-output regular linear system, subject to a globally Lipschitz and non-decreasing actuator nonlinearity and compensated by an integral controller, is known to ensure asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is sufficiently small and (c) the reference value is feasible in a very natural sense. Here we derive lower bounds for the maximal regulating gain for various special cases including systems with non-overshooting step-response and second-order systems with a time-delay in the input or output. The lower bounds are given in terms of open-loop frequency/step response data and the Lipschitz constant of the nonlinearity, and are hence readily obtainable.

Keywords: Integral control; actuator nonlinearities; input saturation; robust tracking; monotone step-response; infinite-dimensional systems; systems with time-delay.

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1. Introduction

The synthesis of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has received considerable attention in the last 20 years. The following principle is well known (see Davison [5], Lunze [15] and Morari [18]): closing the loop around a stable, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}(s)$, compensated by a pure integral controller k/s (see Fig. 1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$. Therefore, if a plant is known to be stable and if the sign of $\mathbf{G}(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design ("tuning regulator theory"[5]) has been successfully applied in process control, see, for example, Coppus et al. [4] and Lunze [14]. The approach has been extended by Logemann et al. [9], Logemann and Owens [10], Logemann and Townley [12], Pohjolainen [20, 21] and Pohjolainen and Lätti [22] to various classes of (abstract) infinite-dimensional systems, and by Jussila and Koivo [7] and Koivo and Pohjolainen [8] to differential delay systems. Furthermore, the problem of tuning the integrator gain adaptively has been addressed recently in a number of papers, see Cook [3] and Miller and Davison [16, 17] for the finite-dimensional case and Logemann and Townley [12, 13] for the infinite-dimensional case.

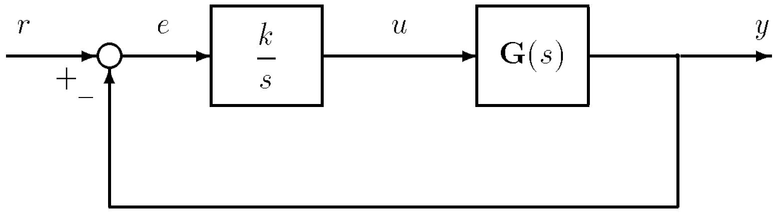

Fig. 1: Low-gain control system

In a recent paper, Logemann, Ryan and Townley [11] have proved that the above principle remains true if the plant to be controlled is a single-input, single-output, regular, infinite-dimensional, linear system subject to an input nonlinearity (see Fig. 2). More precisely, it is shown in [11] that, for an exponentially stable system with $\mathbf{G}(0) > 0$, there exists $K > 0$ such that, for all non-decreasing globally Lipschitz nonlinearities ϕ with Lipschitz constant λ and all $k \in (0, K/\lambda)$, the output $y(t)$ of the closed-loop system shown in Fig. 2 converges to r as $t \to \infty$, provided that $[\mathbf{G}(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\phi)$. In particular, K is the supremum of the set of all $k^* > 0$ such that the function

$$1 + k\,\mathrm{Re}\,\frac{\mathbf{G}(s)}{s}$$

is positive real for all $k \in (0, k^*)$.

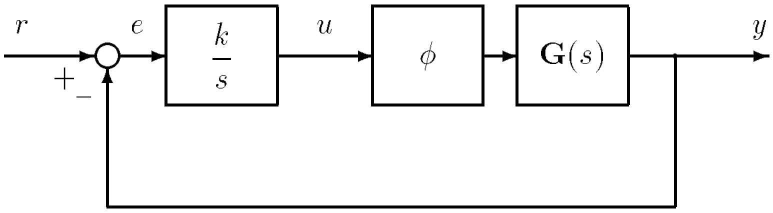

Fig. 2: Low-gain control with input nonlinearity

In this paper, we show that K can be obtained from frequency and step-response experiments performed on the linear part of the plant. Moreover, we present an easily obtainable lower bound for K. For a number of special cases, we show that $K = 1/|\mathbf{G}'(0)|$; determination of $|\mathbf{G}'(0)|$ (and hence of K), in principle, requires only frequency and step-response data. In particular, the latter formula for K applies to systems with non-overshooting step-response and a class of second-order systems with a time-delay in the input or output. We remark that, in the finite-dimensional and linear case, Mustafa [19] has recently derived a formula for the smallest $k > 0$ such that the closed-loop system shown in Fig. 1 is unstable: this formula is in terms of a minimal realization of $\mathbf{G}$ and hence requires exact knowledge of the system.

2. Preliminaries

Let $\mathbb{R}_+ := [0, \infty)$ and, for $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$. The algebra of all holomorphic and bounded functions on $\mathbb{C}_\alpha$ is denoted by H^∞_α. If $f \in H^\infty_\alpha$ for some $\alpha < 0$, we define $\|f\|_\infty = \sup_{s \in \mathbb{C}_0} |f(s)|$ and, as is well-known, we have $\|f\|_\infty = \sup_{\omega \in \mathbb{R}} |f(i\omega)|$. The Hardy space, of order 2, of holomorphic functions defined on $\mathbb{C}_\alpha$ is denoted by H^2_α. Let $L^2_\alpha(\mathbb{R}_+)$ denote the space of all locally square-integrable functions f such that the weighted function $t \mapsto f(t)e^{-\alpha t}$ is in $L^2(\mathbb{R}_+)$. Moreover, let $M(\mathbb{R}_+)$ denote the set of all bounded Borel measures on $\mathbb{R}_+$. For $\alpha \in \mathbb{R}$, let $M_\alpha(\mathbb{R}_+)$ denote the set of all locally bounded Borel measures μ on $\mathbb{R}_+$ such that $e^{-\alpha t}\mu(dt)$ belongs to $M(\mathbb{R}_+)$. The Laplace transform is denoted by $\mathfrak{L}$. If $\mu \in M_\alpha(\mathbb{R}_+)$, then $\mathfrak{L}(\mu) \in H^\infty_\alpha$, and $\mathfrak{L}(\mu)(s)$ exists and is continuous on the closed-right half plane $\mathrm{Re}\, s \geq \alpha$ (see [6] for details).

In the following, let (A, B, C, D) be the generating operators of a linear, single-input, single-output regular system with state space X, a Hilbert space. Let $\mathbf{T}_t$ denote the strongly continuous semigroup generated by A and let $\mathbf{G}(s)$ denote the transfer function of (A, B, C, D). Suppose that the system is subject to an input nonlinearity ϕ. We assume that $\phi \in N(\lambda)$, where $N(\lambda)$ denotes the set of all non-decreasing globally Lipschitz nonlinearities $f : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant λ. Denoting the constant reference signal by r, an application of the integrator

$$u(t) = u_0 + k \int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau \,,$$

where k is a real parameter (see Fig. 2), leads to the following nonlinear system of differential equations

$$\dot{x} = Ax + B\phi(u), \quad x(0) = x_0 \in X \tag{2.1a}$$
$$\dot{u} = k[r - C_L x - D\phi(u)], \quad u(0) = u_0 \in \mathbb{R}, \tag{2.1b}$$

where C_L denotes the so-called Lebesgue extension of C. For a comprehensive treatment of regular systems, see Weiss [24, 25] and the references therein. For a treatment of regular systems specific to low-gain control, the reader is referred to [11, 12]. We remark that most linear distributed parameter systems and time-delay systems arising in control engineering fall within the framework of regular systems.

If, for some $\alpha < 0$, $\mathbf{G} \in H^\infty_\alpha$ (this is true if $\mathbf{T}_t$ is exponentially stable or if $\mathfrak{L}^{-1}(\mathbf{G}) \in M_\alpha(\mathbb{R}_+)$) and $\mathbf{G}(0) > 0$, then it is not difficult to show that

$$1 + k\,\mathrm{Re}\,\frac{\mathbf{G}(s)}{s} \geq 0, \qquad \text{for all } s \in \mathbb{C}_0, \tag{2.2}$$

for all sufficiently small $k > 0$ (see Lemma 3.1 below). We define

$$K := \sup\{k^* > 0 \mid (2.2) \text{ holds for all } k \in (0, k^*)\}. \tag{2.3}$$

With the convention $\inf \emptyset := \infty$, K can be expressed as

$$K = \inf\{k > 0 \mid (2.2) \text{ does not hold}\}. \tag{2.4}$$

We mention, that if $K < \infty$, then supremum in (2.3) and infimum in (2.4) can be replaced by maximum and minimum, respectively. It is easy to construct examples for which $K = \infty$, see Example 3.5, part (1), below.

The following tuning regulator result was proved in [11].

Theorem 2.1 *Let $\lambda > 0$ and $\phi \in N(\lambda)$. Assume that $\mathbf{T}_t$ is exponentially stable, $\mathbf{G}(0) > 0$, $k \in (0, K/\lambda)$ and $r \in \mathbb{R}$ is such that*

$$\phi_r := [\mathbf{G}(0)]^{-1} r \in \mathrm{clos}\,(\mathrm{im}\,\phi).$$

If C is bounded, then for all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of (2.1) exists on $[0, \infty)$ and satisfies

- *(1)* $\lim_{t\to\infty} \phi(u(t)) = \phi_r$,
- *(2)* $\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_r\| = 0$,
- *(3)* $\lim_{t\to\infty}(r - y(t)) = 0$, *where* $y(t) = Cx(t) + D\phi(u(t))$.
- *(4)* *if* $\phi_r \in \mathrm{im}\,\phi$, *then*
$$\lim_{t\to\infty} \mathrm{dist}\,(u(t), \phi^{-1}(\phi_r)) = 0,$$
- *(5)* *if* $\phi_r \in \mathrm{int}\,(\mathrm{im}\,\phi)$, *then* $u(\cdot)$ *is bounded.*

If C is unbounded, then the statements (1)–(5) remain true provided that $\mathfrak{L}^{-1}(\mathbf{G}) \in M(\mathbb{R}_+)$ and x_0 is in the domain of A.

In particular, (4) states that $u(t)$ converges as $t \to \infty$ if the set $\phi^{-1}(\phi_r)$ is a singleton, which, in turn, is true if ϕ_r is not a critical value of ϕ. The conditions imposed in Theorem 2.1 on ϕ are satisfied by saturation and deadzone nonlinearities and combinations of the two, as shown in Fig. 3. The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in M(\mathbb{R}_+)$ is not very restrictive and seems to be satisfied in all practical examples of systems with H^∞-transfer functions. Generally, a measure $\mu \in M(\mathbb{R}_+)$ can be written in the form

$$\mu(dt) = a(t)dt + \sum_{j=0}^{\infty} a_j \delta_{t_j}(dt) + \mu_s(dt)\,,$$

where $a(\cdot) \in L^1(\mathbb{R}_+)$, $\sum_{j=0}^{\infty} a_j \delta_{t_j}$ and μ_s, respectively, represent the absolutely continuous, the discrete and the singular parts of μ. In particular, δ_{t_j} denotes the unit point mass at $t_j \geq 0$ and the a_j are real numbers such that $\sum_{j=0}^{\infty} |a_j| < \infty$. However, in most applications one has $\mu_s = 0$.

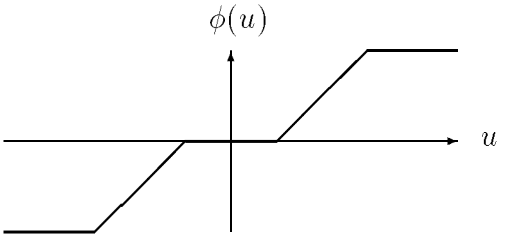

Fig. 3: Nonlinearity with saturation and deadzone

For the application of Theorem 2.1, especially in process control, it is important to develop formulae or lower bounds for K in terms of easily obtainable open-loop data, such as Nyquist diagrams and step-response data. This development will be addressed in the next section.

3. Estimation and determination of K

We shall invoke one or both of the following two assumptions where appropriate:

(A1) $\mathbf{G} \in H^\infty_\alpha$ for some $\alpha < 0$,

(A2) $\bar{\mathbf{G}}(\bar{s}) = \mathbf{G}(s)$ for all $s \in \mathbb{C}_0$.

We mention that (A2) is satisfied for all systems with real parameters. For $k > 0$ set

$$\mathbf{G}_k(s) = \frac{k}{s}\mathbf{G}(s)\left(1 + \frac{k}{s}\mathbf{G}(s)\right)^{-1}.$$

Clearly, $\mathbf{G}_k$ is the transfer function of the feedback system obtained by applying the integral controller k/s to $\mathbf{G}$.

The next lemma is a trivial consequence of results in [12].

Lemma 3.1 *Assume that (A1) holds and that* $\mathbf{G}(0) > 0$*. Then (2.2) holds for all sufficiently small* $k > 0$*, so that* $K > 0$*. Moreover, for given* $k > 0$*, (2.2) holds if and only if* $\|\mathbf{G}_{k/2}\|_\infty = 1$*.*

In Corollary 3.2, we give a graphical characterization of the number K. To this end let $k > 0$ and define D_k to be the open disc in the complex plane of radius k and with centre $(-k, 0)$, i.e.

$$D_k = \{s \in \mathbb{C} \,|\, |s + k| < k\}.$$

The inverse Nyquist curve of $\mathbf{G}(s)/s$ given by

$$N = \left\{\frac{i\omega}{\mathbf{G}(i\omega)} \mid \omega \in \mathbb{R}\right\}.$$

Corollary 3.2 *Assume that (A1) holds and that* $\mathbf{G}(0) > 0$*. Then,* $N \cap D_k = \emptyset$ *for all sufficiently small* $k > 0$*, and*

$$0 < \inf\{k > 0 \,|\, N \cap D_k \neq \emptyset\} = K/2.$$

Proof: Setting

$$\tilde{\mathbf{G}}_k(s) = \left(\frac{s}{\mathbf{G}(s)} + k\right)^{-1},$$

it follows that $\mathbf{G}_k(s) = k\tilde{\mathbf{G}}_k(s)$. Clearly, for any $k > 0$, we have

$$\|\mathbf{G}_k\|_\infty = 1 \quad \Leftrightarrow \quad \|\tilde{\mathbf{G}}_k\|_\infty = 1/k \quad \Leftrightarrow \quad N \cap D_k = \emptyset,$$

and therefore the claim follows from Lemma 3.1. □

It will turn out to be convenient to introduce the following auxiliary transfer function

$$\mathbf{E}(s) := \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0)).$$

The above definition makes sense for all $s \neq 0$ for which $\mathbf{G}(s)$ is defined. If $\mathbf{G}(s)$ is holomorphic at 0 (which is the case if (A1) is satisfied), then we set $\mathbf{E}(0) = \mathbf{G}'(0)$.

Lemma 3.3 *Assume that (A1) holds and that* $\mathbf{G}(0) > 0$ *and let* $k > 0$*. Then the following statements are equivalent*

(1) $1 + k\,\mathrm{Re}\,(\mathbf{G}(s)/s) \geq 0\,,$ for all $s \in \mathbb{C}_0\,,$

(2) $1 + k\,\mathrm{Re}\,(\mathbf{G}(i\omega)/i\omega) \geq 0\,,$ for all $\omega \in \mathbb{R} \setminus \{0\}\,,$

(3) $1 + k\,\mathrm{Re}\,\mathbf{E}(i\omega) \geq 0\,,$ for all $\omega \in \mathbb{R}\,,$

(4) $1 + k\,\mathrm{Re}\,\mathbf{E}(s) \geq 0\,,$ for all $s \in \mathbb{C}_0\,.$

Proof: Trivially, (1) implies (2), and since $\mathbf{G}(0)$ is real, (2) implies (3). In order to show that (4) follows from (3), assume that (3) holds. By considering

$$e^{-(1+k\,\mathrm{Re}\,\mathbf{E}(s))} = \left| e^{-(1+k\,\mathbf{E}(s))} \right|\,,$$

applying the maximum modulus theorem and using the fact that $\mathbf{E}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$, it then follows that

$$1 + k\,\mathrm{Re}\,\mathbf{E}(s) \geq 0\,, \qquad \text{for all } s \in \mathbb{C}_0\,,$$

which is (4). Finally, since $\mathbf{G}(0) > 0$ we have that $\mathrm{Re}\,(\mathbf{G}(0)/s) > 0$ for all $s \in \mathbb{C}_0$, and therefore (1) is implied by (4). □

The following corollary provides a lower bound and an upper bound for K in terms of the transfer function $\mathbf{E}$.

Corollary 3.4 *Assume that (A1) holds and that* $\mathbf{G}(0) > 0$*. Then*

$$1/\|\mathbf{E}\|_\infty \leq K \leq \begin{cases} 1/|\mathrm{Re}\,\mathbf{E}(0)| & \text{if} \quad \mathrm{Re}\,\mathbf{E}(0) < 0 \\ \infty & \text{if} \quad \mathrm{Re}\,\mathbf{E}(0) \geq 0 \end{cases} \,. \tag{3.1}$$

Proof: For $k > 0$ we have that

$$1 + k\,\mathrm{Re}\,\mathbf{E}(s) \geq 1 - k\|\mathbf{E}\|_\infty\,, \qquad \text{for all } s \in \mathbb{C}_0\,.$$

Combining this with Lemma 3.3 we see that $1/\|\mathbf{E}\|_\infty \leq K$, which is the first inequality in (3.1). Moreover, using Lemma 3.3 again, it follows from the definition of K that $1 + K\,\mathrm{Re}\,\mathbf{E}(0) \geq 0$. If $\mathrm{Re}\,\mathbf{E}(0) < 0$, we may conclude that $K \leq 1/|\mathrm{Re}\,\mathbf{E}(0)|$, yielding the second inequality in (3.1). □

The following examples show that if Re $\mathbf{E}(0) =$ Re $\mathbf{G}'(0) \geq 0$, then cases of finite K and infinite K can occur.

Example 3.5 (1) Consider

$$\mathbf{G}(s) = \frac{2s+1}{s+1}\,.$$

Obviously, $\mathbf{G} \in H^\infty_\alpha$ for all $\alpha \in (-1, 0)$, and $\mathbf{G}(0) = \mathbf{G}'(0) = 1 > 0$. An easy calculation yields

$$\mathrm{Re}\,\frac{\mathbf{G}(i\omega)}{i\omega} = \frac{1}{1+\omega^2}\,,$$

showing that $K = \infty$.

(2) As a second example consider

$$\mathbf{G}(s) = \frac{s+1}{(s+2)^2}\,.$$

Then $\mathbf{G} \in H^\infty_\alpha$ for all $\alpha \in (-2, 0)$, $\mathbf{G}(0) = 1/4 > 0$ and $\mathbf{G}'(0) = 0$. Since

$$\mathrm{Re}\,\frac{\mathbf{G}(i\omega)}{i\omega} = \frac{-\omega^2}{(4-\omega^2)^2 + 16\omega^2}\,,$$

we see that $K < \infty$. $\diamond$

In the following we introduce a condition which will guarantee that $K = 1/|\mathbf{G}'(0)|$. To this end let $\sigma(\cdot)$ denote the *step-response* of the regular system (A, B, C, D) and define the *step-response error* $\varepsilon(\cdot)$ by

$$\varepsilon(t) = \sigma(t) - \mathbf{G}(0)\,,$$

with Laplace transforms given by $[\mathfrak{L}(\sigma)](s) = \mathbf{G}(s)/s$ and $[\mathfrak{L}(\varepsilon)](s) = \mathbf{E}(s)$, respectively. Under the assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in M(\mathbb{R}_+)$ it follows trivially that $\lim_{t\to\infty} \varepsilon(t) = 0$. This is in general not true under the assumption (A1). However, we can prove the following lemma.

Lemma 3.6 *If (A1) holds, then there exists $\alpha < 0$ such that $\varepsilon \in L^2_\alpha(\mathbb{R}_+)$.*

Proof: Choose $\alpha < 0$ such that $\mathbf{G} \in H^\infty_\alpha$. Then, $\mathbf{E} \in H^2_\alpha$, and by a well-known theorem of Paley and Wiener $\varepsilon = \mathfrak{L}^{-1}(\mathbf{E}) \in L^2_\alpha(\mathbb{R}_+)$. $\square$

If (A2) is satisfied then the step-response error is real-valued and we say that the system satisfies the *no-overshoot condition* if $\varepsilon(t) \leq 0$ for almost all $t \in \mathbb{R}_+$. We say that the step-response $\sigma(\cdot)$ is *essentially non-decreasing* if there exists a non-decreasing function $\tilde{\sigma}(\cdot)$ such that $\sigma(t) = \tilde{\sigma}(t)$ for almost all $t \in \mathbb{R}_+$. If $\mathfrak{L}^{-1}(\mathbf{G}) \in M(\mathbb{R}_+)$, then the step-response $\sigma(\cdot)$ is continuous, and hence $\sigma(\cdot)$ is essentially non-decreasing if and only if $\sigma(\cdot)$ is decreasing. However, if $\mathfrak{L}^{-1}(\mathbf{G}) \notin M(\mathbb{R}_+)$, then $\sigma(\cdot)$ might be discontinuous, and consequently $\sigma(\cdot)$ might be essentially non-decreasing, but not decreasing. If (A1) and (A2) hold, then it follows from Lemma 3.6 that systems with an essentially non-decreasing step-response satisfy the no-overshoot condition. We mention that systems with monotone step-responses have received some attention in the robust control literature, see e.g. Åström [1].

Corollary 3.7 *Assume that (A1) and (A2) hold and that $\mathbf{G}(0) > 0$. If the system satisfies the no-overshoot condition, then $\mathbf{G}'(0) \leq 0$ and $K = 1/|\mathbf{G}'(0)|$ (where we define $1/0 = \infty$).*

Proof: By the no-overshoot condition, we obtain for $s \in \mathbb{C}_0$

$$-\mathbf{G}'(0) = -\mathbf{E}(0) = -\int_0^\infty \varepsilon(\tau)\, d\tau = \int_0^\infty |\varepsilon(\tau)|\, d\tau \geq |\mathbf{E}(s)|\,.$$

Thus

$$-\mathbf{G}'(0) = \|\mathbf{E}\|_\infty\,, \tag{3.2}$$

and therefore in particular, $\mathbf{G}'(0) \leq 0$. If $\mathbf{G}'(0) < 0$, then the claim follows from (3.2) and Corollary 3.4. If $\mathbf{G}'(0) = 0$, then by (3.2), $\mathbf{E}(s) \equiv 0$, and so $\mathbf{G}(s) \equiv \mathbf{G}(0)$, which in turn implies that $K = \infty$. $\square$

Remark 3.8 The quantity $\mathbf{E}(0) = \mathbf{G}'(0)$ plays an important role in Corollaries 3.4 and 3.7. The following remarks show that this quantity can be obtained from step as well as frequency-response data. We assume that (A1) and (A2) are satisfied.

(1) By Lemma 3.6, $\varepsilon \in L^1(\mathbb{R}_+)$, and hence $\lim_{t\to\infty} \int_0^t \varepsilon(\tau)\, d\tau$ exists. Now $\mathbf{E}(s)/s$ is the Laplace transform of the function $t \mapsto \int_0^t \varepsilon(\tau)\, d\tau$, and hence the final-value theorem yields

$$\mathbf{G}'(0) = \mathbf{E}(0) = \int_0^\infty \varepsilon(\tau)\, d\tau\,,$$

i.e. $\mathbf{G}'(0)$ is equal to the area enclosed between the graphs of $t \mapsto \sigma(t)$ and $t \mapsto \mathbf{G}(0)$.

(2) The curvature κ of N at 0 is given by $\kappa = 2|\mathbf{G}'(0)|$ (this follows from a straightforward calculation which is left to the reader), and hence $|\mathbf{G}'(0)|$ can be obtained from the inverse Nyquist diagram of $\mathbf{G}(s)/s$. $\diamond$

Example 3.9 Assume that $\mathbf{G}$ satisfies (A1) and (A2), $\mathbf{G}(0) > 0$ and the no-overshoot condition holds. Then the same is true for the transfer function

$$\mathbf{H}(s) = \mathbf{G}(s) \sum_{n=0}^\infty \gamma_n e^{-h_n s}\,, \tag{3.3}$$

where $\gamma_n, h_n \geq 0$ and

$$0 < \sum_{n=0}^\infty \gamma_n e^{\delta h_n} < \infty\,, \qquad \text{for some } \delta > 0\,.$$

By Corollary 3.7, the constant K (for $\mathbf{H}$) is then given by

$$K = 1/|\mathbf{H}'(0)| = \left(\sum_{n=0}^\infty \gamma_n |\mathbf{G}'(0) - h_n \mathbf{G}(0)|\right)^{-1}\,.$$

Let us consider two specific examples.

(1) Consider the following first order system with time delay

$$\mathbf{H}(s) = \frac{e^{-hs}}{1+\tau s}\,,$$

where $h \geq 0$ and $\tau > 0$. Then, by the above remarks, $\mathbf{H}$ satisfies the no-overshoot condition and hence

$$K = 1/|\mathbf{H}'(0)| = 1/(h+\tau).$$

(2) Consider

$$\mathbf{H}(s) = \frac{e^{-\alpha s}}{(1+\tau s)(1-\gamma e^{-\beta s})},$$

where $\alpha, \beta \geq 0$, $\tau > 0$ and $\gamma \in (0,1)$. The above transfer function has been used to model a heat circulation process, Blanchini [2]. In this application, $\gamma = 1 - \eta$, where η is a heat exchange efficiency index (which by definition is positive and smaller than 1).

Setting $\gamma_n = \gamma^n$, $h_n = \alpha + n\beta$ and $\mathbf{G}(s) = 1/(1+\tau s)$, $\mathbf{H}$ can be written in the form (3.3). Hence

$$K = \frac{1}{|\mathbf{H}'(0)|} = \frac{(1-\gamma)^2}{(\alpha+\tau)(1-\gamma)+\gamma\beta}.$$

$\diamond$

Example 3.10 Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = [0,1]$, with scalar pointwise control action (applied at point $x_b \in I$) and pointwise scalar observation (output at point $x_c \in I$, $x_c \geq x_b$.). We formally write this single-input, single-output system as

$$z_t(t,x) = a z_{xx}(t,x) + \delta(x - x_b)u(t), \quad y(t) = z(t, x_c)$$
$$z(t,0) = 0 = z(t,1), \qquad \text{for all } t > 0.$$

This example can be represented as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{\sinh\left(x_b\sqrt{s/a}\right)\sinh\left((1-x_c)\sqrt{s/a}\right)}{a\sqrt{s/a}\,\sinh\sqrt{s/a}}.$$

Clearly, a positive injection of heat at x_b produces a non-negative response at x_c, and so the step-response is non-decreasing. Therefore the no-overshoot condition is satisfied, and thus by Corollary 3.7

$$K = \frac{1}{|\mathbf{G}'(0)|} = \frac{6a^2}{x_b(1-x_c)(1-x_b^2-(1-x_c)^2)}.$$

$\diamond$

If (A1) and (A2) are satisfied, then Corollary 3.7 shows that the no-overshoot condition is sufficient for the formula $K = 1/|\mathbf{G}'(0)|$ to hold. However, not surprisingly, the no-overshoot condition is not neccessary for the validity of the latter formula. The next result identifies a class of second-order systems with time-delay for which $K = 1/|\mathbf{G}'(0)|$, but which may have overshoot.

Proposition 3.11 *Let $\mathbf{G}(s) = e^{-sh}/(s^2 + as + b)$, where $a, b > 0$ and $h \geq 0$. If*

$$a^2 \geq 2b - \frac{b^2 h}{a + bh}\,, \tag{3.4}$$

then $K = 1/|\mathbf{G}'(0)| = b^2/(a + bh)$.

Note that the right-hand side of (3.4) is decreasing as a function of h. In particular, if

$$a^2 \geq 2b\,, \tag{3.5}$$

then (3.4) is satisfied for all $h \geq 0$, and consequently the formula $K = 1/|\mathbf{G}'(0)|$ holds independently of the length of the delay h. Clearly, condition (3.5) is satisfied if and only if the poles of $\mathbf{G}(s)$ belong to the sector $\{s \in \mathbb{C} \,|\, 3\pi/4 \leq \arg s \leq 5\pi/4\}$.

Proof of Proposition 3.11: In view of Lemma 3.3, it suffices to prove that the function

$$\omega \mapsto f(\omega) := -\mathrm{Re}\,\mathbf{E}(i\omega) = \frac{p(\omega)}{q(\omega)}$$

attains its maximum at $\omega = 0$, where, for convenience, we have introduced

$$p(\omega) := \begin{cases} a\cos\omega h + (b - \omega^2)\sin(\omega h)/\omega\,, & \omega \neq 0 \\ a + bh\,, & \omega = 0 \end{cases}$$

and

$$q(\omega) := (b - \omega^2)^2 + a^2\omega^2\,.$$

We will first show that $f(\omega) \leq f(0)$ for all ω with $\omega^2 \leq b$.

Let $0 \leq \omega^2 \leq b$. Then $p(\omega) \leq a + (b - \omega^2)h$ and so

$$f(\omega) \leq g(\omega) := \frac{(a + bh) - \omega^2 h}{q(\omega)}.$$

By direct calculation,

$$g'(\omega) = \frac{2\omega[h\omega^4 - 2\omega^2(a + bh) - hb^2 - (a + bh)(a^2 - 2b)]}{q^2(\omega)}\,.$$

Using (3.4),

$$hb^2 + (a + bh)(a^2 - 2b) \geq hb^2 - hb^2 = 0\,,$$

and so we may conclude that

$$\omega g'(\omega) \leq 2\omega^4[h\omega^2 - 2(a + bh)]/q^2(\omega) \leq 0 \quad \text{for all } \omega \in [-\sqrt{b}, \sqrt{b}]\,.$$

Therefore,

$$f(\omega) \leq g(\omega) \leq g(0) = f(0) \qquad \text{for all } \omega \in [-\sqrt{b}, \sqrt{b}]\,.$$

We complete the proof by showing that, for every $n \in \mathbb{N}$, $f(\omega) < f(0)$ for all ω with $nb \leq \omega^2 \leq (n + 1)b$.

Let $n \in \mathbb{N}$ and let $\omega \in \mathbb{R}$ be such that

$$nb \leq \omega^2 \leq (n+1)b.$$

Then,

$$p(\omega) < a + (\omega^2 - b)h = a + nbh - ((n+1)b - \omega^2)h \leq a + nbh \leq n(a + bh)$$

and (again using (3.4))

$$q(\omega) = b^2 + \omega^4 + (a^2 - 2b)\omega^2 \geq b^2 + \omega^4 - b\omega^2 .$$

Since $nb \leq \omega^2 \leq (n+1)b$, we have $-b\omega^2 \geq -\omega^4/n$ and

$$q(\omega) \geq b^2 + \frac{(n-1)\omega^4}{n} \geq b^2(n^2 - n + 1) \geq nb^2 .$$

We may now conclude that

$$f(\omega) < \frac{a + bh}{b^2} = f(0) .$$

This completes the proof. □

Example 3.12 Consider

$$\mathbf{G}(s) = \frac{e^{-hs}}{s^2 + as + b} ,$$

where $h \geq 0$ and $a, b > 0$. Suppose that

$$4b > a^2 \geq 2b .$$

Then, by Proposition 3.11, it follows that

$$K = 1/|\mathbf{G}'(0)| = b^2/(a + bh) , \qquad \text{for all } h \geq 0 .$$

Note that $\mathbf{G}$ does not satisfy the no-overshoot condition. Indeed, since $4b > a^2$, the maximum overshoot is $e^{-a\pi/\sqrt{4b - a^2}}/b > 0$, see for example [23], p. 191. ◇

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