An Asymptotic and Numerical Description of Self-Similar Blow-up in Quasilinear Parabolic Equations

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Abstract

We study the blow-up behaviour of two reaction-diffusion problems with a quasilinear degenerate diffusion and a superlinear reaction. We show that in each case the blow-up is self-similar, in contrast to the linear diffusion limit of each in which the diffusion is only approximately self-similar. We then investigate the limit of the self-similar behaviour and describe the transition from a stable manifold blow-up behaviour (for quasilinear diffusion) to a centre manifold one for the linear diffusion.

1 Introduction

1.1 Blow-up with linear diffusion, a review

Many nonlinear evolution partial differential equations, which act as models for combusting or other processes, have solutions which develop strong singularities in a finite time, see the references in the books [5], [39] and in the survey paper [35]. The prototype of such is the semilinear parabolic equation from combustion theory

$$u_t = u_{xx} + f(u) \quad \text{for} \quad (x,t) \in (-L, L) \times \mathbf{R}_+, \tag{1.1}$$

with the Dirichlet boundary and initial conditions

$$u(-L,t) = u(L,t) = 0 \text{ for } t > 0, \quad u(x,0) = u_0(x) \geq 0 \text{ in } (-L, L).$$

The function $f(u) > 0$ for $u > 0$ (reaction term) is superlinear for $u \gg 1$ and satisfies the well-known necessary blow-up condition

$$\int_1^\infty \frac{du}{f(u)} < \infty. \tag{1.2}$$

Important examples of such functions include

$$f(u) = e^u \text{ Frank-Kamenetskii equation}, \quad \text{and} \quad f(u) = u^\beta, \quad \beta > 1. \tag{1.3}$$

For these equations, analytic methods show that if $u_0(x)$ is sufficiently large and has a single maximum in the interval $[-L, L]$, then there is a time T (the blow-up time) and

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a single point $x_* \in (-L, L)$ such that there exist a sequence of times $t_n \to T^-$ and a sequence of points $x_n \to x_*$ with

$$u(x_n, t_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{1.4}$$

The behaviour of the solution as the blow-up time is approached has been the subject of intensive analytical, asymptotic and numerical studies. See books and papers [1], [4], [11], [5], [6], [7], [10]-[30], [35], [39] addressing such questions as when blow-up occurs, where blow-up occurs and how blow-up occurs. It is the last of these that we consider here. In general, if $f(u)$ is strongly superlinear, say, $f(u) \gg u \ln^{2+\varepsilon} u$ ($\varepsilon > 0$) for $u \gg 1$ (see Sect. 7, Ch. 4 in [39]), blow-up occurs at a single point x_*, so that, in general, there exists a point $x = x_*$ such that $u(x_*, t) \to \infty$ as $t \to T$ whereas $u(x, t) \to u(x, T) < \infty$ as $t \to T$ if $x \neq x_*$. When t is close to T and x is close to x_*, both u and its derivatives become increasingly large in a well pronounced and increasingly narrow blow-up peak, the form of which is essentially independent of the initial conditions. This phenomenon makes studies of the peak useful for testing numerical methods designed to resolve structures with small length scales.

In the two cases $f(u) = e^u$ or $f(u) = u^\beta$ the differential equation (1.1) is invariant under scaling transformations applied to x, t and u. For example, if $f(u) = e^u$ then (1.1) is invariant under a Lie group of transformations

$$u \mapsto u - 2\log\lambda, \quad x \mapsto \lambda x, \quad t \mapsto \lambda^2 t \quad (\lambda > 0), \tag{1.5}$$

and if $f(u) = u^\beta$ it is invariant under

$$u \mapsto \lambda^{-2/(\beta-1)} u, \quad x \mapsto \lambda x, \quad t \mapsto \lambda^2 t. \tag{1.6}$$

In the absence of boundary conditions, we expect that equation (1.1) with the above nonlinear reaction terms should admit self-similar solutions which are themselves invariant under the action of the above transformations. As blow-up is approached, the peak of the function $u(x, t)$ is increasingly narrow and on the length scale of the peak the boundaries appear distant and have apparently little influence. Observing these phenomena, Kapila [32] conjectured that the profile of the peak could be described in terms of such a self-similar solution. In particular when $f(u) = e^u$, Kapila suggested that if x is close to x_* and t is close to T, then there existed a function $\theta(y)$ such that $u(x, t)$ could be expressed in the following form:

$$u(x, t) = -\log(T - t) + \theta(y) \tag{1.7}$$

where the similarity variable y is the invariant of the infinitesimal generator of the Lie Group (1.5):

$$y = |x - x_*|/(T - t)^{1/2}. \tag{1.8}$$

Substituting (1.7), (1.8) into the partial differential equation (1.1) leads to an ordinary differential equation for $\theta(y)$ with a slow-growth condition at infinity to allow for the effect of the boundaries. Similarly, if $f(u) = u^\beta$ then there is a related function $\theta(y)$ such that with y given by (1.8)

$$u(x, t) = (T - t)^{-1/(\beta-1)}\theta(y).$$

Despite the simplicity of these arguments, it was later found that the evolution of the peak could not be described in this self-similar form (since no nontrivial self-similar profiles $\theta(y)$ exist). The solutions instead are *approximately self-similar* and include a slowly varying component. In the above examples, the resulting form of $u(x, t)$ is

$$u(x, t) = -\log(T - t) + \theta(y, s) \quad \text{or} \quad u(x, t) = (T - t)^{-1/(\beta-1)}\theta(y, s)$$

with the new slow-variable s given by

$$s = -\log(T-t) \to \infty \quad \text{as} \quad t \to T.$$

This departure from self-similarity is closely linked to the non-existence of a non-trivial solution of the ordinary differential equation governing the self-similar solution $\theta(y)$.

If $f(u) = e^u$, then uniformly on compact subsets in the variable η defined by

$$\eta = (x - x_*)(T-t)^{-1/2}|\log(T-t)|^{-1/2}, \tag{1.9}$$

there holds

$$\theta(\eta, s) \to -\log\left(1 + \frac{\eta^2}{4}\right) \quad \text{as} \quad s \to \infty. \tag{1.10}$$

Similarly, if $f(u) = u^\beta$, then in the same limit,

$$\theta(\eta, s) \to \gamma \left(1 + \frac{(\beta-1)\eta^2}{4\beta}\right)^{-1/(\beta-1)}, \tag{1.11}$$

where

$$\gamma = (\beta - 1)^{-1/(\beta-1)}. \tag{1.12}$$

These results were first derived in [29] ($\beta = 3$), [25] ($\beta > 1$) and in [17] (for $f(u) = e^u$). Proofs can be found in [6], [7], [10], [19] and [28]. It turns out that for both heat equations the stable generic blow-up behaviour corresponds to the *centre* manifold of the corresponding linearized operator.

1.2 Blow-up with quasilinear diffusion

In many physical systems, the diffusion term is not linear but depends either upon the function u or upon its gradient, and in this paper we consider the effect of this upon blow-up the by studying the two problems

$$u_t = (|u_x|^\sigma u_x)_x + e^u, \tag{1.13}$$

and

$$u_t = (u^\sigma u_x)_x + u^\beta, \tag{1.14}$$

with the same Dirichlet boundary conditions. Here $\sigma > 0$ and $\beta > \sigma + 1$ are fixed constants. Under suitable initial conditions both equations have solutions which blow-up in a finite time, developing peaks which become increasingly narrow as the blow-up time is approached. The first differential equation arises from studies of turbulent diffusion or the flow of a non-Newtonian liquid. The second problem has a porous-medium-type diffusion term, and arises as a model for the temperature profile of a fusion reactor plasma with one source term (see [40], Ch. 4 in [39] and [36] for further references). For $0 < \sigma < \beta - 1$, the solution blows up at a single point given sufficiently large initial data with a single maximum, and the heat interfaces of a compactly supported solution are localised; see [21]-[23], [26] and Ch. 4 in [39].

In Figure 1.1 we present a profile of the evolution of a solution of (1.14) with initial data $u_0 = 10\exp(-x^2/4)$ on the interval $[-7, 7]$. (As the solution is symmetric, the form of

$u(x,t)$ is only shown on the half-interval $[0,7]$.) The form of the profile was computed using an adaptive mesh method which is a generalisation of the methods described in [11], [12]. For this problem $T = 0.106344$ and the solution forms a peak centred on $x = 0$ with $u(x,t)$ ultimately increasing monotonically towards a singular function $u(x,T)$ as $t \to T$. This behaviour can be seen clearly in the figure. An asymptotic form of the singular function $u(x,T)$ is calculated in Section 4 and this is indicated on the figure.

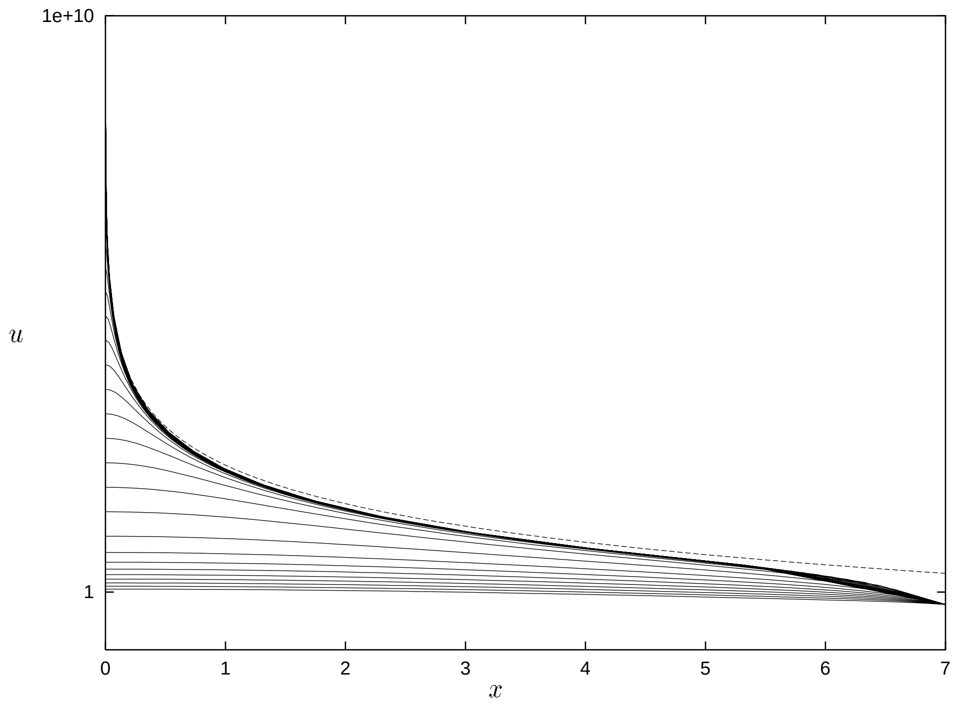

Figure 1.1: Symmetric solution to the PDE with polynomial term and $\sigma = 0.1$ and $\beta = 2$. Solid line is the solution u at various times and dashed line is the asymptotic relation $(8/\sigma)x^{-2/(1-\sigma)}$. The y axis is on a logarithmic scale.

Both of the equations (1.13) and (1.14) are invariant under Lie groups of scaling transformations closely related to those given earlier. Consequently, we again ask whether the profile of the solutions close to the blow-up time and the blow-up point can be described by self-similar solutions of the partial differential equations. Remarkably, unlike the linear diffusion problem, if the initial data has a single maximum, the blow-up profiles for the nonlinear diffusion equations can be closely approximated by the nontrivial self-similar solutions (in the sense that these solutions are global attractors) for arbitrarily small values of $\sigma > 0$. The blow-up history of these problems is thus, in a sense, simpler in its form than that of the linear diffusion equation (1.1). The existence of such self-similar solutions was established rigorously for the two equations above in the respective papers [13] and [1], see also Ch. 4 in [39]. The self-similar solutions are asymptotically stable. In particular, for problem (1.13) as $t \to T$ uniformly on compact subsets in the rescaled space variable y (i.e., if t is close to T and x is close to x_*) then [13]

$$u(x,t) + \log(T-t) \to \theta(y) \quad \text{where} \quad y = |x - x_*|/(T-t)^{1/(\sigma+2)}. \tag{1.15}$$

Similarly for equation (1.14) [23],

$$u(x,t)(T-t)^{1/(\beta-1)} \to \theta(y) \quad \text{where} \quad y = |x - x_*|/(T-t)^{\frac{\beta-(\sigma+1)}{2(\beta-1)}}, \tag{1.16}$$

where $\theta(y)$ are nontrivial nonconstant profiles which in both cases satisfy relatively simple ordinary differential equations described in Section 2.

An interesting question on the nature of blow-up is thus how the true self-similar blow-up observed in these equations evolves into the approximately self-similar blow-up of the linear diffusion equations in the limit of $\sigma \to 0$. It is this question that we will address in this paper with reference to (1.13) and (1.14) above. Our approach will use matched asymptotic expansions to determine the solution of the ordinary differential equations satisfied by $\theta(y)$ in the limit of small σ. Although formal, our results agree very closely with some numerical computations, and are descriptive for values of σ which are not especially small. In particular, suppose that $0 < \sigma \ll 1$ and we consider equation (1.13). Then if $0 \le y \ll \sigma^{-1/2}$

$$\theta(y) = \frac{\sigma}{2}(1 - y^2/2) + \mathcal{O}(\sigma^2 y^4) \tag{1.17}$$

and if $1 \ll y$

$$\theta(y) \sim -\log\left(1 + \frac{\sigma}{4}y^{\sigma+2}\right). \tag{1.18}$$

Similarly, if we consider equation (1.14), then if $0 \le y \ll \sigma^{-1/2}$

$$\theta(y) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}(1 - y^2/2)\right) + \mathcal{O}(\sigma^2 y^4) \tag{1.19}$$

where γ is as given in (1.12). Furthermore, if $y \gg 1$

$$\theta(y) \sim \gamma\left(1 + \frac{\sigma}{4\beta}y^{\frac{2(\beta-1)}{\beta-(\sigma+1)}}\right)^{-1/(\beta-1)}. \tag{1.20}$$

The formulae (1.18), (1.20) can be compared directly with (1.10), (1.11) respectively. We see that they are very similar and both have a term involving y^2 in the limit. For $\sigma > 0$ this term is a multiple of σy^2 which vanishes as $\sigma \to 0$. In contrast, if $\sigma = 0$ the corresponding term is a multiple of y^2/s which vanishes as $1/s \to 0$.

Both of these descriptions of $\theta(y)$ lead directly to time dependent solutions of the partial differential equations which we consider in more detail in Section 5. In particular we show that as $t \to T$ then if $x \neq x_*$ is fixed, then $u(x,t) \to u(x,T)$ given respectively by

$$u(x,T) \sim -(\sigma+2)\log|x - x_*| + \log(4/\sigma), \quad u(x,T) \sim \gamma\left(\frac{4\beta}{\sigma}\right)|x - x_*|^{-2/(\beta-(\sigma+1))}. \tag{1.21}$$

Thus, the asymptotic behaviour of $u(x,t)$ is self-similar and described by the asymptotics above. This description of $u(x,t)$ does not give insight into the transient motion as it evolves toward the self-similar solution. In Section 5 we consider the linear stability of the self-similar solution and in Section 6 we perform some numerical computations to determine this transient effect.

We note at this stage that if the initial data has more than one maximum, then it is shown in [15] for equation (1.13) that there may also be evolving solutions with several maxima, which if σ is small are not described by a self-similar solution.

The layout of the remainder of this paper is as follows. In Section 2 we review the theory for the equations of the form (1.1), (1.13) and (1.14). In Section 3 we apply the method of matched asymptotic expansions to derive a formal description of the profiles of the self-similar solutions for both (1.13) and (1.14) (concentrating our description on the slightly more complex case of (1.14)). In Section 4 we apply the results of Section 3 to derive the time dependent behaviour of $u(x,t)$.

In Section 5 we give a preliminary analysis of the rate of convergence to the self-similar profiles in (1.16) in the quasilinear case $\sigma > 0$ of single point blow-up $\beta > \sigma + 1$. Unlike the semilinear case $\sigma = 0$, for $\sigma > 0$ the asymptotic blow-up behaviour corresponds to the *stable* manifold of the linearized operator. Moreover, using a formal asymptotic analysis we give an estimate of the second eigenvalue, $\lambda_2 = \sigma/(\beta-1) + o(\sigma) > 0$ which governs the exponentially fast convergence. Therefore, $\lambda_2 \to +0$ as $\sigma \to +0$ which corresponds to the transition from a stable manifold to a centre one for $\sigma = 0$ discussed above in the case of linear diffusion.

Finally in Section 6 we use an adaptive mesh method to determine a numerical solution of (1.13) and (1.14) and compare our results with those obtained asymptotically, as well as considering the effect of transients.

2 General theory

In this section we review some of the theory associated with the problems (1.14) and (1.13) looking at both the case $\sigma = 0$ and $\sigma > 0$.

2.1 Local existence

The quasilinear equations (1.13), (1.14) will, in general admit weak solutions for times $t < T$ which are classical at any point for which respectively $u_x \neq 0$ or $u \neq 0$. Such weak solutions $u(x,t)$ are continuous functions satisfying the respective identities

$$\int_{\Omega_{T'}} (|u_x|^\sigma u_x \varphi_x - e^u \varphi - u\varphi_t)\,dx\,dt - \int_{-L}^{L} u_0(x)\varphi(x,0)\,dx = 0, \tag{2.1}$$

$$\int_{\Omega_{T'}} (|u|^\sigma u_x \varphi_x - u^\beta \varphi - u\varphi_t)\,dx\,dt - \int_{-L}^{L} u_0(x)\varphi(x,0)\,dx = 0, \tag{2.2}$$

where φ is any compactly supported test function $\varphi \in C_0^\infty(\Omega_{T'})$, $T' < T$. The local existence and uniqueness of such a solution can be found in Lions [34] and Kalashnikov [31] and the references therein. Furthermore it is shown in [2] and [3] that such solutions of (1.13) have a continuous first derivative u_x. Indeed, the weak solution can be constructed as a limit of the sequence $u_\varepsilon(x,t) \to u(x,t)$ of solutions to the *regularised* forms of (1.13) or (1.14) where the derivative or function terms are respectively replaced by

$$((|u_x|^2 + \varepsilon^2)^{\sigma/2} u_x)_x, \quad \text{or} \quad ((u^2 + \varepsilon^2)^{\sigma/2} u_x)_x.$$

It is not difficult to prove that if u_0 has a single maximum and is suitably large then the weak solutions of (2.1) must blow-up in a finite time and at a single point. See proofs of localization in [13] for equation (1.13) and in [22], [26], [39], p. 269, for (1.14).

2.2 The semilinear case

When $\sigma = 0$ then both (1.13) and (1.14) reduce to (1.1) with $f(u) = e^u$ or $f(u) = u^\beta$. For $f(u) = e^u$ we introduce the similarity variables

$$y = |x - x_*|/(T-t)^{1/2}, \quad \theta(y,s) = u(x,t) + \log(T-t), \quad s = -\log(T-t).$$

Rescaling (1.1) then gives the equivalent partial differential equation for θ:

$$\theta_s = \theta_{yy} - \frac{y}{2}\theta_y + e^\theta - 1. \tag{2.3}$$

This equation needs to be augmented with additional conditions so that the solution matches the original boundary conditions. Observe that at the boundary points $x = \pm L$ we have $y = y_1 = (L - x_*)/(T-t)^{1/2}$ and $y = y_2 = (L + x_*)/(T-t)^{1/2}$ so that $|y_1|$ and $|y_2|$ both become infinite as $t \to T$. At y_1 and y_2 we thus have

$$\theta(y_i, s) = \log(T-t) \to -\infty.$$

The steady-state solution $\theta(y)$ of problem (2.3) satisfies the ordinary differential equation

$$\theta_{yy} - \frac{y}{2}\theta_y + e^\theta - 1 = 0, \tag{2.4}$$

with $\theta_y(0) = 0$. If such a solution exists we have a self-similar solution of (1.1). Observe that as $|x| \to L$, $\log(T-t) \to -2\log(y) + O(1)$ so that for the self-similar solution to match with the boundary conditions in the asymptotic limit of $s \to \infty$ we require that

$$\theta(y) = -2\log(y) + O(1) \quad \text{as} \quad y \to \infty. \tag{2.5}$$

More generally, consider those solutions of problem (2.4) for which $\theta(y)/\log y$ is *bounded* as $y \to \infty$. We term such solutions *slowly-growing*. In contrast *rapidly-growing* solutions of (2.4) also exist which grow like

$$-\frac{1}{y}\exp(y^2/4) \quad \text{as} \quad y \to \infty.$$

A necessary condition for $u(x,t)$ to have self-similar behaviour close to the peak, is that there should be a non-trivial slowly-growing solution of (2.4). Unfortunately, it has been proven in [8] and [18] that the only slowly-growing solution of (2.4) is the trivial solution $\theta(y) = 0$ and that if $\theta(0) \neq 0$ then $\theta(y)$ is necessarily rapidly-growing. It was shown further that $\theta(y,s) \equiv 0$ is an attractor as $s \to \infty$, for the solutions of (2.3) over compact sets in y, see [5] the references therein.

If a similar rescaling in terms of the similarity variables is made for the case of $f(u) = u^\beta$, then the only solution of the resulting ordinary differential equation for the steady state $\theta(y)$ which does not grow exponentially fast as $y \to \infty$ is the constant solution

$$\theta(y) \equiv \gamma.$$

The constant solution $\theta(y,s) \equiv \gamma$ is also an attractor for solutions of (2.4) over compact subsets in y.

In [6], [10], [19] and [28], a centre-manifold reduction was proposed which applies to study the behaviour of the solutions of (2.3) in the neighbourhood of $\theta = 0$. From this reduction it follows that on compact subsets in y a solution which is monotonic evolves in the manner:

$$\theta(y,s) = -\frac{1}{2s}\left(\frac{y^2}{2} - 1\right) + o\left(\frac{1}{s}\right) \quad \text{as} \quad s \to \infty. \tag{2.6}$$

This result clearly shows that such a rescaled solution $\theta(y,s)$ converges *slowly* with the decay rate $O(1/s)$ to the zero solution on such compact subsets. A more general description is obtained by recasting the equation (2.3) in terms of the *ignition* variable η given in (1.9). Rescaling in terms of this variable the effect of the diffusion operator vanishes in the limit of $s \to \infty$ and the system reduces to a first order Hamilton-Jacobi equation. Solving this, it follows that as $s \to \infty$,

$$\theta(y,s) \to -\log\left(1 + \frac{\eta^2}{4}\right), \tag{2.7}$$

uniformly on compact sets in η, giving (1.10).

The profile of $u(x,T)$ for x close to x_* was also determined formally by Dold [17] and proven by Bressan [10] to be of the form

$$u(x,T) = -2\log|x - x_*| + \log|\log|x - x_*|| + \log 8 + o(1) \quad \text{as} \quad x \to x_*. \tag{2.8}$$

For $f(u) = u^\beta$ similar arguments apply to give (1.11) and the final-time profile

$$u(x,T) = \left(\frac{8\beta}{(\beta-1)^2}\frac{|\ln|x - x_*||}{|x - x_*|^2}\right)^{1/(\beta-1)}(1 + o(1)). \tag{2.9}$$

2.3 The quasilinear case

Now consider the quasilinear problems (1.13) or (1.14), $\beta > \sigma + 1$. For both equations it is known [39], that suitably large initial data leads to solutions in which a narrowing peak forms which grows as $t \to T$ at the same asymptotic rate as for the linear diffusion problems. These equations are invariant under the respective Lie groups of transformations:

$$u \mapsto u - \log\,\lambda, \quad x \mapsto \lambda^{1/(\sigma+2)}x, \quad t \mapsto \lambda t, \tag{2.10}$$

or

$$u \mapsto \lambda^{-1/(\beta-1)}u, \quad x \to \lambda^m x, \quad t \to \lambda t, \tag{2.11}$$

where

$$m = \frac{\beta - (\sigma+1)}{2(\beta-1)} = \frac{1}{2} - \frac{\sigma}{2(\beta-1)} > 0. \tag{2.12}$$

Solutions of (1.13) or (1.14) which are invariant under the transformations (2.10) and (2.11) respectively take the form

$$u(x,t) = -\log(T-t) + \theta(y) \quad \text{with} \quad y = |x - x_*|/(T-t)^{1/(\sigma+2)}, \tag{2.13}$$

or

$$u(x,t) = (T-t)^{-1/(\beta-1)}\theta(y) \quad \text{with} \quad y = |x - x_*|/(T-t)^m. \tag{2.14}$$

For the similarity solution to match to a solution of the original partial differential equation with $u(-L,t) = u(L,t) = 0$, it has (as before) to satisfy a matching condition which, after some manipulation is respectively

$$\theta(y) + (\sigma+2)\log(y) \to C \quad \text{as} \quad y \to \infty, \tag{2.15}$$

or

$$\theta(y) y^{1/m(\beta-1)} \to C > 0 \quad \text{as} \quad y \to \infty, \tag{2.16}$$

where C is an appropriate constant in each case, the value of which we determine in the next section. Substituting (2.13) and (2.14) into the corresponding quasilinear equations gives the following respective ordinary differential equations for $\theta(y)$

$$(|\theta_y|^\sigma \theta_y)_y - \frac{y}{\sigma+2}\theta_y + e^\theta - 1 = 0, \tag{2.17}$$

or

$$(\theta^\sigma \theta_y)_y - my\theta_y - \frac{1}{\beta-1}\theta + \theta^\beta = 0, \tag{2.18}$$

with the symmetry condition at the origin

$$\theta_y(0) = 0.$$

Results, proved in [13], [16] for (1.13) and [1], [39] for (1.14), $\beta > \sigma + 1 > 1$, show that equations (2.17), (2.18) have solutions with $\theta(y)$ monotone decreasing for $y > 0$. From this we deduce the existence of a self-similar solution of the partial differential equations. Such solutions are attracting for all monotone initial data, see [13] and [23]. For the solution of (2.17) we have that $\theta(0) \to 0$ as $\sigma \to 0$, and for (2.18) that $\theta(0) \to \gamma + 0$ as $\sigma \to 0$.

If $1 < \beta < \sigma + 1$ then the solutions of (1.14) exhibit global blow-up rather than single point blow-up (observe that in this case we have $m < 0$ which is a clear indication that the blow-up is not localised), and blow-up is regional if $\beta = \sigma + 1$. See Ch. 4 in [39]. When $\beta > \sigma + 1$, an application of the maximum principle implies that $\theta(0) > \gamma$, and moreover the ODE does not admit a compactly supported solution so that $\theta(y) > 0$ for all $y \geq 0$. As $\sigma \to \beta - 1 - 0$, the rate of decay of $\theta(y)$ as $y \to \infty$ becomes greater. In the limit, $\sigma = \beta - 1$, the solution tends to a weak solution with compact support. This latter solution, described in [39], Sect. 1, Ch. 4, takes the form

$$\theta(y) = \left(\frac{2\beta}{\beta^2-1}\cos^2\left(\frac{\beta-1}{2\sqrt{\beta}}y\right)\right)^{1/(\beta-1)} \tag{2.19}$$

with support in $\{y < y_0 \equiv \pi\sqrt{\beta}/(\beta-1)\}$. Observe that $m = 0$ in this case so that $y = |x - x_0|$, and (2.14) has a variables-separable solution. The measure of the support $2y_0 = 2\pi\sqrt{\beta}/(\beta-1)$ is called the *fundamental length* since it gives the measure of blow-up sets for arbitrary bell-shaped initial data [21], [24], [39].

3 Asymptotic calculations

We now calculate an asymptotic description of the solutions of (2.17) or (2.18) when $\sigma > 0$ is small. Our calculations are formal, but the agreement with numerical calculations results is impressive. To perform our calculations, we use the method of matched asymptotic expansions, seeking approximate solutions in an *inner region* $y \ll \sigma^{-1/2}$ and an *outer* *region* $y \gg 1$, and matching these in an intermediate region $1 \ll y \ll \sigma^{-1/2}$ where both approximations are valid. This approach works with minor modifications for both equations. As the solution of the problem (2.18) with the polynomial nonlinearity involves both σ and β we consider this first and in more detail. The arguments for (2.17) are slightly simpler and we present them in less detail, cf. [16].

3.1 The inner behaviour of the solutions of the polynomial equation (2.18)

The solutions that we seek satisfy the ordinary differential equation (2.18) with the condition

$$\theta(y) = C y^{-1/m(\beta-1)}(1+o(1)) \quad \text{as} \quad y \to \infty, \tag{3.1}$$

and the object of our analysis is to determine a formal description of $\theta(y)$ in both the inner and outer regions. In part, we show show that as $\sigma \to 0$

$$\theta(0) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}\right) + \mathcal{O}(\sigma^2) \quad \text{and} \quad C = \gamma\left(\frac{4\beta}{\sigma}\right)^{1/(\beta-1)}(1+o(1)).$$

Accordingly we pose an asymptotic series for $\theta(y)$ of the form

$$\theta(y) = \theta_0(y) + \sigma\theta_1(y) + \sigma^2\theta_2(y) + \dots . \tag{3.2}$$

For $\sigma\theta_1(y)$ to be small in comparison to $\theta_0(y)$, we find presently that y must be restricted to the inner region $y \ll \sigma^{-1/2}$. Now, we substitute (3.2) into (2.18), expand in powers of σ and compare to all orders.

To order $\mathcal{O}(\sigma^0)$ we obtain

$$\theta_0'' - \frac{1}{2}y\theta_0' - \frac{1}{\beta-1}\theta_0 + \theta_0^\beta = 0, \quad \theta_0'(0) = 0. \tag{3.3}$$

To satisfy the decay condition (3.1) we require that $\theta_0(y)$ does not grow exponentially as y increases. The only solution of (3.3) with this property is

$$\theta_0(y) = \gamma. \tag{3.4}$$

Although this solution does not satisfy the condition (3.1), we show presently that it can be matched to an outer solution which does.

To order $\mathcal{O}(\sigma)$ we obtain

$$\theta_1'' - \frac{1}{2}y\theta_1' - \frac{1}{\beta-1}\theta_1 + \beta\theta_0^{\beta-1}\theta_1 = 0. \tag{3.5}$$

Using the explicit expression for θ_0 this gives

$$\mathbf{H}\theta_1 \equiv \theta_1'' - \frac{1}{2}y\theta_1' + \theta_1 = 0, \quad \theta_1'(0) = 0, \tag{3.6}$$

where $\mathbf{H}$ is the Hermite differential operator. Equation (3.6) has linearly independent solutions $\varphi(y)$ and $\psi(y)$ satisfying $\varphi'(0) = 0$ and $\psi(0) = 0$ which are given by

$$\varphi(y) = 1 - y^2/2 \quad \text{and} \quad \psi(y) = \varphi(y)\int_0^y \frac{e^{z^2/4}}{\varphi^2(z)}dz, \tag{3.7}$$

and which have Wronskian

$$W(y) \equiv \varphi\psi' - \varphi'\psi = e^{y^2/4}. \tag{3.8}$$

For large y, using integration by parts, we develop an asymptotic expansion for $\psi(y)$ in the form

$$\psi(y) \sim -4e^{y^2/4}\left(\frac{1}{y^3} + \frac{12}{y^5} + \dots\right). \tag{3.9}$$

The function $\psi(y)$ grows exponentially fast as y increases, and a solution of (3.6) which includes a contribution due to $\psi(y)$ grows too rapidly to be matched with an outer solution satisfying the decay condition (3.1). For $\theta_1(y)$ to satisfy the boundary condition (and simultaneously to grow slowly at infinity) we take

$$\theta_1(y) = A(1 - y^2/2), \tag{3.10}$$

where the constant A is undetermined at this level of the asymptotic calculation.

Observe that $\sigma\theta_1(y)$ is uniformly small in comparison to $\theta_0(y)$ provided that σy^2 is small, equivalently if $y \ll \sigma^{-1/2}$. This motivates our definition of the inner region.

Now consider the terms of order $\mathcal{O}(\sigma^2)$. Expanding we obtain

$$\mathbf{H}\theta_2 + \frac{\beta(\beta-1)}{2}\gamma^{\beta-2}\theta_1^2 + \log\gamma\,\theta_1'' + \frac{y}{2(\beta-1)}\theta_1' = 0. \tag{3.11}$$

Using the explicit form for θ_1 and rearranging this gives

$$\mathbf{H}\theta_2 = f(y) \equiv -\frac{\beta}{2\gamma}A^2(1-\frac{y^2}{2})^2 + A\log\gamma + A\frac{y^2}{2(\beta-1)}, \tag{3.12}$$

with $\theta_2'(0) = 0$. Using the method of variation of constants and enforcing the boundary conditions, we may invert the Hermite operator $\mathbf{H}$ to give

$$\theta_2(y) = p(y)\varphi(y) + q(y)\psi(y)$$

where

$$p(y) = B - \int_0^y e^{-x^2/4}\psi(x)f(x)dx \quad \text{and} \quad q(y) = \int_0^y e^{-x^2/4}\varphi(x)f(x)dx$$

with B a constant undetermined at this level of the calculation. Using the expression for $\psi(y)$, a simple calculation shows that the term $p(y)\varphi(y)$ grows polynomially as y increases and we show presently can be matched with the outer solution. In contrast, if $q(y)$ does not tend to zero for large y, then the term $q(y)\psi(y)$ grows exponentially for large y and no such matching is possible. Exponential growth is avoided only if $q(y) \to 0 \quad \text{as} \quad y \to \infty$, so that

$$\int_0^\infty e^{-x^2/4}\varphi(x)f(x)dx = 0. \tag{3.13}$$

From (3.12) we then have

$$-A\frac{\beta}{2\gamma}\int_0^\infty e^{-x^2/4}(1-x^2/2)^3dx + \int_0^\infty e^{-x^2/4}\left[\log\gamma + \frac{x^2}{2(\beta-1)}\right](1-x^2/2)dx = 0.$$

Evaluating these integrals gives

$$-4A\beta\sqrt{\pi}/2\gamma + 2\sqrt{\pi}/(\beta-1) = 0$$

so that

$$A = \frac{\gamma}{2\beta(\beta-1)}. \tag{3.14}$$

Using this value of A we then have

$$\mathbf{H}\theta_2 = -\frac{\gamma}{2\beta(\beta-1)^2}\left[\frac{y^4}{16} - \frac{3y^2}{4} + \frac{1}{4} + \log(\beta-1)\right]. \tag{3.15}$$

We now complete our calculation by determining θ_2 for large y. As $\theta_2(y)$ grows polynomially for large y we can pose a series expansion for $\theta_2(y)$ of the form

$$\theta_2(y) = ay^4 + by^2 \log\, y + B\varphi(y) + \mathcal{O}(\log\, y).$$

Substituting into (3.15) and balancing terms we find that

$$\theta_2(y) = \frac{\gamma y^4}{32\beta(\beta-1)^2} + B\varphi(y) + \mathcal{O}(\log\, y). \tag{3.16}$$

We can, in principle, continue this expansion to all orders. Observe that in each case the dominant contribution to $\theta_n(y)$ for large y is a term proportional to $\sigma^n y^{2n}$.

Combining results, in the inner region $y \ll \sigma^{-1/2}$ we have

$$\theta(y) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}(1 - y^2/2)\right) + \sigma^2\theta_2(y) + \mathcal{O}(\sigma^3 y^6), \tag{3.17}$$

and if $1 \ll y \ll \sigma^{-1/2}$, then more precisely

$$\theta(y) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}(1 - y^2/2) + \frac{\sigma^2 y^4}{32\beta(\beta-1)^2}\right) + \mathcal{O}(\sigma^2 y^2)) + \mathcal{O}(\sigma^3 y^6). \tag{3.18}$$

Observe that when $y = 0$

$$\theta(0) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}\right) + \mathcal{O}(\sigma^2). \tag{3.19}$$

3.2 The outer solution of the polynomial equation

We now find an outer solution of the problem (2.18) in a region $y \gg 1$ for which $\theta(y)$ satisfies the decay condition (3.1). To construct this solution we are motivated by the observation that if $y \gg 1$ and $\theta(y)$ satisfies (3.1) then

$$|\theta''| \ll \min\{|\theta' y|, \theta, \theta^\beta\}.$$

Consequently, we neglect the terms in (2.18) which involve second derivatives of θ and consider a leading order outer approximation θ_0 to θ satisfying the reduced equation

$$-m\theta_0' y + \theta_0^\beta - \frac{\theta_0}{\beta-1} = 0. \tag{3.20}$$

This has the exact solution

$$\theta_0(y) = \left(\beta - 1 + Ey^{1/m}\right)^{-1/(\beta-1)} \tag{3.21}$$

where E is an arbitrary *positive* constant. Observe that for large y we then have

$$\theta_0(y) = E^{-1/(\beta-1)} y^{-1/m(\beta-1)}(1+o(1)),$$

which has precisely the correct asymptotic behaviour for large y when

$$C = E^{-1/(\beta-1)}. \tag{3.22}$$

Conversely, if we make the assumption (to be justified presently) that the value of E is sufficiently small such that we can find a range of values of y for which y is large and $Ey^{1/m}$ is small, then in this range we have

$$\theta_0(y) = \gamma \left[1 - \frac{Ey^{1/m}}{(\beta-1)^2} \right] + \mathcal{O}(E^2 y^{2/m}).$$

If $\sigma y^2 \ll 1$ then, using the explicit form for m (see (2.12)) we have

$$y^{1/m} = y^2 y^{2\sigma/(\beta-1)} = y^2 + \frac{2\sigma}{\beta-1} y^2 \log\, y + \dots . \tag{3.23}$$

Now compare the above expression for $\theta_0(y)$ with the inner expression of $\theta(y)$ given in (3.17). From inspection we have an immediate correspondence of the leading order terms. Furthermore, the dominant terms involving the function σy^2 and Ey^2 match in the two expressions if

$$\frac{E}{(\beta-1)^2} = \frac{\sigma}{4\beta(\beta-1)} \quad \text{so that} \quad E = \frac{\sigma(\beta-1)}{4\beta}. \tag{3.24}$$

We note that as E is positive, this matching is only possible if σ is also positive, strongly implying that a solution of the underlying differential equation is only possible if $\sigma > 0$. Observe that this matching procedure implies that $E = \mathcal{O}(\sigma)$. Our assumption that $Ey^{1/m}$ is small is thus asymptotically equivalent to the condition $\sigma y^2 \ll 1$ or equivalently $y \ll \sigma^{-1/2}$. Thus we can directly match $\theta_0(y)$ with the inner expression in the range $1 \ll y \ll \sigma^{-1/2}$.

Substituting this value of E we obtain a leading order outer approximation to $\theta(y)$ of the form

$$\theta_0 = \gamma \left(1 + \frac{\sigma}{4\beta} y^{1/m} \right)^{-1/(\beta-1)} \tag{3.25}$$

and a direct calculation using (3.25) gives $|\theta_0''| \ll \min\{|y\theta_0'|, \theta_0, \theta_0^\beta\}$ so the asymptotic assumptions remain self-consistent. Expanding this expression when $1 \ll y \ll \sigma^{-1/2}$ gives

$$\theta_0(y) = \gamma \left[1 - \frac{\sigma y^2}{4\beta(\beta-1)} + \frac{\sigma^2}{2\beta(\beta-1)^2} \left(\frac{y^4}{16} - y^2 \log\, y \right) \right] + \mathcal{O}(\sigma^3 y^6). \tag{3.26}$$

We now develop a refined outer expansion for $\theta(y)$ by considering an expression of the form

$$\theta(y) = \theta_0(y) + \sigma\theta_1(y) + \sigma^2\theta_2(y) \dots$$

and investigating the behaviour of this for $1 \ll y \ll \sigma^{-1/2}$. A simple calculation shows that in this range we have

$$(\theta_0^\sigma \theta_0')' = -\frac{\sigma\gamma}{2\beta(\beta-1)} + \frac{3\gamma}{8} \frac{\sigma^2 y^2}{\beta(\beta-1)^2} + \mathcal{O}(\sigma^2 \log\, y).$$

Substituting the outer expansion for θ in (2.18) and considering terms of order $\mathcal{O}(\sigma)$ we obtain

$$-\frac{\gamma}{2\beta(\beta-1)}-\frac{y}{2}\theta_1'+\theta_1=0.$$

This has the solution

$$\theta_1(y)=\frac{\gamma}{2\beta(\beta-1)}+Fy^2.$$

Matching with the inner solution implies that the constant $F=0$, so that if $1\ll y\ll\sigma^{-1/2}$, θ_1 is simply

$$\theta_1(y)=\frac{\gamma}{2\beta(\beta-1)}. \tag{3.27}$$

To finish this calculation, we estimate $\theta_2(y)$. Expanding to order $\mathcal{O}(\sigma^2)$, we have (after some manipulation)

$$-\frac{y}{2}\theta_2'+\theta_2+\frac{\gamma y^2}{4\beta(\beta-1)^2}+\mathcal{O}(\log\, y)=0.$$

Solving this gives

$$\theta_2(y)=\frac{\gamma}{2\beta(\beta-1)^2}y^2\log\, y+Gy^2+\mathcal{O}(\log\, y)$$

where G is a constant. Combining our results, the inner limit of the outer solution for the range $1\ll y\ll\sigma^{-1/2}$ is given by

$$\gamma\left[1-\frac{\sigma y^2}{4\beta(\beta-1)}+\frac{\sigma^2}{2\beta(\beta-1)^2}(\frac{y^4}{16}-y^2\log\, y)+\frac{\sigma}{2\beta(\beta-1)}+\frac{\sigma^2}{2\beta(\beta-1)^2}y^2\log\, y\right]+G\sigma^2y^2$$

giving

$$\theta(y)=\gamma\left[1+\frac{\sigma}{2\beta(\beta-1)}\left(1-\frac{y^2}{2}\right)+\frac{\sigma^2y^4}{32\beta(\beta-1)^2}\right]+G\sigma^2y^2$$

which exactly matches the outer limit of the inner expansion given in (3.18). The inner and outer expansions are thus both consistent and match to $\mathcal{O}(\sigma^2)$.

We conclude by summarising our results, giving the leading order expressions for $\theta(y)$ as $\sigma\to 0$ for different ranges of y:

$$\theta(y)\sim\gamma\left(1+\frac{\sigma}{2\beta(\beta-1)}(1-y^2/2)\right)\quad\text{for}\quad y\ll\sigma^{-1/2}, \tag{3.28}$$

$$\theta(y)\sim\gamma\left(1+\frac{\sigma}{4\beta}y^{1/m}\right)^{-1/(\beta-1)}\quad\text{for}\quad y\gg 1, \tag{3.29}$$

$$\theta(y)\sim\gamma\left(\frac{4\beta}{\sigma}\right)^{1/(\beta-1)}y^{-1/m(\beta-1)}\quad\text{for}\quad y\gg\sigma^{-1/2}. \tag{3.30}$$

3.3 Asymptotic estimates for the exponential equation (2.17)

We now use similar techniques to establish the asymptotic behaviour of the solutions of the problem (2.17). It is shown in [16] that for this equation as $\sigma \to 0$ we have $\theta(0) \to 0$ on any compact subset in y and hence $\exp(\theta) - 1 = \theta + \theta^2/2 + O(\theta^3)$ on such a set. This we use as the basis of our argument. Again we construct an inner region $y \ll \sigma^{-1/2}$ in which we look for solutions which grow slowly at infinity. Thus we seek solutions of

$$(|\theta'|^\sigma \theta')' - \frac{y}{\sigma+2}\theta' + e^\theta - 1 = 0, \quad y > 0; \quad \theta'(0) = 0, \tag{3.31}$$

with the condition that

$$\theta(y) \sim C - (\sigma+2)\log(y) \quad \text{as} \quad y \to \infty. \tag{3.32}$$

In particular, we establish that as $\sigma \to 0$

$$\theta(0) = \frac{\sigma}{2} + \mathcal{O}(\sigma^2) \quad \text{and} \quad C = \log\left(\frac{4}{\sigma}\right)(1 + o(1)).$$

3.4 The inner region

As before we pose an asymptotic expansion of the solution in the form (3.2). To leading order in σ we then have

$$\theta_0'' - \frac{y}{2}\theta_0' + e^{\theta_0} - 1 = 0$$

with θ_0 slowly growing at infinity. This equation has $\theta_0 = 0$ as the only slowly growing solution. Using this result and considering terms of order $\mathcal{O}(\sigma)$, the expression for $\theta_1(y)$ is simply

$$\mathbf{H}\theta_1 = 0, \quad \theta_1'(0) = 0. \tag{3.33}$$

The slowly growing solution of this is

$$\theta_1(y) = A\varphi(y) \tag{3.34}$$

with $\varphi(y)$ as defined in the last section and A undetermined.

Now proceed to terms of order $\mathcal{O}(\sigma^2)$. Expanding and substituting the expression for $\theta_1(y)$ gives

$$\mathbf{H}\theta_2 = Af(y) + A^2 g(y), \quad \theta_2'(0) = 0, \tag{3.35}$$

where

$$f(y) = -(\log|\varphi'|\varphi')' - \frac{1}{4}y\varphi', \quad g(y) = -\frac{\varphi^2}{2}. \tag{3.36}$$

Exactly as before, we may invert the Hermite operator to give the function θ_2 determined up to an arbitrary multiple of the function $\varphi(y)$. Following the previous analysis, this solution is slowly growing only if

$$\int_0^\infty e^{-x^2/4} f(x)\varphi(x)dx + A\int_0^\infty e^{-x^2/4} g(x)\varphi(x)dx = 0.$$

Both of these integrals can be evaluated exactly to give

$$A = \frac{\sigma}{2}. \tag{3.37}$$

We conclude from this inner calculation that to leading order, if $y \ll 1/\sigma^{1/2}$

$$\theta(y) = \frac{\sigma}{2}(1 - \frac{y^2}{2}) + \mathcal{O}(\sigma^2 y^4). \tag{3.38}$$

3.5 The outer region

In the range of $y \to \infty$ the functional form (3.32) imposes the condition that $(|\theta'|^\sigma \theta')'$ is small compared with the other terms in (3.31). As in the calculations for the polynomial equation, if we consider this to be generally true for an outer region of values for y then (3.31) may be approximated to leading order by the first-order ordinary differential equation

$$-\frac{y\theta'}{\sigma + 2} + e^\theta - 1 = 0. \tag{3.39}$$

This ordinary differential equation has the exact solution

$$\theta(y) = -\log(1 + Ey^{\sigma+2}), \tag{3.40}$$

where $E > 0$ is an arbitrary positive constant. For large y, (3.40) gives

$$\theta(y) \sim -(\sigma + 2)\log(y) - \log(E),$$

so that

$$C = -\log(E).$$

If, in contrast we can take values of y in the inner region (to be justified) such that $Ey^{\sigma+2}$ can be considered small, then from (3.40) we deduce that to leading order

$$\theta(y) = -Ey^{\sigma+2} = -Ey^2 + \mathcal{O}(\sigma E y^2 \log(y)). \tag{3.41}$$

Matching this with the inner expression then gives

$$E = \frac{\sigma}{4}.$$

Note again that E can only be positive if σ is positive, again implying strongly that no solutions of the underlying differential equation will be found for negative values of σ.

For small σ, there holds $(|\theta'|^\sigma \theta')' \approx \theta''$ which in expression (3.41) is approximated by $-2E$. In comparison, the terms $y\theta'$ and $(e^\theta - 1)$ are both of order Ey^2 which dominate $-2E$ for large y^2. So the condition for (3.38) and (3.41) for (3.41) to be valid is simply that

$$y \gg 1,$$

which is the appropriate outer region for y.

A more precise calculation can be made as in the previous section to refine the agreement between the inner and outer expressions. This exercise is purely routine and we do not include it here.

Thus, if σ is small, the function $\theta(y)$ can be described as follows

$$\theta(y) \sim \frac{\sigma}{2}\left(1 - \frac{y^2}{2}\right) \quad \text{for} \quad y \ll \sigma^{-1/2}, \tag{3.42}$$

$$\theta(y) \sim -\log\left(1 + \frac{\sigma}{4}y^{\sigma+2}\right) \quad \text{for} \quad 1 \ll y, \tag{3.43}$$

$$\theta(y) \sim \log\left(\frac{4}{\sigma}\right) - (\sigma+2)\log(y) \quad \text{for} \quad y \gg \sigma^{-1/2}. \tag{3.44}$$

4 Time dependent solutions

The previous section has described the form of $\theta(y)$ and we now consider the resulting profile of the solution $u(x,t)$. In this section we presume that the evolution of $u(x,t)$ is sufficiently advanced so that the behaviour close to the peak can be described in terms of the self-similar solution. Observe that the behaviour can never be self-similar over the whole of the domain due to the effects of the boundary conditions. Thus our description is a good approximation only for t close to T and for x close to x_*.

4.1 The polynomial equation (1.14)

In this case the self-similar solution takes the form

$$u(x,t) = \frac{\theta(y)}{(T-t)^{1/(\beta-1)}}, \quad y = \frac{|x-x_*|}{(T-t)^m}.$$

Observe that if $|x - x_*|$ is small and fixed, then $y \to \infty$ as $t \to T$, so that for this range of values of x it is appropriate to use the expression (3.21), (3.26) for $\theta(y)$. Substituting we then have

$$u(x,t) \sim \frac{\gamma}{(T-t)^{1/(\beta-1)}}\left(1 + \frac{\sigma}{4\beta}\frac{|x-x_*|^{2(\beta-1)/(\beta-(\sigma+1))}}{(T-t)}\right)^{-1/(\beta-1)} \tag{4.1}$$

provided that

$$|x - x_*| \ll 1 \quad \text{and} \quad |x-x_*|/(T-t)^{1/m(\beta-1)} \gg 1.$$

Holding $x \neq x_*$ fixed, with $|x - x_*|$ small, and letting $t \to T$ gives

$$u(x,t) \to u(x,T) \sim \gamma\left(\frac{4\beta}{\sigma}\right)^{1/(\beta-1)}|x-x_*|^{-2/(\beta-\sigma-1)}. \tag{4.2}$$

We see that $u(x,T) \to \infty$ as $x \to x_*$. Observe also that $u(x,T) \to \infty$ as $\sigma \to 0$.

From the expression (4.1) we see that that the function $u(x,t)$ increases monotonically toward $u(x,T)$ which is an outer envelope for the general solution.

If $x = x_*$ we have instead

$$u(x_*,t) \sim \gamma(T-t)^{-1/(\beta-1)}\left(1 + \frac{\sigma}{2\beta(\beta-1)}\right). \tag{4.3}$$

4.2 The exponential equation (1.13)

This equation exhibits very similar behaviour. Substituting

$$u(x,t) = -\log(T-t) + \theta(y), \quad y = \frac{|x-x_*|}{(T-t)^{1/(\sigma+2)}}$$

into the expression (3.43) for $\theta(y)$ when $|x-x_*|$ is small and y is large gives

$$u(x,t) \sim -\log(T-t) - \log\left(1 + \frac{\sigma}{4}\frac{|x-x_*|^{\sigma+2}}{(T-t)}\right) = -\log\left((T-t) + \frac{\sigma}{4}|x-x_*|^{\sigma+2}\right). \tag{4.4}$$

If x is fixed, $|x-x_*|$ is small and $t \to T$ then we have

$$u(x,T) \sim -(\sigma+2)\log|x-x_*| + \log\left(\frac{4}{\sigma}\right). \tag{4.5}$$

Again, $u(x,T) \to \infty$ both as $x \to x_*$ and as $\sigma \to 0$. Furthermore, the solution $u(x,t)$ tends monotonically toward this function $u(x,T)$ as $t \to T$.

When $x = x_*$ then

$$u(x,t) \sim -\log(T-t) + \frac{\sigma}{2}.$$

5 On the spectrum of the linearized operator

In this section we discuss the rate of convergence to self-similar solutions for the quasilinear equation (1.14). The main goal is to justify the conjecture that for $\sigma > 0$ the behaviour corresponds to the evolution on a stable manifold which breaks down as $\sigma \to 0$. We describe the transition between stable manifold for $\sigma > 0$ and the corresponding centre manifold for $\sigma = 0$, and actually we evaluate an asymptotic expansion for the first positive eigenvalue of the linearized operator and show that it vanishes at $\sigma = 0$ leading to a center manifold in this case.

5.1 The linearized problem

Let $\sigma > 0$, $\beta > \sigma + 1$, and let $\theta(y) > 0$ be the monotone solution of the ODE (2.18) which can be constructed as the limit of monotone solutions which vanish at some point, see comments below. According to [23] we fix an evolution rescaled orbit

$$f(y,s) \equiv (T-t)^{1/(\beta-1)} u(y(T-t)^m, t), \quad s = -\log(T-t) \to \infty \tag{5.1}$$

which converges to $\theta(y)$ as $s \to \infty$ uniformly on compact subsets in y. The rescaled function f satisfies the equation

$$f_s = (f^\sigma f_y)_y - myf_y - \frac{1}{\beta-1}f + f^\beta, \quad s > s_0 = -\log T, \tag{5.2}$$

with the corresponding boundary conditions in the expanding domain and a symmetric bounded initial data $f_0(y)$. The $f(y,s)$ is symmetric for all $s > s_0$ so that we restrict our stability analysis to the class of symmetric functions.

By a standard linearization procedure, setting

$$f(y,s) = \theta(y) + Y(y,s),$$

we obtain the equivalent equation with the linearized operator

$$Y_s = \mathbf{A}Y + \mathbf{B}(Y), \quad s > s_0, \tag{5.3}$$

where $\mathbf{A}$ is the linear operator

$$\mathbf{A}Y = (\theta^\sigma(y)Y)'' - myY' + \left[-\frac{1}{\beta-1} + \beta\theta^{\beta-1}(y)\right]Y, \tag{5.4}$$

and $\mathbf{B}$ is a quadratic perturbation satisfying

$$\mathbf{B}(\varepsilon Y) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \quad \text{for any } Y \in C^2.$$

5.2 Spectrum of linearized operator

Consider the operator $\mathbf{A}$ written down in the standard Sturm-Liouville form

$$\mathbf{A}Y = \frac{1}{\rho}(pY')' - qY, \tag{5.5}$$

where

$$p(y) = \theta^{2\sigma}(y)\exp\left\{-m\int_0^y \xi\theta^{-\sigma}(\xi)d s\xi\right\}, \quad \rho = \theta^{-\sigma}p, \tag{5.6}$$

and

$$q(y) = \frac{1}{\beta-1} - \beta\theta^{\beta-1}(y) - (\theta^\sigma(y))''.$$

It is known that the symmetric self-similar profile θ satisfies the properties:

$$\theta(y) > 0, \quad \theta(y) = Cy^{-2/(\beta-\sigma-1)}(1+o(1)) \quad \text{as } y \to \infty, \tag{5.7}$$

where $C = C(\sigma,\beta) > 0$ is a constant [1], [39], Ch. 4. Since positive solutions of a non-singular ODE are C^∞ smooth, using the asymptotic expansion in (5.7) one can conclude that

$$|(\theta^\sigma)''| \le c_0 \quad \text{in } \mathbf{R},$$

so that the coefficients of the linear operator (5.5) satisfy:

$$p > 0, \quad \rho > 0, \quad q \ge q_0 = \frac{1}{\beta-1} - c_0 \quad \text{in } \mathbf{R}, \tag{5.8}$$

and

$$p(y), \rho(y) \sim \exp\{-C^{-\sigma}y^{1/m}\} \quad \text{as } y \to \infty. \tag{5.9}$$

Proposition 5.1. *There exists a self-adjoint extension of* $\mathbf{A}$ *with a discrete spectrum.*

Proof. $\mathbf{A}$ is a symmetric operator in the space $C_0(\mathbf{R})$ of twice continuously differentiable compactly supported functions with the inner product of the weighted space $L^2_\rho(\mathbf{R})$

$$(u,v)_\rho = \int_{-\infty}^{\infty}\rho uv dy.$$

Denote $\|\cdot\|_\rho^2 = (\cdot,\cdot)_\rho$. By (5.8) we have that

$$(\mathbf{A}u, u)_\rho \le -\int \rho q u^2 \le -q_0 \|u\|_\rho^2, \tag{5.10}$$

Therefore, $\tilde{\mathbf{A}} = -\mathbf{A} - q_0 + 1$ is lower semibounded and moreover positive definite. Hence, there exists a unique self-adjoint Friedrich's extension of $\tilde{\mathbf{A}}$, as well as $\mathbf{A}$ (still denoted by $\mathbf{A}$), which is obtained from the quadratic form associated with $\mathbf{A}$ and satisfies the same bound given in (5.10). See [9], p. 228.

Consider this self-adjoint extension $\mathbf{A}$ in $L^2_\rho(\mathbf{R})$ restricted to symmetric (even) functions. It follows from (5.8) that the only singular point for the extended operator is $y = \infty$. Using the asymptotic expansion (due to (5.7)) of the coefficients of $\mathbf{A}$, one can see that for $y \gg 1$ the equation $\mathbf{A}u = 0$ admits two linearly independent solutions

$$u_1 \sim \exp\{C^{-\sigma} y^{1/m}\} \notin L^2_\rho, \quad \text{and} \quad u_2 \sim y^{-2/(\beta-\sigma-1)} \in L^2_\rho.$$

Therefore, $y = \infty$ is in the limit point case of a singular endpoint, and hence $\mathbf{A}$ has a discrete spectrum, see Ch. 2 in [33]. This completes the proof.

Let $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ be the ordered set of (simple) eigenvalues of $\mathbf{A}$ with the corresponding complete orthonormal set of eigenfunctions $\{\psi_k\}$ satisfying

$$\mathbf{A}\psi_k = -\lambda_k \psi_k.$$

The first eigenvalue/eigenfunction pair is easily calculated explicitly from equation (5.5):

$$\lambda_1 = -1, \quad \psi_1(y) = c_1 \left[\frac{1}{\beta - 1}\theta(y) + m y \theta'(y) \right], \tag{5.11}$$

where $c_1 > 0$ is a normalization constant. Observe that $\psi_1 > 0$ in $\mathbf{R}$, [39], p. 197, so that $\psi_1(y)$ does not vanish in $\mathbf{R}$. One can see that the unstable manifold driven by the negative eigenvalue (5.11) corresponds to the natural instability of blow-up solutions to small perturbations of the blow-up time T.

Remark. For nonsymmetric functions from $L^2_\rho(\mathbf{R})$, the explicit representation of the second pair is available:

$$\tilde{\lambda}_2 = -m > \lambda_1, \quad \tilde{\psi}_2(y) = c_2 \theta'(y) \tag{5.12}$$

($\tilde{\psi}_2$ has a single zero at the origin), which obviously corresponds to the unstability upon spatial perturbations of a singular blow-up point $0 \mapsto x_0$.

We now prove the main result of this section

Theorem 5.2. *In the space $L^2_\rho(\mathbf{R})$ of symmetric functions, there holds*

$$\lambda_2 \ge 0. \tag{5.13}$$

Proof. Consider the stationary nonlinear ODE

$$(f^\sigma f')' - m y f' - \frac{1}{\beta - 1} f + f^\beta = 0, \quad y > 0, \tag{5.14}$$

with the boundary condition

$$f(0) = \alpha > \gamma = (\beta - 1)^{-1/(\beta-1)}, \quad f'(0) = 0. \tag{5.15}$$

Let $f = f(y;\alpha) \geq 0$ be a unique solution defined on an interval where it is strictly positive. It is known that the self-similar profile $\theta(y)$ can be constructed as follows: $\theta = f(y;\alpha_*)$ where

$$\alpha_* = \inf\{\bar{\alpha} > \gamma \,|\, f(y;\alpha) \text{ has exactly one intersection with } f \equiv \gamma \text{ for all } \alpha \in (\bar{\alpha}, \infty)\} \tag{5.16}$$

and $\alpha_* > \gamma$. See [1] and [39], p. 194. Moreover, it then follows from the construction of θ ([39], p. 193) that for any $\alpha > \alpha_*$ the solution $f(y;\alpha)$ vanishes at a finite point, and hence intersects $\theta(y)$. The number of intersections of $f(y;\alpha)$ and $\theta(y)$ is exactly one for all $\alpha \gg 1$, since $f(y;\alpha)$ is very "steep", [39], p. 196. Therefore, the number of intersections between f and θ cannot be larger than one for all $\alpha > \alpha_*$. Indeed, if this were not true for some $\alpha = \alpha_1 > \alpha_*$ then by continuity we could choose $\alpha_2 \geq \alpha_1$ such that $f(y;\alpha_2)$ would touch $\theta(y)$ at a finite point contradicting the uniqueness result for the ODE (5.14) in the positivity domain.

We now set $\alpha = \alpha_* + \varepsilon$ and pass to the limit $\varepsilon \to +0$. Since $\theta(y) > 0$ for $y > 0$, by the continuous dependence of solutions of the non-singular ODE upon small perturbations of the boundary data, we conclude that on any compact subset $[0, K]$

$$f(y;\alpha_* + \varepsilon) = \theta(y) + \varepsilon Y(y) + O(\varepsilon^2), \tag{5.17}$$

where Y solves the linearized equation

$$\mathbf{A}Y = 0, \quad y > 0; \quad Y(0) = 1, \quad Y'(0) = 0. \tag{5.18}$$

Recall that Y has exactly one zero for $y > 0$. Since (5.18) is the eigenfunction equation $\mathbf{A}Y = -\lambda Y$ with $\lambda = 0$, by the standard properties of eigenvalues of ordinary self-adjoint operators (see p. 201 in [33]) we conclude that ψ_2 with exactly one zero cannot have $\lambda_2 < 0$ so that (5.13) holds. This completes the proof.

The proof of the strict positivity, $\lambda_2 > 0$, needs extra detailed properties of the function Y in (5.17) (namely, that $Y \notin L^2_\rho(\mathbf{R})$). Below, we give a strong analytical and numerical evidence that λ_2 is positive for any arbitrarily small $\sigma > 0$.

5.3 The transitional behaviour of λ_2 as $\sigma \to +0$

We now show that as $\sigma \to +0$

$$\lambda_2 = \frac{\sigma}{\beta - 1} + \ldots \to +0. \tag{5.19}$$

The eigenvalue problem is now

$$\mathbf{A}\psi = -\lambda\psi, \quad \psi \in L^2_\rho, \quad \psi \text{ is symmetric}, \tag{5.20}$$

where $\mathbf{A}$ is the linearized operator (5.4). To solve this equation approximately, we seek an asymptotic expansion of λ and ψ so that

$$\lambda = \nu_0 + \sigma\nu_1 + \sigma^2\nu_2 + \ldots, \quad \psi = \varphi_0 + \sigma\varphi_1 + \sigma^2\varphi_2 + \ldots. \tag{5.21}$$

We now perform an expansion with respect to $\sigma \approx +0$ by substituting the expression (5.21) into (5.20) and using the asymptotic expressions (3.28) for $\theta(y)$ and

$$m = \frac{1}{2} - \frac{\sigma}{2(\beta - 1)}.$$

To leading order $O(\sigma^0)$ we then have

$$\mathbf{H}\varphi \equiv \varphi_0'' - \frac{1}{2}y\varphi_0' + \varphi_0 = -\nu_0\varphi_0. \tag{5.22}$$

Thus φ_0 is a Hermite polynomial and ν_0 is an eigenvalue of the Hermite operator $\mathbf{H}$. In the space $L^2_w(\mathbf{R})$, $w = e^{-y^2/4}$, of symmetric functions, we thus have $\nu_0 = -1, 0, 1, 2, \dots$. In this sequence, as in the quasilinear case, the first pair $\nu_0 = -1$, $\varphi_0 \equiv 1$, corresponds to an instability arising from a perturbation of the blow-up time T, and we exclude this. The next critical eigenvalue which dominates the stability analysis is then $\nu_0 = 0$ for which $\varphi_0(y) = 1 - y^2/2$.

Now consider terms in (5.20) of order $\mathcal{O}(\sigma)$. This gives

$$\log\gamma\,\varphi_0'' + \varphi_1'' - \frac{1}{2}y\varphi_1' + \frac{1}{2(\beta-1)}\varphi_0' + \varphi_1 + \frac{1}{2(\beta-1)}\varphi_0^2 = -\nu_1\varphi_0,$$

which rearranges to give

$$\mathbf{H}\varphi_1 \equiv \varphi_1'' - \frac{1}{2}\varphi_1' + \varphi_1 = -\nu_1\varphi_0 - \log\gamma\,\varphi_0'' - \frac{1}{2(\beta-1)}\varphi_0' - \frac{1}{2(\beta-1)}\varphi_0^2. \tag{5.23}$$

As φ_1 must be slowly growing, $\varphi_1 \in L^2_w$, we apply exactly the argument of the previous section to give the following orthogonality condition on the right-hand side of (5.23):

$$-\nu_1\int_0^\infty \varphi_0^2 w\,dy = \int_0^\infty \left[\log\gamma\,\varphi_0'' + \frac{1}{2(\beta-1)}\varphi_0' + \frac{1}{2(\beta-1)}\varphi_0^2\right]\varphi_0 w\,dy.$$

All quadratures may be examined explicitly to give

$$\nu_1 = \frac{1}{\beta-1},$$

and hence as $\sigma \to 0$ we have the following expansion of the eigenvalue/eigenfunction pair

$$\lambda_2 = \frac{\sigma}{\beta-1} + \dots, \quad \psi_2 = c_2(1 - y^2/2) + \dots, \tag{5.24}$$

as required. Observe that other eigenvalues are perturbations of $1, 2, \dots$.

6 Numerical results

We now present some numerical results to compare with the asymptotic calculations. In this Section we confine ourselves to the porous medium type equation (1.14) as this presents fewer technical numerical difficulties. The first calculation looks at the numerical solution of the ordinary differential equation (2.18) for a range of values of σ not only restricted to the case of σ small. The second series of calculations looks at the solution of the partial differential equation (1.14) using arbitrary initial data.

6.1 The numerical solution of the ordinary differential equation

To solve the ordinary differential equations (2.18) we use a shooting method. In this the ordinary differential equation is treated as an initial value problem with $\theta(0) = \alpha$. The value of α is then varied until a slowly-growing solution is found. Observe that a similar argument is used for proving the existence of suitable self-similar profiles, cf. [1], [15] and [39], p. 194. See also Section 5. To solve this initial value problem we use the variable step, variable order BDF solver in the package DDASSL [38].

A difficulty with using this approach is the existence of the exponentially growing solutions of the respective ordinary differential equations. To perform the calculations, the value of α is slowly varied. Suppose that $\alpha(\sigma)$ corresponds to the slowly-growing solution. We find that if α is close to, but smaller than $\alpha(\sigma)$ then the corresponding solution of the initial value problem is close to the slowly-growing solution for a bounded set in y but eventually (and rapidly) the solution becomes large and positive (exhibiting exponential growth). Conversely, if α is close to, but larger than $\alpha(\sigma)$ then the solution eventually becomes large and negative. Fixing a value of $y = 12$ and using the method of bisection it was then relatively easy to find an intermediate value of $\alpha = \alpha(\sigma)$ giving the slowly growing solution. As an example of this approach we present in Figure 6.1, a solution of the initial value problem (2.18) with $\beta = 2, \sigma = 0.1, \gamma = 1, 1/m = 2.22222222$ in which α takes a value just above $\alpha(\sigma) = 1.0265620225916$. On the same figure we present three asymptotic solutions. These are the leading order inner solution

$$\theta_I(y) = 1 + \frac{\sigma}{4}\left(1 - \frac{y^2}{2}\right),$$

the leading order outer solution

$$\theta_{O1}(y) = \left(1 + \frac{\sigma}{8} y^{1/m}\right)^{-1}$$

and a second order correction to the outer solution. This is determined by continuing the calculations presented in Section 3 and which we find after some manipulation to be

$$\theta_{O2}(y) = \theta_{O1}(y) + \frac{y^2}{(1+\frac{\sigma}{8}y^2)^2}\left\{\frac{\sigma}{4} + \frac{\sigma^2}{4}y^2\log(y) - \frac{\sigma^2}{8}y^2\left[\log(8) + \log\left(1 + \frac{\sigma}{8}y^2\right)\right]\right\}.$$

From this figure we see that $\theta_I(y)$ is only a good approximation for small values of y and $\theta_{O2}(y)$ is a good approximation for a wide range of values. The approximation $\theta_{O1}(y)$ is reasonable but not especially good, indicating a strong dependence of the solution upon σ.

For a general calculation with σ varying, a value of α close to the asymptotic value of

$$\alpha(\sigma) = \gamma\left(1 + \frac{\sigma}{2\beta(\beta-1)}\right),$$

was taken to start the calculation for σ small. For larger values of σ path following was used. For problem (2.18) σ was increased to $\beta - 1$, at which point the value

$$\alpha(\beta - 1) = \left(\frac{2\beta}{\beta^2 - 1}\right)^{1/(\beta-1)}$$

leads to a solution with compact support. Values of $\alpha(\sigma)$ for $\sigma > \beta - 1$ can also be computed using a very similar method, with the desired solution always having compact support in this case.

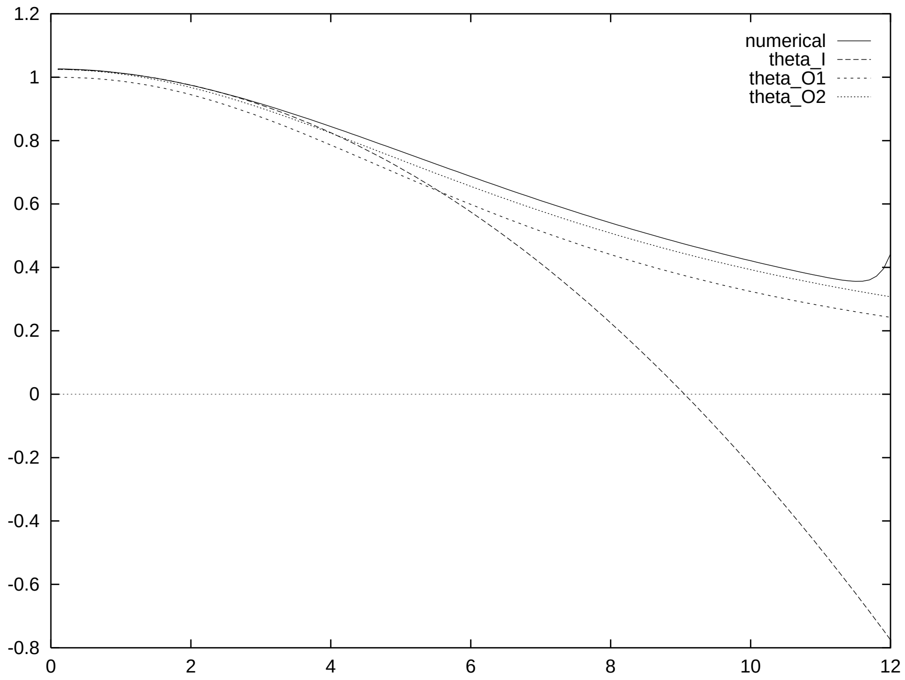

Figure 6.1: A comparison of the numerically computed graph with the inner and outer asymptotic solutions.

In Figure 6.2 we present the resulting computations for (2.18) with $\beta = 2$. In this calculation σ varies between 0 and 1.1, observing that $\alpha(1) = 4/3$. For comparison we also plot the asymptotic curve $1 + \sigma/4$.

The resulting values of $\alpha(\sigma)$ are given in the table below.

σ	$\alpha(\sigma)$	$1 + \sigma/4$
0.0	1.00000	1.000
0.1	1.02656	1.025
0.2	1.05551	1.050
0.3	1.08608	1.075
0.4	1.11805	1.100
0.5	1.15119	1.125
0.6	1.18541	1.150
0.7	1.22073	1.175
0.8	1.27509	1.200
0.9	1.29457	1.225
1.0	1.33333	1.250
1.1	1.37363	1.275

We see from this figure and the tabulated values, that the asymptotic description of $\alpha(\sigma)$ is reasonable provided that $\sigma < 0.1$ but for larger values of σ the numerical value is rather greater than the asymptotic value.

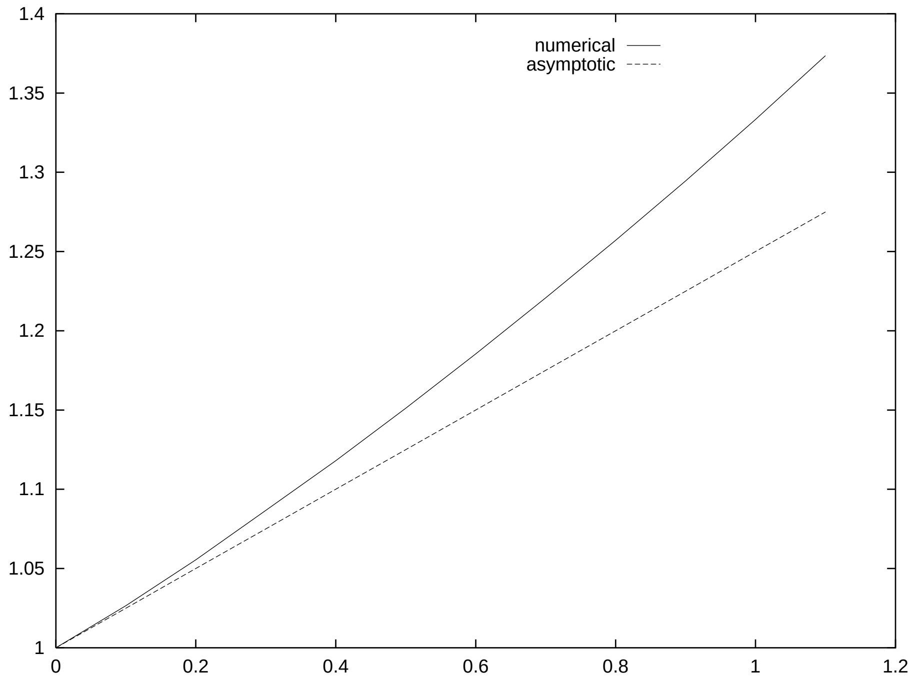

Figure 6.2: A comparison of the numerically computed graph of $\alpha(\sigma)$ with the leading order asymptotic $1+\sigma/4$.

6.2 The numerical solution of the partial differential equations

We now study the evolution of the solutions of the partial differential equation (1.14) from general initial data. Our principle interest will be to see how rapidly the solution evolves towards the self-similar profile, and to compare the numerical results with the analytic conclusions of Section 5. To capture the structure of the narrowing peak as $t \to T$, it is necessary to use an adaptive method which moves mesh points towards the centre of the peak. To do the calculation we use an adaptation of the adaptive method of lines procedure described in [30], [11]. In this method of lines approach we take $U_i(t)$ with $0 \le i \le N$ to be an approximation to the function $u(x,t)$ at the mesh point $X_i(t)$. The equation for U_i is then found by discretising (1.14) in Lagrangian form so that

$$\frac{du}{dt} - \frac{\partial u}{\partial x}\frac{dx}{dt} = \left(\frac{u^{\sigma+1}}{\sigma+1}\right)_{xx} + u^{\beta}. \tag{6.1}$$

The discretisation of $u(x,t)$ in space uses a collocation method with Hermite cubic interpolants. To calculate the location of the mesh points X_i we use the method of moving mesh partial differential equations described in [30]. In particular, we require that X_i satisfies the equation

$$\frac{dX_i}{dt} = -\frac{1}{\tau}\left(\int_{-L}^{X_i} M dx - \frac{i}{N}\int_{-L}^{L} M dx\right). \tag{6.2}$$

Here τ is a small, positive constant. The above has an equilibrium mesh which equidistributes the monitor function M. Following [11] we choose M so that (6.2) is invariant to the same transformation group as (6.1). In particular we take

$$M(u) = (u)_+^{\beta-1}. \tag{6.3}$$

Observe that with this choice of monitor, the mesh points move toward regions where u (and hence M) is large. In practice a smoothed form of (6.2) is used to prevent instabilities and the equation discretised using a collocation method.

The resulting discretisations of the mesh equation and the original partial differential equation when augmented with the boundary conditions

$$U_0(t) = U_N(t) = 0, \quad X_0(t) = -L, \quad X_N(t) = L,$$

leads to a system of ordinary differential equations for U_i and X_i. These are then solved using DDASSL. Typically we take $N = 160$. As the solution approaches the blow-up time, it is essential that the time steps in the ODE solver are adaptive. If this is not the case then it is possible that the scheme steps over the blow-up time and misses the solution completely. Moreover adaptivity is needed in order to calculate the blow-up time accurately. A natural choice of time step Δt is given by the similarity variable Δ given by

$$\Delta = K \max(U_i)_+^{1-\beta}$$

and we instruct the ODE solver to take a time step not smaller than Δ where K is a constant which we set to be 10^{-4}.

If an initially uniform mesh is chosen and $u(x,0)$ taken to be $10\exp(-x^2/4)$ then the resulting solution for $\sigma = 0.1$ and $\beta = 2$ is as given in Figure 1.1 which includes the asymptotic form of the envelope

$$u(x,T) \sim \frac{8}{\sigma}|x|^{-2/(1-\sigma)}.$$

To compare this solution with the self-similar solution, the value of the blow-up time is estimated by running the algorithm until overflow occurs. The function $e(x,t)$ defined by

$$e(x,t) = (T-t)^{1/(\beta-1)}u(x,t) - \theta(x/(T-t)^m)$$

is then computed and we calculate

$$E(t) = \sup_x |e(x,t)|.$$

The resulting form of $E(t)$ as a function of $u(x_*,t)$ is ploted in Figure 6.3 for the values of $\sigma = 0.1$ and $\sigma = 0.2$. Observe that there is good evidence from these figures of a strong scaling structure of $E(t)$. In particular we conclude from these figures that there is a function $\alpha(\sigma) > 0$ which depends upon σ and which satisfies $\alpha(\sigma) \to 0$ as $\sigma \to 0$, such that

$$E(t) \sim e^{-\alpha(\sigma)s}.$$

Furthermore, there is good evidence from the graph that

$$\alpha(0.1) \approx 0.1 \quad \text{and} \quad \alpha(0.2) \approx 0.2.$$

This numerical evidence indicates that the transient evolution towards the self-similar solution is dominated by the second (positive) eigenvalue $\lambda_2 = \alpha(\sigma)$ of the linearized stationary operator corresponding to the rescaled partial differential equation

$$f_s = (f^\sigma f_y)_y - myf_y - \frac{1}{\beta-1}f + f^\beta,$$

linearised about the self-similar profile $\theta(y)$. Furthermore that this eigenvalue tends to zero as $\sigma \to 0$, and is close to the asymptotic estimate of

$$\lambda_2 \sim \sigma/(\beta-1)$$

given in Section 5. This results acts to support both our numerical and asymptotic calculations.

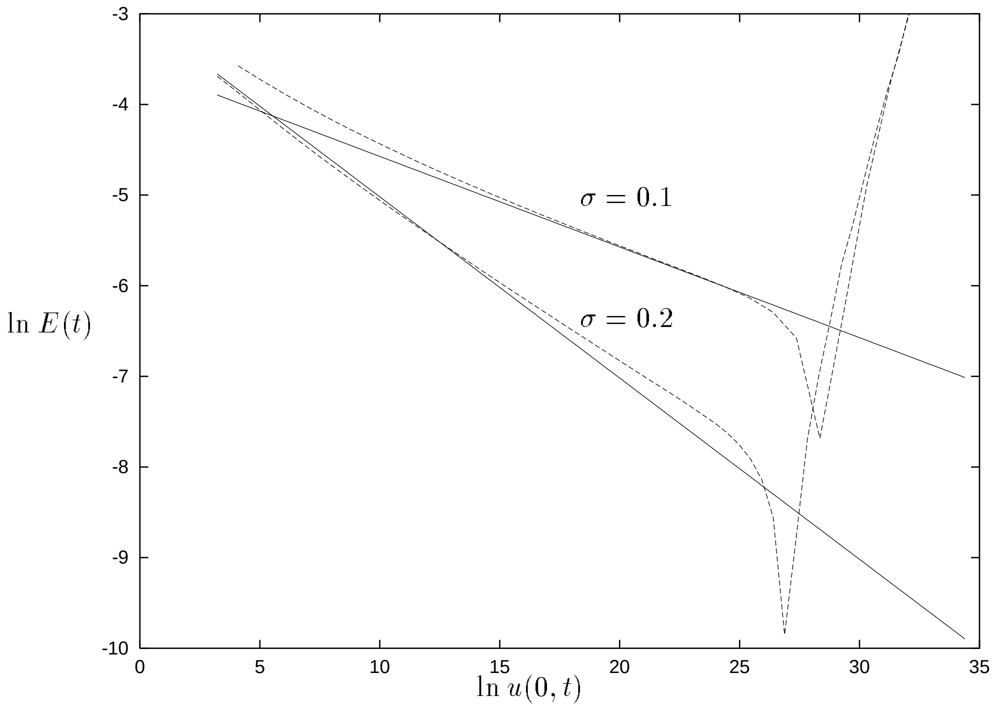

Figure 6.3: Convergence onto the self-similar solution. Dashed lines are the computed errors and solid lines are lines of gradients -0.1 and -0.2 respectively. Initial solution is $u_0 = 10\exp(-x^2/4)$.

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