Title: Estimates on the Green's function of second-order elliptic operators in $\mathbb{R}^N$

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Abstract: Sharp upper and lower pointwise bounds are obtained for the Green's function of the equation

$$(L+\lambda)u \equiv -\Delta u + a(x).\nabla u + \lambda u = f, \qquad x \in \mathbb{R}^N,$$

for $\lambda > 0$. Initially in a Cartesian frame it is assumed that $\|a_i\|_{L_\infty(\mathbb{R}^N)} \leq M_i$, $i = 1, \ldots, N$. Pointwise estimates for the heat kernel of $u_t + Lu = 0$, recently obtained under this assumption, are Laplace-transformed to yield corresponding elliptic results.

In a second approach, the coordinate–free constraint $\||a|_{l_2}\|_{L_\infty(\mathbb{R}^N)} \leq M$ is imposed. Within this class of operators, the equations defining the maximal and minimal Green's functions are found to be simple ODEs when written in polar coordinates, and these are soluble in terms of the singular Kummer confluent hypergeometric function. For both approaches, bounds on $\|(L+\lambda I)^{-1}\|_{L_1(\mathbb{R}^N)}$ are derived as a consequence.

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1 Introduction

In this paper we obtain upper and lower bounds on the pointwise value of the Green's function for second-order elliptic operators with constant uniform diffusion, defined on $\mathbb{R}^N$. As a consequence of these estimates, we deduce bounds on the norm of the solution operator of the elliptic problem, considered as a mapping on $L_1(\mathbb{R}^N)$. It is our particular concern here to derive explicit sharp quantitative estimates of use in various areas of applied analysis.

Specifically, we consider the equation

$$(L+\lambda)u \equiv -\Delta u + a(x).\nabla u + \lambda u = f(x), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

for $\lambda > 0$ and $a \in L_\infty(\mathbb{R}^N)$. Here, u is required to satisfy the global condition,

$$u(x) = \lim_{k\to\infty} u_k(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where, for $k \in \mathbb{N}$, $u_k(x)$ is the solution of the truncated problem

$$(L_k+\lambda)u_k = f_k(x), \qquad x \in B_k; \qquad u_k(x) = 0, \quad |x| = k. \tag{1.3}$$

Here, $B_k = \{x \in \mathbb{R}^N \,:\, |x| < k\}$, and f_k and L_k are restrictions onto B_k.

Since general $a \in L_\infty(\mathbb{R}^N)$ is permitted in (1.1), we consider non-classical Green's functions. Below, we define what we term the *weak Green's function*, convenient for our purposes in this paper, and also quote the more familiar definition of a classical Green's function.

Definition 1.1 *$g(\lambda, x, y)$, the* weak Green's function *for (1.1, 1.2) satisfies*
(i) $g(\lambda, \cdot, \cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N \setminus D)$, $\qquad D \equiv \{(x, x) \,:\, x \in \mathbb{R}^N\}$;
(ii) For $f \in C_0^\infty(\mathbb{R})$,

$$u(x) = \int_{\mathbb{R}^N} g(\lambda, x, y) f(y)\, dy, \qquad x \in \mathbb{R}^N, \tag{1.4}$$

defines a function in $C(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ obeying (1.2) and

$$\int_{\mathbb{R}^N} \nabla u.\nabla\phi + (a.\nabla u)\phi + \lambda u\phi - f\phi\, dx = 0, \qquad \textit{for all } \phi \in C_0^1(\mathbb{R}^N). \tag{1.5}$$

Definition 1.2 *$G(\lambda, x, y)$, the* classical Green's function *for (1.1, 1.2), in the case where $a(x)$ is locally Hölder continuous, satisfies* *(i) $G(\lambda, x, y)$ is a classical solution of the homogeneous form of (1.1) in $\mathbb{R}^N \setminus \{y\}$ for each fixed $y \in \mathbb{R}^N$;*

(ii) For $f \in C_0^\infty(\mathbb{R})$, u given by (1.4), with G replacing g, is the classical solution of (1.1, 1.2).

We remark that a unique classical Green's function always exists when $a(x)$ is locally Hölder continuous, see for instance the book of Itô [9]. Further properties of classical Green's functions, together with references to the original literature are given in [9] and the book of Shimakura [12]. The latter book also summarises the literature on various weaker notions of Green's functions as kernel representations of the solution operator of elliptic boundary value problems defined on Sobolev spaces.

In Sections 2 and 3, we assume, in a Cartesian frame of reference, that, for some $M \in [0, \infty)^N$, $a = (a_i)$ obeys

$$\|a_i\|_\infty \le M_i, \qquad i = 1, 2, \ldots, N. \tag{1.6}$$

Our main result states that the two families $g_{\pm M}(z;\, \lambda, x, y)$ of weak Green's functions for the equations

$$(L_M(z) + \lambda)u \equiv -\Delta u \pm \sum_{i=1}^{N} M_i \operatorname{sign}[x_i - z_i] \frac{\partial u}{\partial x_i} + \lambda u = f, \qquad x \in \mathbb{R}^N, \tag{1.7}$$

defined for $z \in \mathbb{R}^N$, satisfy

$$g_{-M}(y;\, \lambda, x, y) \le g(\lambda, x, y) \le g_M(y;\, \lambda, x, y), \qquad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D,$$

for all weak Green's functions $g(\lambda, x, y)$ corresponding to a problem of the form (1.1, 1.2, 1.6).

This result is essentially a Laplace-transformation of a corresponding theorem recently obtained for weak heat kernels in [8]. Much of the analytical machinery used in the proof is analogous to the work of Aronson [1] on heat kernels. We remark that the Laplace-transform relationship between the heat kernel and the elliptic Green's function has been extensively studied in the classical case by Itô [9]. The idea of a maximal elliptic operator, amongst a class of operators all obeying a coefficient constraint in L_∞, was considered by Pucci [11].

In Section 4, we consider problem (1.1, 1.2) under the alternative spherically symmetric constraint

$$\|a\|_\infty = \left\| \sum_{i=1}^{N} a_i^2 \right\|_\infty^{1/2} \le M, \qquad \text{for some } M \ge 0. \tag{1.8}$$

A theorem on the pointwise approximation of weak Green's functions by classical ones, established in Section 3, is applied here to prove that the maximal and minimal weak Green's functions in this case are given by $w_{\pm M}(\lambda,\, |x-y|)$, where $w_{\pm M}(\lambda,\, r)$ are solutions of the ODEs

$$c_N r^{N-1}\left(-u'' - \frac{(N-1)}{r}u' \pm Mu' + \lambda u\right) = \delta(r), \qquad r \geq 0,$$

$\delta(r)$ is the Dirac distribution, and c_N is the spherical constant. The advantage of considering this case in addition to (1.6) is that the formulae derived here for the optimal weak Green's functions and $\|(L+\lambda I)^{-1}\|_{L_1(\mathbb{R}^N)}$ are considerably simpler and easier to compute explicitly than those for the Cartesian case. More specifically, we find that $w_{\pm M}(\lambda,\, r)$ may be written in terms of the singular Kummer hypergeometric function $U(a,\, b,\, x)$, whilst the optimal bounds in $L_1(\mathbb{R}^N)$ are expressed using the Gauss hypergeometric function $F(a,\, b,\, c,\, x)$.

We remark that our Cartesian results may be carried over to the case where the diffusion is not the same in each coordinate direction, but still a positive constant, using scaling arguments. Also, because the pointwise value of the Green's function for Dirichlet boundary conditions increases monotonically with the size of the domain, the upper bounds on $g(\lambda,\, x,\, y)$ and $\|(L+\lambda I)^{-1}\|_1$ apply to any subdomain of $\mathbb{R}^N$. When combined with the straightforward estimate, $\|(L+\lambda I)^{-1}\|_\infty \leq 1/\lambda$, one may use interpolation to obtain reasonably sharp upper bounds for $\|(L+\lambda I)^{-1}\|_p$, $p \in (1,\, \infty)$. Such quantitative estimates have a variety of applications in the analysis of numerical methods and control systems. Finally, we note that the important case $\lambda = 0$ cannot be considered within the above framework, since, under assumption (1.6), $g(0,\, x,\, y)$ may fail to be the kernel of a bounded operator on $L_1(\mathbb{R}^N)$, or even to exist in a pointwise sense.

2 Estimates in a Cartesian frame

In order to state the upper and lower pointwise and L_1 bounds that are our main results, we introduce the function

$$v_M(t,\, x) = \frac{1}{2\sqrt{\pi t}}\exp\left\{-\frac{(|x|-Mt)^2}{4t}\right\} + \frac{M}{4}\mathrm{erfc}\left(\frac{|x|}{2\sqrt{t}} - \frac{M\sqrt{t}}{2}\right), \qquad (2.1)$$

for $(t,\, x) \in (0,\, \infty) \times \mathbb{R}$ and $M \in \mathbb{R}$, which we first considered in [8]. We recall here, see [2, Chapter 7], that

$$\begin{aligned} \mathrm{i}^n\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{t^n}{n!} \exp(-(t+x)^2)\, dt, \qquad x \in \mathbb{R},\ n = 0,\, 1,\, 2,\, \ldots, \\ \mathrm{erfc}(x) &= \mathrm{i}^0\mathrm{erfc}(x), \qquad \mathrm{erf}(x) = 1 - \mathrm{erfc}(x). \end{aligned}$$

We define

$$V_M(t,\, x) \equiv \prod_{i=1}^N v_{M_i}(t,\, x_i), \qquad (t,\, x) \in (0,\, \infty) \times \mathbb{R}^N,\ M \in \mathbb{R}^N, \tag{2.2}$$

$$\Gamma_M(\lambda,\, x) \equiv \int_0^\infty e^{-\lambda t} V_M(t,\, x)\, dt, \qquad x \in \mathbb{R}^N \setminus \{0\},\ M \in \mathbb{R}^N,\ \lambda > 0. \tag{2.3}$$

We observe that $\Gamma_M(\lambda,\, x)$ exists due to the uniform boundedness of $V_M(t,\, x)$ for $x \in \mathbb{R}^N \setminus \{0\}$.

Our main pointwise theorem is as follows:

Theorem 2.1 *Suppose that a problem of the form (1.1, 1.2), for some $\lambda > 0$, has coefficient $a \in L_\infty(\mathbb{R}^N)$ obeying (1.6) for some $M \in [0,\, \infty)^N$. Then, the corresponding weak Green's function satisfies*

$$\Gamma_{-M}(\lambda,\, x - y) \le g(\lambda,\, x,\, y) \le \Gamma_M(\lambda,\, x - y), \qquad (x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D. \tag{2.4}$$

For each $y \in \mathbb{R}^N$, these bounds are attained for all $x \in \mathbb{R}^N$ by the weak Green's function $g_{\pm M}(z;\, \lambda,\, x,\, y)$ of (1.7) when $z = y$.

Suppose that for fixed $\lambda > 0$ one considers the set of locally Hölder continuous $a(x)$, obeying constraint (1.6) for a certain $M \in [0,\, \infty)^N$. Then, for each $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D$, the supremum and infimum of the corresponding set of classical Green's functions $G(\lambda,\, x,\, y)$ for problems of the form (1.1, 1.2) are given by the upper and lower bounds of (2.4) respectively.

The basis of the proof of Theorem 2.1 is the justification of a Laplace transform pair relationship betweeen $g(\lambda,\, x,\, y)$ and the corresponding weak heat kernel $h(t,\, x,\, y)$. Here, we note that Aronson [1] demonstrated the existence of a weak heat kernel $h(t,\, x,\, y)$ for the equation

$$u_t + Lu = 0, \qquad (t,\, x) \in (0,\, \infty) \times \mathbb{R}^N, \tag{2.5}$$

under assumption (1.2). That is, $h(t, x, y)$ is a continuous function for $t > 0$, $x, y \in \mathbb{R}^N$, and, for $f \in C_0^\infty(\mathbb{R}^N)$,

$$v(t, x) = \int_{\mathbb{R}^N} h(t, x, y) f(y)\, dy \tag{2.6}$$

is the unique weak solution of (2.5) such that $\lim_{t\to 0} \|v(t) - f\|_{L_2(\mathbb{R}^N)} = 0$ and $v(t, x)$ is the pointwise limit of the sequence $(v_k(t, x))$, where v_k is the solution of the heat equation corresponding to (1.3) on B_k.

The relationship between $g(\lambda, x, y)$ and $h(t, x, y)$ is given below by Theorem 2.2. The proof of Theorem 2.1 then essentially follows by Laplace-transforming the inequality

$$V_{-M}(t, x-y) \leq h(t, x, y) \leq V_M(t, x-y), \quad t > 0,\ x, y \in \mathbb{R}^N, \tag{2.7}$$

established in [8].

Theorem 2.2 *Suppose that $\lambda > 0$ and that $a \in L_\infty(\mathbb{R}^N)$. Then,*

$$g(\lambda, x, y) = \int_0^\infty e^{-\lambda t} h(t, x, y)\, dt, \qquad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D. \tag{2.8}$$

A classical Green's function is also a weak Green's function, and there exists a sequence of classical Green's functions $(G^n(\lambda, x, y))$ for elliptic operators $(L^n + \lambda)$, where L^n satisfies (1.6), which converges to $g(\lambda, x, y)$ uniformly on compact subsets of $\mathbb{R}^N \times \mathbb{R}^N \setminus D$.

Before stating Theorem 2.3, we introduce the notation

$$K = \{f \in L_1(\mathbb{R}^N) \ :\ f \geq 0 \quad \text{a.e. in } \mathbb{R}^N\}, \tag{2.9}$$

for the non-negative cone in $L_1(\mathbb{R}^N)$, see e.g. Dautray & Lions [3, p. 188].

Theorem 2.3 *Suppose that $\lambda > 0$ and $M \in [0, \infty)^N$ are fixed and that L is of the form (1.1, 1.6). Then, $(L + \lambda I)^{-1}$ may be extended to a bounded linear mapping on $L_1(\mathbb{R}^N)$, and for all $u, f \in K \setminus \{0\}$ such that $(L + \lambda I)^{-1} f = u$,*

$$\int_{\mathbb{R}^N} \Gamma_{-M}(\lambda, x)\, dx \leq \frac{\|u\|_1}{\|f\|_1} \leq \int_{\mathbb{R}^N} \Gamma_M(\lambda, x)\, dx < \infty. \tag{2.10}$$

For the given parameters λ and M, the upper and lower bounds for $\|u\|_1/\|f\|_1$ are the supremum and infimum over the set of problems of the form (1.1, 1.2, 1.6) for $f \in K$. Furthermore, the upper bound in (2.10) also holds for all $u, f \in L_1(\mathbb{R}^N) \setminus \{0\}$ such that $(L + \lambda I)^{-1} f = u$.

We shall prove these theorems in Section 3. In the remainder of this section we investigate the behaviour of the function $\Gamma_M(\lambda,\, x)$, for $M \in \mathbb{R}^N$ and $\lambda > 0$.

Initially, we consider the pointwise form of $\Gamma_M(\lambda,\, x)$. From (2.2) and (2.3),

$$\Gamma_M(\lambda,\, x) = \int_0^\infty e^{-\lambda t} \prod_{i=1}^N v_{M_i}(t,\, x_i)\, dt, \qquad x \in \mathbb{R}^N \setminus \{0\}. \tag{2.11}$$

This formula can be combined with either (2.1) or the alternative expression,

$$v_M(x,\, t) = \frac{1}{4\sqrt{\pi t^3}} \int_{|x|}^\infty z \exp\left\{ -\left(\frac{z}{2\sqrt{t}} - \frac{M\sqrt{t}}{2} \right)^2 \right\} dz, \qquad x,\, M \in \mathbb{R}, \tag{2.12}$$

which one may deduce to be equal to (2.1) using integration by parts. Identity (2.12) has some computational advantages over (2.1) when $M < 0$, since in this case severe cancellation occurs between the two terms in (2.1) when $|M|t \gg |x|$.

Below, we obtain expressions for $\Gamma_M(\lambda,\, x)$ in which the t-variable has been explicitly integrated in (2.11). Before stating these, we introduce the operator

$$J_i(f(x_1,\, \ldots,\, x_i,\, \ldots,\, x_N)) \equiv \int_{|x_i|}^\infty f(x_1,\, \ldots,\, z,\, \ldots,\, x_N)\, dz,$$

for $i = 1,\, \ldots,\, N$. We follow [2] in using the notation $K_\nu(x)$ for the modified Bessel function of order $\nu \in \mathbb{R}$.

Proposition 2.4 *Suppose $\lambda > 0$ and $M \in \mathbb{R}^N$. Then, when $N = 1$,*

$$\Gamma_M(\lambda,\, x) = \frac{e^{-(\beta - M/2)|x|}}{2\beta - M}, \qquad \beta = \sqrt{\lambda + M^2/4}. \tag{2.13}$$

When $N \geq 2$, $\Gamma_M(\lambda,\, x)$ is given by the multinomial-type expansion

$$\begin{aligned} &\Gamma_M(\lambda,\, x) \\ &\quad = \gamma_M(\lambda,\, x) + \sum_i \frac{M_i}{2} J_i\left(\gamma_M(\lambda,\, x)\right) + \sum_{i<j} \frac{M_i}{2}\frac{M_j}{2} J_i J_j\left(\gamma_M(\lambda,\, x)\right) + \cdots \\ &\qquad \cdots + \left(\prod_{i=1}^N \frac{M_i}{2} \right) \left(\prod_{i=1}^N J_i \right) \left(\gamma_M(\lambda,\, x)\right), \qquad x \in \mathbb{R}^N \setminus \{0\}, \end{aligned} \tag{2.14}$$

where

$$\gamma_M(\lambda,\, x) = \left(\prod_{i=1}^N \frac{e^{M_i|x_i|/2}}{\sqrt{2\pi}} \right) \left(\frac{\beta}{|x|} \right)^\nu K_\nu(\beta|x|), \qquad x \in \mathbb{R}^N \setminus \{0\},$$

$\nu = (N-2)/2$ and $\beta = \sqrt{\lambda + (M_1^2 + M_2^2 + \ldots + M_N^2)/4}$. An alternative expression in the case $N \geq 2$ is

$$\Gamma_M(\lambda,\, x) = \left(\prod_{i=1}^{N} J_i\right) \left\{ \left(\prod_{i=1}^{N} \frac{x_i e^{M_i|x_i|/2}}{\sqrt{2\pi}}\right) \left(\frac{\beta}{|x|}\right)^{\mu} K_\mu(\beta|x|) \right\}, \tag{2.15}$$

where $\mu = (3N-2)/2$.

Proof: (2.13) is the result of an elementary calculation. According to Erdélyi *et al.* [5, Formula (29) p. 146],

$$\int_0^\infty e^{-\lambda t} t^{\nu-1} e^{-\alpha^2/(4t)}\, dt = 2\left(\frac{\alpha}{2\sqrt{\lambda}}\right)^{\nu} K_\nu(\alpha\sqrt{\lambda}), \qquad \alpha,\, \lambda > 0, \qquad \nu \in \mathbb{R}. \tag{2.16}$$

On the other hand, we remark from (2.1) and the definition of $\operatorname{erfc}(x)$ that

$$v_{M_i}(t,\, x_i) = \left(1 + \frac{M_i}{2} J_i\right) \left(\frac{1}{2\sqrt{\pi t}} \exp\left\{\frac{(|x_i| - M_i t)^2}{4t}\right\}\right). \tag{2.17}$$

Substituting this into (2.11), Fubini's theorem implies that t-integration commutes with the operators J_i. Applying (2.16), we obtain (2.14). That all the terms on the right-hand side of (2.14) are individually well-defined may be seen from the formulae [2, pp. 375, 378],

$$\begin{aligned} K_0(x) &\sim -\log(x), \quad K_\nu(x) \sim \frac{1}{2}\Gamma(\nu)(x/2)^{-\nu},\ \nu > 0, \qquad \text{as } x \to 0, \\ K_\nu(x) &\sim \sqrt{\frac{\pi}{2\pi}} e^{-x},\ \nu \geq 0 \qquad \text{as } x \to \infty. \end{aligned}$$

(2.15) is obtained from (2.11) and (2.12) by a similar procedure. $\square$

It is not too difficult to see that each of the terms in (2.14) independently satisfies the equation

$$-\Delta u + \sum_{i=1}^{N} M_i \operatorname{sign}[x_i] \frac{\partial u}{\partial x_i} + \lambda u = 0 \tag{2.18}$$

in the interior of each of the 2^N hyperoctant regions bounded by the coordinate hyperplanes $x_i = 0$, $i = 1, \ldots, N$. Furthermore, cf. [12, p. 25], the principal term $\gamma_M(\lambda,\, x)$ posseses the critical singularity of $O(\log(1/|x|))$ for $N = 2$ and $O(|x|^{N-2})$ for $N \geq 3$. The remaining less singular terms in (2.14) are necessary to ensure that $\Gamma_M(\lambda,\, x)$ satisfies (2.18) across the coordinate hyperplanes, and thus throughout $\mathbb{R}^N \setminus \{0\}$.

In the case where $M \in (0,\,\infty)^N$, the final term in (2.14) dominates the behaviour of $\Gamma_M(\lambda,\,x)$ for large $|x|$. (The case $M \in [0,\,\infty)$ essentially reduces to the strictly positive case after the exclusion of the terms in which $M_i = 0$.) When $M \in (-\infty,\,0]^N$, substantial cancellation occurs between the two components of $v_{M_i}(x_i,\,t)$ when $|M_i|t \gg |x_i|$, for each $i \in [1,\,N]$. In such cases, it is easier to study the behaviour by considering (2.15), or the unintegrated form (2.11) together with (2.12).

The following proposition quantifies the upper and lower bounds given in (2.10) by Theorem 2.3.

Proposition 2.5 *When $N = 1$, $\lambda > 0$ and $M \geq 0$,*

$$\int_{\mathbb{R}} \Gamma_{\pm M}(\lambda,\,x)\,dx = \frac{1}{(\beta \mp M/2)^2}, \qquad \beta = \sqrt{\lambda + M^2/4}. \tag{2.19}$$

When $N \geq 2$, $\lambda > 0$ and $M \in [0,\,\infty)^N$,

$$2^{-N} C_0(N,\,\lambda,\,M) \leq \int_{\mathbb{R}^N} \Gamma_M(\lambda,\,x)\,dx \leq C_0(N,\,\lambda,\,M), \tag{2.20}$$

where $C_0(N,\,\lambda,\,M)$ is given by the multinomial expansion

$$\frac{2^N}{\lambda} + \frac{2^{(N-1)}\sum_i M_i^2}{\lambda^2} + \frac{2^{(N-2)}\sum_{i>j} M_i^2 M_j^2}{\lambda^3} + \cdots + \frac{\left(\prod_{i=1}^N M_i^2\right)}{\lambda^{N+1}}. \tag{2.21}$$

Furthermore,

$$\frac{1}{\beta^2}\left(1 - \frac{\sum_i M_i}{\beta}\mathrm{erf}\left(\frac{\sqrt{\pi}\beta}{2\sum_i M_i}\right)\right) \leq \int_{\mathbb{R}^N} \Gamma_{-M}(\lambda,\,x)\,dx \leq \frac{1}{\beta^2}, \tag{2.22}$$

where $\beta = \sqrt{\lambda + (M_1^2 + \cdots + M_N^2)/4}$.

Proof: The result for $N = 1$ follows immediately from (2.13). When $N \geq 2$, we apply a result of [8], which states that

$$\int_{\mathbb{R}^N} V_M(t,\,x)\,dx = \prod_{i=1}^N p\left(\frac{M_i\sqrt{t}}{2}\right), \tag{2.23}$$

where

$$p(x) \equiv (1 + 2x^2)\mathrm{erfc}(-x) + \frac{2x}{\sqrt{\pi}}\exp(-x^2) = 4.\mathrm{i}^2\mathrm{erfc}(-x), \qquad x \in \mathbb{R}. \tag{2.24}$$

$p(x)$ is increasing for $x \in \mathbb{R}$, whilst $p(x) + p(-x) = 2 + 4x^2$. Thus,

$$1 + 2x^2 \leq p(x) \leq 2 + 4x^2, \qquad x \geq 0. \tag{2.25}$$

Substituting (2.25) into (2.23), multiplying by $e^{-\lambda t}$ and integrating with respect to t, we obtain the upper and lower bounds (2.20, 2.21) for

$$\begin{aligned}\int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^N} V_M(t,\, x)\, dx\, dt &= \int_{\mathbb{R}^N} \int_0^\infty e^{-\lambda t} V_M(t,\, x)\, dt\, dx \\ &= \int_{\mathbb{R}^N} \Gamma_M(\lambda,\, x)\, dx.\end{aligned}$$

For the integral of $\Gamma_{-M}(\lambda,\, x)$, we note [2, p. 300] that

$$\frac{d}{dx}(e^{x^2} \mathrm{i}^n \mathrm{erfc}(x)) = -2(n+1) e^{x^2} \mathrm{i}^{n+1} \mathrm{erfc}(x), \quad x \in \mathbb{R},\ n = 0,\, 1,\, 2,\, \ldots.$$

Defining $q(x) \equiv e^{x^2} p(x)$, we deduce that $q'(x) < 0 < q''(x)$ for $x > 0$. Thus,

$$1 - \frac{4x}{\sqrt{\pi}} = q(0) + x q'(0) \leq q(x) \leq q(0) = 1, \qquad x \geq 0,$$

where we have used $q'(0) = -24.\mathrm{i}^3\mathrm{erfc}(0) = -4\pi^{-1/2}$, see [2]. Consequently,

$$\begin{aligned}\max\left\{ e^{-\sum_i M_i^2 t/4} \left(1 - \frac{2\sum_i M_i \sqrt{t}}{\sqrt{\pi}}\right),\, 0 \right\} &\leq \prod_{i=1}^N p\left(-\frac{M_i \sqrt{t}}{2}\right) \\ &\leq e^{-\sum_i M_i^2 t/4}.\end{aligned}$$

Substituting into (2.23) and Laplace-transforming, we obtain (2.22). □

The accuracy of the upper and lower bounds of (2.20) depends upon the sharpness of the two bounds in (2.25). When $\lambda \gg \sum_i M_i^2/4$, the principal contribution to the Laplace transform of (2.23) comes from small values of $x = M_i\sqrt{t}/2$, where $p(x)$ is closer to $1 + 2x^2$, and thus the lower bound in (2.20) is sharper. On the other hand, when $\lambda \ll \sum_i M_i^2/4$, larger values of $x = M_i\sqrt{t}/2$ are of almost equal significance. Here, we remark that the equation $p(x) = 2 + 4x^2 - p(-x)$, combined with the rapidly decaying asymptotic expansion [2, p. 300]

$$p(-x) \sim \frac{\exp(-x^2)}{\sqrt{\pi}|x|^3} \sum_{m=0}^\infty \frac{(-1)^m (2m+2)!}{2(2x)^{2m} m!},$$

for x large and positive, imply that the upper bound in (2.25) is very sharp for large x. Accordingly, we expect the upper bound in (2.20) to be fairly accurate for small λ. In fact, the results of Section 4 will demonstrate, in the case

$M = (M, M, \ldots, M)^T \in \mathbb{R}^N$ and λ small, that the coefficient of the term of $O(\lambda^{-(N+1)})$ is exact.

Clearly, the bounds of (2.22) are quantitatively accurate when λ is large, but not sharp when λ is small. For particular values of the parameters, one may instead find the Laplace transform of (2.23) by numerical integration, using a high-order Gauss-Laguerre quadrature, following a change of integration variable to $\tau = \lambda t$. Alternatively, one may use the more precise bounds of Section 4, when applicable.

3 Proofs of Cartesian results

We begin with a proof of Theorem 2.2, which uses the upper bound of (2.7), together with a weak convergence argument similar to that used by Aronson [1] for the parabolic case.

Proof of Theorem 2.2: Let B_k and L_k be as in (1.3). It is well-known [7, Chapter 1] that L_k, considered as an operator from $H^2(B_k) \cap H_0^1(B_k)$ to $H(B_k)$ is the generator of the analytic semigroup $e^{-L_k t}$ on $H(B_k)$. By the proof of the Hille-Yosida theorem [10, Section 1.3],

$$(L_k + \lambda I)^{-1} f_k = \int_0^\infty e^{-\lambda t} e^{-L_k t} f_k \, dt, \qquad \text{for all } f_k \in H(B_k).$$

Let $f \in C_0^\infty(\mathbb{R}^N)$ be fixed, and let f_k be its restriction to B_k. The equation

$$u_t + L_k u = 0, \quad (x, t) \in B_k, \tag{3.1}$$

with boundary conditions $u(t, x) = 0$ for $(t, x) \in (0, \infty) \times B_k$, and initial data $u(x, 0) = f_k(x)$ for $x \in B_k$, possesses the solution $v_k = e^{-L_k t} f_k$, which is unique in the space $L^\infty[[0, \infty); H(B_k)] \cap L^2[(0, \infty); H_0^1(B_k)]$. On the other hand, $v_k(t, x)$ has the following representation in terms of the weak heat kernel $h_k(t, x, y)$, (see [1]):

$$v_k(t, x) = \int_{B_k} h_k(t, x, y) f_k(y) \, dy.$$

Hence, the solution u_k of (1.3) satisfies

$$\begin{aligned} u_k(x) &= ((L_k + \lambda I)^{-1} f_k)(x) \\ &= \left(\int_0^\infty e^{-\lambda t} e^{-L_k t} f_k \, dt \right)(x) = \int_0^\infty e^{-\lambda t} v_k(t, x) \, dt \end{aligned}$$

$$= \int_0^\infty e^{-\lambda t} \int_{B_k} h_k(t,\, x,\, y) f_k(y)\, dy\, dt, \qquad x \in B_k.$$

From [1, Lemma 7] and (2.7),

$$0 \le h_k(t,\, x,\, y) \le h(t,\, x,\, y) \le V_M(t,\, x-y), \quad t > 0, \;\; x,\, y \in \mathbb{R}^N. \tag{3.2}$$

[1, Lemma 7] also states that $h_k(t,\, x,\, y) \to h(t,\, x,\, y)$ uniformly over compact subsets of $(0,\, \infty) \times \mathbb{R}^N \times \mathbb{R}^N$. Thus, by Proposition 2.5, the dominated convergence theorem and Fubini's theorem, $u_k(x) \to u(x)$ for each $x \in \mathbb{R}^N$, where

$$u(x) = \int_{\mathbb{R}^N} r(\lambda,\, x,\, y) f(y)\, dy, \qquad r(\lambda,\, x,\, y) = \int_0^\infty e^{-\lambda t} h(t,\, x,\, y)\, dy. \tag{3.3}$$

By (3.2), the uniform boundedness of $V_M(t,\, x-y)$ over $t \in (0,\, \infty)$ for fixed $(x,\, y) \notin D$ also applies to $h(t,\, x,\, y)$. Since, in addition, $h(t,\, x,\, y)$ is continuous for $t > 0$, we deduce that $r(\lambda,\, x,\, y)$ is also continuous in $\mathbb{R}^N \times \mathbb{R}^N \setminus D$.

We note that, by (3.2), (3.3) and (2.3), $r(\lambda,\, x,\, y) \le \Gamma_M(\lambda,\, x-y)$. Thus, fixing $x \in \mathbb{R}^N$ and considering $\epsilon > 0$, there is a $\delta > 0$, such that for all $z \in B(x,\, \delta/2)$,

$$\int_{B(x,\delta)} r(\lambda,\, z,\, y) f(y)\, dy < \epsilon/3.$$

Taking into account the uniform continuity of $r(\lambda,\, z,\, y)$ for $(z,\, y) \in \overline{B}(x,\, \delta/2) \times \overline{B}(0,\, R) \setminus B(x,\, \delta)$, where $\overline{B}(0,\, R) \supset \mathrm{supp}(f)$, there is an $\eta \in (0,\, \delta/2]$ such that

$$\int_{\overline{B}(0,R)\setminus B(x,\delta)} (r(\lambda,\, x,\, y) - r(\lambda,\, z,\, y)) f(y)\, dy < \epsilon/3, \qquad \text{for all } z \in B(x,\, \eta).$$

Thus, we deduce the continuity of $u(x)$ for $x \in \mathbb{R}^N$.

It is well-known [7, Chapter 3] that $v_k = e^{-L_k t} f_k$ is of sufficient regularity that all terms in (3.1) are defined in $H(B_k)$ for $t > 0$. Taking the inner product in $H(B_k)$ with $e^{-\lambda t} v_k(t)$, we obtain the following inequality for $w_k(t) = v_k(t) e^{-\lambda t/2}$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_k\|_2^2 + \|\nabla w_k\|_2^2 &= -\frac{\lambda}{2}\|w_k\|_2^2 - (a.\nabla w_k,\, w_k) \\ &\le \frac{\sum_{i=1}^N M_i^2}{2}\|w_k\|_2^2 + \frac{1}{2}\|\nabla w_k\|_2^2, \qquad t > 0. \end{aligned}$$

Since $w_k(0) = f_k$, we deduce that for $T > 0$,

$$\int_0^T \|\nabla w_k\|_2^2\, dt \le \|f_k\|_2^2 + \sum_{i=1}^N M_i^2 \int_0^T \|w_k(t)\|_2^2\, dt. \tag{3.4}$$

The maximum principle applied to (3.1) implies that

$$\|w_k(t)\|_{L_\infty(B_k)} = e^{-\lambda t/2}\|v_k(t)\|_{L_\infty(B_k)} \leq e^{-\lambda t/2}\|f\|_{L_\infty(\mathbb{R}^N)}.$$

On the other hand, the results of [8, Section 4] imply that

$$\|w_k(t)\|_{L_1(B_k)} = e^{-\lambda t/2}\|v_k(t)\|_{L_1(B_k)} \leq e^{-\lambda t/2}\prod_{i=1}^{N}(2+M_i^2 t)\|f\|_{L_1(\mathbb{R}^N)}.$$

As $\|w_k(t)\|_2^2 \leq \|w_k(t)\|_{L_1(B_k)}\|w_k(t)\|_{L_\infty(B_k)}$, we deduce, on substituting the above bounds into (3.4) and letting $T \to \infty$, that

$$\int_0^\infty \|w_k(t)\|_2^2 + \|\nabla w_k(t)\|_2^2\, dt = C_1(N,\, \lambda,\, M,\, f), \tag{3.5}$$

where $C_1(N,\, \lambda,\, M,\, f)$ is a constant independent of k.

Hence, by the Schwartz inequality,

$$\begin{aligned}
\|u_k\|_{H^1(B_k)}^2 &= \|u_k\|_2^2 + \|\nabla u_k\|_2^2 \\
&\leq \left(\int_0^\infty e^{-\lambda t}\|v_k(t)\|_2\, dt\right)^2 + \left(\int_0^\infty e^{-\lambda t}\|\nabla v_k(t)\|_2\, dt\right)^2 \\
&\leq \int_0^\infty e^{-\lambda t}\, dt \left(\int_0^\infty \|w_k(t)\|_2^2 + \|\nabla w_k(t)\|_2^2\, dt\right) \\
&\leq \frac{C_1(N,\, \lambda,\, M,\, f)}{\lambda}.
\end{aligned} \tag{3.6}$$

Extending the definition of $u_k(x)$ by 0 for $|x| > k$, the sequence $\left(\|u_k\|_{H^1(\mathbb{R}^N)}\right)$ is uniformly bounded. Thus, (u_k) possesses a subsequence converging weakly in $H^1(\mathbb{R}^N)$ and strongly in $H_{\text{loc}}(\mathbb{R}^N)$ to some limit $\overline{u}$. However, $\overline{u}$ must coincide almost everywhere with u given by (3.3). Thus, $u \in H^1(\mathbb{R}^N)$.

Given $\phi \in C_0^1(\mathbb{R}^N)$, the fact that f has compact support implies that, for all k sufficiently large,

$$\int_{\mathbb{R}^N} \nabla u_k.\nabla\phi + (a(x).\nabla u_k)\phi + \lambda u_k\phi - f\phi\, dx.$$

Thus, taking the limit through the weakly $H^1(\mathbb{R}^N)$-convergent subsequence of the last paragraph, u continues to satisfy this relation for all such ϕ. Consequently, $r(\lambda,\, x,\, y)$ is the weak Green's function $g(\lambda,\, x,\, y)$, as required. The uniqueness of $g(\lambda,\, x,\, y)$ follows from that of $u(x)$.

When $a(x)$ is also locally Hölder continuous, integration by parts implies that the classical solution $u(x)$ for $(1.1, 1.2)$ is also the unique weak solution. Both weak and classical solutions may be written in the form (1.4), with $G(\lambda,\, x,\, y)$ replacing $g(\lambda,\, x,\, y)$ in the classical case, and hence these two integral kernels are the same.

Aronson [1, p. 681] showed that, given $h(t,\, x,\, y)$, the weak heat kernel for (2.5), there exists a sequence $(H^n(t,\, x,\, y))$ of classical heat kernels which converges to $h(t,\, x,\, y)$ uniformly on compact subsets of $(0,\, \infty) \times \mathbb{R}^N \times \mathbb{R}^N$. It is not difficult to verify (see [8]) that the H^n may be chosen to be the classical heat kernels for operators L^n satisfying (1.6). Aronson also showed that the classical heat kernel coincides with the weak heat kernel, so that the result just proved for the weak case implies that the Laplace transform of $H^n(t,\, x,\, y)$ is the classical Green's function $G^n(\lambda,\, x,\, y)$ for the corresponding elliptic problem; (see also Itô [9, Theorems 18.1, 19.2]). Hence, for fixed $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D$,

$$\begin{aligned}
& |G^n(\lambda,\, x,\, y) - g(\lambda,\, x,\, y)| \\
\leq\ & \int_0^\infty e^{-\lambda t} |H^n(t,\, x,\, y) - h(t,\, x,\, y)|\, dt \\
\leq\ & 2\int_0^\eta e^{-\lambda t} V_M(t,\, x-y)\, dt + \int_\eta^T e^{-\lambda t} |H^n(t,\, x,\, y) - h(t,\, x,\, y)|\, dt \\
& \quad +2\int_T^\infty e^{-\lambda t} V_M(t,\, x-y)\, dt,
\end{aligned}$$

where $0 < \eta < T$. Given $\epsilon > 0$, one may take η small enough and T large enough so that the first and third integrals on the right-hand side are each bounded by $\epsilon/3$. Then, for sufficiently large n, the second term is also less than $\epsilon/3$. Thus, $G^n(\lambda,\, x,\, y) \to g(\lambda,\, x,\, y)$. As asserted above, $g(\lambda,\, x,\, y)$ is a continuous function of $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D$, and hence the convergence of the G^n's is uniform in compact subsets of this region. $\square$

Proof of Theorem 2.1: (2.4) follows from (2.7) and Theorem 2.2. It was established in [8] that $V_{\pm M}(t,\, x-y) = h_{\pm M}(y;\, t,\, x,\, y)$, where $h_{\pm M}(z;\, t,\, x,\, y)$ are the weak heat kernels for $u_t + L_{\pm M}(z)u = 0$, and $L_{\pm M}(z)u$ are given by (1.7). Hence, Theorem 2.2 shows that $\Gamma_{\pm M}(\lambda,\, x-y) = g_{\pm M}(y;\, \lambda,\, x,\, y)$, where $g_{\pm M}(z;\, \lambda,\, x,\, y)$ are the weak Green's functions for (1.7). Thus, the bounds of (2.4) are attained in the class of weak Green's functions.

Since classical Green's functions are also weak Green's functions, the bounds (2.4) also apply. The last part of Theorem 2.1 follows from the existence, established by Theorem 2.2, of sequences $(G^n_\pm(z;\, \lambda,\, x,\, y))$ of classical Green's functions, for equations satisfying (1.6), converging to $g_{\pm M}(z;\, \lambda,\, x,\, y)$ pointwise.

$\square$

Proof of Theorem 2.3: Let $f \in C_0^\infty(\mathbb{R})$ be non-null and non-negative. By (1.4), the positivity of $g(\lambda,\, x,\, y)$, (which follows from that of $h(t,\, x,\, y)$), Proposition 2.5 and Fubini's theorem, the solution of a problem (1.1, 1.2, 1.6) satisfies

$$\begin{aligned}\|u\|_1 = \int_{\mathbb{R}^N} u(x)\,dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(\lambda,\, x,\, y) f(y)\,dy\,dx \\ &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} g(\lambda,\, x,\, y)\,dx\right) f(y)\,dy.\end{aligned}$$

Hence, we deduce that

$$\inf_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N} g(\lambda,\, x,\, y)\,dx \le \frac{\|u\|_1}{\|f\|_1} \le \sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N} g(\lambda,\, x,\, y)\,dx. \tag{3.7}$$

(2.10) now follows from Theorem 2.1, since $\int_{\mathbb{R}^N} \Gamma_{\pm M}(\lambda,\, x-y)\,dx$ are constant functions of y.

The set of non-negative functions in $C_0^\infty(\mathbb{R})$ is dense in K, (see [3]), and therefore (2.10) implies that the mapping $(L+\lambda I)^{-1}$ may be defined as a mapping from K to itself, where (2.10) continues to hold. Since $K - K = L_1(\mathbb{R}^N)$, we may extend $(L+\lambda I)^{-1}$ to a mapping on $L_1(\mathbb{R}^N)$, and the upper bound in (2.10) still holds.

For $z = 0$, consider the two problems (1.7) and the corresponding weak Green's functions $g_{\pm M}(0;\, \lambda,\, x,\, y)$. Taking a sequence of non-negative functions $f^n \in C_0^\infty(B(0,\, 1/n))$, $n \in \mathbb{N}$, each with unit integral, we denote the corresponding solutions by $u^n_{\pm M}$. Since $g_{\pm M}(0;\, \lambda,\, x,\, 0) = \Gamma_{\pm M}(\lambda,\, x)$,

$$\begin{aligned}\lim_{n\to\infty}\|u^n_{\pm M}\|_1 &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_{\pm M}(0;\, \lambda,\, x,\, y) f^n(y)\,dy\,dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_{\pm M}(0;\, \lambda,\, x,\, y)\,dx\, f^n(y)\,dy \\ &= \int_{\mathbb{R}^N} g_{\pm M}(0;\, \lambda,\, x,\, 0)\,dx = \int_{\mathbb{R}^N} \Gamma_{\pm M}(\lambda,\, x)\,dx.\end{aligned} \tag{3.8}$$

$\square$

4 Estimates under a spherical constraint

In the last part of this paper we consider (1.1, 1.2) under the Euclidean norm constraint (1.8), for $N \ge 2$. (Note that (1.8) coincides with (1.6) when $N = 1$.)

We establish similar results to Theorems 2.1 and 2.3, with the differences here that the sharp bounds in $L_1(\mathbb{R}^N)$ are obtained more explicitly, and the formulae for the optimal Green's functions are simpler. The method of proof here is also more direct, though it relies on Theorem 2.2.

Using the notation of [2], we recall that $U(a,\,b,\,x)$ is the Kummer confluent hypergeometric function of second kind; see [2, Chapter 13] and [4, Chapter VI]. For $\lambda > 0$ and $M \in \mathbb{R}$, we define

$$w_M(\lambda,\,r) = C_M e^{-(\beta - M/2)r} U((N-1)\theta_M,\,(N-1),\,2\beta r), \qquad r > 0, \tag{4.1}$$

where $\beta = \sqrt{\lambda + M^2/4}$,

$$\theta_M = (1/2 - (M/4\beta)), \quad C_M = \frac{(2\beta)^{N-2}\Gamma((N-1)\theta_M)}{c_N (N-2)!}, \quad c_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}; \tag{4.2}$$

i.e. c_N is the spherical constant. For $\lambda > 0$ and $M \in \mathbb{R}$, we define

$$\Phi_M(\lambda,\,x) = w_M(\lambda,\,|x|), \qquad x \in \mathbb{R}^N. \tag{4.3}$$

We shall show that $\Phi_M(\lambda,\,x-y)$ is equal to $g_M(z;\,\lambda,\,x,\,y)$, the weak Green's function for the equation

$$(L_M(z) + \lambda)u \equiv -\Delta u + M\frac{x-z}{|x-z|}.\nabla u + \lambda u = f, \qquad x \in \mathbb{R}^N, \tag{4.4}$$

when $z = y$. Here, $|x-z|$ is the Euclidean norm of the vector and the coefficient $M(x-z)/|x-z|$ is replaced by 0 at the point $x = z$.

The main pointwise theorem in this case is as follows.

Theorem 4.1 *For fixed $\lambda > 0$ and $M \geq 0$, the weak Green's function for a problem of the form (1.1, 1.2, 1.8) satisfies*

$$\Phi_{-M}(\lambda,\,x-y) \leq g(\lambda,\,x,\,y) \leq \Phi_M(\lambda,\,x-y). \tag{4.5}$$

For each $y \in \mathbb{R}^N$, these bounds are attained for all $x \in \mathbb{R}^N$ by $g_{\pm M}(z;\,\lambda,\,x,\,y)$ when $z = y$.

The principal properties of the function Φ_M are given by Lemma 4.3 below. Further properties may be deduced from the references [2, 4] on confluent hypergeometric functions.

We recall, again with the notation of [2], that $F(a,\,b,\,c,\,x)$ is the Gauss hypergeometric function; see [2, Chapter 15] and [4, Chapter II]. The following result is the analogue of Theorem 2.3 for (1.1, 1.2, 1.8).

Theorem 4.2 *Suppose that $\lambda > 0$ and $M \geq 0$ are fixed for the class of problems of the form (1.1, 1.2, 1.8). Then, if $u,\, f \in K \setminus \{0\}$ satisfy $(L+\lambda I)^{-1} f = u$,*

$$\begin{aligned} &\frac{(N-1)\Gamma((N-1)\theta_{-M})}{4\beta^2\Gamma(2+(N-1)\theta_{-M})} F(N,\, 2,\, 2+(N-1)\theta_{-M},\, 1-\theta_{-M}) \\ &\quad \leq \frac{\|u\|_1}{\|f\|_1} \leq \frac{(N-1)\Gamma((N-1)\theta_M)}{4\beta^2\Gamma(2+(N-1)\theta_M)} F(N,\, 2,\, 2+(N-1)\theta_M,\, 1-\theta_M). \end{aligned} \tag{4.6}$$

For the given parameters λ and M, the upper and lower bounds for $\|u\|_1/\|f\|_1$ are the supremum and infimum over the set of problems of the form (1.1, 1.2, 1.8) for $f \in K \setminus \{0\}$. The upper bound in (4.6) also applies to all $u,\, f \in L_1(\mathbb{R}^N) \setminus \{0\}$ such that $u = (L+\lambda I)^{-1} f$.

We discuss some of the implications of (4.6) at the end of the paper.

Before coming to the proof of Theorems 4.1 and 4.2, we need to establish further properties of the functions $w_M(\lambda,\, r)$.

Lemma 4.3 *For fixed $\lambda > 0$ and $M \in \mathbb{R}$, $u(r) = w_M(\lambda,\, r)$ satisfies*

$$u(r) > 0 \;>\; u'(r), \qquad r > 0, \tag{4.7}$$

$$-u''(r) - \frac{(N-1)}{r} u'(r) + M u'(r) + \lambda u(r) = 0, \qquad r > 0, \tag{4.8}$$

$$\lim_{r \to 0} c_N r^{N-1} u'(r) = -1. \tag{4.9}$$

For large r,

$$u(r),\, u'(r) = O(r^{-(N-1)\theta_M} e^{-(\beta - M/2)r}). \tag{4.10}$$

Proof: According to Slater [2, Chapter 13], $U((N-1)\theta_M,\, (N-1),\, x)$ satisfies

$$x U_{xx} + ((N-1) - x) U_x - (N-1)\theta_M U = 0, \qquad x > 0.$$

Changing the variable to $r = x/(2\beta)$, and multiplying by the factor $e^{-(\beta - M/2)r}$, it is readily verified that $u(r)$ satisfies (4.8).

The integral representation [2, p. 505]

$$U(a,\, b,\, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1}\, dt, \qquad x,\, a,\, b > 0, \tag{4.11}$$

implies $u(r) > 0$ for $r > 0$. From the identity [2, p. 507],

$$\frac{d}{dx} U(a,\, b,\, x) = -a U(a+1,\, b,\, x) < 0, \qquad x,\, a,\, b > 0, \tag{4.12}$$

and the fact that $e^{-(\beta-M/2)r}$ is decreasing, we deduce that $u'(r)<0$ for $r>0$.

The asymptotic estimate [2, p. 504]

$$U(a,\,n,\,x)=x^{-a}[1+O(x^{-1})],\qquad a>0,\,n\in\mathbb{N},\quad \text{as } x\to\infty,$$

implies (4.10) in the case of $u(r)$. In combination with (4.12), we also deduce (4.10) for $u'(r)$.

According to [2, p. 504], for $N\geq 2$,

$$\begin{aligned} U(a,\,N-1,\,x) &= \frac{(-1)^{N-1}}{(N-2)!\Gamma(a-(N-2))}\left[\log(x)P_{N,a}(x)+Q_{N,a}(x)\right] \\ &+ \frac{(N-3)!}{\Gamma(a)}x^{-(N-2)}p_{N,a}(x), \qquad x\in\mathbb{R},\,a>0, \end{aligned} \tag{4.13}$$

where the last term is omitted when $N=2$. Here, $P_{N,a}(z)$ and $Q_{N,a}(z)$ are absolutely convergent power series with $P_{N,a}(0)=1$, and $p_{N,a}(x)$ is a polynomial of degree $N-3$ such that $p_{N,a}(0)=1$. Thus,

$$\lim_{r\to 0}c_N r^{N-1}u'(r)=-\frac{C_M c_N(N-2)!}{(2\beta)^{(N-2)}\Gamma((N-1)\theta_M)}=-1.$$

□

Lemma 4.4 *Suppose that for some $\lambda>0$ and $M\geq 0$, L is an elliptic operator of the form (1.1, 1.8) such that $a(x)$ is locally Hölder continuous. Then, for all non-negative $f\in C_0^\infty(\mathbb{R})$ and all $x\in\mathbb{R}^N$,*

$$\begin{aligned} &(L+\lambda)\int_{\mathbb{R}^N}\Phi_{-M}(\lambda,\,x-y)f(y)\,dy \\ &\qquad\leq f(x)\leq (L+\lambda)\int_{\mathbb{R}^N}\Phi_M(\lambda,\,x-y)f(y)\,dy. \end{aligned} \tag{4.14}$$

Proof: Let $R>0$ be so large that $B(x,\,R)$ contains the support of f, and let $\Omega_\epsilon=B(x,\,R)\setminus\overline{B}(x,\,\epsilon)$, for $\epsilon>0$. We may now apply a standard argument using Green's identity and (4.9), cf. [12, p. 24], to obtain

$$\begin{aligned} &(-\Delta+\lambda)\int_{\mathbb{R}^N}\Phi_M(\lambda,\,x-y)f(y)\,dy \\ &\quad=\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}(-\Delta_x+\lambda)\Phi_M(\lambda,\,x-y)f(y)\,dy+f(x),\qquad x\in\mathbb{R}^N, \end{aligned}$$

for $M\geq 0$. (4.9) also implies that $\nabla\Phi_M=O(|x-y|^{-(N-1)})$ and is therefore integrable with respect to y over compact subsets of $\mathbb{R}^N$, with an integral over $B(x,\,\epsilon)$

which vanishes in the limit as $\epsilon \to 0$. One x-derivative therefore commutes with y-integration, and we deduce that

$$\begin{aligned}(L+\lambda)\int_{\mathbb{R}^N}&\Phi_M(\lambda,\, x-y)f(y)\,dy\\ &=\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}(L_x+\lambda)\Phi_M(\lambda,\, x-y)f(y)\,dy+f(x).\end{aligned}$$

However, for $y\in\Omega_\epsilon$, (4.7) implies that

$$\begin{aligned}&(L+\lambda)\Phi_M(\lambda,\, x-y)\\ &\quad=(L_M(y)+\lambda)\Phi_M(\lambda,\, x-y)+\left(a(x)-M\frac{x-y}{|x-y|}\right).\nabla\Phi_M(\lambda,\, x-y)\\ &\quad=\left(a(x)-M\frac{x-y}{|x-y|}\right).\nabla\Gamma_M(\lambda,\, x-y)\geq 0,\end{aligned}$$

and therefore the right-hand inequality of (4.14) holds; the proof for the left-hand inequality is similar. $\square$

Proof of Theorem 4.1: For each non-negative $f\in C_0^\infty(\mathbb{R}^N)$, there is an $R_0>0$ such that $B(0,\, R_0)\supset\operatorname{supp}(f)$. Thus, for $|x|>R_0$,

$$\begin{aligned}\int_{\mathbb{R}^N}\Phi_M(\lambda,\, x-y)f(y)\,dy &= \int_{B(0,R_0)}\Phi_M(\lambda,\, x-y)f(y)\,dy\\ &\leq \|f\|_\infty c_N R_0^N w_M(\lambda,\, |x|-R_0),\end{aligned}\tag{4.15}$$

and so, by (4.10), the right-hand side tends to 0 as $|x|\to\infty$.

Let us now take an operator L with coefficient $a(x)$ locally Hölder continuous and satisfying (1.8). By Theorem 2.1, the corresponding Green's function satisfies $G(\lambda,\, x,\, y)\leq\Gamma_M(\lambda,\, x-y)$, for $M=[M,\, M,\, \ldots,\, M]^T$. A similar estimate to (4.15) may be deduced for $\Gamma_M(\lambda,\, x-y)$ from the fact, evident from Proposition 2.4, that $\Gamma_M(\lambda,\, x)\to 0$, as $|x|\to\infty$, to show that $\int_{\mathbb{R}^N}G(\lambda,\, x,\, y)f(y)\,dy\to 0$, as $|x|\to\infty$. Thus, given $\epsilon>0$, there is an $R_\epsilon\geq 1/\epsilon$ such that

$$\sup_{x\in\partial B(0,R_\epsilon)}\left|\int_{\mathbb{R}^N}(\Phi_M(\lambda,\, x-y)-G(\lambda,\, x,\, y))f(y)\,dy\right|\leq\epsilon.$$

By Lemma 4.4 and Definition 1.2,

$$(L+\lambda)\int_{\mathbb{R}^N}(\Phi_M(\lambda,\, x-y)-G(\lambda,\, x,\, y))f(y)\,dy\geq 0.$$

Applying the maximum principle over $B(0,\, R_\epsilon)$, we deduce that

$$\int_{\mathbb{R}^N} (\Phi_M(\lambda,\, x-y) - G(\lambda,\, x,\, y))f(y)\, dy \geq -\epsilon, \qquad x \in B(0,\, R_\epsilon).$$

Thus, letting $\epsilon \to 0$,

$$\int_{\mathbb{R}^N} (\Phi_M(\lambda,\, x-y) - G(\lambda,\, x,\, y))f(y)\, dy \geq 0, \qquad x \in \mathbb{R}^N. \tag{4.16}$$

From Lemma 4.1 and Definition 1.2, any $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D$ is a point of continuity and thus of non-negativity of $\Phi_M(\lambda,\, x-y) - G(\lambda,\, x,\, y)$. A similar argument in the case of $\Phi_{-M}(\lambda,\, x-y)$ establishes that

$$\Phi_{-M}(\lambda,\, x-y) \leq G(\lambda,\, x,\, y) \leq \Phi_M(\lambda,\, x-y), \qquad (x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus D. \tag{4.17}$$

By Theorem 2.2, this inequality extends to the weak Green's function $g(\lambda,\, x,\, y)$ for arbitrary L satisfying (1.8), and so (4.5) holds.

It only remains to prove that

$$\Phi_{\pm M}(\lambda,\, x-y) = g_{\pm M}(y;\, \lambda,\, x,\, y). \tag{4.18}$$

From the translational invariance of (4.4), $g_{\pm M}(z;\, \lambda,\, x,\, y) = g_{\pm M}(0;\, \lambda,\, x-z,\, y-z)$; thus we need only show $\Phi_{\pm M}(\lambda,\, x) = g_{\pm M}(0;\, \lambda,\, x,\, 0)$.

For $n \in \mathbb{N}$, let $f^n \in C_0^\infty(B(0,\, 1/n))$ be a spherically symmetric non-negative function with unit integral. The corresponding solution of (4.4) for $z = 0$ is given by

$$u^n(x) = \int_{\mathbb{R}^N} g_M(0;\, \lambda,\, x,\, y) f^n(y)\, dy. \tag{4.19}$$

By the spherical symmetry of f^n and the differential operator, $u^n(x) = u^n(Sx)$, where S is any coordinate rotation in $\mathbb{R}^N$; i.e. any $S \in SO(N)$. Taking the limit as $n \to \infty$ in (4.19), we conclude that $g_M(0;\, \lambda,\, x,\, 0) = g_M(0;\, \lambda,\, Sx,\, 0)$, and thus there exists a function $\hat{g}_M$ such that

$$g_M(0;\, \lambda,\, x,\, 0) = \hat{g}_M(\lambda,\, |x|), \qquad x \in \mathbb{R}^N.$$

Writing the operator $(L_M(0) + \lambda)$ of (4.4) in polar coordinates, we deduce that $\hat{g}_M(\lambda,\, r)$ satisfies (4.8). Since (4.8) transforms into Kummer's equation, as shown in the proof of Lemma 4.3, the general solution of (4.8) may be inferred from [4, Section 6.7.1]. This yields

$$\begin{aligned} \hat{g}_M(\lambda,\, r) \;&=\; Ae^{-(\beta - M/2)r} M((N-1)\theta_M,\, (N-1),\, 2\beta r) \\ &\qquad + Be^{-(\beta - M/2)r} U((N-1)\theta_M,\, (N-1),\, 2\beta r), \end{aligned} \tag{4.20}$$

for $A,\, B \in \mathbb{R}$, where $M(a,\, b,\, x)$ is the Kummer function of first kind. However, from the upper bound in (4.5), the asymptotic behaviour of $\Phi(\lambda,\, r)$ given by (4.10), and the asymptotic result [2, p. 504],

$$M(a,\, b,\, x) = O(e^x x^{a-b}), \qquad \text{as } x \to \infty,\ \ a,\, b > 0,$$

we conclude that $A = 0$.

Considering the asymptotic behaviour near $r = 0$, the upper bound for g_M in (4.5), together with the results of Lemma 4.3 implies that $B \le C_M$. The lower bound in (4.5), together with (4.13), shows that, for $N \ge 3$,

$$\begin{aligned} 1 &\ge \lim_{r\to 0} \frac{B r^{N-2} e^{-(\beta - M/2)r} U((N-1)\theta_M,\, (N-1),\, 2\beta r)}{r^{N-2} C_{-M} e^{-(\beta-M/2)r} U((N-1)\theta_{-M},\, (N-1),\, 2\beta r)} \\ &= \frac{B\Gamma((N-1)\theta_{-M})}{C_{-M}\Gamma((N-1)\theta_M)} = \frac{B}{C_M}, \end{aligned} \tag{4.21}$$

and thus $B = C_M$. The case $N = 2$ follows similarly, and we conclude that $\hat{g}_{\pm M}(\lambda,\, r) = w_M(\lambda,\, r)$. The same analysis follows for $-M$ in the place of M, and thus (4.18) is proved. $\square$

Proof of Theorem 4.2: In the same way that Theorem 2.3 followed from Theorem 2.1, we obtain from Theorem 4.1 that, for $f \in K \setminus \{0\}$ and $u = (L + \lambda I)^{-1} f$,

$$\int_{\mathbb{R}^N} \Phi_{-M}(\lambda,\, x)\, dx \le \frac{\|u\|_1}{\|f\|_1} \le \int_{\mathbb{R}^N} \Phi_M(\lambda,\, x)\, dx, \tag{4.22}$$

and that these bounds are sharp. Again, the upper bound applies to arbitrary $f,\, u \in L_1(\mathbb{R}^N) \setminus \{0\}$ such that $u = (L + \lambda I)^{-1} f$. Writing θ for θ_M, $\int_{\mathbb{R}^N} \Phi_M(\lambda,\, x)\, dx$ equals

$$\begin{aligned} & C_M c_N \int_0^\infty r^{N-1} e^{(\beta - M/2)r} U((N-1)\theta,\, N-1,\, 2\beta r)\, dr \\ = \;& \frac{C_M c_N}{\Gamma((N-1)\theta)} \int_0^\infty r^{N-1} e^{(\beta-M/2)r} \int_0^\infty e^{-2\beta r t} t^{(N-1)\theta - 1} (t+1)^{(N-1)(1-\theta)-1}\, dt\, dr \\ = \;& \frac{C_M c_N}{\Gamma((N-1)\theta)} \int_0^\infty t^{(N-1)\theta-1} (t+1)^{(N-1)(1-\theta)-1} \int_0^\infty r^{N-1} e^{-(\beta - M/2 + 2\beta t)r}\, dr\, dt \\ = \;& \frac{C_M c_N}{\Gamma((N-1)\theta)} \int_0^\infty t^{(N-1)\theta-1} (t+1)^{(N-1)(1-\theta)-1} \frac{(N-1)!}{(\beta - M/2 + 2\beta t)^N}\, dt \\ = \;& \frac{C_M c_N}{(2\beta)^N \Gamma((N-1)\theta)} \int_0^\infty t^{(N-1)\theta-1} (t+1)^{(N-1)(1-\theta)-1} (t+\theta)^{-N}\, dt. \end{aligned} \tag{4.23}$$

Noting the value of C_M from (4.2), and the identity [4, p. 60]

$$\int_0^\infty x^{\alpha-1}(1+x)^{-\beta+\gamma}(x+\delta)^{-\gamma}\,dx = \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)}F(\gamma,\,\gamma-\alpha,\,\beta,\,1-\delta), \quad (4.24)$$

valid for $\beta > \alpha > 0$, we deduce that $\displaystyle\int_{\mathbb{R}^N} \Phi_M(\lambda,\,x)\,dx$ is equal to the right-hand side of (4.6). The same calculation, with $-M$ instead of M, shows that $\displaystyle\int_{\mathbb{R}^N} \Phi_{-M}(\lambda,\,x)\,dx$ equals the left-hand side of (4.6). $\square$

In discussing inequality (4.6), we may be more precise than was possible following Proposition 2.5, since $\Phi_M(\lambda,\,x)$ has been integrated exactly. The main parameter in (4.6) is $\theta_M(\lambda) = (1/2-(M/4\beta(\lambda))) \in (0,\,1)$ for $\lambda > 0$ and $M \in \mathbb{R}$. Given $M > 0$, this has the limiting behaviour

$$\lim_{\lambda\to 0}\frac{\theta_M(\lambda)}{\lambda} = \frac{1}{M^2}, \quad \lim_{\lambda\to 0}\theta_{-M}(\lambda) = 1, \quad \lim_{\lambda\to\infty}\theta_{\pm M}(\lambda) = \frac{1}{2}. \quad (4.25)$$

We note that, as a consequence of analyticity results, see [5, p. 2 and p. 68], $\Gamma(x)$ is continuous for $x > 0$ and $F(a,\,b,\,c,\,x)$ is continuous with respect to all parameters when $c > 0$ and $x \in (0,\,1)$.

Denoting the upper and lower bounds of (4.6) by $L(\lambda)$ and $l(\lambda)$, we have

$$\begin{aligned}\lim_{\lambda\to\infty}(\lambda+M^2/4)L(\lambda) &= \lim_{\lambda\to\infty}(\lambda+M^2/4)l(\lambda)\\ &= \frac{(N-1)\Gamma((N-1)/2)}{4\Gamma((N+3)/2)}F(N,\,2,\,(N+3)/2,\,1/2) = 1, \end{aligned}\quad (4.26)$$

where the last equality follows from [4, Formula (50) p. 104]. Taking appropriate limits in (4.6), we obtain

$$\lim_{\lambda\to 0} l(\lambda) = \frac{(N-1)\Gamma(N-1)}{4(M^2/4)\Gamma(N+1)}.1 = \frac{1}{NM^2}, \qquad M > 0. \quad (4.27)$$

Considering $L(\lambda)$ for small λ, we note the identities

$$\begin{aligned} F(a,\,b,\,c,\,1-x) &= x^{c-a-b}F(c-a,\,c-b,\,c,\,1-x), \qquad |x| < 1,\\ \lim_{x\to 0} F(c-a,\,c-b,\,c,\,1-x) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \qquad c \neq 0,\,-1,\,-2,\,\ldots, \end{aligned}$$

from [4, Formulae (23) p. 64, (46) p. 104], the second of which requires $a+b > c$. These formulae, together with (4.24) and the result that $\lim_{x\to 0} x\Gamma(x) = 1$, imply that

$$L(\lambda) = \frac{M^{2N}}{\lambda^{N+1}} + o(\lambda^{-(N+1)}), \qquad M > 0, \quad (4.28)$$

for small λ. Comparing (4.28) with the upper bound of (2.20, 2.21) in the case $M_i = M$ for $i = 1, \ldots, N$, we see that the leading order behaviour for small λ is identical, despite the fact that the constraint (1.8) is stricter than (1.6). We conclude that the leading order term in (2.21) is sharp, at least when all of the M_i's are equal, and that we may change $o(\lambda^{-(N+1)})$ to $O(\lambda^{-N})$ in (4.28).

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