Discrete Boundary Element Methods on General Meshes in 3D*

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Dedicated to Professor Dr. R. Bulirsch on the occasion of his 65th birthday

Abstract

This paper is concerned with the stability and convergence of fully discrete Galerkin methods for boundary integral equations on bounded piecewise smooth surfaces in $\mathbb{R}^3$. Our theory covers equations with very general operators, provided the associated weak form is bounded and elliptic on H^μ, for some $\mu \in [-1, 1]$. In contrast to other studies on this topic, we do not assume our meshes to be quasiuniform, and therefore the analysis admits locally refined meshes. To achieve such generality, standard inverse estimates for the quasiuniform case are replaced by appropriate generalised estimates which hold even in the locally refined case. Since the approximation of singular integrals on or near the diagonal of the Galerkin matrix has been well-analysed previously, this paper deals only with errors in the integration of the nearly singular and smooth Galerkin integrals which comprise the dominant part of the matrix. Our results show how accurate the quadrature rules must be in order that the resulting discrete Galerkin method enjoys the same stability properties and convergence rates as the true Galerkin method. Although this study considers only continuous piecewise linear basis functions on triangles, our approach is not restricted in principle to this case. As an example, the theory is applied here to conventional "triangle-based" quadrature rules which are commonly used in practice. A subsequent paper [14] introduces a new and much more efficient "node-based" approach and analyses it using the results of the present paper.

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1 Introduction

Boundary element methods are popular in numerical engineering and physical sciences for the solution of classical partial differential equations on geometrically complicated and/or infinite domains. Their advantages are well-documented ([2], [4], [6], [16] [17]). From the implementation point of view, the main computational tasks which arise from these methods are, firstly, the stiffness matrix assembly process, which requires the computation of (iterated) integrals with singular, nearly singular and smooth integrands over flat or curved surfaces and, secondly, the solution of the resulting algebraic systems, which are usually fully populated and often unstructured.

For the second task, the construction of efficient solvers has been the subject of intense recent activity and iterative methods based on multi-grid/multi-level techniques and related preconditioners have been widely reported. Complementary to these are fast techniques for handling dense matrices using panel clustering [19], [33], [18] or multipole expansions [28] or, alternatively, conversion to sparse matrices using wavelet bases [34], [26].

On the other hand, work on the assembly of the stiffness matrices has tended to concentrate on the specific problem of computing singular integrals [36], [29], [20], [3], [11], while less has been done on the nearly singular or smooth (far field) integrals which form the bulk of the stiffness matrix. Since the process of matrix assembly can dominate the overall solution time in some applications, the generation of efficient methods for the first task remains an important problem.

This paper is the first of a series of two which have the overall aim of reducing the complexity of the stiffness matrix assembly process, while still preserving the stability and convergence properties which are known when the integrals in the stiffness matrix are computed without error. (Note that in practice numerical integration is almost always necessary.) The second paper in the series [14] (see also [13], which gave a preliminary discussion) presents a new quadrature scheme which considerably reduces the number of kernel evaluations required for the numerical integration of the stiffness matrix. (As we shall see later, the number of kernel evaluations is a reasonable measure of the complexity of the matrix assembly algorithm.)

The present paper provides a general theory of quadrature approximation in boundary element methods which (a) extends existing analysis to non-quasiuniform meshes; (b) covers, as a special case, conventional quadrature approaches which are commonly used and (c) provides a convenient framework for the analysis of the new quadrature method to be presented in [14].

The inclusion of non-quasiuniform meshes is especially important and novel here; the class of equations which we can analyse includes those on non-smooth boundaries which normally require locally refined meshes for good resolution of the solution. In this context it does not make sense to require the meshes to be quasiuniform. However there are technical problems with removing the quasiuniformity, since this property is normally required for the standard inverse estimates often used in the analysis (e.g., [1], [31], [37]). One of the key contributions of this paper is the proof of new estimates, valid for non-quasiuniform meshes, which play the rôle of inverse estimates in the stability and convergence theory.

At the end of this section we shall give an overview of the contents of this paper. Before that, we explain and motivate the type of boundary element method which we include in our study.

The first question which usually arises in the application of boundary elements is the choice of discretisation method. Galerkin ("weighted residual") methods have a strong theory based on variational arguments, but they require double integration (one from the integral operator itself and one from the inner product arising in the variational form). Collocation ("point-matching") is cheaper since it requires only one level of integration, but for some problems (e.g., hypersingular equations) its implementation may not be obvious. Moreover, despite considerable work on this topic, the gap between the theory of Galerkin and collocation methods remains huge, especially in 3D. In the present work we consider methods which are based on approximating the Galerkin formulation. Then (in contrast to collocation theory) we can include a very wide class of integral equation reformulations of boundary value problems for second order elliptic partial differential equations in three dimensional domains with piecewise smooth boundaries which may contain corners or edges. The key starting point to our analysis is the fact that *singular* Galerkin integrals (i.e., those which involve integration across a singularity) can be computed to arbitrary accuracy, and using relatively few kernel evaluations by transformation techniques (see, e.g., [29], [20]). So our quadrature analysis need only cover the *nearly-singular* and *non-singular* integrals which comprise most of the Galerkin matrix entries. This means that the stability question (usually a sticking point for collocation theory) can be settled by perturbation arguments applied to the standard Galerkin theory and thus we are able to obtain very general results. Interestingly, the new quadrature approach introduced in the subsequent paper [14] fits into this approximate Galerkin framework but its computational complexity is comparable to that of the collocation method.

The second question which arises is the choice of the finite element space. Here we restrict the discussion to the standard continuous piecewise linear functions on triangular surface meshes. This is not a fundamental restriction but, since considerable generality is obtained in this work with respect to integral equations, surfaces and meshes, we feel that to aim for further generality would impede the clarity of the presentation and reduce the utility of the paper.

To give the flavour of the type of result which we shall prove, suppose the first-kind integral equation with the single-layer potential kernel arising from Laplace's equation is solved on a smooth closed bounded surface by the Galerkin method with continuous piecewise-linear basis functions. Then the Galerkin method is stable and converges with order $O(h^{5/2})$ (where h is the mesh diameter) in the energy norm $H^{-1/2}$. Our results show that, provided Nh^2 is bounded above as $h \to 0$ (where N is the number of nodes in the mesh) then, in order to preserve the stability and this convergence rate, we should compute the entries of the Galerkin matrix using quadrature rules which have degree of precision 2 in the far field (i.e., when the supports of the corresponding basis functions are well-separated) and the degree of precision should increase only logarithmically in h when the supports of the basis functions are closer together. On the other hand, for the double layer potential and for the hypersingular equation on such a surface, only degree of precision 1 is required in the far field to preserve the stability and convergence properties of the Galerkin method in the energy norm. As we shall see below, the results mentioned here are only a sample and we can in fact analyse a much wider class of equations than Laplace's equation and also problems on non-smooth boundaries approximated using locally refined meshes, even those for which Nh^2 is unbounded above.

The layout of this paper is as follows. In §2 we state in detail the assumptions which we shall make on the surfaces, integral equations and meshes and give some elementary deductions from these assumptions. In §3 we obtain the generalisation of standard inverse estimates to non-quasiuniform meshes and then in §4 we give our stability and consistency theory for approximate Galerkin methods. Numerical experiments are postponed to the end of [14], where we compare the performance of standard quadrature and our new quadrature method on some three-dimensional model problems.

2 Assumptions and elementary results

We shall consider integral equations of the form

$$\lambda u(\mathbf{x}) + \mathcal{K}u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Gamma, \tag{2.1}$$

where Γ - an orientable closed bounded surface in $\mathbb{R}^3$, $\mathcal{K}$ - a linear integral operator, λ - a real scalar and f - a real-valued function on Γ are all given and $u : \Gamma \to \mathbb{R}$ is to be found. (We restrict to the case of real-valued integral equations simply to avoid notational technicalities.) We assume that the integral operator $\mathcal{K}$ in (2.1) can be written in the form

$$\mathcal{K}u(\mathbf{x}) = \int_\Gamma k(\mathbf{x},\mathbf{y})\, u(\mathbf{y})\, d\mathbf{y} \tag{2.2}$$

with a *kernel function* $k : \Gamma \times \Gamma \to \mathbb{R}$. Here, $d\mathbf{y}$ denotes the surface measure on Γ at $\mathbf{y} \in \Gamma$. We include the possibility that $k(\mathbf{x},\mathbf{y})$ contains a non-integrable singularity at $\mathbf{x} = \mathbf{y}$, in which case we assume that the integral in (2.2) is understood in the regularised sense of Hadamard (see [16], [35]). For simplicity we avoid using a special notation for the integral in this case.

Equation (2.2) will be solved by writing it in its weak form and applying a discrete Galerkin method with continuous piecewise linear basis functions. In the following four subsections we will describe the assumptions which we shall make on Γ, k, the weak form of (2.2) and the meshes which will be used in the discretisation.

2.1 The surface Γ

We assume that Γ consists of a finite number of smooth surface pieces joined together at non-cuspoid corners and edges. More precisely, we assume that Γ is parametrised by a the surface of a polyhedron $\tilde{\Gamma}$ via a bijective mapping

$$\boldsymbol{\eta} : \tilde{\Gamma} \to \Gamma,$$

which is assumed bi-Lipschitz continuous (i.e., $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^{-1}$ are Lipschitz continuous).

Denote the (open and plane polygonal) faces of $\tilde{\Gamma}$ by $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_L$. Without loss of generality, we can identify each $\tilde{\Gamma}_\ell$ with a polygonal domain in $\mathbb{R}^2$ described by the coordinate system $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2)$. The images $\Gamma_\ell = \boldsymbol{\eta}\left(\tilde{\Gamma}_\ell\right)$ are disjoint and $\Gamma = \cup \overline{\Gamma_\ell}$. Then, for any measurable $F : \Gamma \to \mathbb{R}$, we have

$$\int_\Gamma F(\mathbf{x})\, d\mathbf{x} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} F(\mathbf{x})\, d\mathbf{x} = \sum_{\ell=1}^{L} \int_{\tilde{\Gamma}_\ell} (F \circ \boldsymbol{\eta})(\widetilde{\mathbf{x}})\, g(\widetilde{\mathbf{x}})\, d\widetilde{\mathbf{x}}.$$

Here, we have made the change of variable $\mathbf{x} = \boldsymbol{\eta}(\tilde{\mathbf{x}})$ and the function $g(\tilde{\mathbf{x}})$ is such that the surface element transforms as $d\mathbf{x} = g(\widetilde{\mathbf{x}})\, d\widetilde{\mathbf{x}}$. For $\widetilde{\mathbf{x}} \in \tilde{\Gamma}_\ell$, $g(\widetilde{\mathbf{x}})$ can be computed by the square root of the "Gram determinant":

$$g(\widetilde{\mathbf{x}}) = \left\{ \det \left[\left\langle \frac{\partial \boldsymbol{\eta}}{\partial \widetilde{\mathbf{x}}_i}(\widetilde{\mathbf{x}}), \frac{\partial \boldsymbol{\eta}}{\partial \widetilde{\mathbf{x}}_j}(\widetilde{\mathbf{x}}) \right\rangle \right]_{1 \le i,j \le 2} \right\}^{1/2}, \tag{2.3}$$

with $\langle \cdot, \cdot \rangle$ denoting the usual inner product on $\boldsymbol{R}^3$.

In order to define precisely the smoothness properties of Γ, we introduce the following spaces of piecewise smooth functions on $\tilde{\Gamma}$ and Γ.

$$\begin{aligned} \mathcal{C}^\infty_{unif}\left(\tilde{\Gamma}_\ell\right) &:= \left\{ u : \tilde{\Gamma}_\ell \to \mathbb{R} \mid \text{all partial derivatives are } \textit{uniformly} \text{ continuous} \right\}, \\ \mathcal{C}^\infty_{pw}\left(\tilde{\Gamma}\right) &:= \left\{ u : \tilde{\Gamma} \to \mathbb{R} \mid u|_{\tilde{\Gamma}_\ell} \in \mathcal{C}^\infty_{unif}\left(\tilde{\Gamma}_\ell\right),\ 1 \le \ell \le L \right\}, \\ \mathcal{C}^\infty_{pw}(\Gamma) &:= \left\{ u : \Gamma \to \mathbb{R} \mid u \circ \boldsymbol{\eta} \in \mathcal{C}^\infty_{pw}\left(\tilde{\Gamma}\right) \right\}. \end{aligned} \tag{2.4}$$

We will assume that $\boldsymbol{\eta}$ is infinitely continuously differentiable on each $\tilde{\Gamma}_\ell$, from which it follows that

$$g \in \mathcal{C}^\infty_{pw}\left(\tilde{\Gamma}\right). \tag{2.5}$$

Moreover we shall also assume that $\left(\tilde{\Gamma}, \boldsymbol{\eta}\right)$ is a good parametrisation of Γ in the sense that there exists a constant $C_1 > 0$ such that

$$\min_{\ell=1,\cdots,L} \inf_{\widetilde{\mathbf{x}} \in \tilde{\Gamma}_\ell} g(\widetilde{\mathbf{x}}) \ge C_1 > 0. \tag{2.6}$$

Note that (2.5) also implies that there is a constant $C_2 > 0$ such that

$$\max_{\ell=1,\cdots,L} \sup_{\widetilde{\mathbf{x}} \in \tilde{\Gamma}_\ell} g(\widetilde{\mathbf{x}}) \le C_2 < \infty. \tag{2.7}$$

With this description of Γ we can now make precise our assumptions on k.

2.2 Properties of the kernel function

Throughout the paper $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$.

We will make the following assumptions on the kernel k of (2.1). First we introduce the *local kernel functions* $\tilde{k}_{\ell,\ell'} : \tilde{\Gamma}_\ell \times \tilde{\Gamma}_{\ell'} \to \mathbb{R}$, defined by

$$\tilde{k}_{\ell,\ell'}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) := k(\boldsymbol{\eta}(\widetilde{\mathbf{x}}), \boldsymbol{\eta}(\widetilde{\mathbf{y}})), \qquad (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in \tilde{\Gamma}_\ell \times \tilde{\Gamma}_{\ell'}.$$

For these we assume that there exists $\alpha \in \mathbb{R}$ and, for each $m \in \mathbb{N}_0$, a constant $B_m > 0$, such that

$$\left|\left(D^m \tilde{k}_{\ell,\ell'}\right)(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\right| \le B_m \left|\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\right|^{-\alpha-m}, \quad \tilde{\mathbf{x}} \ne \tilde{\mathbf{y}} \tag{2.8}$$

for all $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in \tilde{\Gamma}_\ell \times \tilde{\Gamma}_{\ell'}$ and all $1 \le \ell, \ell' \le L$. Here D^m represents any partial differential operator of order m with respect to $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$.

Remark 2.1 *Note that (2.8) implies that*

$$k_{\ell,\ell'} \in \mathcal{C}^\infty(\tilde{\Gamma}_\ell \times \tilde{\Gamma}_{\ell'}) \quad \textit{when} \quad \ell \ne \ell' \; . \tag{2.9}$$

As we shall now see, assumption (2.8) is satisfied by kernels arising from applying the integral equation method to a wide range of homogeneous, elliptic boundary value problems with constant coefficients. To be more precise, let L denote a scalar, linear, elliptic differential operator of order $2m$ with constant coefficients operating on a domain in $\mathbb{R}^3$. The fundamental solution S of this problem is a (distributional) solution to the equation

$$LS = \delta_0 \; ,$$

where δ_0 is the Dirac delta function centred at the origin. In [24], [21], [12] it is proved that S satisfies

$$S(\mathbf{z}) = A\left(\frac{\mathbf{z}}{|\mathbf{z}|}, |\mathbf{z}|\right) |\mathbf{z}|^{2m-3} \tag{2.10}$$

where the function $A(\boldsymbol{\xi}, r)$ is real analytic with respect to $\boldsymbol{\xi}$ in a neighbourhood of the unit sphere and with respect to r in a neighbourhood of 0.

Now the homogeneous problem

$$Lu = 0$$

in a domain Ω, subject to general boundary conditions on Γ (the boundary of Ω) can be reduced to a system of boundary integral equations using several standard techniques. By using the so-called *direct* approach (see, e.g., [39]), the kernel functions which arise are Gâteau (i.e., directional) derivatives of $S(\mathbf{x}-\mathbf{y})$ of order up to $2m-1$, in directions normal and/or tangential to Γ with respect to either $\mathbf{x}$ or $\mathbf{y}$. The following elementary arguments show that these derivatives satisfy estimates of the form (2.8).

If $\boldsymbol{\nu} : \Gamma \to S_2 := \{\mathbf{x} \in \boldsymbol{R}^3 : |\mathbf{x}| = 1\}$ is any vector field, its is easily shown that with S as in (2.10), the derivative of $S(\mathbf{x}-\mathbf{y})$ in the direction $\boldsymbol{\nu}(\mathbf{y})$ has the form

$$\frac{\partial}{\partial \nu(\mathbf{y})} S(\mathbf{x}-\mathbf{y}) = \left\{ \sum_{i=1}^{3} \nu_i(\mathbf{y}) \mathcal{A}_i \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, |\mathbf{x}-\mathbf{y}| \right) \right\} |\mathbf{x}-\mathbf{y}|^{2m-4} ,$$

where each $\mathcal{A}_i$ has the same analyticity properties as A. Generalising to $2m-1$th order derivatives with respect to $\mathbf{x}$ or $\mathbf{y}$, it follows that the kernel functions which arise in the direct method are sums of functions of the form:

$$c(\mathbf{x}) d(\mathbf{y}) \mathcal{A} \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, |\mathbf{x}-\mathbf{y}| \right) |\mathbf{x}-\mathbf{y}|^s , \tag{2.11}$$

where $s \in \{2m-3, 2m-4, \ldots, -2m-1\}$, and $\mathcal{A}$ has the same analyticity properties as A. The coefficients c and d depend on the tangetial and normal vector fields for Γ and their derivatives of order up to $2m-2$ and (because of our assumptions on the smoothness of Γ in §2.1) we have $c, d \in \mathcal{C}^\infty_{pw}(\Gamma)$.

Analogous estimates can be proved for kernel functions arising from other boundary integral equation methods (such as the classical *ansatz* approach) .

Kernel functions of the form (2.11) have two main characteristic properties, namely, the *singular* behaviour for $\mathbf{x} = \mathbf{y}$ and the *piecewise smooth* behaviour on the rest of the surface. In particular they satisfy (2.8).

We illustrate these calculations with two examples.

Example 2.2 (Elliptic boundary value problem of 2nd order - see, e.g., [25], [17]). Let $A \in \mathbb{R}^{3\times 3}$ be a symmetric and positive definite matrix, let $\boldsymbol{\lambda} \in \mathbb{R}^3$, let $c \in \mathbb{R}$ and consider the differential operator

$$Lu = -\operatorname{div}(A \nabla u) + 2 \langle \boldsymbol{\lambda}, \nabla u \rangle + cu.$$

If $B \in \mathbb{R}^{3\times 3}$ is a matrix satisfying $B^T B = A^{-1}$, then the fundamental solution of L is given by

$$S(\mathbf{z}) = \frac{|\det B|}{4\pi |B\mathbf{z}|} \exp(\langle B\boldsymbol{\lambda}, B\mathbf{z} \rangle - \sqrt{c + |B\boldsymbol{\lambda}|^2} |B\mathbf{z}|).$$

In the case of the Laplace operator, i.e., $A = I$, $\boldsymbol{\lambda} = \mathbf{0}$, $c = 0$, we obtain the well-known fundamental solution

$$S(\mathbf{z}) = \frac{1}{4\pi |\mathbf{z}|}. \tag{2.12}$$

□

Example 2.3 The kernel of the classical double layer potential for Laplace's equation in 3D is the normal derivative of (2.12):

$$\frac{\partial S(\mathbf{x}-\mathbf{y})}{\partial n(\mathbf{y})} = -\frac{\langle n(\mathbf{y}), \mathbf{y}-\mathbf{x}\rangle}{4\pi|\mathbf{y}-\mathbf{x}|^3}.$$

□

Finally we emphasize that the range of applicability of assumption (2.8) is by no means restricted to scalar boundary value problems but includes also systems of equations. Each entry of the fundamental tensor of the Lamé equation and the fundamental velocity tensor of the Stokes equation also satisfies (2.8). However we do not explicitly consider systems of boundary integral equations in this work.

2.3 Weak form of the problem

To define the Galerkin method for (2.1) we first reformulate it in weak form. To do this we introduce the space $L_2(\Gamma) = H^0(\Gamma)$ of measurable functions which are square-integrable with respect to surface measure dy. The space $H^1(\Gamma)$ is defined as usual by employing a Lipschitz atlas of Γ and a partition of unity. The intermediate spaces $H^t(\Gamma)$ for $t \in (0,1)$ are defined by interpolation ($H^t(\Gamma) = [H^0(\Gamma), H^1(\Gamma)]_t$ – see, e.g., [5])), while for $t \in [-1,0]$, $H^t(\Gamma)$ is the dual of $H^{-t}(\Gamma)$ with respect to the $L_2(\Gamma)$ inner product. For simplicity we often write L_2 and H^t instead of $L_2(\Gamma)$ and $H^t(\Gamma)$.

We assume that for some $\mu \in [-1,1]$

$$\lambda I + \mathcal{K} : H^\mu \to H^{-\mu} \tag{2.13}$$

is bounded. This allows us to define the bilinear form $a : H^\mu \times H^\mu \to \mathbb{R}$

$$a(u,v) := ((\lambda I + \mathcal{K})\, u, v)_{-\mu\times\mu},$$

where $(\cdot,\cdot)_{-\mu\times\mu}$ denotes the duality pairing between $H^{-\mu}$ and H^μ. We assume that $a(\cdot,\cdot)$ is H^μ-elliptic:

$$a(u,u) \geq C_3 \|u\|_{H^\mu}^2, \qquad \forall u \in H^\mu. \tag{2.14}$$

Then, the weak formulation of (2.1) is the following problem: Given $f \in H^{-\mu}$, find $u \in H^\mu$ such that, for all $v \in H^\mu$,

$$a(u,v) = (f,v)_{-\mu\times\mu}. \tag{2.15}$$

The Lax-Milgram theorem then guarantees that this problem has a unique solution depending continuously on the data.

If $a(\cdot,\cdot)$ satisfies a Gårding inequality instead of (2.14) and if, for all $v \in H^\mu\backslash\{0\}$,

$$\sup_{u\in H^\mu} a(u,v) > 0 \tag{2.16}$$

holds, then, the above statements remain true. If (2.16) is violated, one obtains existence, uniqueness, and continuous dependence on the data by restricting H^μ to an appropriate subspace or by introducing Lagrange multipliers (see [39], [16, Sections 4.8.4, 8.2.6, 8.3]). For simplicity, we restrict here to the situation (2.14) but emphasize that the proposed approach can directly be applied to this more general situation.

2.4 Meshes on Γ

We assume that $\tilde{\Gamma}$ is triangulated by a family of conforming triangular meshes $\widetilde{\mathcal{T}}_h$ parametrised by the mesh diameter h and that edges of $\tilde{\Gamma}$ consist entirely of edges of triangles. This means that, for each h, $\tilde{\mathcal{T}}_h$ is a collection of open planar triangles with the property that

$$\tilde{\Gamma} = \overline{\cup\{\tilde{\tau} : \tilde{\tau} \in \tilde{\mathcal{T}}_h\}},$$

and such that if $\tilde{\tau} \neq \tilde{\tau}'$ are any two triangles in $\widetilde{\mathcal{T}}_h$, then $\overline{\tilde{\tau}} \cap \overline{\tilde{\tau}'}$ is either empty, a common point, or a common edge. The meshes $\widetilde{\mathcal{T}}_h$ induce curvilinear triangular meshes $\mathcal{T}_h$ on Γ consisting of the collection of all $\tau = \{\boldsymbol{\eta}(\tilde{\mathbf{x}}) : \tilde{\mathbf{x}} \in \tilde{\tau}\}$, as $\tilde{\tau}$ ranges over all of $\tilde{\mathcal{T}}_h$. The nodes of a typical mesh on $\tilde{\Gamma}$ (respectively Γ) are denoted by $\{\widetilde{\mathbf{x}}_p\}$ (respectively $\{\mathbf{x}_p\}$), where $\mathbf{x}_p = \boldsymbol{\eta}(\widetilde{\mathbf{x}}_p)$. For each mesh $\mathcal{T}_h$, we introduce the index set

$$\mathcal{N} := \{p : \mathbf{x}_p \text{ is a node of } \mathcal{T}_h\} \ ,$$

and we let N denote the number of elements in $\mathcal{N}$. For any (curvilinear or planar) triangle τ we introduce its diameter

$$h_\tau = \sup\{|\mathbf{x}-\mathbf{y}| : \mathbf{x}, \mathbf{y} \in \tau\} \tag{2.17}$$

and its nodes

$$\mathcal{N}(\tau) = \{p \in \mathcal{N} : \mathbf{x}_p \in \mathcal{T}_h\} \ . \tag{2.18}$$

The quantity h which parametrises the mesh sequences $\tilde{\mathcal{T}}_h$ and $\mathcal{T}_h$ is

$$h = \max\{h_\tau : \tau \in \mathcal{T}_h\} .$$

Various quantities like $\mathcal{N}$, $\mathcal{N}(\tau)$ and N depend on h, but we suppress this dependence in the notation.

For any triangle $\tau \in \tilde{\mathcal{T}}_h$ or $\mathcal{T}_h$ we let $|\tau|$ denote its area. We assume that the meshes $\mathcal{T}_h$ are *shape regular* in the sense that there exists a constant $C_4 > 0$ such that for all $\tau \in \mathcal{T}_h$ and all h

$$|\tau| \geq C_4 h_\tau^2. \tag{2.19}$$

We note that since $\boldsymbol{\eta}$ is smooth on each face $\tilde{\Gamma}_\ell$ and each $\tilde{\tau} \in \widetilde{\mathcal{T}}_h$ lies entirely within one face, the family $\mathcal{T}_h$, $h \to 0$ could just as well be parametrised by $\tilde{h} := \max\left\{h_{\tilde{\tau}} : \tilde{\tau} \in \widetilde{\mathcal{T}}_h\right\} \to 0$.

Associated with the mesh $\widetilde{\mathcal{T}}_h$ on $\tilde{\Gamma}$ we introduce the usual continuous, piecewise linear "hat" functions with the property

$$\tilde{\phi}_p\left(\widetilde{\mathbf{x}}_q\right) = \begin{cases} 1, & p = q \\ 0, & p \neq q \ . \end{cases}$$

From these we define basis functions ϕ_p on Γ by

$$\phi_p\left(\mathbf{x}\right) = \tilde{\phi}_p\left(\boldsymbol{\eta}^{-1}\left(\mathbf{x}\right)\right), \qquad \mathbf{x} \in \Gamma, \ \ p \in \mathcal{N}.$$

For each h we can then introduce the space

$$\mathcal{S}_h = \operatorname{span}\left\{\phi_p : p \in \mathcal{N}\right\}$$

in which we shall approximate (2.15).

For $p \in \mathcal{N}$ the supports of $\tilde{\phi}_p$ and ϕ_p are defined by

$$\operatorname{supp}\tilde{\phi}_p = \overline{\left\{\widetilde{\mathbf{x}} \in \tilde{\Gamma} : \tilde{\phi}_p\left(\widetilde{\mathbf{x}}\right) \neq 0\right\}}, \quad \operatorname{supp}\phi_p = \overline{\left\{\mathbf{x} \in \Gamma : \phi_p\left(\mathbf{x}\right) \neq 0\right\}}$$

and the mesh diameter "local" to $\mathbf{x}_p$ is defined to be

$$h_p := \max\left\{h_\tau : \tau \subset \operatorname{supp}\left(\phi_p\right)\right\}, \tag{2.20}$$

In additon, we shall let $\mathcal{T}_h(\phi_p)$ (respectively $\tilde{\mathcal{T}}_h(\tilde{\phi}_p)$) denote the set of triangles of $\mathcal{T}_h$ (respectively $\tilde{\mathcal{T}}_h$) which are in supp ϕ_p (respectively supp $\tilde{\phi}_p$).

2.5 Quadrature rules

The key ingredient of the discrete Galerkin method which we shall discuss in Section 4 is the introduction of a class of quadrature rules for integrals of the form

$$\int_\Gamma F\left(\mathbf{x}\right)\phi_p\left(\mathbf{x}\right)d\mathbf{x}, \qquad p \in \mathcal{N}, \tag{2.21}$$

which work well for any piecewise smooth integrand $F \in \mathcal{C}^\infty_{pw}\left(\Gamma\right)$. Such rules are obtained by first employing $\boldsymbol{\eta}$ to pull this integral back onto $\tilde{\Gamma}$:

$$\int_\Gamma F\phi_p = \int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p, \tag{2.22}$$

where $\tilde{F} = (F \circ \boldsymbol{\eta})\, g$ is now a function in $\mathcal{C}^\infty_{pw}(\tilde{\Gamma})$ (see (2.3)). (Here and from now on, when no confusion can arise, we shall omit the explicit mention of independent variables such as $\mathbf{x}$ and $\tilde{\mathbf{x}}$ and their corresponding surface elements $d\mathbf{x}$ and $d\tilde{\mathbf{x}}$ when discussing integrals and integrands.)

"Standard" methods for (2.22) would now simply split it into a sum of integrals over the triangles in $\tilde{\mathcal{T}}_h(\tilde{\phi}_p)$ and apply a triangle-based rule of appropriately high order to each. This approach - while being adequate - may not be best possible, since it ignores the fact that $\tilde{F}$ is in fact smooth over the union of several $\tilde{\tau} \in \tilde{\mathcal{T}}_h(\tilde{\phi}_p)$. In fact $\tilde{F}$ is smooth over *all* of supp $\tilde{\phi}_p$ unless $\tilde{\mathbf{x}}_p$ lies on the boundary of one of the faces $\tilde{\Gamma}_l$. To exploit this fact (and this is crucial to our new approach in [14]), let us consider then a more general approach than the usual triangle-based method for (2.22).

For any node index $p \in \mathcal{N}$, we introduce the set of face indices

$$\ell(p) = \{\ell : \tilde{\mathbf{x}}_p \in \overline{\tilde{\Gamma}_\ell}\} \ .$$

In most cases $\tilde{\mathbf{x}}_p$ will lie inside one $\tilde{\Gamma}_\ell$ and $\ell(p)$ will contain exactly one ℓ. However $\ell(p)$ will contain several ℓ when $\tilde{\mathbf{x}}_p$ lies on the boundary of one (and hence several) $\tilde{\Gamma}_\ell$. With this notation we can then write

$$\int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p = \sum_{\ell \in \ell(p)} \int_{\tilde{\Gamma}_\ell} \tilde{F}\tilde{\phi}_p \ . \tag{2.23}$$

Then, since we have assumed that $F \in \mathcal{C}^\infty_{pw}(\Gamma)$, it follows that $\tilde{F}$ is smooth in each of the terms on the right-hand side of (2.23). Our approach exploits this by considering quadrature rules for each of these terms which are exact when $\tilde{F}$ is restricted to $\tilde{\Gamma}_\ell$ is a polynomial of a specified degree. This includes - as a special case - the triangle-based approach described above, but is more general. In particular it allows for the possibility that $\tilde{\phi}_p$ can be treated as a weight function in each of the terms in (2.23), which is the starting point for the new quadrature approach which we shall introduce in [14].

Let us now give more detail of our approach to computing (2.23). To compute a typical term in (2.23), we shall employ rules of the form

$$\int_{\tilde{\Gamma}_\ell} \tilde{F}\tilde{\phi}_p \ \approx \sum_{j \in J_{p,l}} \tilde{w}_j \tilde{F}(\tilde{\mathbf{x}}_j) \ . \tag{2.24}$$

with weights $\{\tilde{w}_j : j \in J_{p,\ell}\} \subset \mathbb{R}$ and points $\{\tilde{\mathbf{x}}_j : j \in J_{p,\ell}\} \subset \tilde{\Gamma}_\ell \cup \partial\tilde{\Gamma}_\ell$, where $J_{p,\ell}$ is a certain index set which we shall make precise using the following notational convention.

Notation 2.4 In (2.24), the index set $J_{p,\ell}$ consists of a set of triples $j = (p, \ell, i)$, where i is a counting index which ranges over a finite subset of $\mathbb{N}$. Through this mechanism the points $\{\tilde{\mathbf{x}}_j \ : \ j \in J_{p,\ell}\}$ are *associated with* $p \in \mathcal{N}$ and $\ell \in \{1, \ldots L\}$. When $\tilde{F} \in \mathcal{C}^\infty_{\mathrm{pw}}(\tilde{\Gamma})$, $\tilde{F}$ may be discontinuous across the boundaries $\partial\tilde{\Gamma}_\ell$ of the faces $\tilde{\Gamma}_\ell$. Here and from now on we adopt the convention that if $\tilde{\mathbf{x}}_j \in \partial\tilde{\Gamma}_\ell$ for $j \in J_{p,l}$, then $\tilde{F}(\tilde{\mathbf{x}}_j)$ is understood as **the limit** of $\tilde{F}(\tilde{\mathbf{x}})$ as $\tilde{\mathbf{x}} \to \tilde{\mathbf{x}}_j$ in $\tilde{\Gamma}_\ell$. $\square$

For the later analysis we shall require the family of rules (2.24) to have certain polynomial exactness and stability properties and to have quadrature points $\{\tilde{\mathbf{x}}_j :$

$j \in J_{p,\ell}\}$ which are not too far from $\tilde{\mathbf{x}}_p$. (Note that these quadrature points are not constrained to be in supp $\tilde{\phi}_p$.) These assumptions are encapsulated in the following statement.

We will assume that there exist constants $\sigma \geq 1$ and $\delta \geq 1$ such that, for all $d \in \mathbb{N}_0$, h, $p \in \mathcal{N}$ and $\ell \in \ell(p)$, the quadrature formula (2.24) has the following properties:

$$\text{(2.24) is } \textit{exact} \quad \text{when } \tilde{F}\Big|_{\tilde{\Gamma}_\ell} \in \mathcal{P}_{d,\ell} \ , \tag{2.25}$$

where $\mathcal{P}_{d,\ell}$ denotes the space of bivariate polynomials of degree $\leq d$ with respect to the local coordinates on $\tilde{\Gamma}_\ell$. Secondly the weights $\tilde{w}_j$ satisfy

$$\sum_{j \in J_{p,l}} |\tilde{w}_j| \leq \sigma \int_{\tilde{\Gamma}_\ell} \tilde{\phi}_p, \tag{2.26}$$

and thirdly the points $\tilde{\mathbf{x}}_j$ satisfy

$$\sup_{j \in J_{p,\ell}} |\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_p| \leq \delta \tilde{h}_p \ . \tag{2.27}$$

Note that since (2.25) is assumed true for $d \geq 0$ and hence for $d = 0$, (2.26) is equivalent to

$$\sum_{j \in J_{p,l}} |\tilde{w}_j| \leq \sigma \sum_{j \in J_{p,l}} \tilde{w}_j.$$

Then, since

$$\sum_{j \in J_{p,l}} |\tilde{w}_j| \geq \sum_{j \in J_{p,l}} \tilde{w}_j = \int_{\tilde{\Gamma}} \tilde{\phi}_p > 0 \ ,$$

it follows that $\sigma \geq 1$ and that we can take $\sigma = 1$ in (2.26) when (2.24) has non-negative weights.

Example 2.5 As an example let us show that the standard triangle-based approach satisfies the assumptions (2.25), (2.26) and (2.27). This approach employs the splitting:

$$\int_{\tilde{\Gamma}_\ell} \tilde{F}\tilde{\phi}_p = \sum_{\tilde{\tau}} \int_{\tilde{\tau}} \tilde{F}\tilde{\phi}_p, \tag{2.28}$$

where the sum is over all triangles $\tilde{\tau} \subset \operatorname{supp} \tilde{\phi}_p \cap \tilde{\Gamma}_\ell$. Since the integrand is now smooth on each $\tilde{\tau}$, we can apply any rule on $\tilde{\tau}$ of the following general form:

$$\int_{\tilde{\tau}} G \approx \sum_{j \in J_{\tilde{\tau}}} \alpha_{\tilde{\tau},j} G(\widetilde{\mathbf{x}}_{\tilde{\tau},j}) \tag{2.29}$$

with index set $J_{\tilde{\tau}}$, weights $\alpha_{\tilde{\tau},j}$, and points $\widetilde{\mathbf{x}}_{\tau,j} \in \overline{\tilde{\tau}}$. Such rules exist with positive weights $\alpha_{\tilde{\tau},j}$ and which integrate bivariate polynomials G of any chosen degree exactly (see [38], [9]). For these we have

$$\sum_{j\in J_{\tilde{\tau}}} |\alpha_{\tilde{\tau},j}| = \sum_{j\in J_{\tilde{\tau}}} \alpha_{\tilde{\tau},j} = \int_{\tilde{\tau}} 1 = |\tilde{\tau}| \,. \tag{2.30}$$

Employing such rules in (2.28) one then obtains

$$\int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p \approx \sum_{\tilde{\tau}} \sum_{j\in J_{\tilde{\tau}}} \hat{\alpha}_{\tilde{\tau},j} \tilde{F}\left(\widetilde{\mathbf{x}}_{\tilde{\tau},j}\right)$$

with $\hat{\alpha}_{\tilde{\tau},j} = \alpha_{\tilde{\tau},j}\tilde{\phi}_p\left(\widetilde{\mathbf{x}}_{\tilde{\tau},j}\right)$. Recalling that $\tilde{\phi}_p$ is a degree 1 polynomial on each $\tilde{\tau}$, we see that such rules satisfy the exactness criterion (2.25) provided (2.29) is exact for $G \in \mathcal{P}_{d+1,\ell}$. To prove the stability (2.26) in this case, note that since $\alpha_{\tilde{\tau},j} > 0$ and $\tilde{\phi}_p \geq 0$, we have

$$\sum_{\tilde{\tau}} \sum_{j\in J_{\tilde{\tau}}} |\hat{\alpha}_{\tilde{\tau},j}| = \sum_{\tilde{\tau}} \sum_{j\in J_{\tilde{\tau}}} \alpha_{\tilde{\tau},j} \phi_p\left(\widetilde{\mathbf{x}}_{\tilde{\tau},j}\right) = \sum_{\tilde{\tau}} \int_{\tilde{\tau}} \tilde{\phi}_p$$

(i.e., (2.26) holds with $\sigma = 1$), whereas (2.27) holds with $\delta = 1$, since all the quadrature points are inside (supp $\tilde{\phi}_p) \cap \tilde{\Gamma}_\ell$ when we use the triangle-based approach. $\square$

Since the parameters σ and δ in (2.26) and (2.27) in some sense characterise the generalised nature of the quadratures being considered here, we explicitly carry them through the analysis in this paper (see also Notation 2.6 below).

Once we have constructed rules (2.24) which satisfy assumptions (2.25)-(2.27), we now have a rule for (2.23):

$$\int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p \approx \sum_{\ell\in\ell(p)} \sum_{j\in J_{p,\ell}} \tilde{w}_j \tilde{F}\left(\widetilde{\mathbf{x}}_j\right). \tag{2.31}$$

By introducing a new index set

$$J_p = \{j : j \in J_{p,l},\ \ell \in \ell(p)\}\ ,$$

we can write this more succinctly as

$$\int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p \approx \sum_{j\in J_p} \tilde{w}_j \tilde{F}\left(\widetilde{\mathbf{x}}_j\right). \tag{2.32}$$

Note (recalling Notation 2.4) that with this construction the set $\{\tilde{\mathbf{x}}_j : j \in J_p\}$ may contain some repeated points. If $\tilde{F}$ is continuous at any such point, then we may amalgamate all the corresponding terms in (2.32) into a single term. But if $\tilde{F}$ is discontinuous there then the terms should be treated separately since in each case the value of $\tilde{F}$ is obtained as a limit in some one of the $\tilde{\Gamma}_\ell$ (see again Notation 2.4).

To complete this subsection, recall that the quantity we actually want to compute is (2.21). For this integral, (2.22) and (2.32) can be combined to obtain the rule:

$$\int_\Gamma F\varphi_p \approx \sum_{j\in J_p} w_j F(\mathbf{x}_j), \tag{2.33}$$

where

$$w_j = g(\widetilde{\mathbf{x}}_j)\,\tilde{w}_j, \qquad \mathbf{x}_j = \boldsymbol{\eta}(\widetilde{\mathbf{x}}_j). \tag{2.34}$$

Error estimates for these rules are given in the next subsection.

2.6 Elementary estimates

Here we gather some elementary results which follow from the assumptions in §§2.1–2.5. First we introduce a notational convention which will be used throughout the paper.

Notation 2.6 We will often be concerned with estimates on various quantities which depend on the meshes $\mathcal{T}_h$ through parameters such as $\tau \in \mathcal{T}_h$, $p \in \mathcal{N}$, h_τ or h_p and on the constants σ and δ arising in the quadrature estimates (2.26), (2.27) as well as other parameters. For any two such quantities A and B, the notation

$$A \lesssim B \tag{2.35}$$

will mean that

$$A \leq const\ B \tag{2.36}$$

with *const* denoting a constant independent of the meshes $\mathcal{T}_h$ and any related quantities and also independent of σ and δ. We write $A \cong B$ to mean both $A \lesssim B$ and $A \gtrsim B$. □

Making use of Notation 2.6, we can now make a number of observations. Since $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^{-1}$ are Lipschitz-continuous, it is easy to see that

$$|\tau| \cong |\tilde{\tau}| \quad \text{and} \quad h_\tau \cong h_{\tilde{\tau}} \qquad \text{where } \tau = \boldsymbol{\eta}(\tilde{\tau}). \tag{2.37}$$

Moreover, short calculations show that the shape regularity assumption (2.19) implies firstly that the mesh is *locally quasiuniform*, i.e.,

$$\max_{\tau,\tau'\in\mathcal{T}_h(\phi_p)} \{h_\tau/h_{\tau'}\} \lesssim 1, \tag{2.38}$$

and secondly that there exists an integer $N_{\max}$ independent of h and p such that

$$\#\{\tau \in \mathcal{T}_h \mid \tau \subset \operatorname{supp}\phi_p\} \leq N_{\max}, \tag{2.39}$$

where # denotes the number of elements in a set.

Then, using (2.38), we have

$$h_p \cong h_\tau \qquad \text{for } p \in \mathcal{N}(\tau)\,, \tag{2.40}$$

which then implies that

$$h_q \cong h_p \qquad \text{for } p, q \in \mathcal{N}(\tau)\,,\ \tau \in \mathcal{T}_h. \tag{2.41}$$

Thus by (2.37),

$$h_p \cong \tilde{h}_p, \ \text{ where } \ \tilde{h}_p = \max\{h_{\tilde{\tau}} : \tilde{\tau} \in \tilde{\mathcal{T}}_h(\tilde{\phi}_p)\}\ . \tag{2.42}$$

Finally observe that by (2.19) and the definition of h_τ,

$$h_\tau^2 \cong |\tau| \tag{2.43}$$

which implies that

$$|\Gamma| = \sum_{\tau\in\mathcal{T}_h} |\tau| \cong \sum_{\tau\in\mathcal{T}_h} h_\tau^2 \lesssim N h^2\ , \tag{2.44}$$

where the final inequality arises since the number of triangles in $\mathcal{T}_h$ can be bounded in terms of the number of nodes. Hence, for shape regular meshes, we always have

$$N \gtrsim h^{-2}\ . \tag{2.45}$$

If the mesh is, in addition, globally quasiuniform, then it satisfies the stronger condition

$$N \cong h^{-2}\ . \tag{2.46}$$

However (2.46) is not equivalent to quasiuniformity, since even if (2.46) holds the meshes can become more refined in some regions than in others (see also the discussion at the end of §4). In more extreme cases N may be asymptotically greater than $O(h^{-2})$. Our estimates cater for all these eventualities by carrying N and h as separate parameters which are constrained to satisfy (2.45) but not necessarily (2.46). Our estimates can be reduced to estimates involving h only in the special case when (2.46) holds.

In the remainder of this subsection we obtain the following estimates for the rules (2.24), (2.32) and (2.33). Before doing so we we need to introduce the following notation. For each $p \in \mathcal{N}$, and $\ell \in \{1, \dots L\}$, let $B_{p,\ell}$ denote the smallest starlike region containing the sets $(\operatorname{supp}\tilde{\phi}_p) \cap \tilde{\Gamma}_\ell$ and $\{\widetilde{\mathbf{x}}_j,\ j \in J_{p,\ell}\}$. For any $n \in \mathbb{N}$ and any function $\tilde{F} : \tilde{\Gamma} \to \mathbb{R}$, set

$$|\tilde{F}|_{n,\infty,B_{p,\ell}} = \max_{|\nu|\le n}\ \sup_{\tilde{\mathbf{x}}\in B_{p,\ell}} |(D^\nu \tilde{F})(\tilde{\mathbf{x}})|\ ,$$

where ν is a multiindex of order $|\nu|$ and D^ν is the corresponding differential operator with respect to local coordinates on $\tilde{\Gamma}_\ell$.

Lemma 2.7 *Consider rule (2.24). Then there exists a constant C_5 independent of $\tilde{F}$, $\tilde{h}$, p, σ and δ such that*

$$\left| \int_{\tilde{\Gamma}_\ell} \tilde{F} \tilde{\phi}_p - \sum_{j \in J_{p,\ell}} \tilde{w}_j \tilde{F}\left(\widetilde{\mathbf{x}}_j\right) \right| \leq C_5 \sigma (\delta \tilde{h}_p)^{d+1} |\tilde{F}|_{d+1,\infty,B_{p,\ell}} \int_{\tilde{\Gamma}_\ell} \tilde{\phi}_p. \tag{2.47}$$

Proof. By (2.25) and (2.26), the left-hand side of (2.47) can be written, for any $\tilde{p} \in \mathcal{P}_{d,\ell}$,

$$\left| \int_{\tilde{\Gamma}_\ell} \left(\tilde{F} - \tilde{p}\right) \tilde{\phi}_p - \sum_{j \in J_{p,\ell}} \tilde{w}_j \left(\tilde{F} - \tilde{p}\right) \left(\widetilde{\mathbf{x}}_j\right) \right| \leq \left\| \tilde{F} - \tilde{p} \right\|_{0,\infty,B_{p,\ell}} (1+\sigma) \int_{\tilde{\Gamma}_\ell} \tilde{\phi}_p.$$

Taylor's theorem and (2.27) complete the proof. □

Corollary 2.8 *If rule (2.32) is constructed as in (2.31) then we have the stability estimate:*

$$\sum_{j \in J_p} | \tilde{w}_j | \leq \sigma \int_{\tilde{\Gamma}} \tilde{\phi}_p \ , \quad p \in \mathcal{N} \tag{2.48}$$

and the error estimate:

$$\left| \int_{\tilde{\Gamma}} \tilde{F} \tilde{\phi}_p - \sum_{j \in J_p} \tilde{w}_j \tilde{F}\left(\widetilde{\mathbf{x}}_j\right) \right| \leq C_5 \sigma (\delta \tilde{h}_p)^{d+1} \max_{\ell \in \ell(p)} |\tilde{F}|_{d+1,\infty,B_{p,\ell}} \int_{\tilde{\Gamma}} \tilde{\phi}_p. \tag{2.49}$$

Proof. From (2.26) and the definition of J_p, we have

$$\sum_{j \in J_p} | \tilde{w}_j | = \sum_{\ell \in \ell(p)} \sum_{j \in J_{p,l}} | \tilde{w}_j | \leq \sigma \sum_{\ell \in \ell(p)} \int_{\tilde{\Gamma}_\ell} \tilde{\phi}_p = \sigma \int_{\tilde{\Gamma}} \tilde{\phi}_p.$$

Moreover,

$$\int_{\tilde{\Gamma}} \tilde{F} \tilde{\phi}_p - \sum_{j \in J_p} \tilde{w}_j \tilde{F}(\tilde{\mathbf{x}}_j) = \sum_{\ell \in \ell(p)} \left\{ \int_{\tilde{\Gamma}_\ell} \tilde{F} \phi_p - \sum_{j \in J_{p,l}} \tilde{w}_j \tilde{F}(\tilde{\mathbf{x}}_j) \right\}$$

and the required estimate follows by summing the result of Lemma 2.7 over $\ell \in \ell(p)$. □

For this rule we have the additional corollary:

Corollary 2.9 *For the rule (2.33) specified by (2.34), we have the stability and convergence estimates:*

$$\sum_{j \in J_p} | w_j | \leq \left(\frac{C_2}{C_1}\right) \sigma \int_{\Gamma} \phi_p \ . \tag{2.50}$$

Moreover there exists a constant C_6 independent of $\tilde{F}, \tilde{h}, p, \sigma$ and δ such that

$$\left| \int_\Gamma F\phi_p - \sum_{j\in J_p} w_j F(\mathbf{x}_j) \right| \leq C_5 C_1^{-1} C_6^{d+1} \sigma(\delta h_p)^{d+1} \max_{\ell\in\ell(p)} |\tilde{F}|_{d+1,\infty,B_{p,\ell}} \int_\Gamma \phi_p. \qquad (2.51)$$

Proof. By properties (2.6), (2.7) of g, we have

$$\max\{g(\mathbf{x}_j) : j \in J_p\} \leq C_2 \quad \text{and} \quad \int_\Gamma \phi_p = \int_{\tilde{\Gamma}} \tilde{\phi}_p g \geq C_1 \int_{\tilde{\Gamma}} \tilde{\phi}_p \ .$$

Then, using Corollary 2.8 ,

$$\sum_{j\in J_p} | w_j | \leq C_2 \sum_{j\in J_p} | \tilde{w}_j | \leq C_2 \sigma \int_{\tilde{\Gamma}} \tilde{\phi}_p \leq \left(\frac{C_2}{C_1}\right) \sigma \int_\Gamma \phi_p \ .$$

Moreover,

$$\left| \int_\Gamma F\phi_p - \sum_{j\in J_p} w_j F(\mathbf{x}_j) \right| = \left| \int_{\tilde{\Gamma}} \tilde{F}\tilde{\phi}_p - \sum_{j\in J_p} \tilde{w}_j \tilde{F}(\tilde{\mathbf{x}}_j) \right| \ .$$

Because of the Lipschitz continuity of $\boldsymbol{\eta}^{-1}$, we have $\tilde{h}_p \leq C_6 h_p$ and so (2.51) then follows.

□

Corollary 2.9 is the main result of this subsection and will play a key rôle in the estimates of §4.

3 Estimates for surface triangulations with non-quasiuniform meshes

The purpose of this section is to obtain estimates relating the H^t norm ($t \in [-1, 1]$) of functions in $\mathcal{S}_h$ to certain weighted ℓ_2 norms of their nodal values. These estimates, obtained in Theorem 3.5, can be thought of as generalisations of standard inverse estimates for piecewise linear finite elements to the non-quasiuniform case. Indeed (as we shall show), in the special case of quasiuniform meshes, Theorem 3.5 implies these inverse estimates.

To obtain these results we need some additional notation. Introduce the interpolation operator $\mathcal{I} : \mathbb{R}^N \to \mathcal{S}_h \subset H^1$ defined by

$$\mathcal{I}\mathbf{V} = \sum_{p\in\mathcal{N}} V_p \phi_p \qquad \text{for } \mathbf{V} = (V_p)_{p\in\mathcal{N}} \in \mathbb{R}^N.$$

Let $\mathbf{h} = (h_p)_{p\in\mathcal{N}} \in \mathbb{R}^N$ and introduce the function

$$\mathfrak{h} := \mathcal{I}\mathbf{h} \subset H^1.$$

Since $h_p > 0$ for all $p \in \mathcal{N}$, it follows that $\mathfrak{h} > 0$ on Γ and thus $\mathfrak{h}^t \subset H^1$ is uniquely defined for all $t \in \mathbb{R}$. For the rest of this section, the range of t will be restricted to a compact set Σ in $\mathbb{R}$. Note that $\mathcal{I}$ and $\mathfrak{h}^t$ both depend on h, but this is suppressed in the notation. In this context, the symbols $\lesssim$, $\gtrsim$, and $\cong$ are understood as in Notation 2.6, with the additional meaning that the constants of proportionality are also independent of $t \in \Sigma$.

Our first result estimates the L_2 and H^1 norms of $\mathfrak{h}^t\mathcal{I}\mathbf{V}$ (for any $\boldsymbol{V} \in \mathbb{R}^N$) on each $\tau \in \mathcal{T}_h$ in terms of a weighted ℓ_2 norm of $\mathbf{V}$ at nodes in $\overline{\tau}$.

For this estimate recall the definition of $\mathcal{N}(\tau)$ given in (2.18).

Lemma 3.1 *For all h, $\tau \in \mathcal{T}_h$, $\mathbf{V} \in \mathbb{R}^N$, and t in a compact subset Σ of $\mathbb{R}$,*

$$\int_\tau \left|\mathfrak{h}^t\mathcal{I}\mathbf{V}\right|^2 \;\cong\; h_\tau^{2+2t} \sum_{p\in\mathcal{N}(\tau)} \left|V_p\right|^2, \tag{3.1}$$

$$\int_\tau \left|\nabla\left(\mathfrak{h}^t\mathcal{I}\mathbf{V}\right)\right|^2 \;\lesssim\; h_\tau^{2t} \sum_{p\in\mathcal{N}(\tau)} \left|V_p\right|^2 \;. \tag{3.2}$$

Proof. We begin with the first estimate. To prove it observe that (2.40) and (2.41) imply that

$$\sup\{|\ \mathfrak{h}(\mathbf{x})\ |: \mathbf{x} \in \tau\} \cong h_\tau \;, \tag{3.3}$$

and so

$$\sup\{|\ \mathfrak{h}(\mathbf{x})\ |^t: \mathbf{x} \in \tau\} \cong h_\tau^t \;,$$

for all $t \in \Sigma$. This implies that $\int_\tau |\mathfrak{h}^t\mathcal{I}\mathbf{v}|^2 \cong h_\tau^{2t}\int_\tau |\mathcal{I}\mathbf{v}|^2$, and thus the result (3.1) follows provided we prove it for $t = 0$.

To do this, observe that, since $\tau = \boldsymbol{\eta}\left(\tilde{\tau}\right)$, for some $\tilde{\tau} \in \tilde{\mathcal{T}}_h$, we have

$$\int_\tau |\mathcal{I}\mathbf{V}|^2 = \int_{\tilde{\tau}} |(\mathcal{I}\mathbf{V})\circ\boldsymbol{\eta}|^2\, g.$$

This integral is over the flat triangle $\tilde{\tau}$ of size $h_{\tilde{\tau}} \cong h_\tau$ (cf. (2.37)). Assumptions (2.6), (2.7) imply that $g \cong 1$ and hence

$$\int_\tau |\mathcal{I}\mathbf{V}|^2 \cong \int_{\tilde{\tau}} |(\mathcal{I}\mathbf{v})\circ\boldsymbol{\eta}|^2 \;. \tag{3.4}$$

To estimate the right-hand side of (3.4) let $\boldsymbol{\kappa} : \tilde{\tau} \to \tau_{ref}$ be the affine mapping of $\tilde{\tau}$ onto the reference triangle

$$\tau_{ref} = \left\{\mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 < 1\right\}$$

which maps vertices of $\tilde{\tau}$ to vertices of τ_{ref} (both ordered anticlockwise) and denote the composition composition $\boldsymbol{\eta}\circ(\boldsymbol{\kappa}^{-1})$ by $\boldsymbol{\eta}_{\boldsymbol{\kappa}}$. Then, by a simple calculation,

$$\int_{\tilde{\tau}} |\mathcal{I}\mathbf{v}\circ\boldsymbol{\eta}|^2 \;=\; 2\,|\tilde{\tau}|\int_{\tau_{ref}} |\mathcal{I}\mathbf{v}\circ\boldsymbol{\eta}_\kappa|^2 \tag{3.5}$$

and using (3.4) and (3.5), together with (2.19) and (2.37), we obtain

$$\int_\tau |\mathcal{I}\mathbf{V}|^2 \cong \; h_\tau^2 \int_{\tau_{ref}} |\mathcal{I}\mathbf{v} \circ \boldsymbol{\eta}_\kappa|^2 . \tag{3.6}$$

Now, for any vector $\boldsymbol{\Psi} \in \mathbb{R}^3$, let Ψ be the linear function on τ_{ref} with values Ψ_1, Ψ_2, Ψ_3 at its three vertices. The functional

$$\left\{ \int_{\tau_{ref}} \mid \Psi \mid^2 \right\}^{1/2}$$

defines a norm on $\boldsymbol{\Psi} \in \mathbb{R}^3$ and, by equivalence of norms, the ratio

$$\left\{ \int_{\tau_{ref}} \mid \Psi \mid^2 \right\}^{1/2} / \left\{ \sum_{j=1}^{3} \mid \Psi_j \mid^2 \right\}^{1/2}$$

is bounded above and below independently of $\boldsymbol{\Psi} \in \mathbb{R}^3$. Combining this with (3.6) we have

$$\int_\tau |\mathcal{I}\mathbf{V}|^2 \cong h_\tau^2 \int_{\tau_{ref}} \mid \mathcal{I}\boldsymbol{V} \circ \boldsymbol{\eta_\kappa} \mid^2 \cong h_\tau^2 \sum_{p \in \mathcal{N}(\tau)} \mid V_p \mid^2$$

which is (3.1) for $t = 0$, as required.

For the proof of (3.2), first observe that, for any $p \in \mathcal{N}(\tau)$, (defined in (2.18)), we have $|\nabla \phi_p| \lesssim h_\tau^{-1}$ on τ. Therefore, for any $\boldsymbol{V} \in \mathbb{R}^N$ and any $\mathbf{x} \in \tau$, we have

$$|\nabla (\mathcal{I}\mathbf{V})(\mathbf{x})| \le \sum_{p \in \mathcal{N}(\tau)} |V_p| \, |\nabla \phi_p (\mathbf{x})| \lesssim h_\tau^{-1} \left\{ \sum_{p \in \mathcal{N}(\tau)} |V_p|^2 \right\}^{1/2} , \tag{3.7}$$

which, by (2.43), implies that

$$\int_\tau |\nabla (\mathcal{I}\mathbf{V})(\mathbf{x})|^2 \lesssim \sum_{p \in \mathcal{N}(\tau)} \mid V_p \mid^2 \; . \tag{3.8}$$

Because $\mathfrak{h} = \mathcal{I}\boldsymbol{h} > 0$, the function $\mathfrak{h}^t$ is differentiable on each τ for any $t \in \Sigma$, and, using (3.3) and (3.7) (with $\boldsymbol{V}$ replaced by $\boldsymbol{h}$), we have, for $\mathbf{x} \in \tau$,

$$\left|\nabla \mathfrak{h}^t (\mathbf{x})\right| = |t| \left|\mathfrak{h}^{t-1} (\mathbf{x})\right| |\nabla \mathfrak{h} (\mathbf{x})| \lesssim |t| \, h_\tau^{t-2} \left\{ \sum_{p \in \mathcal{N}(\tau)} h_p^2 \right\}^{1/2} .$$

Thus by (2.40) and the fact that Σ is compact,

$$\left|\nabla \mathfrak{h}^t (\mathbf{x})\right| \lesssim h_\tau^{t-1} \tag{3.9}$$

Now, noting that

$$\nabla(\mathfrak{h}^t \mathcal{I}\boldsymbol{V}) = (\nabla\mathfrak{h}^t)(\mathcal{I}\boldsymbol{V}) + (\mathfrak{h}^t)(\nabla\mathcal{I}\boldsymbol{V}) ,$$

we can use the triangle inequality and (3.9), (3.3) and then (3.1) and (3.8) to obtain:

$$\begin{aligned}
\left\{\int_\tau |\nabla(\mathfrak{h}^t\mathcal{I}\mathbf{V})|^2\right\}^{1/2} &\leq \left\{\int_\tau |\nabla\mathfrak{h}^t|^2 |\mathcal{I}\mathbf{V}|^2\right\}^{1/2} + \left\{\int_\tau |\mathfrak{h}^t|^2 |\nabla\mathcal{I}\mathbf{V}|^2\right\}^{1/2} \\
&\lesssim h_\tau^{t-1}\left\{\int_\tau |\mathcal{I}\mathbf{V}|^2\right\}^{1/2} + h_\tau^t\left\{\int_\tau |\nabla\mathcal{I}\mathbf{V}|^2\right\}^{1/2} \\
&\lesssim \left(h_\tau^{t-1}h_\tau + h_\tau^t\right)\left\{\sum_{p\in\mathcal{N}(\tau)} |V_p|^2\right\}^{1/2} \cong h_\tau^t\left\{\sum_{p\in\mathcal{N}(\tau)} |V_p|^2\right\}^{1/2} ,
\end{aligned}$$

completing the proof of (3.2). □

In the rest of this section we will be concerned with estimates for the following weighted ℓ_2-norm on $\mathbb{R}^N$, defined, for $t \in \mathbb{R}$ by

$$\|\mathbf{V}\|_{\ell_2,t} := \left\{\sum_{p\in\mathcal{N}} h_p^{2(1+t)} |V_p|^2\right\}^{1/2} . \tag{3.10}$$

Our second result compares the value of this weighted ℓ_2 norm for any $\boldsymbol{V} \in \mathbb{R}^N$ with the L_2 and H^1 norms of $\mathfrak{h}^t\mathcal{I}\boldsymbol{V}$ for various t.

Lemma 3.2 *For all t in a compact set $\Sigma \subset \boldsymbol{R}$ and $\mathbf{V} \in \mathbb{R}^N$,*

$$\left\|\mathfrak{h}^t\mathcal{I}\mathbf{V}\right\|_{L_2} \cong \|\mathbf{V}\|_{\ell_2,t} , \tag{3.11}$$

and

$$\|\mathbf{V}\|_{\ell_2,t} \lesssim \left\|\mathfrak{h}^t\mathcal{I}\mathbf{V}\right\|_{H^1} \lesssim \|\mathbf{V}\|_{\ell_2,t-1} . \tag{3.12}$$

Proof. First we prove (3.11). Note first that

$$\sum_{\tau\in\mathcal{T}_h}\sum_{p\in\mathcal{N}(\tau)} h_\tau^{2(1+t)} |V_p|^2 = \sum_{p\in\mathcal{N}}\sum_{\tau\in\mathcal{T}_h(\phi_p)} h_\tau^{2(1+t)} |V_p|^2$$

and by (2.39), (2.40) both these sums are equivalent to $\|\mathbf{V}\|_{\ell_2,t}^2$. So (3.11) follows by summing both sides in (3.1) over $\tau \in \mathcal{T}_h$. Similarly, summation of (3.2) yields the right-hand inequality in (3.12). Since L_2 is embedded in H^1, the left-hand inequality in (3.12) follows directly from (3.11). □

Lemma 3.3 *For all t in a compact set $\Sigma \subset \mathbb{R}$, $s \in [0,1]$, and $\mathbf{V} \in \mathbb{R}^N$,*

$$\|\mathbf{V}\|_{\ell_2,t} \lesssim \left\|\mathfrak{h}^t\mathcal{I}\mathbf{V}\right\|_{H^s} \lesssim \|\mathbf{V}\|_{\ell_2,t-s} . \tag{3.13}$$

Note that equivalence ($\cong$ instead of $\lesssim$) holds only for $s = 0$.

Proof. By (3.11) and (3.12) we have

$$\left\|\mathfrak{h}^t \mathcal{I}\mathbf{V}\right\|_{L_2} \lesssim \|\mathbf{V}\|_{\ell_2,t} ,$$

and

$$\left\|\mathfrak{h}^t \mathcal{I}\mathbf{V}\right\|_{H^1} \lesssim \|\mathbf{V}\|_{\ell_2,t-1} .$$

Now recall that for $s \in [0,1]$, H^s is defined to be the interpolation space $[L_2, H^1]_s$. Moreover, if $\ell_{2,t}$ denotes the space of all $\boldsymbol{V} \in \mathbb{R}^N$ with the property that $\|\boldsymbol{V}\|_{\ell_{2,t}} < \infty$, then it follows from a fairly simple calculation that for $t \in \mathbb{R}$, $s \in [0,1]$,

$$[\ell_{2,t} \ , \ \ell_{2,t-1}]_s = \ell_{2,t-s} \ , \tag{3.14}$$

with equivalent norm $\|\boldsymbol{V}\|_{\ell_{2,t-s}}$. Then the right-hand side of (3.13) follows from the Operator Interpolation Theorem (e.g., [23, §5] or [5, §12]). The left-hand side of (3.13) follows from (3.11) and the fact that L_2 is embedded in H^s, for $s \in [0,1]$. □

For negative s, the norm H^s is defined by duality (see §2.3). In this case we can prove the following bounds which complement those in the previous lemma.

Lemma 3.4 *For all t in a compact set $\Sigma \subset \mathbb{R}$, $s \in [0,1]$, and $\mathbf{V} \in \mathbb{R}^N$,*

$$\|\mathbf{V}\|_{\ell_2,t+s} \lesssim \left\|\mathfrak{h}^t \mathcal{I}\mathbf{V}\right\|_{H^{-s}} \lesssim \|\mathbf{V}\|_{\ell_2,t} .$$

Proof. The right-hand inequality follows by combining (3.11) with the fact that H^{-s} is embedded in L_2 for all $s \in [0,1]$. For the left inequality, recall that by definition,

$$\left\|\mathfrak{h}^t \mathcal{I}\mathbf{V}\right\|_{H^{-s}} = \sup_{0 \neq u \in H^s} \frac{(\mathfrak{h}^t \mathcal{I}\mathbf{V}, u)_{L_2}}{\|u\|_{H^s}}$$

Suppose $\mathbf{0} \neq \boldsymbol{V} \in \mathbb{R}^N$ and note that, since $\mathfrak{h}^{t+2s}\mathcal{I}\boldsymbol{V} \in H^1 \subset H^s$, we have

$$\left\|\mathfrak{h}^t \mathcal{I}\mathbf{V}\right\|_{H^{-s}} \geq \frac{(\mathfrak{h}^t \mathcal{I}\mathbf{V}, \mathfrak{h}^{t+2s}\mathcal{I}\mathbf{V})_{L_2}}{\|\mathfrak{h}^{t+2s}\mathcal{I}\mathbf{V}\|_{H^s}}. \tag{3.15}$$

But also note that

$$\left(\mathfrak{h}^t \mathcal{I}\mathbf{V}, \mathfrak{h}^{t+2s}\mathcal{I}\mathbf{V}\right)_{L_2} = \left(\mathfrak{h}^{t+s} \mathcal{I}\mathbf{V}, \mathfrak{h}^{t+s}\mathcal{I}\mathbf{V}\right)_{L_2} = \left\|\mathfrak{h}^{t+s}\mathcal{I}\mathbf{V}\right\|_{L_2}^2 \cong \|\mathbf{V}\|_{\ell_2,t+s}^2 \ , \tag{3.16}$$

(where the last step uses (3.11)). Also, from (3.12),

$$\left\|\mathfrak{h}^{t+2s}\mathcal{I}\mathbf{V}\right\|_{H^s} \lesssim \|\mathbf{V}\|_{\ell_2,(t+2s)-s} = \|\mathbf{V}\|_{\ell_2,t+s} \tag{3.17}$$

Combining (3.16), (3.17) with (3.15) yields the left-hand inequality, at least for $\boldsymbol{V} \neq \mathbf{0}$. But this is trivially true when $\boldsymbol{V} = \mathbf{0}$, completing the proof. □

By putting $t = 0$ in Lemmata 3.3 and 3.4 we obtain the main result of this section:

Theorem 3.5 *For all $s \in [0,1]$ and $\mathbf{V} \in \mathbb{R}^N$,*

$$\|\mathbf{V}\|_{\ell_2,s} \lesssim \|\mathcal{I}\mathbf{V}\|_{H^{-s}} \lesssim \|\mathbf{V}\|_{\ell_2,0} \lesssim \|\mathcal{I}\mathbf{V}\|_{H^s} \lesssim \|\mathbf{V}\|_{\ell_2,-s} .$$

□

We conclude this section with two remarks.

Remark 3.6 Note that Theorem 3.5 is truly a generalisation of standard inverse estimates to the non-quasiuniform case. For, if the mesh is quasiuniform, then $h \lesssim h_p$ for all $p \in \mathcal{N}$ and the left-most inequality of Theorem 3.5 tells us that

$$h^s \|\boldsymbol{V}\|_{\ell_2,0} \lesssim \|\mathcal{I}\boldsymbol{V}\|_{H^{-s}} .$$

Using (3.11), this yields the standard inverse inequality

$$\|\mathcal{I}\boldsymbol{V}\|_{L_2} \lesssim h^{-s}\|\mathcal{I}\boldsymbol{V}\|_{H^{-s}} ,$$

for $s \in [0,1]$. In addition, the right-most inequality in Theorem 3.5 leads to another standard inverse inequality:

$$\|\mathcal{I}\boldsymbol{V}\|_{H^s} \lesssim h^{-s}\|\mathcal{I}\boldsymbol{V}\|_{L_2} ,$$

for $s \in [0,1]$. □

Remark 3.7 The results of Lemmata 3.2, 3.3, 3.4 and Theorem 3.5 generalise to the case when Γ is a m-dimensional manifold in $\mathbb{R}^{m+1}$ for all $m \geq 2$, provided the weighted norm $\|\mathbf{V}\|_{\ell_2,t}$ is defined by

$$\|\mathbf{V}\|_{\ell_2,t} := \left\{ \sum_{p \in \mathcal{N}} h_p^{2(m/2+t)} \mid V_p \mid^2 \right\}^{1/2} .$$

□

4 Discrete Galerkin methods

Now we are ready to introduce and analyse the discrete Galerkin method for (2.1). We consider this problem in its weak form: Find $u \in H^\mu$ such that

$$a(u,v) = (f,v)_{-\mu\times\mu} \qquad \text{for all } v \in H^\mu ,$$

as described in §2. Since $\mathcal{S}_h \subset H^\mu$, for all $\mu \in [-1,1]$, we can define the standard Galerkin method: Find $U \in \mathcal{S}_h$ such that

$$a(U,V) = (f,V)_{-\mu\times\mu} \qquad \text{for all } V \in \mathcal{S}_h. \tag{4.1}$$

Equivalently, we can seek $\boldsymbol{U} \in \mathbb{R}^N$ such that $U := \sum_{q\in\mathcal{N}} U_q\phi_q$, satisfies

$$\sum_{q\in\mathcal{N}} a\left(\phi_q,\phi_p\right)U_q = \left(f,\phi_p\right)_{-\mu\times\mu}, \qquad p\in\mathcal{N}. \tag{4.2}$$

Let M denote the mass matrix with $M_{p,q} = \left\{(\phi_q,\phi_p)_{L_2(\Gamma)}\right\}_{p,q\in\mathcal{N}}$ and $K = (K_{p,q})$ denote the stiffness matrix with

$$K_{p,q} = (\mathcal{K}\phi_q,\phi_p)_{L_2(\Gamma)} = \int_\Gamma\int_\Gamma k(\mathbf{x},\mathbf{y})\phi_q(\mathbf{y})\phi_p(\mathbf{x})dydx\ , \tag{4.3}$$

where, from now on, $(\cdot,\cdot)$ denotes the L_2 inner product on Γ. Then (4.2) can be written as the system of linear equations:

$$\sum_{q\in\mathcal{N}} \left(\lambda M_{p,q} + K_{p,q}\right)U_q = F_p, \qquad p\in\mathcal{N}\ , \tag{4.4}$$

where (assuming $f\in L_2$) $F_p = (f,\phi_p)_{-\mu\times\mu} = (f,\phi_p)$.

The assembly of the equations (4.4) requires computation of M, K and the load vector F, with the most time-consuming part being the computation of K. The computation of the sparse matrix M is relatively straightforward, so we shall restrict attention here to the computation of K, although the method we propose can also be used to compute F.

As described in the introduction, we shall consider only the approximation of $K_{p,q}$ in the nearly singular and nonsingular cases. For this reason we introduce the following notation. For $p,q\in\mathcal{N}$ we write

$$\begin{aligned}\rho_{p,q} &= \min\left\{|\mathbf{x}-\mathbf{y}| : \mathbf{x}\in\operatorname{supp}(\phi_p)\cup\{\mathbf{x}_j\ : j\in J_p\},\ \mathbf{y}\in\operatorname{supp}(\phi_q)\cup\{\mathbf{x}_j\ : j\in J_q\}\right\}\ ,\\ h_{p,q} &= \max\left\{h_p,h_q\right\}\ ,\end{aligned}$$

where h_p is as given in (2.20). Let $\theta\in[0,1)$. Then for (4.3) we can distinguish two cases:

$$\textbf{Case I:} \qquad \rho_{p,q} \geq h_{p,q}^\theta, \tag{4.5}$$

$$\textbf{Case II:} \qquad \rho_{p,q} < h_{p,q}^\theta. \tag{4.6}$$

When p and q satisfy Case I, $\mathbf{x}$ remains bounded away from $\mathbf{y}$ in the double integral (4.3), and we can expect to be able to approximate (4.3) well by the quadrature rules introduced in §2.5 (although we may expect that a higher degree of precision will be needed when $\mathbf{x}_p$ is moderately near $\mathbf{x}_q$ compared to the case when $\mathbf{x}_p$ and $\mathbf{x}_q$ are far apart). Case I is much more common than Case II: For example, in the case of a uniform mesh, it is easily seen that, for h sufficiently small, there exists θ (close to 1) such that there are $O(N^2)$ pairs (p,q) satisfying Case I but only $O(N)$ pairs (p,q) satisfying Case II. In this paper we shall discuss only the computation of $K_{p,q}$ in Case I. This restriction to Case I can be made without loss of generality since in Case II the quadrature can be performed via a triangle-based approach analogous to that in Example 2.5 (see [29], [20], [30], [11]), with an error analysis, including discussion of the required degree of exactness for these singular quadratures, given in [26] and [30]. Our aim here is to study the application of the general rule (2.33) to (4.3) in the more computationally significant Case I. By Example 2.5, the application of standard triangle-based rules to (4.3) is covered as a special case of the theory which we shall present here.

So, let us suppose that, in Case I, we use (2.33) to approximate the double integral (4.3) with respect to both $\mathbf{x}$ and $\mathbf{y}$, i.e., we replace $K_{p,q}$ by

$$\tilde{K}_{p,q} = \sum_{j\in J_p} \sum_{j'\in J_q} w_j k\left(\mathbf{x}_j, \mathbf{x}_{j'}\right) w_{j'}. \tag{4.7}$$

In Case II we shall assume no numerical error, i.e., $\tilde{K}_{p,q} = K_{p,q}$. Moreover, we assume also that M and $\mathbf{F}$ in (4.4) are computed exactly.

The resulting system corresponds to a discrete Galerkin method: Find $\tilde{U} \in \mathcal{S}_h$ such that

$$\tilde{a}\left(\tilde{U}, V\right) = (f, V)_{-\mu\times\mu}, \qquad \text{for all } V \in \mathcal{S}_h, \tag{4.8}$$

where $\tilde{a} : \mathcal{S}_h \times \mathcal{S}_h \to \mathbb{R}$ is the discrete bilinear form

$$\tilde{a}(U, V) = \sum_{p\in\mathcal{N}} \sum_{q\in\mathcal{N}} U_q \left(\lambda M_{p,q} + \tilde{K}_{p,q}\right) V_p. \tag{4.9}$$

Analogously to (4.4), the nodal values of the discrete Galerkin solution satisfy the linear system

$$\left(\lambda M + \tilde{K}\right) \tilde{\mathbf{U}} = \mathbf{F}$$

An abstract estimate of the difference between solutions of (4.2) and (4.8) is available in the literature. The following result is a simple consequence of the "first Strang Lemma" (see [7, Theorem 4.1.1]).

Lemma 4.1 *Suppose, in addition to the assumptions on a in §2.3, $\tilde{a}$ satisfies the stability condition:*

$$\sup_{V,W\in\mathcal{S}_h} \frac{|a(V,W) - \tilde{a}(V,W)|}{\|V\|_{H^\mu} \|W\|_{H^\mu}} \to 0 \tag{4.10}$$

as $h \to 0$. Then, for sufficiently small h, the approximate Galerkin method (4.8) has a unique solution $\tilde{U} \in \mathcal{S}_h$ and we have the error estimate

$$\left\|u - \tilde{U}\right\|_{H^\mu} \lesssim \inf_{V\in\mathcal{S}_h} \left\{ \|u - V\|_{H^\mu} + \sup_{W\in\mathcal{S}_h} \frac{|a(V,W) - \tilde{a}(V,W)|}{\|W\|_{H^\mu}} \right\}. \tag{4.11}$$

□

In Theorem 4.4 we shall obtain bounds on the quantity appearing in (4.10), allowing us to deduce in Theorem 4.6 (and using (4.11)) conditions under which the solution of the discrete Galerkin system (4.8) has the same rate of convergence as the solution of the true system (4.1). As a prerequisite we first prove Lemma 4.3 below, which gives an estimate for the error in individual entries of K.

Notation 4.2 In what follows $\tilde{K}_{p,q}$ given in (4.7) is obtained by applying a rule of form (2.33) with respect to both $\mathbf{x}$ and $\mathbf{y}$ in (4.3). We will assume, for simplicity only, that for each pair (p,q), a rule with the *same* precision is used both for the integration with respect to the $\mathbf{x}$ variable and the $\mathbf{y}$ variable. We denote this precision by $d_{p,q}$. However we allow $d_{p,q}$ to vary as p and q vary; a smaller $d_{p,q}$ may be sufficient when $\mathbf{x}_p$ and $\mathbf{x}_q$ are well-separated. $\square$

With this notation we have the following result:

Lemma 4.3 *For all $p, q \in \mathcal{N}$ and degree $d_{p,q}$ as introduced above, we have*

$$\left|K_{p,q} - \tilde{K}_{p,q}\right| \leq C_7 \sigma^2 (\delta h_{p,q})^{d_{p,q}+1} \rho_{p,q}^{-(d_{p,q}+1+\alpha)} \int_\Gamma \phi_p \int_\Gamma \phi_q,$$

where $C_7 = B_{d_{p,q}+1} C_5 C_1^{-1} C_6^{d_{p,q}+1}(1 + C_1/C_2)$, and where σ is the stability constant appearing in assumption (2.26), δ is the parameter in (2.27) and α and B_m are as in (2.8).

Proof. Since $\tilde{K}_{p,q} = K_{p,q}$ in Case II, we need only consider Case I. Within this proof (only) we set $d_{p,q} = d$ for convenience. Then, by (4.3) and (4.7),

$$\begin{aligned}
K_{p,q} - \tilde{K}_{p,q} &= \int_\Gamma \int_\Gamma k(\mathbf{x},\mathbf{y})\,\phi_q(\mathbf{y})\,\phi_p(\mathbf{x})\,d\mathbf{y}d\mathbf{x} - \sum_{j\in J_p}\sum_{j'\in J_q} w_j k(\mathbf{x}_j,\mathbf{x}_{j'})\,w_{j'} \\
&= \int_\Gamma \int_\Gamma k(\mathbf{x},\mathbf{y})\,\phi_q(\mathbf{y})\,\phi_p(\mathbf{x})\,d\mathbf{x} - \sum_{j\in J_p} w_j \int_\Gamma k(\mathbf{x}_j,\mathbf{y})\,\phi_q(\mathbf{y})\,d\mathbf{y} \\
&\quad + \sum_{j\in J_p} w_j \left[\int_\Gamma k(\mathbf{x}_j,\mathbf{y})\,\phi_q(\mathbf{y})\,d\mathbf{y} - \sum_{j'\in J_q} k(\mathbf{x}_j,\mathbf{x}_{j'})\,w_{j'}\right].
\end{aligned}$$

Applying the convergence estimate in Corollary 2.9 to each of these terms yields

$$\begin{aligned}
&\left|K_{p,q} - \tilde{K}_{p,q}\right| \\
&\leq C_5 C_1^{-1} C_6^{d+1} \sigma \left\{ (\delta h_p)^{d+1} \max_{\ell\in\ell(p)} \left| \int_\Gamma k(\boldsymbol{\eta}(\cdot),\mathbf{y})\,\phi_q(\mathbf{y})\,d\mathbf{y} \right|_{d+1,\infty,B_{p,\ell}} \int_\Gamma \phi_p \right. \\
&\quad \left. + \left(\sum_{j\in J_p} |w_j|\right) (\delta h_q)^{d+1} \max_{j\in J_p} \max_{\ell\in\ell(q)} |k(\mathbf{x}_j,\boldsymbol{\eta}(\cdot))|_{d+1,\infty,B_{q,\ell}} \int_\Gamma \phi_q \right\}.
\end{aligned}$$

Employing the properties of k given in (2.8), together with the stability estimate in Corollary 2.9, and the fact that $\sigma \geq 1$, we obtain the result. □

The next theorem determines the size of the required local degree of precision $d_{p,q}$ in terms of a global accuracy parameter χ. For simplicity we suppose from now on that $h < 1$. Since all our results are asymptotic as $h \to 0$, this represents no loss of generality. Here we also make use of Notation 2.6

Theorem 4.4 *Let $\chi \geq 0$ and suppose, for each p, q in Case I, the degree of precision $d_{p,q}$ is chosen to be the smallest integer such that*

$$d_{p,q} \geq \frac{\chi + (1+\alpha)(\log \rho_{p,q}/\log h_{p,q})}{1 - (\log \rho_{p,q}/\log h_{p,q})} \ . \tag{4.12}$$

Then, for all $h < 1$, $t \in [-1,1]$ and all $V,\ W \in \mathcal{S}_h$,

$$\frac{|a\,(V,W) - \tilde{a}\,(V,W)|}{\|V\|_{H^t}\,\|W\|_{H^t}} \lesssim C(\delta)\sigma^2 h^{\chi+1+t^*} N^{-t^*/2} \tag{4.13}$$

and

$$\frac{|a\,(V,W) - \tilde{a}\,(V,W)|}{\|V\|_{H^0}\,\|W\|_{H^t}} \lesssim C(\delta)\sigma^2 h^{\chi+1} N^{-t^*/2}, \tag{4.14}$$

where $t^* = \min\{t, 0\}$ and

$$C(\delta) \lesssim \delta^{(\chi+\alpha\theta+1)/(1-\theta)+1} \ . \tag{4.15}$$

Here α is as in (2.8) and θ is as in (4.5).

Proof. First we perform some elementary manipulations on the definition of $d_{p,q}$. Observe that by (4.5) and the fact that $h_{p,q} \leq h < 1$, we have

$$\log \rho_{p,q}/\log h_{p,q} \leq \theta < 1 \ , \tag{4.16}$$

and so

$$\frac{\chi + (1+\alpha)(\log \rho_{p,q}/\log h_{p,q})}{1 - (\log \rho_{p,q}/\log h_{p,q})} \leq \frac{\chi + (1+\alpha)\theta}{1-\theta} \ .$$

Hence, because of the choice of $d_{p,q}$, we have

$$d_{p,q} \leq \frac{\chi + (1+\alpha)\theta}{1-\theta} + 1 = \frac{\chi + \alpha\theta + 1}{1-\theta} \ . \tag{4.17}$$

Also observe that (4.12) and (4.16) imply that

$$(1 - (\log \rho_{p,q}/\log h_{p,q}))d_{p,q} \geq \chi + (1+\alpha)(\log \rho_{p,q}/\log h_{p,q}) \ ,$$

and since $\log h_{p,q} < 0$, a short calculation shows that

$$h_{p,q}^{d_{p,q}-\chi} \leq \rho_{p,q}^{d_{p,q}+1+\alpha} \ . \tag{4.18}$$

Now we are ready to estimate $a(V,W)-\tilde{a}(V,W)$. Since we have assumed that the mass matrix M was computed exactly, the definition of the bilinear forms a and $\tilde{a}$ leads to

$$|a(V,W)-\tilde{a}(V,W)| \le \sum_{p\in\mathcal{N}}\sum_{q\in\mathcal{N}} |V_q| \left| K_{p,q}-\tilde{K}_{p,q} \right| |W_p| .$$

Hence by Lemma 4.3 (and recalling that (2.43) and (2.40) imply $\int_\Gamma \phi_p \lesssim h_p^2$), we have

$$|a(V,W)-\tilde{a}(V,W)| \lesssim C(\delta)\sigma^2 \sum_{p\in\mathcal{N}}\sum_{q\in\mathcal{N}} \{h_q^2 |V_q|\}\ h_{p,q}^{d_{p,q}+1}\rho_{p,q}^{-(d_{p,q}+1+\alpha)}\ \{h_p^2|W_p|\}, \quad (4.19)$$

where the bound (4.15) is obtained from (4.17). To estimate (4.19), we note that (4.18) implies that

$$h_{p,q}^{d_{p,q}+1}\rho_{p,q}^{-(d_{p,q}+1+\alpha)} \le h_{p,q}^{\chi+1}. \quad (4.20)$$

Inserting (4.20) into (4.19) yields

$$|a(V,W)-\tilde{a}(V,W)| \lesssim C(\delta)\sigma^2 \sum_{p\in\mathcal{N}}\sum_{q\in\mathcal{N}} \{h_q^2 |V_q|\}\ h_{p,q}^{\chi+1}\ \{h_p^2|W_p|\}. \quad (4.21)$$

In our proof we shall use (4.21) to first obtain (4.14) and then (4.13) is obtained by a refinement of the argument. First, since $h_{p,q}\le h$, we have

$$|a(V,W)-\tilde{a}(V,W)| \lesssim C(\delta)\sigma^2 h^{\chi+1}\left(\sum_{q\in\mathcal{N}} h_q^2|V_q|\right)\left(\sum_{p\in\mathcal{N}} h_p^2|W_p|\right). \quad (4.22)$$

Now for any $t\in[-1,1]$, set $t^*=\min\{t,0\}\in[-1,0]$ and observe that the Cauchy-Schwarz inequality and Theorem 3.5 imply

$$\begin{aligned}\sum_{p\in\mathcal{N}} h_p^2|V_p| = \sum_{p\in\mathcal{N}} h_p^{1+t^*}h_p^{1-t^*}|V_p| &\le \left\{\sum_{p\in\mathcal{N}} h_p^{2(1+t^*)}\right\}^{1/2}\left\{\sum_{p\in\mathcal{N}} h_p^{2(1-t^*)}|V_p|^2\right\}^{1/2} \\ &\lesssim \left\{\sum_{p\in\mathcal{N}} h_p^{2(1+t^*)}\right\}^{1/2} \|V\|_{H^{t^*}} . \end{aligned} \quad (4.23)$$

Also, using Hölder's inequality, (2.40) and (2.44), we obtain (if $t^*\in(-1,0)$),

$$\sum_{p\in\mathcal{N}} h_p^{2(1+t^*)} \le \left\{\sum_{p\in\mathcal{N}} h_p^{\frac{2(1+t^*)}{1+t^*}}\right\}^{1+t^*}\left\{\sum_{p\in\mathcal{N}} 1^{-1/t^*}\right\}^{-t^*} = N^{-t^*}\left\{\sum_{p\in\mathcal{N}} h_p^2\right\}^{1+t^*} \lesssim N^{-t^*} \quad (4.24)$$

and this inequality is also true for $t^*=-1$ or 0. Thus, inserting (4.24) into (4.23), we have

$$\sum_{p\in\mathcal{N}} h_p^2|V_p| \lesssim N^{-t^*/2}\|V\|_{H^{t^*}} \le N^{-t^*/2}\|V\|_{H^t} . \quad (4.25)$$

In particular,

$$\sum_{q\in\mathcal{N}} h_p^2 \mid V_p \mid \lesssim \|V\|_{H^0} \ . \tag{4.26}$$

Then (4.22), (4.25) and (4.26) yield (4.14).

To obtain (4.13), return to (4.21) and write

$$|a\left(V,W\right)-\tilde{a}\left(V,W\right)| \lesssim C(\delta)\sigma^2 \sum_{p\in\mathcal{N}}\sum_{q\in\mathcal{N}} \left(h_q^{1-t^*}\left|V_q\right|\right)\Lambda_{p,q}\left(h_p^{1-t^*}\left|W_p\right|\right),$$

where the matrix Λ is defined by

$$\Lambda_{p,q} = h_p^{1+t^*} h_{p,q}^{\chi+1} h_q^{1+t^*}.$$

Hence, using Cauchy Schwarz and Theorem 3.5 again,

$$|a\left(V,W\right)-\tilde{a}\left(V,W\right)| \lesssim \|\Lambda\|_2 \|V\|_{H^{t^*}} \|W\|_{H^{t^*}} \le \|\Lambda\|_2 \|V\|_{H^t} \|W\|_{H^t} \ , \tag{4.27}$$

where $\|\cdot\|_2$ denotes the spectral norm. Introducing the matrix S given by $S_{p,q} = h_p^{\chi+2+t^*} h_q^{1+t^*}$, we have

$$0 < \Lambda_{p,q} \le h_p^{\chi+2+t^*} h_q^{1+t^*} + h_p^{1+t^*} h_q^{\chi+2+t^*} =: S_{p,q} + S_{q,p}$$

and from [15, Theorem 6.3.10]

$$\|\Lambda\|_2 \le \left\|S+S^T\right\|_2 \le 2\,\|S\|_2\,.$$

Now

$$\|S\|_2 \le \left\{\sum_{p\in\mathcal{N}} h_p^{2(\chi+2+t^*)}\right\}^{1/2} \left\{\sum_{q\in\mathcal{N}} h_q^{2(1+t^*)}\right\}^{1/2} \lesssim \left\{\sum_{p\in\mathcal{N}} h_p^{2(\chi+2+t^*)}\right\}^{1/2} N^{-t^*/2}$$

as in (4.24).

Also, using $h_p^{2(\chi+2+t^*)} \le h^{2(\chi+1+t^*)} h_p^2$ and (2.44), we obtain

$$\sum_{p\in\mathcal{N}} h_p^{2(\chi+2+t^*)} \le h^{2(\chi+1+t^*)} \sum_{p\in\mathcal{N}} h_p^2 \lesssim h^{2(\chi+1+t^*)} \ .$$

So $\|\Lambda\|_2 \lesssim h^{\chi+1+t^*} N^{-t^*/2}$, and inserting this in (4.27) yields (4.13). □

Remark 4.5 Suppose $\mathbf{x}_p, \mathbf{x}_q$ are nodes of $\mathcal{T}_h$ as $h \to 0$ which have the property that $\rho_{p,q} \ge C$ for some constant C independent of h, then we say that (p,q) are a **far-field pair** of nodes. Since $\log h_{p,q} \to -\infty$ as $h \to 0$, (4.12) tells us that for far field pairs we require $d_{p,q}$ to be the smallest integer satisfying $d_{p,q} \ge \chi$ in order to retain the estimates (4.13), (4.14) as $h \to 0$. □

Combining Lemma 4.1 and Theorem 4.4 we have the following consequence.

Corollary 4.6 *Set $\mu^* = \min\{\mu, 0\}$ and suppose $\chi \geq 0$. Suppose, in addition that for each p, q in Case I the degree of precision $d_{p,q}$ is chosen as in (4.12). Then, provided*

$$h^{\chi+1+\mu^*} N^{-\mu^*/2} \to 0 \quad \text{as } h \to 0, \tag{4.28}$$

the discrete Galerkin solution $\tilde{U}$ in (4.8) exists and satisfies the error estimate

$$\|u - \tilde{U}\|_{H^\mu} \lesssim \inf_{V \in \mathcal{S}_h} \{\|u - V\|_{H^\mu} + C(\delta)\sigma^2 h^{\chi+1} N^{-\mu^*/2} \|V\|_{H^0}\} \ . \tag{4.29}$$

□

In practice, for each particular problem, the parameter χ should be chosen in order to ensure that the error in the second term inside the braces in (4.29) balances the first. The size of the first term in the braces will be determined by various factors such as (a) the smoothness of Γ, (b) the smoothness of u, and (c) the ability of the mesh $\mathcal{T}_h$ to accommodate any lack of smoothness of u. Here we shall restrict ourselves to giving three applications of Corollary 4.6 under the assumptions that u is sufficiently smooth and the mesh sequence satisfies (2.46). Note that quasiuniformity is sufficient but not necessary for (2.46). Indeed it is easy to construct, by *a priori* formulae, meshes which satisfy (2.46) but which are highly non-quasiuniform - see, e.g., [27]. In this case it is known that there exists $V \in \mathcal{S}_h$ such that

$$\|u - V\|_{H^\mu} \lesssim C_\mu(u) h^{2-\mu} \ ,$$

and

$$\|V\|_{H^0} \lesssim C_0(u),$$

where $C_\mu(u)$ and $C_0(u)$ are constants which depend u. For simplicity, let us assume that the parameters δ and σ are bounded independently of h and N, which tells us that, under the conditions of Corollary 4.6 (and assuming u is sufficiently smooth), we have

$$\|u - \tilde{U}\|_{H^\mu} \lesssim h^{2-\mu} + h^{\chi+1} N^{-\mu^*/2} \ . \tag{4.30}$$

Now we consider the implications of (4.30) in the context of three standard integral equations arising from Laplace's equation.

Example 4.7 <u>**Single Layer Potential Equation**</u>

Here $\lambda = 0$ in (2.1) and the equation is:

$$\frac{1}{4\pi} \int_\Gamma \frac{1}{|\mathbf{x} - \mathbf{y}|} u(\mathbf{y}) dy = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Gamma \ .$$

The assumptions of §2.2 are true with $\alpha = 1$ and §2.3 with $\mu = -1/2$ (see [8]). So $\mu = \mu^* = -1/2$ in (4.30) and to balance the terms we require (recalling (2.46)),

$$h^{\chi+1/2} \cong h^{5/2} \ .$$

So $\chi = 2$ will suffice, and we need $d_{p,q} = 2$ for far field pairs (recall Remark 4.5). □

Example 4.8 Double Layer Potential Equation

Here (2.1) is

$$u(\mathbf{x}) - \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n(\mathbf{y})}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) u(\mathbf{y})dy = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Gamma \ .$$

The assumptions of §2.2 hold with $\alpha = 1$ and the assumptions of §2.3 hold with $\mu = 0$ at least when Γ is a Lyapunov surface (see, e.g., [2]). In the polyhedral case the theory of the double layer potential operator is more delicate [10]. However on the assumption that Galerkin's method is stable then, to balance the terms in (4.30) we require

$$h^{\chi+1} \cong h^2 \ ,$$

so that $\chi = 1$ and $d_{p,q} = 1$ for far field pairs.

$\square$

Example 4.9 Hypersingular Equation

Here (2.1) is

$$\frac{1}{4\pi}\frac{\partial}{\partial n(\mathbf{x})}\int_\Gamma \frac{\partial}{\partial n(\mathbf{y})}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) u(\mathbf{y})dy = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Gamma \ .$$

The assumptions of §2.2 hold with $\alpha = 3$ and the assumptions of §2.3 hold with $\mu = 1/2$ (see [8]). To balance the terms in (4.30) we require

$$h^{\chi+1} \cong h^{3/2} \ .$$

So we can take $\chi = 1/2$ and $d_{p,q} = 1$ for far field pairs. $\square$

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