A Variational Theory of Stokes Waves and their Sub-Harmonic Bifurcations

B. Buffoni * E. N. Dancer † J.F. Toland ‡

Abstract

A formulation of the classical hydrodynamic problem of periodic waves on the free-surface of an infinitely deep, irrotational flow, which moves steadily under the influence of gravity and possibly surface tension, is introduced. It is a quasi-linear equation involving the conjugate operator in the theory of Fourier series and has variational structure, being the Euler-Lagrange equation of a simple functional.

With non-negative coefficient of surface tension, the equation is studied and regularity properties of its solutions are established. A bifurcation theory for gradient operators is used to prove the existence of small amplitude waves at bifurcation points of potentially high multiplicity.

In the absence of surface tension (the so-called Stokes wave problem) the governing equation is quadratic (with no higher order terms). In this case, the global structure of its solution set is studied using elements of the theory of real-analytic varieties. It is shown that there bifurcate from the first eigenvalue of the linearised problem a uniquely-defined, arc-wise connected set of solutions which, although it is not necessarily maximal as a connected set of solutions and may self-intersect, has a local real-analytic parametrisation and contains a wave of greatest height in its closure (suitably defined).

Further, on this particular branch of Stokes waves there are infinitely many values of the Froude number (a dimensionless parameter reflecting the wavelength when the wave speed is fixed) which are either turning points or bifurcation points. It is also shown that there are infinitely many values of the Froude number at which there bifurcates waves of period an arbitrarily large integer multiple of that of the fundamental period; these are the sub-harmonic bifurcation points.

In 1925 Levi-Civita conjectured that the minimal wavelength of a Stokes wave propagating with speed c does not exceeds $2\pi c^2/g$. We disprove this by showing that for Stokes waves with bounded propagation speed the minimal wavelength may be arbitrarily large.

The mathematical theory involves local bifurcation theory, the theory of germs of real-analytic functions, the spectral theory (in the spirit of that for Hill's equation) of some linear pseudo-differential operators with periodic coefficients, and Morse critical point theory.

*Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. bb1@maths.bath.ac.uk

†School of Mathematics and Statistics, University of Sydney, Sydney NSW 2006, Australia. dancer_n@maths.su.oz.au

‡Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. jft@maths.bath.ac.uk

Contents

1 Introduction

The purpose of this article is to introduce a novel variational formulation of the classical periodic water-wave problem for steady irrotational free surface flows in the presence of gravity, with and without surface tension. It will then be used to give a mathematical explanation of Stokes waves on deep water and their secondary sub-harmonic bifurcations. In this formulation, waves are given by non-zero 2π-periodic solutions w of the equation

$$\begin{aligned}\mu\mathcal{C}w' &= w + w\mathcal{C}w' + \mathcal{C}(ww')\\ &\quad - \beta\left\{\frac{w'}{\sqrt{w'^2+(1+\mathcal{C}w')^2}}\right\}' + \beta\mathcal{C}\left\{\frac{1+\mathcal{C}w'}{\sqrt{w'^2+(1+\mathcal{C}w')^2}}\right\}',\end{aligned} \tag{1.1}$$

which is the Euler-Lagrange equations of the functional

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \frac{1}{2}\left\{w^2(1+\mathcal{C}w') - \mu w\mathcal{C}w'\right\} + \beta\left\{\sqrt{w'^2+(1+\mathcal{C}w')^2}\right\} dt$$

defined on a Sobolev space of periodic functions. Here $\mathcal{C}$ denotes the classical conjugate (or, perhaps more correctly, conjugation) operator in the theory of Fourier series [66], $'$ denotes differentiation, and μ and β are dimensionless physical parameters representing the speed (the Froude number squared) and the surface tension (the Bond number), respectively. (Note that $w \mapsto \mathcal{C}w'$ is an unbounded, self-adjoint operator.)

In the case of zero surface tension, the problem is strikingly simple to write down:

$$\mu\mathcal{C}w' = w + w\mathcal{C}w' + \mathcal{C}(ww'), \tag{1.2}$$

which is the Euler-Lagrange equation of the functional

$$\int_{-\pi}^{\pi} \frac{1}{2}\left\{w^2(1+\mathcal{C}w') - \mu w\mathcal{C}w'\right\} dt.$$

The paper begins with a statement of the classical steady periodic water-wave problem (2.2) in Section 2.1 and its characterisation in Section 2.2 as the conditions for a certain energy functional to have extremals under constraint. Motivated by this observation, the Euler-Lagrange principle (1.1) is developed in two stages. At first, in Section 2.3, the treatment is heuristic and seeks to obtain a physical interpretation for $\mathcal{J}$. Then, in Section 2.4, the connection between critical points of $\mathcal{J}$ and (2.2) is established rigorously. (Strange to say, Section 2.4 is not just a mathematically tidier version of Section 2.3 and, without an explanation of their origins in terms of physical quantities (Section 2.3), the relationship between equation (1.1) and (2.2) might remain rather mysterious.)

The free-boundary problem (2.2) is formulated with minimal smoothness requirements on the surface profile, which smoothness is automatic for solutions of (1.1). In Section 3 is shown that solutions of (1.1) with this level of minimal smoothness are in fact real-analytic functions.

Note that $w = 0$ is a solution of (1.1) for all values of μ and the study turns first to a variational treatment of small-amplitude solutions of (1.1) where the surface-tension parameter $\beta \geq 0$ is regarded as fixed, and μ is considered as a bifurcation parameter. For all non-negative values of surface tension it is shown in Section 4, using a version of Krasnosel'skii's theorem [33, page 332] due to Stuart [57], that bifurcation occurs at every eigenvalue of the problem linearised at the zero solution. Since there are values of the surface tension parameter at which the linearised problem has a four-dimensional null space and the analysis of the associated mode interactions (known as Wilton ripples) by other methods is quite complicated [28, 62], the simplicity and generality of the variational method is significant. The remainder of the paper is concerned with (1.2) when surface tension is zero, the so-called Stokes wave problem.

In 1925, Levi-Civita [35] conjectured that, when calculated using its natural minimal period, the Froude number (see (2.10)) of a Stokes wave always exceeds 1. But, in 1980, Saffman [53] and Chen & Saffman [15] produced persuasive numerical evidence that this is false. Further numerical work, see, for example, Aston [6], indicated that a highly complex, period-multiplying, secondary bifurcation phenomenon occurs for large amplitude Stokes waves. Baesens & MacKay [8] have offered a very perceptive evaluation of these numerical findings by analogy with the bifurcation behaviour of finite-dimensional Hamiltonian dynamical systems, the bifurcation tree for the Hénon maps being perceived as offering particularly significant intuition. The key to this interpretation of the numerical evidence in terms of dynamical systems theory is based on a two-fold assumption: first that a finite-dimensional centre manifold reduction in the spirit of Kirchgässner [32] and Mielke [44] may be possible, and second that Floquet multipliers at large amplitude waves behave in a certain fashion.

From Section 5 onwards, we are concerned with the first steps in a rigorous mathematical theory to explain some of the same numerical observations and, in the process, to give the first proof that Levi-Civita's conjecture about Stokes waves is wrong. In place of assumptions about centre manifolds and Floquet multipliers, our approach uses the Euler-Lagrange equation (1.2) and properties of band spectra of some simple first-order pseudo-differential operators with periodic coefficients. These results on spectra are similar to those for Hill's equation [40], although the proofs in Section 6 are in the spirit of spectral theory of Schrödinger operators on lattices in $\mathbb{R}^d$ [54]. The perturbation theory of such spectra, which is developed in Section 6.3, leads in Section 6.4 to a proof of the conjecture made by Aston [6, Section 5.4, last sentence] on the local non-existence of secondary sub-harmonic bifurcations of Stokes waves. Note that if there were a centre manifold reduction of the Stokes wave problem on infinite depth, in the manner of Kirchgässner [32] and Mielke [44], we could give an easier proof of Theorem 5.2. See also the remark at the end of Section 6.4.

The global theory leads to a qualitative picture of turning points and secondary sub-harmonic bifurcations in the limit as the highest wave is approached. By contrast, Baesens & MacKay are mainly concerned with an explanation of some subtle phenomena at the onset of secondary bifurcation. The present mathematical contribution is in agreement with, but somewhat complementary to, their compelling interpretation of the numerical data.

For the global method to work, the existence theory for Stokes waves [30, 61], which is based on topological degree theory [21, 50], is not sufficiently refined. So instead, in Section 7 and an Appendix, ideas of Dancer [20, 22] are developed to give the existence of a global *arc-wise connected* set $\mathcal{H} \subset \mathbb{R} \times X$, of solutions of a real-analytic bifurcation problem $F(\lambda, x) = 0$ in a real Banach space X, which is the image of a continuous mapping $h : (0, \infty) \to \mathbb{R} \times X$. (Note that $\mathcal{H}$ is not *maximal* in the sense of Rabinowitz [50] and h is not required to be globally injective.) When interpreted in the context of the Stokes-wave problem the approach of s to ∞ corresponds to the approach [43, 60, 61] of the corresponding Stokes wave to the wave of greatest height [2, 3, 42, 43, 48, 56, 60, 61]. In Section 7.2, standard results on the bifurcation of solutions of equations involving gradient mappings at points where a Morse index changes (results due to Chow & Lauterbach[16] and Kielhöfer [31], for example) are used to give a criterion for secondary bifurcation from $\mathcal{H}$.

Armed with these tools, the variational theory of symmetric Stokes waves and their sub-harmonic bifurcations is within reach. The final ingredient involves an idea which Plotnikov [49] introduced to water-wave theory in his proof of non-uniqueness for solitary waves. He had noted that along the global branch of solitary waves given by [4, 5], the Morse index, when calculated using eigenvalues with even eigenfunctions, can be arbitrarily large and thereby deduced the existence of either a turning point or a secondary bifurcation point in the symmetric solitary-wave problem.

Here we develop this observation in the context of symmetric periodic waves on infinitely deep flows. We conclude that the mathematical reason for the remarkable sub-harmonic bifurcation phenomena observed numerically for symmetric Stokes waves is the behaviour of the spectrum (with respect to even functions) of the linearised problem as the Stokes wave of greatest height is approached. In physical terminology: *the linear instability of Stokes waves with respect to even disturbances increases without bound when the highest wave is approached, and sub-harmonic bifurcations are then inevitable.*

By contrast, in Theorem 8.1, we show that the Morse index of symmetric solutions on the primary branch, when calculated using eigenvalues with *odd* eigenfunctions, is always zero. This means that there is no symmetry-breaking secondary bifurcation on the primary branch of symmetric waves. This observation is consistent with Garabedian's theory of non-existence of asymmetric Stokes waves on flows of finite depth [25]. Note that our theorem does not preclude the possibility of asymmetric bifurcation at waves which have emerged as sub-harmonic bifurcations; it is relevant only to the primary branch.

In the light of the real-analytic parametrisation of the branch $\mathcal{H}$ of Stokes waves (Theorem 8.2), it is shown in Section 8.3, that there are infinitely many distinct points on $\mathcal{H}$ which are either turning points or bifurcation points (see Definition 7.5 on page 46), because in this case the Morse index tends to ∞ as $s \to \infty$ Then, in Section 8.4, sub-harmonic bifurcations (spontaneous increases in the minimal wavelengths of Stokes waves as parameters are varied) are shown to occur frequently for large amplitude Stokes waves. The proof combines the spectral theory developed in Section 6.1 and the bifurcation theory for gradient operators in Section 7.2 with the following simple idea.

When calculating a Morse index change using a linearised problem with periodic coefficients, we note that arbitrarily close to every eigenvalue whose eigenfunction has the fundamental period, there are *always* eigenvalues of the same problem the eigenfunctions of which have periods an integer multiple of the fundamental period. Therefore, if the original problem is posed in the space of functions with the longer period, the Morse index defined there also changes *at a nearby parameter value*. The gradient structure then ensures that bifurcation occurs. Provided simultaneous bifurcation of solutions with the fundamental period is impossible (as it is here because of considerations of analyticity) the existence of sub-harmonic bifurcations has been established.

Since the pioneering work of Benjamin & Feir [11] engineers have believed Stokes waves on deep water to be unstable (see also [13]). Nevertheless, the Stokes wave problem remains of central importance in theoretical hydrodynamics. The tools used here to study it, apart form the new formulation (1.2), are local bifurcation theory, the theory of germs of real-analytic functions, the spectral theory of linear pseudo-differential operators with periodic coefficients, and the theory of Morse, all combined with the deep influence of Plotnikov's paper [49].

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2 A Variational Viewpoint

This section is devoted to developing a theory of steady, irrotational, periodic water waves from a variational standpoint. We begin by recalling two aspects of classical theory: the water-wave problem formulated as a free boundary-value problem for a harmonic function (Section 2.1) and the equivalent formulation as a constrained stationary energy principle (Section 2.2). Then, in Section 2.3, an informal argument is used to infer that there is a more elementary, equivalent variational principle in which extrema are sought in a space of real-valued, periodic functions of a real variable. This is the background from which, in Section 2.4, the formulation (1.1) for the steady water-wave problem emerges.

Anyone who wants the variational principle without its physical motivation may ignore Sections 2.2 and 2.3, and the case when $\sigma = 0$ ($\beta = 0$, zero surface tension) is particularly straightforward: the wave is given by a 2π-periodic function of a real variable which satisfies (1.2), an equation which has only two nonlinear terms each of which is a quadratic monomial. It is, however, quasi-linear and involves the conjugate operator (which is the non-local Hilbert transform in the theory of Fourier series). (For global treatments of the Stokes-wave theories due to Levi-Civita [35] and Nekrasov [47] (see also [45, Chapter XIV]), see [34] and [30]; neither is variational.)

After establishing the conclusions of Sections 2.2, 2.3 and 2.4, we learnt that Plotnikov [49] had found a change of variables which led him, apparently without physical motivation, from the solitary wave problem (without surface tension) to a variational problem analogous to ours. Unlike the solitary-wave problem, which is singular because it is defined on a domain of unbounded horizontal extent and finite depth, our problem is periodic on a flow of infinite depth. Consequently, the exposition here is systematic and easily explained in the familiar language of the conjugate operator $\mathcal{C}$, even when surface tension is acting. However, the idea behind Plotnikov's Lemma (Section 8.3) is very important in our theoretical development.

A set of elegant identities among the Fourier coefficients of Stokes waves on infinite depth, found by Longuet-Higgins [38], may be written in concise form as (1.1) with $\beta = 0$. He subsequently spotted a variational interpretation of these identities, which he used in numerical calculations of Stokes wave bifurcations [39]. The present contribution therefore suggests an extension of the Longuet-Higgins analysis to the case of positive surface tension, although the nonlinearity in (1.1) is then no longer simply quadratic and there are no obvious identities among the Fourier coefficients.

2.1 The Free-Boundary Problem

We begin with the classical water-wave problem (see, for example, [10]). Suppose that water has unit density and zero viscosity, that $\sigma \geq 0$ is its coefficient of surface tension and that $g > 0$ is the acceleration due to gravity. We seek steady waves of period 2Λ moving to the left without change of form with speed c on the surface of an infinitely deep irrotational flow. Relative to a reference frame moving with the velocity of the wave, the surface S is a stationary curve given parametrically by

$$S = \{(u(t), v(t)) : \ t \in \mathbb{R}\}, \tag{2.1}$$

Let Ω denote the open region of the (x, y)-plane below S. A 2Λ-periodic steady wave with free surface S and speed c exists if there is a function $\omega = \phi + i\psi$ which is holomorphic in Ω and

$$\left.\begin{array}{c} \text{(i) } (u, v) \text{ is injective and absolutely continuous,} \\ \text{(ii) } t \mapsto (u(t) - t, v(t) \text{ is } 2\Lambda\text{-periodic,} \\ \text{(iii) the classical derivative } (u'(t), v'(t)) \text{ exists almost everywhere} \\ \text{and } u'(t)^2 + v'(t)^2 \neq 0 \text{ for almost all } t, \\ \text{(iv) when } \sigma > 0 \text{ in (2.2f), then } (u', v') \text{ is absolutely continuous;} \end{array}\right\} \tag{2.2a}$$

$$\psi \text{ is continuous on } \overline{\Omega}; \tag{2.2b}$$

$$\psi \equiv 0 \text{ on } S; \tag{2.2c}$$

$$\psi \text{ is } 2\Lambda\text{-periodic in } x; \tag{2.2d}$$

$$\psi(x,y) - cy \to 0 \text{ uniformly as } y \to -\infty; \tag{2.2e}$$

$$\lim_{\substack{(x,y)\in\Omega \\ (x,y)\to(u(t),v(t))}} \left\{\frac{1}{2}|\nabla\psi(x,y)|^2\right\} + gv(t)$$

$$= \sigma\left\{\frac{u'(t)v''(t) - v'(t)u''(t)}{(u'(t)^2 + v'(t)^2)^{3/2}}\right\} + \text{ constant} \tag{2.2f}$$

for almost all t. Here, and in the proof of Theorem 2.3, the notation lim in (2.2f) means the limit as (x,y) approaches the boundary in *non-tangential directions* at points where the tangent to the boundary is well-defined (which is the case for almost all t by part (iii) of (2.2a)).

Here ω is the complex potential, ϕ velocity potential and ψ the stream function (see [45]). In this free-boundary problem, equations (2.2c) and (2.2f), the kinematic and dynamic boundary conditions, determine the unknown free boundary S, which is a non-self-intersecting periodic curve because (u,v) is required to be injective. While ϕ is determined only up to an additive constant, (2.2) is a full specification of the stream function ψ. From classical potential theory and (2.2e) one can infer that

$$(-\phi_y, \phi_x) = \nabla\psi \to (0,c) \text{ as } y \to -\infty, \tag{2.3}$$

and hence that

$$|\nabla\psi| \text{ is bounded as } y \to -\infty. \tag{2.4}$$

In fact the theory ensures that the rates of convergence in (2.2e) and (2.3) are both exponential as $y \to -\infty$. The value of the unspecified constant in (2.2f) is determined by the Lagrange multiplier for a constrained stationary energy principle where the mean height is prescribed (Section 2.2). An unconstrained variational principle for (2.2) will also be developed. This is the key construction which is motivated in Section 2.3 and, in dimensionless co-ordinates, is established rigorously in Section 2.4.

Formulation (2.2) of the steady water-wave problem is sufficiently delicate in terms of its regularity requirements (2.2a), (2.2b) and (2.2f), to include as a special case the Stokes wave of greatest height where $\sigma = 0$ and S is not smooth at the highest point of the highest wave. (See [2, 3, 42, 43, 48, 56, 60, 61].) For the Stokes highest wave, the parametrisation (u,v) given by Theorem 2.3 is absolutely continuous but not differentiable at $t = 0$. But according to Theorem 3.1, Stokes waves below the wave of greatest height ($w < \frac{1}{2}\mu$ in (1.2)) are real-analytic: in (2.2), (u,v) is real-analytic on $\mathbb{R}$ and ψ is real-analytic on $\overline{\Omega}$.

2.2 A Constrained Stationary Energy Principle

Suppose that (u, v) and $\phi + i\psi$ satisfy the equations for a steady wave. Let $\Omega_\Lambda = \{(x, y) \in \Omega : -\Lambda < x < \Lambda\}$ and let the mean height be defined by

$$M_\Lambda(u, v) = \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} v(t)u'(t)dt.$$

Then in a fixed reference frame (one in which the flow speed at infinite depth is zero and in which the wave profile moves without change of form with speed c) the velocity potential is $\phi(x, y) - cx$. Therefore the total kinetic energy in one period is

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\Lambda} \{\phi_x^2 - 2c\phi_x + c^2 + \phi_y^2\}dV &= \frac{1}{2}\int_{\Omega_\Lambda} \{|\nabla \phi|^2 - c^2\}dV + c\int_{\Omega_\Lambda} (cy - \psi)_y dV \\ &= \frac{1}{2}\int_{\Omega_\Lambda} \{|\nabla \phi|^2 - c^2\}dV + c^2 \int_{-\Lambda}^{\Lambda} v(t)u'(t)dt,\end{aligned}$$

where we have used the Cauchy-Riemann equations, (2.2c), (2.2d), (2.2e) and (2.3). Let the straight line $\{y = h\} \subset \Omega$ and let $\underline{\Omega_\Lambda} = \{(x, y) \in \Omega_\Lambda : y > h\}$. Then the potential energy of the water in $\underline{\Omega_\Lambda}$ is

$$\int_{\underline{\Omega_\Lambda}} gy dV = \frac{g}{2}\int_{-\Lambda}^{\Lambda} v(t)^2 u'(t)dt - \Lambda g h^2.$$

Finally the energy due to stretching of the free surface is

$$\sigma \left\{ \int_{-\Lambda}^{\Lambda} \sqrt{u'(t)^2 + v'(t)^2} dt - 2\Lambda \right\}.$$

Therefore the total energy of a wave (up to an additive constant independent of u, v and ϕ in bounded sets) is

$$\begin{aligned}E_{c,\Lambda}(u, v, \phi) = \frac{1}{2}\int_{\Omega_\Lambda} \{|\nabla \phi|^2 - c^2\}dV + c^2 \int_{-\Lambda}^{\Lambda} v(t)u'(t)dt \\ + \frac{g}{2}\int_{-\Lambda}^{\Lambda} v(t)^2 u'(t)dt + \sigma \int_{-\Lambda}^{\Lambda} \sqrt{u'(t)^2 + v'(t)^2}dt. \qquad (2.5)\end{aligned}$$

Now we give a brief informal explanation of how solutions of the constrained variational principle

find extremals (u, v, ϕ) of $E_{c,\Lambda}(u, v, \phi)$ subject to $M_\Lambda(u, v) = \text{constant}$ (2.6)

give solutions of the steady water-wave problem. To this end, let the set of admissible (u, v) be those described by (2.2a), define S, Ω and Ω_Λ using u and v and let admissible $\phi : \Omega_\Lambda \to \mathbb{R}$ be those with $\phi(\Lambda, y) = 2c\Lambda + \phi(-\Lambda, y)$ for which

$E_{c,\Lambda}$ is finite. Suppose that $(\underline{u}, \underline{v}, \underline{\phi})$ is a solution of this constrained variational principle.

First, with (u, v) fixed at $(\underline{u}, \underline{v})$, arbitrary admissible variations of ϕ, where the perturbations of $\underline{\phi}$ are compactly supported in the interior of Ω_Λ, gives that $\underline{\phi}$ is weakly harmonic, and hence harmonic, in Ω_Λ. Then variations of admissible ϕ, when the perturbations of $\underline{\phi}$ have supports which intersect $\partial\Omega_\Lambda$ only at points where $x = \pm\Lambda$, yields the natural boundary condition that $\nabla\underline{\phi}(-\Lambda, y) = \nabla\underline{\phi}(\Lambda, y)$ in $\partial\Omega_\Lambda$. Hence $\underline{\phi}(x, y) - cx$ has a 2Λ-periodic (with respect to x) weakly harmonic, and hence harmonic, extension to Ω. Now variations in admissible ϕ, where the supports of the perturbations intersect S, gives that $\partial\underline{\phi}/\partial\eta = 0$, where η denotes the outward normal to S. Therefore, if $\underline{\psi}$ is a harmonic conjugate of $\underline{\phi}$, it is constant on S and the constant can be chosen to be zero. Hence $\underline{\psi}$ is 2Λ-periodic in Ω. Thus the function $\underline{\omega} = \underline{\phi} + i\underline{\psi}$ is holomorphic in Ω, satisfies the boundary conditions (2.2c) and (2.2d), and satisfies the condition at infinite depth (2.3) in an integrated sense because of the admissibility criterion. From potential theory, strong versions of (2.3) and (2.4) can be inferred.

Now with $\underline{\phi}$ fixed, consider admissible variations in (u, v) which satisfy the constraint $M_\Lambda(u, v) = \text{constant}$. Both from variations in u for fixed $v = \underline{v}$, and from variations in v for fixed $u = \underline{u}$, it follows that $(\underline{u}, \underline{v}, \underline{\phi})$ satisfy the same equation, namely (2.2f) with the constant on the right-hand side given by $(\frac{\gamma}{2\Lambda} - c^2)$, where γ is the Lagrange multiplier due to the constraint on the mean height. A judicious choice of the origin of the y-axis now gives (2.2e), while changing the constant in (2.2f).

Thus (2.6) gives solutions of the water-wave problem. Here is where we depart from previous expositions, in which (2.6) is developed to derive a Hamiltonian formulation of the steady water-wave problem (see [8, 12, 26, 44]). Instead we seek an alternative Euler-Lagrange formulation.

2.3 Another Stationary Energy Principle

The Euler-Lagrange equations for the stationary energy principle of the preceding section are equivalent to

$$\partial_\phi E_{c,\Lambda}(\underline{u}, \underline{v}, \underline{\phi}) = 0 \text{ and } \partial_{(u,v)} E_{c,\Lambda}(\underline{u}, \underline{v}, \underline{\phi}) = \gamma \partial M_\Lambda(\underline{u}, \underline{v}). \tag{2.7}$$

We observe that formally $\partial_\phi E_{c,\Lambda}(\underline{u}, \underline{v}, \underline{\phi}) = 0$ is equivalent $\triangle\underline{\phi} = 0$ in Ω_Λ with $\underline{\phi} + i\underline{\psi}$ satisfying (2.2c), (2.2d), (2.3) and (2.4). But when $(\underline{u}, \underline{v})$ is given, $\underline{\phi}$ is determined by these conditions; in other words $\partial_\phi E_{c,\Lambda}(\underline{u}, \underline{v}, \underline{\phi}) = 0$ implies that $\underline{\phi} = \Phi(\underline{u}, \underline{v})$, say. Let

$$\mathcal{E}_{c,\Lambda}(u, v) = E_{c,\Lambda}(u, v, \Phi(u, v)).$$

Now consider the constrained variational principle

$$\text{find extremals } (\underline{u}, \underline{v}) \text{ of } \mathcal{E}_{c,\Lambda}(u, v) \text{ subject to } M_\Lambda(u, v) = \text{constant}. \tag{2.8}$$

The critical points of (2.8) automatically give critical points of (2.6) and clearly (2.8) is also a constrained stationary energy principle which may be further manipulated as follows.

An integration by parts, using the fact that Φ is harmonic and (2.2c), (2.2d), (2.3) and (2.4) hold, gives that

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_\Lambda}\{|\bigtriangledown\Phi|^2-c^2\}dV &= \frac{1}{2}\int_{\Omega_\Lambda}\bigtriangledown(\Phi-cx)\cdot\bigtriangledown(\Phi+cx)dV\\
=\frac{1}{2}\int_{S_\Lambda}(\Phi-cx)\frac{\partial}{\partial\eta}(\Phi+cx)dS &= \frac{c}{2}\int_{S_\Lambda}(\Phi-cx)\frac{\partial x}{\partial\eta}dS\\
=-\frac{c}{2}\int_{-\Lambda}^{\Lambda}\{\Phi(u(t),v(t))-cu(t)\}v'(t)dt &= \frac{c}{2}\int_{-\Lambda}^{\Lambda}\frac{d}{dt}\{\Phi(u(t),v(t))-cu(t)\}v(t)dt,
\end{aligned}$$

where η denotes the outward normal from Ω_Λ at points of $S_\Lambda=\{(x,y)\in S: -\Lambda<x<\Lambda\}$.

Now let

$$\tau(t)=c^{-1}\Phi(u(t),v(t)) \text{ and } (u(t),v(t))=(\tilde{u}(\tau),\tilde{v}(\tau)).$$

Then, formally as before, with (u,v) given,

$$\frac{1}{2}\int_{\Omega_\Lambda}\{|\bigtriangledown\Phi|^2-c^2\}dV=\frac{c^2}{2}\int_{-\Lambda}^{\Lambda}\{\tilde{v}(\tau)-\tilde{v}(\tau)\tilde{u}'(\tau)\}d\tau$$

where $\tilde{u}'=d\tilde{u}/d\tau$. Hence from (2.5)

$$\begin{aligned}
\mathcal{E}_{c,\Lambda}(u,v)=&\frac{c^2}{2}\left\{\int_{-\Lambda}^{\Lambda}\tilde{v}(\tau)d\tau+\int_{-\Lambda}^{\Lambda}\tilde{u}'(\tau)\tilde{v}(\tau)d\tau\right\}\\
&+\frac{g}{2}\int_{-\Lambda}^{\Lambda}\tilde{v}(\tau)^2\tilde{u}'(\tau)d\tau+\sigma\int_{-\Lambda}^{\Lambda}\sqrt{\tilde{u}'(\tau)^2+\tilde{v}'(\tau)^2}d\tau
\end{aligned}$$

and

$$M_\Lambda(u,v)=\frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda}\tilde{v}(\tau)\tilde{u}'(\tau)d\tau.$$

Finally, observe that if $(\underline{u},\underline{v})$ is a solution of (2.7) and τ is a monotone function of t then $\tilde{u}(\tau)-\tau+i\tilde{v}(\tau)$, which is defined for $\tau\in(-\Lambda,\Lambda)$, has a bounded holomorphic extension to the semi-infinite region $(-\Lambda,\Lambda)\times(-\infty,0)$ in the complex plane. We therefore write $\tilde{u}(\tau)-\tau=\mathcal{C}_{\Lambda/\pi}\tilde{v}(\tau)$ where $\mathcal{C}_{\Lambda/\pi}$ is the conjugate operator defined by $\mathcal{C}_{\Lambda/\pi}(1)=0$,

$$\mathcal{C}_{\Lambda/\pi}(e^{i\pi k/\Lambda})=-i\operatorname{sgn}(k)(e^{i\pi k/\Lambda})$$

for all $k\in\mathbb{Z}$, and by extension for all 2Λ-periodic functions. Therefore the energy and the mean height may be regarded as functions of $\tilde{v}$ alone, as follows:

$$
\begin{aligned}
\widetilde{\mathcal{E}}_{c,\Lambda}(\tilde v) = \frac{c^2}{2}\left\{\int_{-\Lambda}^{\Lambda}\tilde v(\tau)d\tau + \int_{-\Lambda}^{\Lambda}(1+\mathcal{C}_{\Lambda/\pi}\tilde v'(\tau))\tilde v(\tau)d\tau\right\} \\
+\frac{g}{2}\int_{-\Lambda}^{\Lambda}\tilde v(\tau)^2(1+\mathcal{C}_{\Lambda/\pi}\tilde v'(\tau))d\tau + \sigma\int_{-\Lambda}^{\Lambda}\sqrt{(1+\mathcal{C}_{\Lambda/\pi}\tilde v'(\tau))^2+\tilde v'(\tau)^2}d\tau
\end{aligned}
$$

and

$$
\widetilde{M_\Lambda}(\tilde v) = \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda}\tilde v(\tau)(1+\mathcal{C}_{\Lambda/\pi}\tilde v'(\tau))d\tau.
$$

Since $\mathcal{C}_{\Lambda/\pi}$ commutes with differentiation, these expressions are unambiguous. Finally, let

$$
\tau = \frac{\Lambda s}{\pi} \quad \text{and} \quad \frac{\pi}{\Lambda}\tilde v(\frac{\Lambda s}{\pi}) = v(s), \quad s \in (\pi, \pi).
$$

(There should be no confusion between this v and the original one.) Then

$$
\begin{aligned}
\mathcal{E}_{c,\Lambda}(v) = \frac{\Lambda^2 c^2}{2\pi^2}\left\{\int_{-\pi}^{\pi} v(s)ds + \int_{-\pi}^{\pi}(1+\mathcal{C}v'(s))v(s)ds\right\} \\
+\frac{\Lambda^3 g}{2\pi^3}\int_{-\pi}^{\pi} v(s)^2(1+\mathcal{C}v'(s))ds + \frac{\sigma\Lambda}{\pi}\int_{-\pi}^{\pi}\sqrt{(1+\mathcal{C}v'(s))^2+v'(s)^2}ds
\end{aligned}
$$

and

$$
M_\Lambda(v) = \frac{\Lambda}{2\pi^2}\int_{-\pi}^{\pi} v(s)(1+\mathcal{C}v'(s))ds
$$

where $\mathcal{C} = \mathcal{C}_1$. This suggests that the following problem is equivalent to the original one:

$$
\text{find } 2\pi\text{-periodic extremals } v \text{ of } \mathcal{E}_{c,\Lambda}(v) \text{ subject to } M_\Lambda(v) = \text{constant}. \tag{2.9}
$$

In the next section we make this observation rigorous.

2.4 Analytic Formulation

First define the parameters

$$
\mu = \frac{\pi c^2}{g\Lambda} \text{ and } \beta = \frac{\pi^2\sigma}{g\Lambda^2}, \tag{2.10}
$$

the Froude number squared and the Bond number, respectively. Throughout Λ is regarded as fixed, and therefore β is a fixed constant for water, whereas μ, or equivalently the wave speed c, is part of the solution to be found. When $\beta = 0$, let H_β denote the Hilbert space of absolutely continuous functions v on $[-\pi,\pi]$, the weak derivatives of which are square-integrable, with $v(-\pi) = v(\pi)$. When $\beta > 0$, let H_β denote the Hilbert space of continuously differentiable functions v on $[-\pi,\pi]$, the derivatives of which are absolutely continuous and the second weak derivatives of which are square-integrable, with $v(-\pi) = v(\pi)$ and $v'(-\pi) = v'(\pi)$. Let H_β have the usual Sobolev-space norms. Let

$$\mathcal{D}_\beta = \begin{cases} H_\beta & \text{if } \beta = 0 \\ \{v \in H_\beta : (1+\mathcal{C}v'(s))^2 + v'(s)^2 > 0\} & \text{if } \beta > 0 \end{cases}$$

The variational principle (2.9) is formally equivalent to the following:

$$\text{find extremals in } \mathcal{D}_\beta \text{ of } E_{\beta,\mu}(v) \text{ subject to } M(v) = \text{constant}, \tag{2.11}$$

where

$$E_{\beta,\mu}(v) = \frac{\mu}{2}\int_{-\pi}^{\pi} v(s)ds + \frac{1}{2}\int_{-\pi}^{\pi} v(s)^2(1+\mathcal{C}v'(s))ds + \beta\int_{-\pi}^{\pi}\sqrt{v'(s)^2 + (1+\mathcal{C}v'(s))^2}ds \tag{2.12}$$

and

$$M(v) = \frac{1}{2}\int_{-\pi}^{\pi} v(s)(1+\mathcal{C}v')ds. \tag{2.13}$$

Note that $\mathcal{D}_\beta$ is open in H_β and the functionals $E_{\beta,\mu}$ and M are smooth in $\mathcal{D}_\beta$. Also, $\mathcal{C}$ is a bounded linear operator on H_β, $\mathcal{C}$ commutes with weak differentiation, $[\mathcal{C}v] = 0$ and $\mathcal{C}^2(v) = -v + [v]$, where $[v]$ denotes the mean value of v on $[-\pi, \pi]$. Before establishing the main result of this Section, we need some notation. Suppose f is a real-valued, absolutely continuous, 2π-periodic function of a real variable whose Fourier series is

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{-ikx} \text{ where } a_{-k} = \overline{a_k}.$$

Define $\mathcal{R}f$ on the lower complex half plane to be the holomorphic function given by

$$\mathcal{R}f(x+iy) = [f] + 2\sum_{k=1}^{\infty} a_k e^{-ik(x+iy)}. \tag{2.14}$$

Then $\mathcal{R}f$ is a function which is continuous on the closed lower half plane, the real and imaginary parts of which coincide with f and $-\mathcal{C}f$, respectively, on the real axis and $|\mathcal{R}f(x+iy)| \to [f]$ as $y \to -\infty$.

We will need a familiar result under slightly weaker hypotheses than is usual.

Lemma 2.1. *Let D denote the unit disc in $\mathbb{C}$ and suppose that $f : \overline{D} \to \mathbb{C}$ has the following properties:*

$f : \overline{D} \to \mathbb{C}$ is continuous and f is analytic in D;

$f|_{\partial D}$ is absolutely continuous;

$f|_{\partial D}$ is injective so that $f(\partial D)$ is a simple closed Jordan curve Γ. Let Ω be the interior of Γ, that is Ω is the bounded component of $\mathbb{C} \setminus \Gamma$.

Then

(a) $\Omega = f(D)$ and (b) $f : \overline{D} \to \overline{\Omega}$ is a bijection.

Proof. Since f is non-constant on ∂D, it is non-constant on D. Since it is analytic on D, $f(D)$ is an open set. Therefore $\partial(f(D)) \subset f(\partial D) = \Gamma$. Therefore $f(D) = \Omega$. (Otherwise there would be boundary points of $f(D)$ which are not in Γ.) This proves part (a). (See also [65, (3.11), page 74].)

To prove (b), note again that if $z_1 \in \partial D$ and $z_2 \in D$, then $f(z_1) \neq f(z_2)$, as in the argument for part (a). Suppose then that z_1, $z_2 \in D$, $z_1 \neq z_2$ and $f(z_1) = f(z_2) = \omega \in \Omega$. Let $r \in (0,1)$ be such that $|z_i| < r$, $i = 1, 2$. Then by [58, 3.4, page 115],

$$\frac{1}{2\pi i}\int_{\partial D_r} \frac{f'(z)dz}{f(z)-\omega} \geq 2,$$

where $D_r = \{z \in D : |z| < r\}$. Since $|f(z) - \omega|$ is bounded away from 0 on ∂D, and $f' \in L_1(\partial D)$, it follows (see, for example, [7, Thm. 6.6, page 100]) that

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)dz}{f(z)-\omega} \geq 2.$$

However,

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)dz}{f(z)-\omega} = \frac{1}{2\pi i}\int_{\Gamma} \frac{d\zeta}{\zeta-\omega} = 1,$$

since $\omega \in \Omega$, the interior of Γ. This contradiction implies (b) and the Lemma is proven. □

Lemma 2.2. *Suppose that h is a function which is bounded and continuous on the closed complex lower half plane $\overline{\mathbb{H}}$, analytic on the open lower half plane $\mathbb{H} = \{(x,y) : y < 0\}$ with the property that h is 2Λ-periodic in x, (locally) absolutely continuous on $\mathbb{R} + i0$ and tends to a limit as $y \to -\infty$. Let $\mathcal{T}(\zeta) = \zeta + h(\zeta)$ for $\zeta \in \overline{\mathbb{H}}$ and suppose that $\mathcal{T}$ is injective on $\mathbb{R} + i0$ with $\mathcal{T}(\mathbb{R} + i0) = S$, say.*

Then $\mathcal{T} : \overline{\mathbb{H}} \to \mathbb{C}$ is a bijection onto the closure of the infinite component of $\mathbb{C} \setminus S$ which contains $(0, y)$ for all large negative values of y.

Proof. Let N be a natural number and, in the notation of the preceding Lemma, let $f_N : \overline{D} \to \mathbb{C}$ be defined by

$$f_N(z) = \exp\{-(i\pi/N\Lambda)\mathcal{T}((i\Lambda N/\pi)\log(z))\},\ z \in \overline{D},\quad -\pi < \arg z \leq \pi.$$

Then f_N satisfies the hypotheses of the preceding Lemma. Let $f_N(D) = \Omega_N$ and $\Gamma_N = \partial\Omega_N$. Then f_N is a bijection from ∂D onto Γ_N and from D onto Ω_N, and Ω_N is the interior of the Jordan curve Γ_N.

Suppose now that $\mathcal{T}(\zeta_1) = \mathcal{T}(\zeta_2)$ where $\zeta_i \in \overline{\mathbb{H}}$, $i = 1, 2$. Then for some natural number N, $\zeta_1 = iN\log(z_1)$ and $\zeta_2 = iN\log(z_2)$, $z_i \in \overline{D}$. This implies that $f_N(z_1) = f_N(z_2)$, which contradicts the bijectivity of f_N. Therefore $\mathcal{T}$ is injective on $\overline{\mathbb{H}}$.

Since $\mathcal{T}$ is injective, no point of $\mathbb{H}$ is mapped by $\mathcal{T}$ to a point of $\mathcal{T}(\mathbb{R}+i0) = S$, and therefore $H(\mathbb{H})$ lies on one side of S. That $\mathcal{T}(\mathbb{H})$ is the entire component of $\mathbb{C}\backslash S$ which contains $(0, y)$ for large negative y, is immediate from connectedness and the behaviour of $\mathcal{T}$ (a bounded perturbation of the identity). This completes the proof. $\square$

Theorem 2.3. *Suppose that for given $\beta \geq 0$ and $\mu > 0$, $\underline{v} \in \mathcal{D}_\beta$ is a solution of the constrained variational problem* (2.11) *with $t \mapsto (t + \mathcal{C}\underline{v}(t), \underline{v}(t))$ injective on $[-\pi, \pi]$. Then there exists a solution of the steady water-wave problem* (2.2) *the speed and wavelength of which satisfy* (2.10).

Proof. Suppose that $\underline{v} \in \mathcal{D}_\beta$ is a solution of (2.11). Then it is straightforward to observe that there is a Lagrange multiplier γ such that

$$\gamma(\frac{1}{2} + \mathcal{C}\underline{v}') = \frac{\mu}{2} + \underline{v} + \underline{v}\mathcal{C}\underline{v}' + \mathcal{C}(\underline{v}\,\underline{v}')$$
$$- \beta \left\{ \frac{\underline{v}'}{\sqrt{\underline{v}'^2 + (1 + \mathcal{C}\underline{v}')^2}} \right\}' + \beta\mathcal{C} \left\{ \frac{1 + \mathcal{C}\underline{v}'}{\sqrt{\underline{v}'^2 + (1 + \mathcal{C}\underline{v}')^2}} \right\}'. \tag{2.15}$$

Let

$$U = \frac{\mu}{2} + (\underline{v} - \frac{\gamma}{2})(1 + \mathcal{C}\underline{v}') - \beta \left\{ \frac{\underline{v}'}{\sqrt{\underline{v}'^2 + (1 + \mathcal{C}\underline{v}')^2}} \right\}'. \tag{2.16}$$

Since $[U] = 0$, because U is the conjugate of a function by (2.15), and $\mathcal{C}^2 = -I + [\,\cdot\,]$,

$$\mathcal{C}U = \underline{v}'(\underline{v} - \frac{\gamma}{2}) + \beta \left\{ \frac{1 + \mathcal{C}\underline{v}'}{\sqrt{\underline{v}'^2 + (1 + \mathcal{C}\underline{v}')^2}} \right\}'.$$

Now let $V = \mathrm{Imag}\{(\mathcal{R}U - \mu/2)(1 + i\mathcal{R}\underline{v}')\}$. Then V is a harmonic function on the lower complex half plane and is 2π-periodic in x. It is the imaginary part of the product of two complex analytic functions, V_1 and V_2 say, each of which was obtained by the action of $\mathcal{R}$ on functions which are square-integrable on $[-\pi, \pi]$. Consequently, $\|V_j(\cdot, y) - V_j(\cdot, 0)\|_{L_2[-\pi,\pi]} \to 0$ as $y \nearrow 0$. (See [7, Chapter 6].) Consequently $\|V(\cdot, y) - V(\cdot, 0)\|_{L_1[-\pi,\pi]} \to 0$ as $y \nearrow 0$. However, on the real axis,

$$V(x,0) = \text{Imag}\{(\mathcal{R}U - \mu/2)(1+i\mathcal{R}\underline{v}')\} = \text{Imag}\{(U-\mu/2-i\mathcal{C}U)(1+\mathcal{C}\underline{v}'+i\underline{v}')\}$$

$$= \underline{v}'(U-\mu/2) - \mathcal{C}U(1+\mathcal{C}\underline{v}') = \underline{v}'(\underline{v}-\frac{\gamma}{2})(1+\mathcal{C}\underline{v}') - \beta\underline{v}'\left\{\frac{\underline{v}'}{\sqrt{\underline{v}'^2+(1+\mathcal{C}\underline{v}')^2}}\right\}'$$

$$-(1+\mathcal{C}\underline{v}')\underline{v}'(\underline{v}-\frac{\gamma}{2}) - \beta(1+\mathcal{C}\underline{v}')\left\{\frac{1+\mathcal{C}\underline{v}'}{\sqrt{\underline{v}'^2+(1+\mathcal{C}\underline{v}')^2}}\right\}' = 0.$$

At any point (x^*, y^*) of the lower half plane, $V(x^*, y^*)$ can be represented by the Poisson integral (with periodic kernel) of its value on any line $\{y = \epsilon,\ y^* < \epsilon < 0\}$. But $\|V(\cdot,\epsilon) - V(\cdot,0)\|_1 \to 0$ as $\epsilon \nearrow 0$ and $V(\cdot,0) = 0$ then implies that $V(x^*, y^*) = 0$. (See [7, Chapter 6].)

Thus $V \equiv 0$ on the lower half-plane and consequently, by the Cauchy-Riemann equations,

$$\text{Real}\{(\mathcal{R}U - \mu/2)(1 + i\mathcal{R}\underline{v}')\} \text{ is constant on the lower half plane .}$$

Since $[U]$ and $[\underline{v}']$ are both zero, it follows that the constant is the asymptotic value of this function for large negative y, namely $-\mu/2$. Therefore, almost everywhere on the real axis,

$$-\mu/2 = \text{Real}\{(\mathcal{R}U - \mu/2)(1 + i\mathcal{R}\underline{v}')\}$$

$$= (\underline{v} - \gamma/2)\{\underline{v}'^2 + (1+\mathcal{C}\underline{v}')^2\} - \beta\left\{\frac{(1+\mathcal{C}\underline{v}')\underline{v}'' - \underline{v}'(1+\mathcal{C}\underline{v}')'}{\sqrt{\underline{v}'^2+(1+\mathcal{C}\underline{v}')^2}}\right\}. \tag{2.17}$$

That $0 < \underline{v}'^2 + (1+\mathcal{C}\underline{v}')^2 < \infty$ everywhere when $\beta > 0$ follows since $\underline{v} \in \mathcal{D}_\beta$. When $\beta = 0$, $0 < \underline{v}'^2 + (1+\mathcal{C}\underline{v}')^2 < \infty$ holds almost everywhere on $[-\pi, \pi]$ since $\underline{v} \in H_0$ and (2.17) holds. Hence

$$0 < \underline{v}'^2 + (1+\mathcal{C}\underline{v}')^2 < \infty \tag{2.18}$$

and

$$\frac{\mu}{2}\left\{\frac{1}{\underline{v}'^2+(1+\mathcal{C}\underline{v}')^2}\right\} + \underline{v} - \beta\left\{\frac{(1+\mathcal{C}\underline{v}')\underline{v}'' - \underline{v}'(1+\mathcal{C}\underline{v}')'}{\{\underline{v}'^2+(1+\mathcal{C}\underline{v}')^2\}^{3/2}}\right\} = \frac{\gamma}{2}, \tag{2.19}$$

for any $\beta \geq 0$, almost everywhere on the real axis. Now for ζ in the lower half plane let

$$\mathcal{T}(\zeta) = \zeta + \frac{i\Lambda}{\pi}(\mathcal{R}\underline{v}(\frac{\pi\zeta}{\Lambda}) - [\underline{v}])$$

and for $t \in \mathbb{R}$ let $(u(t), v(t)) = \mathcal{T}(t)$. Then

$$(\frac{\pi}{\Lambda})(u(t), v(t)) = (\frac{\pi t}{\Lambda} + \mathcal{C}\underline{v}(\frac{\pi t}{\Lambda}), \underline{v}(\frac{\pi t}{\Lambda}) - [\underline{v}]),$$

and so $\mathcal{T}$ is injective on the real axis, by hypothesis. It follows from Lemma 2.2 that $\mathcal{T}$ is a bijection from the closed lower half plane onto the closure of an infinite open region Ω the boundary of which is the infinite curve $S = \{(u(t), v(t)) : t \in \mathbb{R}\}$. According to (2.18), the curve S has a well-defined tangent almost everywhere and fits the description (2.2a).

Define a complex potential [45, page 152] $\omega = \phi + i\psi$ on $\overline{\Omega}$ by

$$\omega(x+iy) = c\mathcal{T}^{-1}(x+iy).$$

Then

$$\omega(u(t), v(t)) = c\mathcal{T}^{-1}(u(t), v(t)) = ct + i0.$$

Therefore ψ is zero on S since $\psi(u(t), v(\underline{t})) \equiv 0$ on $\mathbb{R}$, and by definition ψ is 2Λ-periodic in x on Ω and continuous on $\overline{\Omega}$. Clearly, from the properties of $\mathcal{R}\underline{v}$, $\mathcal{T}(\zeta) - \zeta \to 0$ as $y \to \infty$, and therefore $\psi(x,y) - cy \to 0$ as $y \to -\infty$. Also for $z \in \Omega$

$$\psi_y(z) - i\psi_x(z) = \omega'(z) = \frac{c}{\mathcal{T}'(\frac{\omega(z)}{c})} \to c + i0 \text{ as } y \to -\infty.$$

Finally, recall from the remarks following (2.2) that lim means a non-tangential limit at points of the boundary where the tangent is well-defined, and note from the definition of ω that for all such t

$$\lim_{\substack{(x,y)\in\Omega \\ (x,y)\to(u(t),v(t))}} \frac{1}{2}|\nabla\psi(x,y)|^2 = \lim_{\substack{(x,y)\in\Omega \\ (x,y)\to(u(t),v(t))}} \frac{c^2}{2}|\mathcal{T}'(\frac{\omega(z)}{c})|^{-2}$$

$$= \frac{c^2}{2}|\mathcal{T}'(\frac{\omega(u(t),v(t))}{c})|^{-2}, \text{ by [66, (7.9), Vol.I, page 101] },$$

$$= (\frac{c^2}{2})\frac{1}{(1+\mathcal{C}\underline{v}'(\frac{\pi t}{\Lambda}))^2 + \underline{v}'(\frac{\pi t}{\Lambda})^2} = (\frac{g\Lambda\mu}{2\pi})\frac{1}{(1+\mathcal{C}\underline{v}'(\frac{\pi t}{\Lambda}))^2 + \underline{v}'(\frac{\pi t}{\Lambda})^2}, \tag{2.20}$$

by the definition of $\mathcal{T}$,

$$= (\frac{-g\Lambda}{\pi})\left\{\underline{v}(s) - \beta\frac{(1+\mathcal{C}\underline{v}'(s))\underline{v}''(s) - \underline{v}'(s)(1+\mathcal{C}\underline{v}'(s))'}{\{(1+\mathcal{C}\underline{v}'(s))^2 + \underline{v}'(s)^2\}^{3/2}} - \frac{\gamma}{2}\right\}\Bigg|_{s=\frac{\pi t}{\Lambda}},$$

by (2.19). Hence, by the Cauchy Riemann equations,

$$\lim_{\substack{(x,y)\in\Omega \\ (x,y)\to(u(t),v(t))}} \left\{\frac{1}{2}|\nabla\psi(x,y)|^2\right\} + gv(t)$$

$$= \sigma\left\{\frac{u'(t)v''(t) - v'(t)u''(t)}{(u'(t)^2 + v'(t)^2)^{3/2}}\right\} + \frac{g\Lambda\gamma}{2\pi} - \frac{g\Lambda}{\pi}[\underline{v}] \text{ on } S. \tag{2.21}$$

This completes the proof of the theorem. □

Remark. There is nothing special about the use of the Hilbert space H_β in the formulation of the preceding theorem. The result is true, with the same proof, if analogous Banach spaces of continuously differentiable functions are used instead. $\square$

Suppose that $\underline{v}$ is a solution of (2.15) and let

$$\underline{w} = \underline{v} + \frac{1}{2}\{\mu - \gamma\}.$$

Then $\underline{w}$ satisfies

$$\mu \mathcal{C}\underline{w}' = \underline{w} + \underline{w}\mathcal{C}\underline{w}' + \mathcal{C}(\underline{w}\,\underline{w}') \\ - \beta \left\{ \frac{\underline{w}'}{\sqrt{\underline{w}'^2 + (1+\mathcal{C}\underline{w}')^2}} \right\}' + \beta\mathcal{C}\left\{ \frac{1+\mathcal{C}\underline{w}'}{\sqrt{\underline{w}'^2 + (1+\mathcal{C}\underline{w}')^2}} \right\}', \qquad (2.22)$$

which is the Euler-Lagrange equation, without constraint, of the functional defined on $\mathcal{D}_\beta$ by

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \frac{1}{2}\left\{ w^2(1+\mathcal{C}w') - \mu w \mathcal{C}w' \right\} + \beta \left\{ \sqrt{w'^2 + (1+\mathcal{C}w')^2} \right\} dt. \quad (2.23)$$

Thus we reach an unconstrained formulation of the steady water-wave problem which is equivalent to that in Theorem 2.3, namely:

$$\text{find extremals } \underline{w} \text{ of the functional } \mathcal{J} \text{ on } \mathcal{D}_\beta. \qquad (2.24)$$

A direct proof that solutions of (2.24) give solutions of the steady water-wave problem is identical to that of Theorem 2.3 (just put $\gamma = \mu$ in (2.15).)

As was remarked earlier, once the wavelength 2Λ is chosen, β (the dimensionless coefficient of surface tension) should be be regarded as fixed for water whereas the parameter μ, which is the one defined in (2.10), represents the *a priori* unknown speed of the wave.

When $\beta = 0$, a limiting case of our theory arises when waves approach the Stokes wave of greatest height [56] which has a stagnation point on the surface [60] (at a stagnation point $\nabla\psi = 0$). From (2.19), with $\beta = 0$ and $\gamma = \mu$, it follows that the corresponding solutions $\underline{w}$ of (2.22) have solutions whose maximum approaches $\mu/2$. Therefore $\max \underline{w} = \mu/2$ characterises the Stokes highest wave in this formulation.

3 Regularity

The goal of this section is to establish the smoothness of solutions of (2.22) in $\mathcal{D}_\beta$ when $\beta > 0$ and to prove that that when $\beta = 0$ all solutions up to, but not including, the wave of greatest height are smooth.

Recall that the solution of (2.22) with $\beta = 0$ which corresponds to the Stokes highest wave has $\mu/2 = \max w$. This gives the hypotheses of the first regularity result its special significance.

Theorem 3.1. *Suppose that $w \in H_0$ satisfies (2.22) with $\beta = 0$,*

$$\mu \mathcal{C} w' = w + w \mathcal{C} w' + \mathcal{C}(ww') \tag{3.1}$$

and $w < \frac{1}{2}\mu$. Then w is a real-analytic function and ${w'}^2 + (1 + \mathcal{C}w')^2 \neq 0$ on $[-\pi, \pi]$.

In the presence of surface tension effects, the regularity theorem has the following form.

Theorem 3.2. *Suppose that $v \in \mathcal{D}_\beta$ satisfies (2.22) with $\beta > 0$. Then v is a real-analytic function.*

The proofs use a bootstrap argument and the Cauchy principal value integral formula [66, Vol.I, page 51] for the operator $\mathcal{C}$ acting on $v \in L_p[-\pi, \pi]$, $1 < p < \infty$:

$$\mathcal{C}(v)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v(t)}{\tan((x-t)/2)}\, dt.$$

First we record two standard results from harmonic analysis, [66, Vol. I, page 153], [9, Vol. II, page 99].

M. Riesz's Theorem. *$\mathcal{C} : L_p[-\pi, \pi] \to L_p[-\pi, \pi]$ is a bounded linear operator for all $p \in (1, \infty)$.*

Let C^α denote space of those functions defined on $[-\pi, \pi]$, periodic extensions of which are Hölder continuous with index $\alpha \in (0, 1)$ on $\mathbb{R}$. Let $C^{k,\alpha}$ be the space of functions on $[-\pi, \pi]$ the first k derivatives of which are in C^α. Thus $C^\alpha = C^{0,\alpha}$.

Privalov's Theorem. *$\mathcal{C} : C^\alpha \to C^\alpha$ is a bounded linear operator for all $\alpha \in (0, 1)$.*

Lemma 3.3. *Suppose that $g \in L_2[-\pi, \pi]$ with $[g] = 0$. Then the function G defined by*

$$G(x) = \int_{-\pi}^{\pi} \frac{g(y) \int_y^x g(t)dt}{\tan((x-y)/2)}\, dy$$

is 2π-periodic and uniformly bounded, independent of $x \in [-\pi, \pi]$. (The functions in the integrals are extended by periodicity in order that they are defined on the ranges of integration.)

Proof. Note that G is 2π-periodic because $[g] = 0$. Choose x and fix it. Then the function $h(x, \cdot)$ defined by

$$h(x, y) = \frac{\int_y^x g(t)dt}{\tan((x-y)/2)}$$

is in $L_2[-\pi, \pi]$, and $\|h(x, \cdot)\|_{L_2[-\pi,\pi]} \leq \text{const}\|g\|_{L_2[-\pi,\pi]}$, by Hardy's inequality [52, Ch. 3, Ex. 14]. Since g is square-integrable, the result follows from the Cauchy-Schwarz inequality. □

Observe that if $g \in L_p[-\pi,\pi]$ with $p > 2$, then

$$\int_y^x g(t)dt = o(|x-y|^{\frac{1}{2}}) \text{ as } y \to x.$$

Therefore, for such g,

$$\begin{aligned} G(x) &= -\frac{1}{2}\int_{-\pi}^{\pi} \frac{\frac{\mathrm{d}}{\mathrm{d}y}\left\{\int_y^x g(t)dt\right\}^2}{\tan((x-y)/2)}dy = \frac{1}{4}\int_{-\pi}^{\pi} \frac{\left\{\int_y^x g(t)dt\right\}^2}{\sin^2((x-y)/2)}dy \\ &= \frac{1}{4}\int_{-\pi}^{\pi} \frac{\left\{\int_{x-y}^x g(t)dt\right\}^2}{\sin^2(y/2)}dy. \end{aligned} \tag{3.2}$$

By a limiting argument it now follows that, for every function in $g \in L_2[-\pi,\pi]$, the formula (3.2) is valid for almost all x. Let $\mathcal{G} : L_2[-\pi,\pi] \to L_\infty[-\pi,\pi]$ be defined by $\mathcal{G}(g) = G$, where G is given by (3.2). Note that G is positive. This leads to the following.

Theorem 3.4. *Let $Y = \{g \in L_2[-\pi,\pi] : [g] = 0\}$. The mapping $\mathcal{G}$ is sequentially continuous from Y, with the weak L_2-topology, into $L_p[-\pi,\pi]$, $1 \le p < \infty$, with the strong L_p-topology.*

Proof. Suppose that $g_n \in Y$ and $g_n \rightharpoonup g$ in $L_2[-\pi,\pi]$. It will suffice to show for a subsequence that $\mathcal{G}(g_n) \to \mathcal{G}(g)$ in $L_p[-\pi,\pi]$, $1 < p < \infty$, as $n \to \infty$. Let $u_n(x) = \int_0^x (g_n(t) - g(t))dt$, so that $u_n{}' = g_n - g$ and $u_n \to 0$ in $L_2[-\pi,\pi]$. Then

$$\begin{aligned} &u_n(x)\mathcal{C}u_n{}'(x) - \mathcal{C}(u_n u_n{}')(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u_n{}'(y)\{u_n(x) - u_n(y)\}}{\tan((x-y)/2)}dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u_n'(y)\int_y^x u_n'(t)dt}{\tan((x-y)/2)}dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(g_n(y) - g(y))\int_y^x (g_n(t) - g(t))dt}{\tan((x-y)/2)}dy \\ &= \frac{1}{2\pi}\mathcal{G}(g_n - g)(x). \end{aligned}$$

Therefore, since the integral of any conjugate function is zero,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{G}(g_n - g)(x)dx = \int_{-\pi}^{\pi} u_n(x)\mathcal{C}(u_n{}')(x)dx = \int_{-\pi}^{\pi} u_n(x)\mathcal{C}(g_n - g)(x)dx \to 0$$

as $n \to \infty$ since $\mathcal{C}$ is bounded on $L_2[-\pi,\pi]$, $g_n \rightharpoonup g$ and $u_n \to 0$ in $L_2[-\pi,\pi]$.

Thus, since $\mathcal{G}(g_n - g)$ is non-negative, it follows that $\mathcal{G}(g_n - g) \to 0$ in $L_1[-\pi,\pi]$ and so a subsequence converges pointwise almost everywhere. Let

$$\phi_n^x(y) = \frac{\int_{x-y}^x g_n(t)dt}{\sin(y/2)} \text{ and } \phi^x(y) = \frac{\int_{x-y}^x g(t)dt}{\sin(y/2)}$$

Then we have just shown for a subsequence that, for almost all x, $\phi_n^x \to \phi^x$ in $L_2[-\pi,\pi]$ as $n \to \infty$, and hence that $\|\phi_n^x\|_{L_2[-\pi,\pi]} \to \|\phi^x\|_{L_2[-\pi,\pi]}$. Another way of saying this is that, for almost all x, $\mathcal{G}(g_n)(x) \to \mathcal{G}(g)(x)$ as $n \to \infty$. Since $\{\mathcal{G}(g_n)\}$ is bounded in $L_\infty[-\pi,\pi]$, by Lemma 3.3, the result of the theorem follows from the Dominated Convergence theorem. $\square$

Lemma 3.5. *Suppose that, in Lemma 3.3, the function $g \in L_p[-\pi,\pi]$ for $2 < p < \infty$. Then $G \in C^{1-\frac{2}{p}}$.*

Proof. Since G is periodic it will suffice to show that $|G(x) - G(0)| \le$ const $|x|^{1-\frac{2}{p}}$, for all x with $|x| < \pi/2$, where the constant depends only on $\|g\|_p$ and p. Without loss of generality, suppose $x > 0$. Let $\frac{1}{p} + \frac{1}{q} = 1$ and let I denote the set $[-\pi,\pi] \setminus [-2x, 2x]$. Then

$$\begin{aligned}|G(x) - G(0)| \le \frac{1}{4}\int_{-2x}^{2x} \frac{\left\{\int_{x-y}^{x} g(t)dt\right\}^2}{\sin^2(y/2)}dy + \frac{1}{4}\int_{-2x}^{2x} \frac{\left\{\int_{-y}^{0} g(t)dt\right\}^2}{\sin^2(y/2)}dy \\ + \left|\frac{1}{4}\int_I \frac{\left\{\int_{x-y}^{x} g(t)dt\right\}^2 - \left\{\int_{-y}^{0} g(t)dt\right\}^2}{\sin^2(y/2)}dy\right|.\end{aligned}$$

Since $g \in L_p[-\pi,\pi]$, $p > 2$, it follows from Hölder's inequality that

$$\frac{1}{4}\int_{-2x}^{2x} \frac{\left\{\int_{x-y}^{x} g(t)dt\right\}^2}{\sin^2(y/2)}dy + \frac{1}{4}\int_{-2x}^{2x} \frac{\left\{\int_{-y}^{0} g(t)dt\right\}^2}{\sin^2(y/2)}dy \le \text{ const } \|g\|_p^2 |x|^{\frac{2}{q}-1},$$

where the constant depends only on p. Also,

$$\begin{aligned}&\left|\frac{1}{4}\int_I \frac{\left\{\int_{x-y}^{x} g(t)dt\right\}^2 - \left\{\int_{-y}^{0} g(t)dt\right\}^2}{\sin^2(y/2)}dy\right| \\ &\le \left|\frac{1}{4}\int_I \frac{\left\{\int_{x-y}^{x} g(t)dt + \int_{-y}^{0} g(t)dt\right\}\left\{\int_{x-y}^{x} g(t)dt - \int_{-y}^{0} g(t)dt\right\}}{\sin^2(y/2)}dy\right| \\ &\le \text{ const } \|g\|_p \int_I \frac{|y|^{\frac{1}{q}}\left|\int_{x-y}^{x} g(t)dt - \int_{-y}^{0} g(t)dt\right|}{\sin^2(y/2)}dy \\ &= \text{ const } \|g\|_p \int_I \frac{|y|^{\frac{1}{q}}\left|\int_{0}^{x} g(t)dt - \int_{-y}^{x-y} g(t)dt\right|}{\sin^2(y/2)}dy \\ &\le \text{ const } \|g\|_p^2 |x|^{\frac{1}{q}} \int_I |y|^{\frac{1}{q}-2}dy \le \text{ const } \|g\|_p^2 |x|^{\frac{2}{q}-1}.\end{aligned}$$

Thus

$$|G(x) - G(0)| \le \text{ const } \|g\|_p^2 |x|^{\frac{2}{q}-1} = \text{ const } \|g\|_p^2 |x|^{1-\frac{2}{p}}.$$

This completes the proof. $\square$

Lemma 3.6. *Suppose that in Lemma 3.3, $g \in C^\alpha$, $\alpha \in (0,1)$. Then $G \in C^{1,\delta}$, $0 < \delta < \alpha$.*

Proof. Since $g \in C^\alpha$, it follows from first principles and the Dominated Convergence Theorem that G given by (3.2) is differentiable, and its derivative is given by

$$G'(x) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{(g(x)-g(x-y))\int_{x-y}^{x} g(t)dt}{\sin^2(y/2)}dy.$$

To see that G' is in C^δ, $0<\delta<\alpha$, it suffices, as in the proof of Lemma 3.5, to show that $|G'(x)-G'(0)| \leq$ const $|x|^\delta$, where the constant depends only on α, δ and $\|g\|_{C^\alpha}$. Let $I = [-\pi,\pi] \setminus [-2x, 2x]$ with $x \in (0, \pi/2)$. Note first that

$$\int_{-2x}^{2x} \left| \frac{g(x)-g(x-y)}{\sin^2(y/2)} \int_{x-y}^{x} g(t)dt \right| dy \leq \text{ const } |x|^\alpha,$$

because $g \in C^\alpha$, and similarly that

$$\int_{-2x}^{2x} \left| \frac{g(0)-g(0-y)}{\sin^2(y/2)} \int_{-y}^{0} g(t)dt \right| dy \leq \text{ const } |x|^\alpha.$$

Next note that because of the definition of I and the fact that $g \in C^\alpha$.

$$\begin{aligned}
&\left| \int_I \frac{(g(x)-g(x-y))}{\sin^2(y/2)} \int_{x-y}^{x} g(t)dtdy - \int_I \frac{(g(0)-g(-y))}{\sin^2(y/2)} \int_{-y}^{0} g(t)dtdy \right| \\
&\leq \int_I \left| \frac{(g(x)-g(0))-(g(x-y)-g(-y))}{\sin^2(y/2)} \int_{x-y}^{x} g(t)dt \right| dy \\
&+ \int_I \left| \frac{g(0)-g(-y)}{\sin^2(y/2)} \left(\int_{x-y}^{x} g(t)dt - \int_{-y}^{0} g(t)dt \right) \right| dy \leq \text{ const } |x|^\alpha \log|x|.
\end{aligned}$$

Combining these estimates gives that $G' \in C^\delta$, $0<\delta<\alpha$, as required. □

3.1 Regularity with Zero Surface Tension

Theorem 3.7. *Suppose that $w \in H_0$ satisfies (3.1), and $w < \frac{1}{2}\mu$. Then $w \in C^{2,\alpha}$, for all $\alpha \in (0,1)$.*

Proof. With w as in the statement of the theorem, let

$$\begin{aligned}
G(x) =& w(x)\mathcal{C}w'(x) - C(ww')(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w'(y)\{w(x)-w(y)\}}{\tan((x-y)/2)}dy \\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w'(y)\int_y^x w'(t)dt}{\tan((x-y)/2)}dy.
\end{aligned}$$

Since $w' \in L_2[-\pi.\pi]$, it follows from Lemma 3.3 that $G \in L_\infty[-\pi,\pi]$. Now

$$\mu\mathcal{C}w' - 2w\mathcal{C}w' = w - G \in L_\infty[-\pi,\pi] \tag{3.3}$$

since w is bounded, and hence $\mu\mathcal{C}w' - 2w\mathcal{C}w' \in L_\infty[-\pi,\pi]$. But $\mu - 2w > 0$ and is continuous on $[-\pi,\pi]$. Therefore $\mathcal{C}w' \in L_\infty[-\pi,\pi]$. Since $\mathcal{C}^2 w' = -w'$, and $\mathcal{C}w' \in L_p[-\pi,\pi]$, it follows from the theorem of M. Riesz that $w' \in L_p[-\pi,\pi]$, for all $p \in (1,\infty)$. From Lemma 3.5 it now follows that $G \in C^\alpha$, for all $\alpha \in (0,1)$. Hence, by (3.3), $\mathcal{C}w' \in C^\alpha$ and it follows from Privalov's Theorem that $w' \in C^\alpha$. Now an application of Lemma 3.6 gives that $G \in C^{1,\alpha}$, and hence that $w \in C^{2,\alpha}$, since $w' \in C^{1,\alpha}$. This completes the proof. □

It is now possible to show, directly from equation (3.1) and a bootstrap argument, using the formula (3.2) and Privalov's theorem, that $w \in C^{k,\alpha}$ for all $k \in \mathbb{N}$, and hence that that w is infinitely differentiable. Here we prove the real-analyticity of w, using a famous result of H. Lewy and equation (2.17) with $\beta = 0$.

Proof of Theorem 3.1. Since w satisfies (2.17) with $\beta = 0$ and $\gamma = \mu$ (see the penultimate paragraph of Section 2.4),

$$(w - \frac{1}{2}\mu)\{{w'}^2 + (1+\mathcal{C}w')^2\} = -\frac{1}{2}\mu \text{ on } [-\pi,\pi],$$

from which it is obvious that ${w'}^2 + (1+\mathcal{C}w')^2 > 0$. Having established that $w \in C^{2,\alpha}$ and that ${w'}^2 + (1+\mathcal{C}w')^2 > 0$, an application of Lewy's theorem on the smoothness of the free boundary [36] gives that w is real-analytic. (See also [61, Section 7] for an account of Lewy's theorem in the present notation. In fact $C^{1,\alpha}$ would ensure sufficient regularity up to the boundary of $\mathcal{R}w$ on the lower half-plane for Lewy's theorem to be applied here.) This completes the proof of the theorem. □

3.2 Regularity with Positive Surface Tension

To see that v is real-analytic in the case of positive surface tension we need the following lemma, the proof of which is omitted because it is a straight-forward extension of that of Lewy's theorem [36].

Lemma 3.8. *Let D denote a half-disc $\{(x,y) : |x|^2+|y|^2 < r^2,\ y<0\}$. Suppose that $F = U + iV$ is a holomorphic function on D such that both $U, V \in C^2(\overline{D})$ and suppose that*

$$V_{xx}(x,0) = \mathcal{A}(U(x,0), V(x,0), U_x(x,0), V_x(x,0), U_{xx}(x,0)), \quad |x| < r\ , \tag{3.4}$$

where $\mathcal{A}$ is a complex-valued analytic function of all its arguments in a neighbourhood of $(U(0,0), V(0,0), U_x(0,0), V_x(0,0), U_{xx}(0,0))$ in $\mathbb{C}^5$, which is real when its arguments are real. Then there exists a disc $\widetilde{D}$ centred at $(0,0)$, and a holomorphic function $\tilde{U} + i\tilde{V} : \widetilde{D} \to \mathbb{C}$ such that

$$U + iV = \tilde{U} + i\tilde{V} \text{ on } D \cap \widetilde{D}\ .$$

□

Proof of Theorem 3.2. Suppose that $\beta > 0$ and that $v \in \mathcal{D}_\beta$ satisfies (2.22). Then v satisfies (2.17) with $\mu = \gamma$. When (2.17) is re-arranged, it emerges that

$$\begin{aligned}\beta v'' =& \beta\{v'\mathcal{C}v'' - v''\mathcal{C}v'\} \\ &+ \sqrt{{v'}^2 + (1+\mathcal{C}v')^2}\left\{\frac{\mu}{2} + (v - \frac{\mu}{2})({v'}^2 + (1+\mathcal{C}v')^2)\right\}.\end{aligned} \tag{3.5}$$

Now $v'' \in L_2[-\pi, \pi]$ and so it follows from Lemma 3.3 that $v'\mathcal{C}v'' - v''\mathcal{C}v' \in L_\infty[-\pi, \pi]$. Hence, by (3.5), $v'' \in L_\infty[-\pi, \pi]$. Therefore, by Lemma 3.5, $v'\mathcal{C}v'' - v''\mathcal{C}v' \in C^\alpha$, for any $\alpha \in (0, 1)$. Hence, from (3.5) $v \in C^{2,\alpha}$ for all $\alpha \in (0, 1)$. A bootstrap argument gives that v is infinitely differentiable.

Now to see that v is real-analytic, observe that ${v'}^2 + (1 + \mathcal{C}v')^2 \neq 0$, since $v \in \mathcal{D}_\beta$ and hence, by (2.17) with $\mu = \gamma$, either $V = \mathcal{R}v$ or $V = \mathcal{R}\mathcal{C}v$ satisfies the hypotheses of Lemma 3.8. (The operator $\mathcal{R}$ is defined by (2.14).) The real-analyticity of v is immediate, and the proof is complete. $\square$

4 Bifurcation Theory

Note that $\underline{w} = 0$ is a solution of (2.22) for all values of the Froude number $\sqrt{\mu}$. The general question of existence of 'small amplitude' solutions of equation (2.22) can be regarded as a bifurcation problem involving gradient operators in Hilbert space (see [16, 31, 33, 57]). Because the eigenvalue μ appears linearly in (2.22), an elementary version of the theory is sufficient. We deal with this aspect next.

4.1 Variational Bifurcation Theory

The linearisation of (2.22) about its solution $\underline{w} = 0$ is

$$\mu\mathcal{C}w' = w - \beta w'' \tag{4.1}$$

which has 2π-periodic solutions $\cos nx$ and $\sin nx$ when $\mu = \mu_n = \beta n + n^{-1}$. Note that $\mu_n = \mu_m$ if $\beta = (mn)^{-1}$, in which case (4.1) has a four-dimensional space of solutions if $m \neq n$. (Note that β depends on Λ, the wavelength, as well as on σ, the surface tension, so this can happen for any liquid for certain wavelengths.)

We now observe that μ_n is a bifurcation point for (2.22) for all $n \in \mathbb{R}$ because of the variational structure. An account of elementary theory used here is to be found in [57]. (Previous treatments of the bifurcation problem for capillary-gravity waves when $\mu_n = \mu_m$ and $n \neq m$ involve careful implicit-function-theoretic calculations. See, for example, [28] and [62].)

With $n \in \mathbb{N}$ arbitrary and $\beta \geq 0$ fixed, let L be defined by

$$Lw = \mu_n\mathcal{C}w' - w + \beta w''.$$

Let X denote the space $L_2[-\pi, \pi]$ and let

$$\mathcal{D}(L) = \{w \in X : Lw \in X,\ w(\pi) = w(-\pi)\}$$

Then L is closed and densely defined on X and therefore $\mathcal{D}(L)$, equipped with its graph norm, is a Hilbert space which we denote by X_1. (In fact for $\beta \geq 0$, X_1 is equivalent to H_β defined in Section 2.4. Note X_1 for $\beta = 0$ and $\beta > 0$ are different spaces.) Now X is the topological direct sum of $N(L)$ and $\text{Im}(L) \cap X_1$, where $N(L)$ and $\text{Im}(L)$ denote the kernel and range of L, respectively.

Let $\lambda = \mu - \mu_n$ and re-write equation (2.22) in the form

$$Lw + \lambda Cw + \lambda R(w) = 0 \tag{4.2}$$

where $Cw = \mathcal{C}w'$ and

$$\begin{aligned} R(w) &= -w\mathcal{C}w' - \mathcal{C}(ww') \\ &\quad + \beta \left\{ \frac{w'}{\sqrt{w'^2 + (1+\mathcal{C}w')^2}} - w' \right\}' - \beta\mathcal{C} \left\{ \frac{1+\mathcal{C}w'}{\sqrt{w'^2 + (1+\mathcal{C}w')^2}} \right\}' \end{aligned}$$

Then hypothesis (H1) of [57, page 81] is satisfied as are those of [57, Section 7, page 97]. (In our case $R_1 = R$, $R_2 = 0$.)

Definition 4.1. *Let $\mathcal{S} = \{(w,\mu) \in X_1 \times \mathbb{R} : (2.22)$ holds and $w \neq 0\}$. Then $\hat{\mu}$ is a bifurcation point for (4.2) if $(\hat{\mu}, 0)$ lies in the closure of $\mathcal{S}$ in $\mathbb{R} \times X_1$.*

Theorem 4.2. *The points μ_n, $n \in \mathbb{N}$, are the only bifurcation point for (2.22).*

Proof. It is classical [57] that all bifurcation points are points μ where L is not injective on X_1. Thus $\{\mu_n : n \in \mathbb{N}\}$ contains all the bifurcation points. To see that each μ_n is a bifurcation point it suffices to confirm that all the hypotheses of [57, Theorem 7.2, page 99] are satisfied by the terms in (4.2). Those pertaining to L and R have been checked already. Note that

$$N(L) = \text{span}\{\sin nx, \cos nx, \sin mx, \cos mx\} \text{ for some } m,\ n \in \mathbb{N}.$$

Hence, for $w \in N(L) \setminus \{0\}$, $\langle \mathcal{C}w', w \rangle > 0$, where $\langle \, , \rangle$ is the inner product in X. Thus all the hypotheses of that theorem are satisfied and it follows that μ_n is a bifurcation point. $\square$

5 Zero Surface Tension

The remainder of this paper is concerned with the existence of steady, periodic waves in the absence of surface tension and the focus henceforth is on the case $\beta = 0$ is (2.22). We begin with the classical bifurcation theory for small amplitude waves and their linearised stability.

5.1 Bifurcation from a Simple Eigenvalue

Define $F : \mathbb{R} \times H_0 \to L_2[-\pi, \pi]$ by

$$F(\lambda, w) = \mathcal{C}w' - \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww')). \tag{5.1}$$

Let $Z_0 \subset H_0$ denote the Sobolev space of absolutely continuous even functions w on $[-\pi, \pi]$ with $w' \in L_2[-\pi, \pi]$, and let Z denote the space of even elements of $L_2[-\pi, \pi]$. Then

$$F(\lambda, w) = 0, \quad (\lambda, w) \in \mathbb{R} \times Z_0, \tag{5.2}$$

is the Euler-Lagrange equation satisfied by critical points of the smooth functional J_λ defined on Z_0 by

$$J_\lambda(w) = \int_{-\pi}^{\pi} \{w(t)\mathcal{C}w'(t) - \lambda w^2(t)(1 + \mathcal{C}w'(t))\} dt. \tag{5.3}$$

Let $\mathcal{E}$ denote the set of all solutions $(\lambda, w) \in \mathbb{R} \times Z_0$ of (5.2) with $w \neq 0$. Here are some observations about symmetry. Suppose that $w \in Z_0$. Define $\hat{w} : \mathbb{R} \to \mathbb{R}$ by extending w periodically to $\mathbb{R}$ and let

$$w_{(k)}(t) = k^{-1}\hat{w}(kt) \text{ and } \tilde{w}(t) = \hat{w}(t + \pi), \quad t \in [-\pi, \pi].$$

Observe that $w_{(k)}$ and $\tilde{w}$ are in Z_0,

$$(\mathcal{C}\tilde{w}')(t) = (\widetilde{\mathcal{C}w'})(t) \text{ and } (\mathcal{C}w')(kt) = \mathcal{C}((w_{(k)})')(t).$$

Therefore when $(\lambda, w) \in \mathbb{R} \times Z_0$ is a solution of (5.2), so also are $(\lambda, \tilde{w})$ and $(k\lambda, w_{(k)})$, for all $k \in \mathbb{N}$. Note that $w_{(k)} = kw$, $k > 1$, implies w is a constant, and that $w = \tilde{w}$ implies that $\langle \Phi_n, w \rangle = 0$ for all odd n, where $\Phi_n(t) = \cos nt$.

We turn first to the bifurcation of even solutions from the line of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ of (5.2). The linearisation about $w = 0$ is

$$\mathcal{C}w' - \lambda w = 0, \quad \lambda \in \mathbb{R},\ w \in Z_0,$$

which has a non-zero solution w if and only if $\lambda = N \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$, in which case $w(t) = \alpha\Phi_N$ for some non-zero real α. Each such λ is a simple eigenvalue in the sense of [17] and [18], and classical bifurcation theory applies. It predicts that from each $(N, 0)$, $N \in \mathbb{N}_0$, on the line of trivial solutions there bifurcates a smooth curve of non-trivial solutions

$$\Gamma_N = \{(\lambda_s^N, w_s^N) : s \in (-1, 1)\} \text{ say },\tag{5.4}$$

with

$$(\lambda_0^N, w_0^N) = (N, 0) \text{ and } \dot{w}^N = (\mathrm{d}/\mathrm{d}s)w_s^N|_{s=0} = \Phi_N. \tag{5.5}$$

(Here and elsewhere $\dot{}$ denotes differentiation with respect to s at $s = 0$.) Moreover, the theory predicts that in a neighbourhood of the bifurcation point $(N, 0)$ there are no non-trivial solutions of (5.2) other than those on the bifurcating curve. Let $\mathcal{E}_N$ denote the maximal connected subset in $\mathbb{R} \times Z_0$ of $\overline{\mathcal{E}}$ which contains Γ_N, $N \in \mathbb{N}_0$.

Because of symmetry,

$$\{(N\lambda, w_{(N)}) : (\lambda, w) \in \Gamma_1\} \subset \mathcal{E}_N \text{ for } N \in \mathbb{N}.$$

Thus, by the uniqueness of Γ_N, the shape of the curve Γ_N, $N \geq 1$, in a neighbourhood of the bifurcation point is determined by that of Γ_1.

Now we observe that in a neighbourhood of $(1, 0)$ the curve Γ_1 must be symmetric in $\mathbb{R} \times Z_0$ about the λ-axis. This follows since $(\lambda, \tilde{w}) \in \mathcal{E}_1$ whenever $(\lambda, w) \in \Gamma_1$ and, for $|s|$ sufficiently small, $w_s^1 \neq \widetilde{w_s^1}$ because $(\mathrm{d}/\mathrm{d}s) w_s^1|_{s=0} = \Phi_1$ and $\Phi_1(t) = \cos t$. Hence $\dot{\lambda}^N = (\mathrm{d}/\mathrm{d}s)\lambda_s^N|_{s=0} = 0$. In other words the tangents to all the bifurcating curves Γ_N, $N \geq 1$, are vertical in $\mathbb{R} \times Z_0$ at the bifurcation point. It may be assumed therefore that

$$w_{-s}^N = \widetilde{w_s^N} \text{ and } \lambda_s^N = \lambda_{-s}^N \tag{5.6}$$

in (5.4). We shall show presently that the whole curve Γ_N is not vertical for $N \geq 1$, unlike the case of $N = 0$ where clearly Γ_0 is vertical, because any constant w is a solution when $\lambda = 0$.

It suffices to study the behaviour of Γ_1 in a neighbourhood of $(1, 0)$. With $Y_1 = \operatorname{span}\{\Phi_1\}$ and Y_2 the closure in Z_0 of $\operatorname{span}\{\Phi_k : k = 0, 2, 3, \dots\}$, write (λ_s, w_s) for convenience instead of (λ_s^1, w_s^1). Then we may suppose that

$$w_s = s(\Phi_1 + \Psi_s) \text{ where } \Psi_s \in Y_2,\ \Psi_0 = 0 \text{ and } s \mapsto \Psi_s \text{ is smooth.}$$

When substituted into (5.2) this gives

$$\begin{aligned} 0 = {} & (1 - \lambda_s)\Phi_1 + \mathcal{C}\Psi_s' - \lambda_s \Psi_s \\ & - s\lambda_s\{(\Phi_1 + \Psi_s)\mathcal{C}(\Phi_1' + \Psi_s') + \mathcal{C}((\Phi_1 + \Psi_s)(\Phi_1 + \Psi_s)')\}. \end{aligned} \tag{5.7}$$

A differentiation with respect to s at $s = 0$, using the fact that $\dot{\lambda} = 0$ gives

$$\begin{aligned} \mathcal{C}\dot{\Psi}' - \dot{\Psi} &= \Phi_1 \mathcal{C}\Phi' + \mathcal{C}(\Phi_1 \Phi_1') \\ &= \Phi_1^2 + \frac{1}{2}\Phi_2 = \frac{1}{2} + \Phi_2. \end{aligned}$$

Therefore $\dot{\Psi} = \Phi_2 - \frac{1}{2}$. Differentiating (5.7) again with respect to s at $s = 0$ gives

$$\begin{aligned} 0 &= -\ddot{\lambda}\Phi_1 + \mathcal{C}\ddot{\Psi}' - \ddot{\Psi} - 2\{\dot{\Psi}\mathcal{C}\Phi_1' + \Phi_1\mathcal{C}\dot{\Psi}' + \mathcal{C}(\Phi_1\dot{\Psi}' + \dot{\Psi}\Phi_1')\} \\ &= -\ddot{\lambda}\Phi_1 + \mathcal{C}\ddot{\Psi}' - \ddot{\Psi} - 2\Phi_1 - 6\Phi_3. \end{aligned}$$

Hence

$$\ddot{\lambda} = -2 \text{ and } \ddot{\Psi} = 3\Phi_3.$$

Therefore each curve Γ_N, $N \geq 1$ is a symmetric, sub-critical pitchfork bifurcation at $(N, 0)$.

Now consider the eigenvalue problem for the linearisation with respect to w at (λ_s, w_s) of equation (5.2)

$$\partial_w F(\lambda_s, w_s)v = \nu v, \quad v \in Z_0. \tag{5.8}$$

At $s=0$, $\nu=0$ is an eigenvalue with eigenfunction Φ_1. In [18] the implicit function theorem is used to show that, for s in a neighbourhood of 0, all solutions of (5.8) in a sufficiently small neighbourhood of $(\nu, v)=(0,\Phi_1)$ (up to a normalisation of v) are given by a smooth mapping $s \mapsto (\nu_s, v_s)$ such that $(\nu_0, v_0) = (0,\Phi_1)$ and

$$\lim_{s\to 0} \frac{s d\lambda_s/ds}{\nu_s} = 1$$

(See [18, Theorem 1.16, page 165].) Further, from (5.6), we may infer that ν_s is even is s and $v_{-s} = \tilde{v}_s$.

Our conclusions so far may be summarised as follows:

$$\left.\begin{array}{rcl} w_s^1 &=& s\Phi_1 + s^2(\Phi_2 - \frac{1}{2}) + \frac{3}{2}s^3\Phi_3 \\ \lambda_s^1 &=& 1-s^2 \\ \nu_s &=& -2s^2 \end{array}\right\} + O(|s|^4) \text{ as } |s|\to 0. \tag{5.9}$$

Suppose that v_s denotes the eigenfunction corresponding to the eigenvalue ν_s, normalised so that $\int_{-\pi}^{\pi} v_s(t)\cot t dt = 1$. Then

$$(\lambda_s^1 + \nu_s)\int_{-\pi}^{\pi} v_s(t)dt + 2\int v_s(t)\mathcal{C}{w_s^1}' dt = 0,$$

whence it follows from (5.9) that

$$\int_{-\pi}^{\pi} v_s(t)dt = -2s + O(|s|^2) \text{ as } s\to 0. \tag{5.10}$$

For s, with $|s|$ sufficiently small, all the eigenvalues of $\partial_w F(\lambda_s^1, w_s^1)$ except for two are greater than $3/2$, since that is true when $s=0$ and the eigenvalues ν of (5.8) depend continuously on s. Of the other two, one is close to -1 (because -1 is an eigenvalue of (5.8) when $s=0$), and the other is negative by (5.9). Thus, when $|s|$ is sufficiently small, equation (5.8) has exactly two negative eigenvalues and all the others exceed $3/2$. (Note that this is false for the linearisation of (2.22) with $\beta=0$ in the space of all functions, not just even ones. There $(w_s^1)'$ is an odd eigenfunction of that equation linearised at (λ_s, w_s^1) with eigenvalue 0. This follows by differentiating (2.22) with respect to t when $\beta=0$.)

Finally note that there is a neighbourhood V of $(1,0)$ in $\mathbb{R}\times H_0$ in which all solutions of $F(\lambda, w)=0$, where F is defined by (5.1), have the property that $\lambda = \lambda_s^1$ and $w(t) = w_s^1(t+\underline{t})$, $t\in\mathbb{R}$, for some translation $\underline{t}$. This is immediate since, in a sufficiently small neighbourhood V of $(1,0)$ in $\mathbb{R}\times H_0$, solutions (λ, w) are uniquely determined, through the Lyapunov-Schmidt procedure, by the three parameters λ, $\int_{-\pi}^{\pi} w(t)\sin t dt$ and $\int_{-\pi}^{\pi} w(t)\cos t dt$. In particular, by the uniqueness implicit in this remark, w must be even if (λ, w) is a solution in V with $\int_{-\pi}^{\pi} w(t)\sin t dt = 0$. However, if w is any 2π-periodic function there is $\underline{t}\in(-\pi,\pi]$ such that $\int_{-\pi}^{\pi} w(t+\underline{t})\sin(t)dt = 0$ and the norm of any translate of w is equal to that of w. Hence (5.9) describes all solutions $(\lambda, w)\in\mathbb{R}\times H_0$ of

$F(\lambda, w) = 0$ in a neighbourhood of $(1,0)$, up to translations. Let $\mathcal{S}$ denote the set of all solutions $(\lambda, w) \in \mathbb{R} \times H_0$ of $F(\lambda, w) = 0$ and let

$$\Upsilon_N(\delta) = \{(N\lambda, w_{(N)}) : (\lambda, w) = (\lambda_s^1, w_s^1) \in \Gamma_1,\ s \in (-\delta, \delta)\},$$

regarded as a subset of $\mathcal{S}$.

Theorem 5.1 below is an observation, written in the present notation, which Plotnikov[49, Theorem 4.2] made in the context of solitary waves. It plays an important role in the development of the theory which follows by making the analysis of spectra of certain linearised operators rather straightforward.

Suppose that $(\mu, w) \in \mathbb{R} \times H_0$ satisfies (3.1) with $w < \mu/2$. Then the regularity theory of Section 3 is relevant and, by (2.19) with $\beta = 0$ and $\gamma = \mu$,

$$|W|^2(\mu - 2w) \equiv \mu \quad \text{on } [-\pi, \pi] \tag{5.11}$$

where

$$W = 1 + i\mathcal{R}w' \neq 0 \text{ on } [-\pi, \pi].$$

Here $\mathcal{R}$ is defined by (2.14), $\mathcal{R}f(x) = f(x) - i\mathcal{C}f(x)$, $x \in [-\pi, \pi]$, and $(\mathcal{R}f)' = \mathcal{R}(f')$. Let the linear operator $L : H_0 \to L_2[-\pi, \pi]$, the linearisation of (3.1) at this solution, be defined by

$$Lu = \mu\mathcal{C}u' - u - w\mathcal{C}u' - u\mathcal{C}w' - \mathcal{C}(uw)'. \tag{5.12}$$

Then for $u, v \in H_0$ we have

$$\begin{aligned}
\int_{-\pi}^{\pi} vLu dx &= \int_{-\pi}^{\pi} \left\{ (\frac{1}{2}\mu - w)(v\mathcal{C}u' + u\mathcal{C}v') - uv(1 + \mathcal{C}w') \right\} dx \\
&= \int_{-\pi}^{\pi} (\frac{1}{2}\mu - w)(v\mathcal{C}u' - u'\mathcal{C}v)dx + \int_{-\pi}^{\pi} u \{w'\mathcal{C}v - v(1 + \mathcal{C}w')\} dx \\
&= \int_{-\pi}^{\pi} (w - \frac{1}{2}\mu)\mathrm{Imag}\{\overline{\mathcal{R}v}\mathcal{R}u'\}dx - \int_{-\pi}^{\pi} u\mathrm{Real}\{\overline{\mathcal{R}v}W\}dx \\
&= \int_{-\pi}^{\pi} \frac{-\mu}{2|W|^2}\mathrm{Imag}\{\overline{\mathcal{R}v}\mathcal{R}u'\}dx - \int_{-\pi}^{\pi} u\mathrm{Real}\{\overline{\mathcal{R}v}W\}dx.
\end{aligned} \tag{5.13}$$

Now define an operator $A : H_0 \to H_0$ by

$$Af = \mathrm{Real}\{W\mathcal{R}f\} = f(1 + \mathcal{C}w') + w'\mathcal{C}f, \quad f \in H_0. \tag{5.14}$$

Then

$$\mathcal{R}(Af) = W\mathcal{R}f \text{ for all } f \in H_0, \tag{5.15}$$

and hence A is a homeomorphism because $f = \mathrm{Real}\frac{\mathcal{R}(Af)}{W}$ and $W \neq 0$.

Theorem 5.1. (Plotnikov's transformation.) *For all $u, v \in H_0$,*

$$\int_{-\pi}^{\pi} Av\, LAu dx = \int_{-\pi}^{\pi} \{\mu v\mathcal{C}u' - quv\}dx$$

where

$$q(x) = \mu \ \ \mathrm{Imag}\frac{W'}{W} + |W|^2(1+\mathcal{C}w').$$

Proof. From (5.11), (5.13) and (5.15)

$$\begin{aligned}
&\int_{-\pi}^{\pi} Av\ LAudx \\
&= \int_{-\pi}^{\pi} \frac{-\mu}{2|W|^2}\mathrm{Imag}\{\overline{W\mathcal{R}v}(W\mathcal{R}u)'\}dx - \int_{-\pi}^{\pi} \mathrm{Real}(W\mathcal{R}u)\mathrm{Real}(|W|^2\overline{\mathcal{R}v})dx \\
&= \int_{-\pi}^{\pi} -\frac{\mu}{2}\mathrm{Imag}\left\{\overline{\mathcal{R}v}\left[\mathcal{R}u' + \frac{W'\mathcal{R}u}{W}\right]\right\}dx - \int_{-\pi}^{\pi} |W|^2 v\mathrm{Real}(W\mathcal{R}u)dx \\
&= -\mu\int_{-\pi}^{\pi} \mathrm{Real}(\mathcal{R}v)\mathrm{Imag}\left(\mathcal{R}u' + \frac{W'\mathcal{R}u}{W}\right)dx - \int_{-\pi}^{\pi} |W|^2 v(u(1+\mathcal{C}w') + w'\mathcal{C}u)dx
\end{aligned}$$

since $\int_{-\pi}^{\pi}(\mathcal{R}v)\{\mathcal{R}u' + (W'\mathcal{R}u)/W\}dx = 0$, by Cauchy's theorem, the periodicity and the convergence to zero at infinity of the integrand,

$$\begin{aligned}
&= \int_{-\pi}^{\pi} \mu v\left\{\mathcal{C}u' - \mathrm{Imag}\frac{W'\mathcal{R}u}{W}\right\} - |W|^2(1+\mathcal{C}w')uv - |W|^2 w' v\mathcal{C}udx \\
&= \int_{-\pi}^{\pi} v\left\{\mu\mathcal{C}u' - \mu u\mathrm{Imag}\frac{W'}{W} + \mu\mathcal{C}u\mathrm{Real}\frac{W'}{W} - |W|^2(1+\mathcal{C}w')u - |W|^2 w'\mathcal{C}u\right\}dx.
\end{aligned}$$

However, differentiating equation (5.11) gives

$$\mathrm{Real}(\overline{W}W')(\mu - 2w) \equiv |W|^2 w',$$

whence

$$|W|^2 w' = \mu\mathrm{Real}\frac{W'}{W}.$$

Therefore

$$\int_{-\pi}^{\pi} Av\ LAudx = \int_{-\pi}^{\pi} \mu v\mathcal{C}u' - \left\{\mu\mathrm{Imag}\frac{W'}{W} + |W|^2(1+\mathcal{C}w')\right\}uvdx,$$

as required. □

5.2 A Question of Local Secondary Bifurcation

We have already observed that $\Upsilon_N(\delta) \subset \mathcal{E}_N$, for all $|\delta| \le 1$. A special case of a question raised by Levi-Civita is whether $\mathcal{E}_N$ is a re-scaled copy of $\mathcal{E}_1$ (using $(\lambda, w) \in \mathcal{E}_1 \mapsto (N\lambda, w_{(N)}) \in \mathcal{E}_N$. This is false, as we will see in Section 8.4. Here we confine ourselves to the local question: does there exist $\delta > 0$ such that there are no secondary bifurcation points (ignoring translations of course) on $\Upsilon_N(\delta)$ for all $N > 1$? This question, to which the answer is yes, is not answered by the theory of bifurcation from a simple eigenvalue summarised above since, *for fixed $\delta > 0$ and very large N*, Γ_N need only be a small subset of $\Upsilon_N(\delta)$.

We will show that there are no secondary bifurcation points on $\Upsilon_N(\delta)$ when δ is sufficiently small, irrespective of the value of N. The result is the following.

Theorem 5.2. *There exists $\delta > 0$ (independent of N) such that if $(\underline{\lambda}, \underline{w}) \in \Upsilon_N(\delta)$, $\underline{w} \neq 0$ and there is a sequence $\{(\lambda_k, w_k)\} \subset \mathcal{S}$ with*

$$(\lambda_k, w_k) \to (\underline{\lambda}, \underline{w}) \text{ in } \mathbb{R} \times H_0 \text{ as } k \to \infty,$$

then for sufficiently large k there exists t_k and $(\lambda_k, \underline{w}_k) \in \Upsilon_N(\delta)$ with

$$w_k(t) = \underline{w}_k(t + t_k).$$

Inevitably the proof depends on an analysis of a linearised problem, but some preparatory work is needed first. Suppose that $\{(\lambda_k, w_k)\} \subset \mathcal{S}$ is the sequence in the statement of the theorem and let $\underline{s}$ be such that $(\underline{\lambda}, \underline{w}) = (N\lambda^1_{\underline{s}}, w^1_{\underline{s}(N)})$. Now let $t_k \in [-\pi/N, \pi/N]$ be such that

$$\int_{-\pi}^{\pi} w_k(t + t_k) \sin Nt dt = 0.$$

Such a sequence $\{t_k\}$ clearly exists; without loss of generality suppose that $t_k \to \underline{t}$. By continuity, therefore,

$$\begin{aligned} 0 = \int_{-\pi}^{\pi} \underline{w}(t + \underline{t}) \sin Nt dt &= \int_{-\pi}^{\pi} \underline{w}(t) \sin N(t - \underline{t}) dt \\ &= \cos N\underline{t} \int_{-\pi}^{\pi} \underline{w}(t) \sin Nt dt - \sin N\underline{t} \int_{-\pi}^{\pi} \underline{w}(t) \cos Nt dt \\ &= -\sin N\underline{t} \int_{-\pi}^{\pi} \underline{w}(t) \cos Nt dt, \text{ since } \underline{w} \text{ is even,} \\ &\qquad = -N^{-1} \sin N\underline{t}\{\pi \underline{s} + O(\underline{s}^2)\} \quad \text{as} \quad |\underline{s}| \to 0, \end{aligned}$$

by (5.9) and the fact that $(\underline{\lambda}, \underline{w}) \in \Upsilon_N(\delta)$.

Therefore if $\underline{s}$ is sufficiently small (independently of the value of N), $\underline{t} \in \{0, \pm\pi/N\}$. Hence $\underline{w}(t + \underline{t})$ is an even function of t and it may be assumed from the outset, without any loss of generality, that the sequence $\{(\lambda_k, w_k)\}$ in the theorem has the additional property that

$$\int_{-\pi}^{\pi} w_k(t) \sin Nt dt = 0. \tag{5.16}$$

Since $(\lambda_k, w_k) \to (\underline{\lambda}, \underline{w})$ in $\mathbb{R} \times H_0$ and $(\lambda_k, w_k) \subset \mathcal{S}$, it follows, in the usual manner, that the linear problem

$$\mathcal{C}f - \underline{\lambda}\{f + f\mathcal{C}\underline{w}' + \underline{w}\mathcal{C}f' + \mathcal{C}(\underline{w}f)'\} = 0 \tag{5.17}$$

has a solution $f \in H_0$ with

$$f \not\equiv 0 \text{ and } \int_{-\pi}^{\pi} f(t) \sin Nt dt = 0. \tag{5.18}$$

The objective now is to establish the existence of $\delta > 0$ with the property that when $(\underline{\lambda}, \underline{w}) = (N\lambda^1_{\underline{s}}, w^1_{\underline{s}(N)})$ for $|\underline{s}| < \delta$, no such f exists. To do so we use Theorem 5.1 to observe that a solution of (5.17) and (5.18) exists if and only if

$$\mathcal{C}v' - \underline{\lambda}\underline{q}v = 0 \tag{5.19}$$

where $v = Af$ and $\underline{q}$ is calculated using the formula in Theorem 5.1 with $\mu = \underline{\lambda}^{-1}$ and $w = \underline{w}$. Note that

$$\begin{aligned}\int_{-\pi}^{\pi} \sin Nt\, v(t) dt &= \int_{-\pi}^{\pi} \sin Nt \{f(1 + \mathcal{C}\underline{w}') + \underline{w}'\mathcal{C}f\} dt \\ &= \int_{-\pi}^{\pi} \sin Nt \{f\mathcal{C}\underline{w}' + \underline{w}'\mathcal{C}f\} dt \text{ (since } f \text{ satisfies (5.18)),} \\ &= \int_{-\pi}^{\pi} \underline{w}' \{(\mathcal{C}f) \sin Nt - \mathcal{C}(f \sin Nt)\} dt.\end{aligned}$$

Now $\underline{w}'$ is an odd function of period $2\pi/N$ and a calculation reveals that the odd part of the function $\{(\mathcal{C}f)\sin Nt - \mathcal{C}(f \sin Nt\}$ is a trigonometric polynomial of the form $\sum_{k=1}^{N-1} B_k \sin kt$. Since the only non-zero coefficients in the Fourier sine series of $\underline{w}'$ are those of $\sin mNt$, $m \in \mathbb{N}$, we have shown that

$$v \not\equiv 0 \text{ and } \int_{-\pi}^{\pi} v(t) \sin Nt dt = 0, \tag{5.20}$$

when $v = Af$ and f satisfies (5.17) and (5.18). Since $\underline{q}$ is smooth (indeed real-analytic) a simple bootstrap argument gives that v is smooth (in fact also real-analytic, by Lewy's theorem [36]). Also v has a smooth 2π-periodic extension to $\mathbb{R}$.

Note that since $\underline{q}$ is (5.19) is even, the even and odd parts of any solution of (5.19) and (5.20) are also a solutions of (5.19) and (5.20). If even solutions are regarded as real and imaginary solutions as odd, then any non-zero solution of (5.19) and (5.20) can be considered as a complex-valued solution with the property that $u(-t) = \overline{u(t)}$, the real and imaginary parts of which are even and odd, respectively.

To proceed, we introduce a discrete family of Banach spaces of functions on $\mathbb{R}$ of different periods $2N\pi$, and a corresponding conjugate operator $\mathcal{C}_N$.

For every natural number N, let Y^N be the real Banach space of all $2N\pi$-periodic functions u such that $u(t) = \overline{u(-t)}$ for $t \in \mathbb{R}$ that are in $L_2^{loc}(\mathbb{R})$, and let $X^N \subset Y^N$ be the real Banach space of all $2N\pi$-periodic functions in $H^1_{loc}(\mathbb{R})$. The norms on X^N and Y^N are the usual H^1 and L_2 norms over $(-N\pi, N\pi)$, and we denote the associated inner-product on Y^N by $\langle \cdot, \cdot \rangle_N$.

The conjugate operator acting on Y^N is defined in terms of Fourier series (see also page 11) by $\mathcal{C}_N(e^{ikt/N}) = -i\mathrm{sgn}(k)e^{ikt/N}$, $t \in \mathbb{R}$, and is a bounded linear operator from Y^N into itself. In particular, $\mathcal{C}_1$ is the natural operator on X^1 defined by $\mathcal{C}$ on H_0 when elements of H_0 are interpreted as defining elements of X^1, and *vice versa.*

Note that if $q_{\underline{s}}$ is obtained from the formula for q in Theorem 5.1 using $\mu = (\lambda^1_{\underline{s}})^{-1}$ and $w = w^1_{\underline{s}}$, then

$$\underline{q}(t) = q_{\underline{s}}(Nt),$$

and hence

$$\underline{\lambda q}(t) = N\lambda^1_{\underline{s}}q_{\underline{s}}(Nt), \quad t \in \mathbb{R}.$$

Now let

$$u(t) = v(t/N).$$

Then

$$\mathcal{C}_N u'(t) = \lambda^1_{\underline{s}}q_{\underline{s}}(t)u(t), \tag{5.21a}$$

$$\int_{-N\pi}^{N\pi} u(t)\sin t dt = 0, \tag{5.21b}$$

where $q_{\underline{s}}$ is a 2π-periodic even function of x and $u \in X^N$ is $2N\pi$-periodic. We need to demonstrate that no such non-zero u exists, for any N.

As already noted, for $|s|$ sufficiently small and positive, the linearised problem

$$\partial_w G(\lambda^1_s, w^1_s)w = \nu w, \quad w \in H_0,$$

has an eigenvalue 0 (the corresponding eigenfunction being the odd function $w^{1\prime}_s$) and an eigenvalue $\nu_s < 0$, with small absolute value, the eigenfunction of which is even. (Recall from the paragraph following (5.9) that there is one other negative eigenvalue close to -1 when s is small.) According to Theorem 5.1, therefore, the eigenvalue problems

$$\mathcal{C}u' - q_s u = \gamma u, \tag{5.22}$$

and (5.21a) with $N = 1$, have 0 as an eigenvalue and a negative eigenvalue close to 0 for all s sufficiently small. Now we will develop some spectral theory associated with (5.21a) in order to understand the possible bifurcations of water waves of very long period from the curve Γ_1.

At this stage we record that

$$q_s(t) = 1 + 2s\cos t + s^2\{1 + 4\cos 2t\} + s^3\{9\cos 3t + 6\cos t\} + O(|s|^4) \tag{5.23}$$

as $|s| \to 0$.

6 Bloch Functions for Conjugate Operators

The purpose of this section is to develop a theory of Bloch functions for conjugate operator equations analogous to the classical theory for Hill's equation [23, 40] and for the Schrödinger operator [54, 59].

Let $\tilde{L}^2$ denote the real linear space of square-integrable functions u on $(-\pi, \pi)$ with the property that $u(-t) = \overline{u(t)}$, $t \in (-\pi, \pi)$. For $\tau \in (-1, 1]$,

$$D_\tau = \{u \in \tilde{L}^2 : u \in H^1(-\pi, \pi) \text{ and } u(\pi) = e^{i\tau\pi} u(-\pi)\}.$$

Note that $\{\exp(ikt + i(\tau/2)t) : k \in \mathbb{Z}\}$ is a basis for $\tilde{L}^2$ and using it let a linear operator C_τ be defined on $\tilde{L}^2$ by

$$C_\tau \exp(ikt + i(\tau/2)t) = -i \operatorname{sgn}(k + (\tau/2)) \exp(ikt + i(\tau/2)t), \; k \in \mathbb{Z},$$

with the convention that $\operatorname{sgn}(0) = 0$. Note that when $\tau = 0$ the operators C_0 and $\mathcal{C}$ coincide. The operator $u \mapsto C_\tau u'$ is self-adjoint and densely defined with domain $D_\tau \in \tilde{L}^2$. For any real-analytic, 2π-periodic, real-valued, even function Q on $\mathbb{R}$, consider the corresponding eigenvalue problem

$$C_\tau u' + Qu = \gamma u \text{ on } \mathbb{R}, \; u \in D_\tau. \tag{$\mathcal{P}_\tau$}$$

(Note that since functions satisfying $u(-x) = \overline{u(x)}$ form a real, but not a complex, linear space, all real eigenfunctions are even and all imaginary eigenfunctions are odd.)

Suppose that $\tau \in (-1, 1]$ and $u \neq 0$ satisfy $(\mathcal{P}_\tau)$. Let

$$v(t) = \exp(-i\tau t/2)u(t). \tag{6.1}$$

Then v is 2π-periodic and hence

$$v(t) = \sum_{k=-\infty}^{\infty} a_k \exp(ikt)$$

where a_k is real because $\overline{v(-t)} = v(-t)$. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-imt)Q(t)v(t)dt$$
$$= \sum_{k\in\mathbb{Z}} \frac{a_k}{2\pi}\int_{-\pi}^{\pi} Q(t)\exp(i(k-m)t)dt = \sum_{k\in\mathbb{Z}} a_k b_{m-k}$$

where $Q(t) = \sum_{k\in\mathbb{Z}} b_k \exp(ikt)$, $b_{-k} = b_k$ and b_k is real (since Q is even and real). Therefore

$$Q(t)v(t) = \sum_{m\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_k b_{m-k}\right) \exp(imt)$$

and $(\mathcal{P}_\tau)$ may be formulated equivalently as a problem for the sequence of Fourier coefficients of v as follows:

$$\left|\frac{\tau}{2}+k\right|a_k+\sum_{m\in\mathbb{Z}}a_m b_{m-k}=\gamma a_k,\quad k\in\mathbb{Z}. \tag{6.2}$$

For each τ, this is an eigenvalue problem for the sequence $\{a_k\}\in l_2(\mathbb{Z},\mathbb{R})$, where the given sequence $\{b_k\}\in l_2(\mathbb{Z},\mathbb{R})$ is even in k. Clearly for each $\tau\in(-1,1]$ this is a self-adjoint eigenvalue problem on $l_2(\mathbb{Z},\mathbb{R})$, and the spectrum is a monotone non-decreasing sequence $\{\gamma_n^\tau\}_{n\in\mathbb{N}_0}$, each eigenvalue of which has finite multiplicity, with $\gamma_n^\tau\to\infty$ as $n\to\infty$. Suppose that each eigenvalue occurs in the sequence according to its multiplicity.

Clearly the corresponding eigenfunctions u of problem $(\mathcal{P}_\tau)$ have bounded extensions, $u(t)=\sum_{k\in\mathbb{Z}}a_k e^{i(k+\frac{\tau}{2})t}$, to $\mathbb{R}$ which satisfy, for all $t\in\mathbb{R}$, $u(t+2\pi)=e^{i\tau\pi}u(t)$, and hence u is periodic if τ is a rational number. For each $m\in\mathbb{N}$ with $-N<2m\le N$ let $\tau_m=2m/N$.

For fixed N and m, let $\{u_n^m\}_{n\in\mathbb{N}_0}$ be the orthonormal (in $\tilde{L}^2$) sequence of eigenfunctions of $(\mathcal{P}_{\tau_m})$, the corresponding eigenvalues being $\{\gamma_n^{2m/N}\}_{n\in\mathbb{N}_0}$. Note that every eigenfunction u of $\mathcal{P}_{\tau_m}$ with eigenvalue γ, when extended to $\mathbb{R}$, satisfies the self-adjoint eigenvalue problem

$$\mathcal{C}_N u'+Qu=\gamma u, \tag{6.3}$$

with the same eigenvalue. (This is because of the definitions of the operators $\mathcal{C}_N$ and C_{τ_m}.)

To settle the question described in Section 5.2, it will be necessary to discuss $2\pi N$-periodic functions. When necessary, elements of $\tilde{L}^2$ can be extended by periodicity and regarded as elements of Y^N, and similarly elements of Y^N can be regarded as periodic functions on $\mathbb{R}$.

Lemma 6.1. *If, for $k=1,2$, $n_k\in\mathbb{N}_0$, m_k satisfies $-N<2m_k\le N$ and $\gamma_{n_1}^{2m_1/N}\neq\gamma_{n_2}^{2m_2/N}$, then*

$$\langle u_{n_1}^{m_1},u_{n_2}^{m_2}\rangle_{Y^N}=0. \tag{6.4}$$

Moreover

$$Y^N=\overline{\operatorname{span}\{u_n^m:-N<2m\le N,\ n\in\mathbb{N}_0\}}. \tag{6.5}$$

Proof. Note that $u_{n_k}^{m_k}$ for $k=1,2$, are eigenfunctions, for distinct eigenvalues, of the problem (6.3); (6.4) follows immediately from standard theory of self-adjoint operators.

Let $\tilde{L}_\tau=\{u\in L^2(\mathbb{R}):u(t)=\exp(\tau\pi i)u(t+2\pi),\ t\in\mathbb{R}\}$. By standard theory of self-adjoint operators, $\{u_n^m\}_{n\in\mathbb{N}_0}$ is complete in $\tilde{L}_{2m/N}$. It suffices therefore to observe that if $v : \mathbb{R} \to \mathbb{C}$ is $2N\pi$-periodic then

$$\begin{aligned} v(t) &= \sum_{n\in\mathbb{Z}} A_n e^{\frac{int}{N}} \\ &= \sum_{\{m\in\mathbb{Z}: -N<2m\leq N\}} \sum_{n\in\mathbb{Z}} A_{(m+nN)} e^{\frac{i(m+nN)t}{N}} \\ &= \sum_{\{m\in\mathbb{Z}: -N<2m\leq N\}} v_m(t) \end{aligned}$$

where $v_m \in \tilde{L}_{2m/N}$. This completes the proof. □

6.1 Spectral Bands and Conjugate operators

Now we show how the spectra of these self-adjoint linear operators with periodic coefficients depend on a parameter $\tau \in (-1,1]$ in a way which is analogous to the theory of band spectra for Hill's equation in $L^2(\mathbb{R})$ or Schrödinger operators on periodic lattices in $L^2(\mathbb{R}^d)$. We begin with the standard definition.

Definition 6.2. *Define the n^{th} spectral band to be $I_n(Q)$ where*

$$\mathcal{I}_n(Q) = \{\gamma_n^\tau : \tau \in (-1,1]\}.$$

Since

$$|k| - \frac{1}{2} \leq \left|\frac{\tau}{2} + k\right| \leq |k| + \frac{1}{2} \quad \text{for } \tau \in (-1,1],$$

it is immediate that

$$\gamma_n^0 - \frac{1}{2} \leq \gamma_n^\tau \leq \gamma_n^0 + \frac{1}{2} \text{ for } \tau \in (-1,1] \text{ and all } n \in \mathbb{N}_0.$$

Then

$$\mathcal{I}_n(Q) \subset [\gamma_n^0 - 1/2, \gamma_n^0 + 1/2], \tag{6.6}$$

and therefore the points of $\mathcal{I}_n(Q)$ tend to $+\infty$ as $n \to \infty$, uniformly for $\tau \in (-1,1]$. Note also that, for $\tau \in (-1,1)$, $\gamma_n^{-\tau} = \gamma_n^\tau$ and the corresponding eigenvectors are complex conjugates of one another. Thus

$$\mathcal{I}_n(Q) = \{\gamma_n^\tau : \tau \in [0,1]\}. \tag{6.7}$$

Next we show that $\mathcal{I}_n(Q)$ is a closed interval, and not a single point, for each $n \in \mathbb{N}_0$.

Recall (6.7), let $S = (-3/2, 3/2) \times i\mathbb{R} \subset \mathbb{C}$ and for $\tau \in S$ consider the eigenvalue problem for $\{a_k\} \in l_2(\mathbb{Z}, \mathbb{C}) \setminus \{0\}$ given by

$$\left.\begin{array}{l} \frac{\tau}{2}a_k + ka_k + \sum_{m\in\mathbb{Z}} a_m b_{m-k} = \gamma a_k,\ k \geq 0, \\ -\frac{\tau}{2}a_k - ka_k + \sum_{m\in\mathbb{Z}} a_m b_{m-k} = \gamma a_k,\ k < 0. \end{array}\right\} \tag{6.8}$$

For real γ and $\tau \in [0,1]$, (6.2) and (6.8) coincide. Note that for $\tau \in S$ and $k \in \mathbb{Z}$,

$$\text{Imag } (1 \pm \frac{\tau}{2} \pm k) = \text{Imag } (\frac{\tau}{2}).$$

and hence, if $\{a_k\} \in l_2(\mathbb{Z}, \mathbb{C})$ satisfies (6.8) for some γ, then

$$\frac{1}{2}|\text{Imag}(\tau)| \|\{a_k\}\|_2 \leq |\gamma| \, \|\{a_k\}\|_2 + \|\{b_k\}\|_1 \|\{a_k\}\|_2$$

since $\{b_k\} \in l_1(\mathbb{Z}, \mathbb{R})$, because Q is smooth. In particular, $|\gamma| \to \infty$ as $|\text{Imag } (\tau)| \to \infty$.

Let $\tau_0 \in [0,1]$, $n \in \mathbb{N}_0$ and let $\gamma_n^{\tau_0}$ be an eigenvalue of (6.8) of multiplicity h. Then there are h power series with values in $\mathbb{C}$

$$\rho^{(1)}(\tau - \tau_0), \ldots, \rho^{(h)}(\tau - \tau_0),$$

and h power series with values in $l_2(\mathbb{Z}, \mathbb{C})$

$$\varrho^{(1)}(\tau - \tau_0), \ldots, \varrho^{(h)}(\tau - \tau_0),$$

convergent for τ in a neighbourhood of τ_0 with $\rho^{(j)}(0) = \gamma_n^{\tau_0}$, and $\varrho^{(j)}(0)$ is an eigenvector in $l_2(\mathbb{Z}, \mathbb{C})$ of (6.8) with $\gamma = \gamma_n^{\tau_0}$, such that there are no other solutions of (6.8) for τ near τ_0 and γ near $\gamma_n^{\tau_0}$. (See [51, page 74].) Suppose now that one of the power series for the eigenvalue is locally constant: say $\rho^{(1)}(\tau - \tau_0)$ is independent of τ in a real neighbourhood of τ_0, Then $\rho^{(1)}(\tau - \tau_0)$ is independent of τ in a complex neighbourhood of τ_0. Since (6.8) depends analytically on τ, [29, Theorem 1.18, Chapter XIII] applies to give that $\gamma_n^{\tau_0}$ is an eigenvalue of (6.8) for all $\tau \in S$. But this would mean that for all $\tau \in S$ (including those with large imaginary parts)

$$\frac{1}{2}|\text{Imag } (\tau)| \leq |\gamma_n^{\tau_0}| + \|\{b_k\}\|_1,$$

which is false. Hence, none of the functions $\rho^{(j)}(\tau - \tau_0)$ is locally constant. Therefore we have proved the following theorem. (Recall (6.6).)

Theorem 6.3. *For each $n \in \mathbb{N}_0$, the n^{th} band $\mathcal{I}_n(Q)$ is a non-trivial closed interval of length at most one, and $\min \mathcal{I}_n(Q) \to \infty$ as $n \to \infty$. For all $t \in \mathbb{R}$, the set $\{(n,\tau) \in \mathbb{N}_0 \times [0,1] : \gamma_n^\tau = t\}$ is finite.*

6.2 Spectral Bands when $Q \equiv -1$

Because of Theorem 5.1 we are interested in spectral bands when $Q = -q_s$. Recall from the last paragraph of Section 5.1 that q_s is obtained from q in Theorem 5.1 by substituting $w = w_s^1$ from (5.4), and note from (5.23) that $q_0 \equiv 1$.

We deal first with $Q = -q_0 \equiv -1$, in which case the eigenvalue problem (6.2) has the form

$$\left|\frac{\tau}{2} + k\right| a_k - a_k = \gamma a_k, \quad k \in \mathbb{Z}. \tag{6.9}$$

Clearly for $\tau \in (-1, 1]$, $\gamma = \left|\frac{\tau}{2} + k\right| - 1$ is an eigenvalue which is simple except when $\tau = 0, 1$, when it is double. There are no other eigenvalues. It follows that

$$\mathcal{I}_n(-1) = [-1 + n/2, -1 + (n+1)/2], \quad n \in \mathbb{N}_0.$$

The point $0 \in \mathcal{I}_1(-1) \cap \mathcal{I}_2(-1)$ is a double eigenvalue the eigenspace of which is $\text{span}\{\mathbf{e}_1, \mathbf{e}_{-1}\}$, where $\{\mathbf{e}_k\}_{k\in\mathbb{Z}}$ is the standard basis for $l_2(\mathbb{Z}, \mathbb{R})$ ($\mathbf{e}_k$ is the element whose k^{th} component is 1 and all other components are zero).

6.3 The Bands $\mathcal{I}_n(-q_s)$, $s \neq 0$, $n = 1, 2$.

Since, for $\tau \in [0, 1]$, problems (6.2) and (6.8) are equivalent and since (6.7) holds, it suffices to examine the spectrum of the self-adjoint operator which is densely defined on the Hilbert space $X = l_2(\mathbb{Z}, \mathbb{R})$ by the left-hand side of (6.8). Let its spectrum, for general Q, be given by the non-decreasing sequence $\{\sigma_n^\tau(Q)\}_{n\in\mathbb{N}_0}$. Therefore

$$\mathcal{I}_n(Q) = \{\sigma_n^\tau(Q) : \tau \in [0, 1]\}.$$

Our goal is to prove that for $|s| > 0$ sufficiently small, $\mathcal{I}_1(-q_s) \cap \mathcal{I}_2(-q_s) = \emptyset$ and that $0 \in \mathcal{I}_n(-q_s)$ implies that $n = 2$ and that $0 = \sigma_2^\tau(-q_s)$ only if $\tau = 0$. Since

$$\sigma_1^\tau(-1) = -\tau/2 \text{ and } \sigma_2^\tau(-1) = \tau/2, \;\; \tau \in [0, 1],$$

and for each n, $\sigma_n^\tau(-q_s)$ depends continuously on s and τ, it suffices, to show that there exists $\epsilon > 0$ such that 0 is not an eigenvalue of (6.2) with $|s| < \epsilon$ and $0 < \tau < \epsilon$.

When $Q = q_s$ let (6.8) be re-written as

$$(\tau A + D + B_s)x = \gamma x \tag{6.10}$$

where $x \in X$ (X is defined in the opening paragraph of this section). Here, for $k \in \mathbb{Z}$,

$$(Ax)_k = \begin{cases} \frac{1}{2}a_k, & k \geq 0, \\ -\frac{1}{2}a_k, & k < 0, \end{cases} \quad (Dx)_k = (|k| - 1)a_k, \text{ and } (B_s x)_k = \sum_{m\in\mathbb{Z}} a_m b^s_{m-k},$$

where $\{b^s_k\}_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of $1 - q_s$ and $x = \{a_k\}_{k\in\mathbb{Z}}$. Note in particular that $B_0 = 0$

Recall also that for any s sufficiently small, (6.10) with $\tau = 0$ has an eigenvalue $\gamma = 0$ the eigenvector of which is an odd sequence in X (corresponding to an odd eigenfunction of (5.22)), and a negative eigenvalue of order $-s^2$ the eigenvector of which is even sequence in X (corresponding to an even eigenfunction of (5.22)).

Let

$$E = \ker(D) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_{-1}\} \text{ and } F = E^{\perp} \subset X.$$

Let P be the orthogonal projection from X onto E and let $X_1 = \{x \in X : Dx \in X\}$, a Hilbert space endowed with the norm $\|x\|_1 = \{\|Dx\|^2 + \|Px\|^2\}^{\frac{1}{2}}$. Now A, D, and B_s are bounded linear operators from X_1 to X and the operator B_s depends smoothly on s for $|s|$ sufficiently small. Let $F_1 = F \cap X_1$ and note that $E \subset X_1$.

Following the Lyapunov-Schmidt procedure we now re-write (6.10) as an equivalent system of equations for $(e, f) \in E \times F_1$:

$$P(\tau A + D + B_s - \gamma I)(e + f) = 0, \tag{6.11a}$$

$$(I - P)(\tau A + D + B_s - \gamma I)(e + f) = 0. \tag{6.11b}$$

Notice that A, D and I are given by infinite diagonal matrices and therefore map F_1 into F and E into itself, and $D : F_1 \to F$ is a homeomorphism. Hence (6.11b) can be re-written as

$$\left[I + D^{-1}(\tau A - \gamma I + (I - P)B_s\right] f = D^{-1}(P - I)B_s(e),$$

for $e \in E$. Let $L(\tau, \gamma, s) = -D^{-1}(\tau A - \gamma I + (I - P)B_s)$, so that for $\tau \geq 0$, $|\gamma|$ and $|s|$ sufficiently small

$$f = \sum_{k \in \mathbb{N}_0} L(\tau, \gamma, s)^k D^{-1}(P - I)B_s(e) = \mathcal{L}(\tau, \gamma, s)(e), \text{ say.}$$

Now (6.11a) can be re-written as

$$(\tau A - \gamma I)e + PB_s(e + \mathcal{L}(\tau, \gamma, s)(e)) = 0,$$

a problem in the two-dimensional space E, and the eigenvalues γ satisfy the equation

$$\mathcal{D}_s(\tau, \gamma) = \operatorname{Det}\{(\tau A - \gamma I) + PB_s(I + \mathcal{L}(\tau, \gamma, s)\} = 0. \tag{6.12}$$

For each fixed s sufficiently small, (6.12) is an equation for two real unknowns, τ and γ, in a neighbourhood of zero. When $\tau = 0$ there are two solutions $\gamma = 0$ and $\gamma = \gamma_-^s$, say. (γ_-^s corresponds, via Lemma 5.1, to ν_s is (5.9).) Each is a simple eigenvalue of $D + B_s$, and so, by the implicit function theorem, each lies on a curve of solutions giving γ as a function of τ in a neighbourhood of zero. Let the curve through 0 at $\tau = 0$ be given by $\gamma = \underline{\gamma}^s(\tau)$ locally, and let $\underline{x}^s(\tau)$ denote a corresponding smooth parametrised family of eigenvectors of (6.10) so that, for fixed s sufficiently small and τ in a neighbourhood of zero,

$$(\tau A + D + B_s)\underline{x}^s(\tau) = \underline{\gamma}^s(\tau)\underline{x}^s(\tau).$$

Differentiation of this identity at $\tau = 0$ where $\underline{\gamma}^s(0) = 0$ gives

$$A\underline{x}^s(0) + (D + B_s)\underline{\dot{x}}^s(0) = \dot{\underline{\gamma}}^s(0)\underline{x}^s(0) \tag{6.13}$$

and, after taking the inner product of (6.13) with $\underline{x}^s(0)$,

$$\langle A\underline{x}^s(0), \underline{x}^s(0)\rangle = \dot{\underline{\gamma}}^s(0)\langle \underline{x}^s(0), \underline{x}^s(0)\rangle,$$

since $(D + B_s)\underline{x}^s(0) = 0$. Now recall that $x^s(0)$ is an odd sequence, and so

$$\langle A\underline{x}^s(0), \underline{x}^s(0)\rangle = 0$$

because of the definition of $\{a_k\}$. Therefore $\dot{\underline{\gamma}}^s(0) = 0$.

Remark. We note in passing that if $\overline{\gamma}(\tau)$ is the curve of eigenvalues with eigenvector the even sequence $\overline{y}^s$ at $\tau = 0$ and $\overline{\gamma}(0) = -O(s^2) < 0$ (corresponding to the even eigenfunction v_s with eigenvalue ν_s in (5.10)), an identical argument gives that

$$\langle A\overline{y}^s(0), \overline{y}^s(0)\rangle = \dot{\overline{\gamma}}^s(0)\langle \overline{y}^s(0), \overline{y}^s(0)\rangle.$$

Now $(\overline{y}_s)_0$ is non-zero for s sufficiently small, by (5.10), and therefore $\dot{\overline{\gamma}}(0) > 0$ since $a_0 > 0$. This is not important in the present argument, but shows that some elements of $\mathcal{I}_1(-q_s)$ correspond to two different values of τ. □

Since $(\tau, \underline{\gamma}^s(\tau))$ is a solution of the equation $\mathcal{D}_s(\tau, \gamma) = 0$, a differentiation with respect to τ at $\tau = 0$ yields that $\frac{\partial}{\partial \tau}\mathcal{D}_s(0,0) = 0$, since $\dot{\underline{\gamma}}^s(0) = 0$. Now suppose that there are positive sequences $\{s_n\}$ and $\{\tau_n\}$ converging to zero such that $\mathcal{D}_{s_n}(\tau_n, 0) = 0$. Then, since $\mathcal{D}_{s_n}(0,0) = 0$, there exists $t_n \in (0, \tau_n)$ such that $\frac{\partial}{\partial \tau}\mathcal{D}_{s_n}(t_n, 0) = 0$, by the mean-value theorem. Again by the mean-value theorem there exists $\hat{\tau}_n \in (0, t_n)$ such that $\frac{\partial^2}{\partial \tau^2}\mathcal{D}_{s_n}(\hat{\tau}_n, 0) = 0$. Therefore, taking the limit as $n \to \infty$, we find that

$$\frac{\partial^2}{\partial \tau^2}\mathcal{D}_0(0,0) = 0.$$

However this is false, as $\mathcal{D}_0(\tau, \gamma) = \gamma^2 - (\frac{\tau}{2})^2$ (see (6.12)). We conclude that there exists $\epsilon > 0$ such that (6.12) has no solutions of the form $(\tau, 0)$, $0 < \tau < \epsilon$ when $|s| < \epsilon$.

Combining these observations in a neighbourhood of $\tau = \gamma = s = 0$ yields the required result: for $|s| > 0$ sufficiently small,

(i) $\mathcal{I}_1(-q_s) \cap \mathcal{I}_2(-q_s) = \emptyset$;

(ii) $0 \in \mathcal{I}_n(-q_s)$ implies that $n = 2$;

(iii) $0 = \sigma_2^\tau(-q_s)$ only if $\tau = 0$.

To complete the picture, note that limiting argument similar to the one just given shows that there exists $\epsilon > 0$ such that (i) if $|s| < \epsilon$ and $|\gamma| < \epsilon$ are fixed, then there are at most two solutions τ of (6.12) with $|\tau| < \epsilon$, and (ii) if $|s| < \epsilon$ and $|\tau| < \epsilon$ are fixed then there are at most two solutions γ of (6.12) with $|\gamma| < \epsilon$.

6.4 Proof of Theorem 5.2

Suppose that a solution u_* of (5.21) exists. Then an expansion of u_* using the orthogonal basis of Y^N given by Lemma 6.1 yields that u_* is a linear combination of eigenfunctions u_n^m, $-N < 2m \leq N$ with eigenvalue $\gamma = 0$ when $Q = -q_{\underline{s}}$. However we have already shown that for $|\underline{s}|$ sufficiently small this implied that $\tau_m = 0$. Since the only eigenvalue when $\tau = \gamma = 0$ is $\sin x$, the constraint (5.21b) cannot be satisfied when $|s|$ is sufficiently small. This contradiction means that Theorem 5.2 has been proved. □

Remark. While this theorem excludes the possibility of certain secondary bifurcation points on branches of Stokes waves, it leaves open the question of whether there exist Stokes waves which are small in some sense (either their amplitudes or the maximum angle of inclination of their free surfaces to the horizontal might be small) which do not arise as secondary bifurcations. (Had there been a satisfactory centre manifold theory for Stokes waves on infinitely deep flows, then this question could be tackled directly.) □

7 Global Bifurcation and Secondary Bifurcation

In this Section we develop a version of global bifurcation theory for real-analytic nonlinear eigenvalue problems in real Banach spaces. This is a more specialised theory of global bifurcation than that of Rabinowitz [50], which relies on rather general topological degree theoretical arguments. (See [21, 30, 61] for developments of that theory in the Stokes wave problem.) Instead, pursuing a line initiated by Dancer [20, 22], the following discussion depends on a real-analytic, finite-dimensional reduction procedure, followed by a detailed application of the theory of germs of real-analytic varieties. For convenience, we have collected in an appendix some results in, and references for, this theory.

Once the global results have been formulated in Section 7.1, we add an additional ingredient, gradient structure, to obtain abstract results about global secondary bifurcation in Section 7.2.

7.1 Global Analytic Bifurcation Theory

Here we study globally some real-analytic features of the solution set of a nonlinear eigenvalue problem $F(\lambda, x) = 0$ in Banach spaces. The setting is the following.

Let X and Y be real Banach spaces, $F : \mathbb{R} \times X \to Y$ a real-analytic function and U a subset of $\mathbb{R} \times X$ such that

(C1) the set S defined by

$$\begin{aligned} S = \{(\lambda, x) \in U :\ & F(\lambda, x) = 0 \text{ and the partial Fréchet derivative} \\ & \partial_x F(\lambda, x) : X \to Y \text{ is invertible with bounded inverse}\} \end{aligned}$$

is not empty and

(C2) the set $\{(\lambda, x) \in U : F(\lambda, x) = 0\}$ is open relative to the set $\{(\lambda, x) \in \mathbb{R} \times X : F(\lambda, x) = 0\}$.

We also suppose that locally the problem can be reduced to a finite dimensional setting by assuming that

for every $(\lambda_0, x_0) \in U$ such that $F(\lambda_0, x_0) = 0$, there exist $n \in \mathbb{N}_0$, an embedded n-dimensional real-analytic submanifold $\Gamma \subset \mathbb{R} \times X$ containing (λ_0, x_0) and real-analytic functions $f, g : \Gamma \to \mathbb{R}$, such that

(C3) if (λ, x) is close enough to (λ_0, x_0) in $\mathbb{R} \times X$ and $F(\lambda, x) = 0$, then $(\lambda, x) \in \Gamma$ and $f(\lambda, x) = 0$;

(C4) if $(\lambda, x) \in \Gamma$ and $f(\lambda, x) = 0$, then $F(\lambda, x) = 0$;

(C5) if $(\lambda, x) \in \Gamma$ and $f(\lambda, x) = 0$, then $(\lambda, x) \in S$ is equivalent to $g(\lambda, x) \neq 0$.

Remark. In real-analytic bifurcation problems, such as the water-wave problem, the above hypotheses may be verified using a Lyapunov-Schmidt reduction (see for example [63]) as follows.

For all $(\lambda_0, x_0) \in U$, suppose that the partial Fréchet derivative, $L := \partial_x F(\lambda_0, x_0)$ is a Fredholm operator of index 0. (This means that the range $R(L)$ of L is closed in Y and its codimension, $n \in \mathbb{N}_0$, is finite and equal to the dimension of the null space $N(L)$ of L.) Without loss of generality, let $x_0 = 0$. The problem $F(\lambda, x) = 0$ is then equivalent to the system

$$z = Px, \; w = (I - P)x, \tag{7.1a}$$

$$QF(\lambda, z + w) = 0, \tag{7.1b}$$

$$(I - Q)F(\lambda, z + w) = 0, \tag{7.1c}$$

where P and Q are any bounded projections onto $N(L)$ and $R(L)$, respectively.

By the analytic implicit function theorem (see Appendix), the solutions $(z, w) \in N(L) \times (I - P)X$ of (7.1b) are locally, in a neighbourhood of $(\lambda_0, 0)$, of the form $w = \psi(\lambda, z)$ with $\psi(\lambda_0, 0) = 0$, where ψ is a real-analytic function defined in some neighbourhood V of $(\lambda_0, 0)$ in $\mathbb{R} \times N(L)$. Hypotheses (C3) and (C4) are now fulfilled with

$$\Gamma = \{(\lambda, z + \psi(\lambda, z)) : (\lambda, z) \in V\}$$

and $f(\lambda, x) = ||(I - Q)F(\lambda, x)||^2$ for $(\lambda, x) \in \Gamma$.

Observe that if $(\lambda, x) \in \Gamma$, $F(\lambda, x) = 0$ and $(u, v) \in N(L) \times (I - P)X$ satisfy $\partial_x F(\lambda, x)(u + v) = 0$, then, by differentiating (7.1) and setting $z = Px$, we obtain $v = \partial_z \psi(\lambda, z)u$ and

$$(I - Q)\partial_x F(\lambda, z + \psi(\lambda, z))(u + \partial_z \psi(\lambda, z)u) = 0.$$

Therefore defining $g(\lambda, x)$ when $x = z + \psi(\lambda, z)$ to be the determinant of the matrix $\partial_z (I - Q)F(\lambda, z + \psi(\lambda, z))$ guarantees hypothesis (C5). $\square$

Definition 7.1. *A maximal path-connected subset of S will be referred to as a distinguished arc (or d-arc).*

By the analytic implicit function theorem every d-arc $\mathcal{A}$ has the form

$$(\lambda, x) \in \mathcal{A} \text{ if and only if } \lambda_{\min} < \lambda < \lambda_{\max} \text{ and } x = h(\lambda)$$

for some real-analytic function $h : (\lambda_{\min}, \lambda_{\max}) \to X$ where $-\infty \leq \lambda_{\min} < \lambda_{\max} \leq +\infty$.

If $\lim_{\lambda \searrow \lambda_{\min}} h(\lambda) = x_{\min}$ with $(\lambda_{\min}, x_{\min}) \in U$ and $\lambda_{\min} > -\infty$, then $(\lambda_{\min}, x_{\min}) \in (\bar{S} \cap U)\backslash S$ will be called the *left-end of $\mathcal{A}$ in U*. Similarly, if it exists, a *right-end of $\mathcal{A}$ in U* is defined. We refer to these as *end-points of $\mathcal{A}$ in U*. If $\mathcal{A}$ has at most one end-point in U, it will be said to be *unlimited in U*. (Note that an unlimited d-arc may be bounded, even in finite dimensions.)

If $\lambda_n \searrow \lambda_{\min} > -\infty$, $\mu_n \searrow \lambda_{\min} > -\infty$, and the limits $\lim_{n\to\infty}(\lambda_n, h(\lambda_n))$ and $\lim_{n\to\infty}(\mu_n, h(\mu_n))$ exist in U and are distinct, then $\mathcal{A}$ has no left-end.

Theorem 7.2. *Under hypotheses (C1) to (C5), the following conclusions hold.*

(a) Every $(\lambda_0, x_0) \in (\bar{S} \cap U)\backslash S$ is an end-point in U of a finite, even number of d-arcs, and every $\xi \in S$ sufficiently close to (λ_0, x_0) belongs to one of them. These d-arcs can be pairwise associated in such a way that, if $\mathcal{A}_1$ and $\mathcal{A}_2$ denote any two associated d-arcs, the union $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{(\lambda_0, x_0)\}$ is a one-dimensional topological manifold that admits a real-analytic parametrisation (whose derivative may vanish at one point).

(b) Every $(\lambda_0, x_0) \in (\bar{S} \cap U)\backslash S$ is isolated in $\bar{S}\backslash S$.

(c) If $\mathcal{B}$ is a topological component of $\bar{S} \cap U$ such that $\mathcal{B}\backslash S$ is finite and non-empty, then $\mathcal{B}$ contains a finite, even (possibly 0) number of unlimited d-arcs in U.

Proof. Let $(\lambda_0, x_0) \in (\bar{S} \cap U)\backslash S$ and let $n \in \mathbb{N}$, Γ, f, g be given by hypotheses (C3)-C(5). Without loss of generality, we identify Γ with $\mathbb{R}^n$ and, in keeping with the notation in the Appendix, introduce the abbreviation $(\lambda_0, x_0) = a \in \mathbb{R}^n$.

Let $\alpha, \beta \in \mathcal{V}_a(\mathbb{R}^n)$ be the germs of $A = \mathrm{var}(\mathbb{R}^n, \{f\})$ and $B = \mathrm{var}(\mathbb{R}^n, \{f, g\})$, respectively. Note that $\gamma_a(A\backslash B) \neq \emptyset$ because $(\lambda_0, x_0) \in \bar{S}$, and that $A\backslash B$ consists of 1-regular points of A.

An application of Proposition 9.4 to $M = A\backslash B$ yields the existence of a 1-dimensional germ $\mu \in \mathcal{V}_a(\mathbb{R}^n)$ with the property that $\alpha\backslash\beta \subset \mu \subset \alpha$.

Let $\mu_1, \ldots, \mu_\nu$, $\nu \in \mathbb{N}$, be the irreducible components of μ (Proposition 9.2(c)). Then, by Proposition 9.2(h), $\dim \mu_j = 1$ and $\mu_k \cap \mu_l = \{a\}$ for all $1 \leq j, k, l \leq \nu$ with $k \neq l$ (Corollary 9.3(d)). Moreover, for all $j \in \{1, \ldots, \nu\}$, $\mu_j \not\subset \beta$, otherwise $\alpha\backslash\beta \subset \cup_{k\neq j}\mu_k \subset \alpha$, which would contradict the minimality of μ given by Proposition 9.4. Thus $\mu_j \cap \beta = \{a\}$ (Corollary 9.3(d)) and $\mu_j\backslash\{a\} \subset \alpha\backslash\beta$ $(j = 1, \ldots, \nu)$. We conclude that $\mu\backslash\{a\} = \alpha\backslash\beta$.

Applying Proposition 9.5 and its corollary to each μ_j, we deduce that $\alpha\backslash\beta$ admits a representative W that has a finite and even number of topological components, each being a 1-dimensional real-analytic submanifold, and that a is isolated in $\overline{W}\backslash W$. This proves parts (a) and (b) of the theorem.

(c) For $b \in \mathcal{B}\backslash S$, the total number of d-arcs of $\mathcal{B}$ with b as an end-point is even. Summing over $b \in \mathcal{B}\backslash S$, the total number of d-arcs so counted is even. Since limited d-arcs are counted twice, we deduce that the number of *unlimited* d-arcs is even. $\square$

Remarks.

(i) The first assertion in Theorem 7.2 does not attempt to describe, even in a neighbourhood of $(\lambda_0, x_0) \in U$, the set of all solutions of the equation $F(\lambda, x) = 0$. For example, there may be submanifolds of solutions of dimension ≥ 2 that contain (λ_0, x_0). The theorem is only concerned with describing those solutions which lie on one-dimensional manifolds of solutions in a neighbourhood of (λ_0, x_0).

(ii) The result does not ensure global smoothness of the solution set which it describes. A simple example to illustrate this is given by $X = Y = \mathbb{R}$, $U = \mathbb{R}^2$, $F(\lambda, x) = \lambda^3 - x^2$ and $(\lambda_0, x_0) = (0, 0)$. The set of solutions of $F = 0$ is the *cusp* with real-analytic parametrisation $\{\lambda(s) = s^2,\ x(s) = s^3,\ s \in \mathbb{R}\}$.

$\square$

From now on, we shall work under the following additional conditions.

There exist $\bar{\lambda} \in \mathbb{R}$ and a function $\nu : U \to [0, \infty)$ with the following properties:

(C6) every subset of $\bar{S} \cap U$ on which ν is bounded is relatively compact in $\mathbb{R} \times X$;

(C7) $(\bar{\lambda}, 0) \notin U$ and $(\bar{\lambda}, 0)$ is in the closure of some d-arc $\mathcal{A}_0$; note that $\mathcal{A}_0$ is unlimited in U since $(\bar{\lambda}, 0) \notin U$;

(C8) if the sequence $(\xi_n) \subset \bar{S} \cap U$ is such that $\xi_n \to \xi \notin U$ and $\sup \nu(\xi_n) < \infty$, then $\xi = (\bar{\lambda}, 0)$ and $\xi_n \in \mathcal{A}_0 \subset S$ for all n sufficiently large.

Remarks. In the water-wave problem, $F(\lambda, 0) = 0$ for every $\lambda \in \mathbb{R}$ and a subset of $(0, \infty) \times (K\backslash\{0\})$ for some cone $K \subset X$ (see [61]) will play the role of U. (The choice of U is important when checking condition (C8).) The existence of the d-arc $\mathcal{A}_0$ is established using the theory of bifurcation from a simple eigenvalue (see Section 5.1).

The function ν will be the so-called Nekrasov parameter in the water-wave problem, where the limiting cases $\nu = \infty$ corresponds to waves of extreme form, with stagnation points at some crests. $\square$

Theorem 7.3. *Under hypotheses (C1) to (C8), ν is unbounded on the following sets:*

(i) all unlimited d-arcs other than $\mathcal{A}_0$;

(ii) $\mathcal{A}_0$ *if* $\mathcal{A}_0$ *has no end-points;*

(iii) all infinite subsets of $(\bar{S} \cap U) \backslash S$*;*

(iv) on the topological component $\mathcal{B}$ *of* $\bar{S} \cap U$ *which contains* $\mathcal{A}_0$.

Proof. We argue by contradiction in each case.

(i) Let $\mathcal{A} \neq \mathcal{A}_0$ be a d-arc on which ν is bounded. From (C6), (C8) and the first part of Theorem 7.2, we deduce that $\mathcal{A}$ has both left- and right-ends.

(ii) If ν is bounded on $\mathcal{A}_0$, then (C6), (C8) and the first part of Theorem 7.2 again ensure the existence of either a left-end or a right-end.

(iii) Let D be an infinite subset of $(\bar{S} \cap U) \backslash S$ on which ν is bounded. By (C6), D therefore contains a non-constant convergent sequence whose limit is not only in U (by (C8)) but also in $(\bar{S} \cap U) \backslash S$ (since S is open in $\bar{S}$), in contradiction with Theorem 7.2(b).

(iv) Suppose that $\sup_{\mathcal{B}} \nu < \infty$. By (iii), $\mathcal{B} \backslash S$ is finite. If $\mathcal{B} \backslash S \neq \emptyset$, Theorem 7.2(c) asserts that $\mathcal{B} \backslash S$ contains an even number of unlimited d-arcs in U. Therefore there exists an unlimited d-arc $\mathcal{A}_1 \neq \mathcal{A}_0$. Hence, from (i), $\sup_{\mathcal{A}_1} \nu = \infty$. If $\mathcal{B} \backslash S = \emptyset$, then $\mathcal{B} = \mathcal{A}_0$ and we also conclude, from (ii), that $\sup_{\mathcal{A}_0} \nu = \infty$. □

Theorem 7.4. *Suppose hypotheses (C1)-(C8) hold. Then there exist continuous functions* $h_1 : (0, \infty) \to \mathbb{R}$ *and* $h_2 : (0, \infty) \to X$ *such that if* $h = (h_1, h_2)$ *then*

(i) $h : (0, \infty) \to \bar{S} \cap U$*;*

(ii) $h((0,1)) \subset \mathcal{A}_0$ *and* $\lim_{s \to 0} h(s) = (\bar{\lambda}, 0)$*;*

(iii) $\lim_{s \to \infty} \nu(h(s)) = \infty$ *and* h *is injective on* $h^{-1}(S)$*;*

(iv) at all points $s \in h^{-1}(S)$*,* h *is real-analytic with* $h_1'(s) \neq 0$*;*

(v) the set $h^{-1}(\bar{S} \backslash S) \subset (0, \infty)$ *consists of isolated values;*

(vi) locally near every point $s_0 \in h^{-1}(\bar{S} \backslash S)$*, there exists an injective and continuous re-parametrisation of the set originally parametrised by* h*, such that* $s = \gamma(\sigma)$*,* $|\sigma| \leq 1$ *(*σ *is the new parameter),* $s_0 = \gamma(0)$ *and* $h \circ \gamma$ *is a real-analytic function (whose derivative may vanish only at* 0*).*

Proof. We consider $\bar{S} \cap U$ as a non-oriented graph whose vertices are the points in $(\bar{S} \cap U) \backslash S$ and whose edges are the d-arcs of S. By an *admissible trail* of length $N \in \mathbb{N} \cup \{\infty\}$, we mean a sequence $(\mathcal{A}_n : 0 \leq n < N)$ of distinct d-arcs such that $\mathcal{A}_0$ is the d-arc of condition (C7) and two consecutive d-arcs are adjacent to a common vertex and associated in the sense of Theorem 7.2. It is easily proved that there exists a unique maximal admissible trail.

If its length N is finite, then $\mathcal{A}_{N-1}$ is unlimited in U and, by Theorem 7.3, ν is unbounded on $\mathcal{A}_{N-1}$. If $N = \infty$, then the maximal admissible trail must have infinitely many vertices (by Theorem 7.2) and so ν is unbounded on the set of adjacent vertices (Theorem 7.3). Theorem 7.2(a) prevents the possibility of a maximal admissible trail of finite length stopping at a limited d-arc. An appropriate parametrisation $h : (0, \infty) \to \mathbb{R} \times X$ of the union of a maximal admissible trail and the set of its adjacent vertices satisfies the conclusions of the theorem. □

Remark. Caution in line with the previous remark is necessary when interpreting the maximal admissible trail in this theorem. In particular, h is *not* required to be globally injective, and a maximal admissible trail can have self-intersection points as well as cusps, while at the same time enjoying a local real-analytic parametrisation. □

Definition 7.5. *Let $h = (h_1, h_2)$ be given by Theorem 7.4. If, for some $\epsilon > 0$, every neighbourhood of $h(s)$ in $\mathbb{R} \times X$ contains a solution of $F = 0$ that is not in $h([s-\epsilon, s+\epsilon])$, then $h(s)$ is called a bifurcation point.*

For $s > 0$, $h(s)$ is called a turning point if $h(s)$ is not a bifurcation point and h_1 is not injective in a neighbourhood of s. (Clearly, in both cases, $h(s) \in \bar{S} \backslash S$.)

We now look for hypotheses which ensure the existence of turning points and bifurcation points on the maximal admissible trail of Theorem 7.4.

7.2 Gradient Structure and Secondary Bifurcation

Bifurcation theory for gradient operators has been thoroughly investigated (for example, [33, page 332] and [16, 31, 49, 41]). Here we state a corollary of a theorem of Kielhöfer [31], and apply it to the global bifurcation theory of the previous section.

Let X be a real Banach space and Y a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ such that the set X is a dense subspace of the Hilbert space Y and the injection of X into Y is continuous from the Banach space into the Hilbert space. Suppose that

(G1) F and U satisfy (C1) and (C2) and there exists a real-valued real-analytic functional $\mathcal{J} : \mathbb{R} \times X \to \mathbb{R}$ such that

$$\partial_x \mathcal{J}(\lambda, x)u = < F(\lambda, x), u > \text{ for all } \lambda \in \mathbb{R}, \quad x \in X \text{ and } u \in X.$$

As a consequence, for all $(\lambda, x) \in \mathbb{R} \times X$, $\partial_x F(\lambda, x) : X \to Y$ is a symmetric operator and therefore its eigenvalues are real.

(G2) for all $(\lambda, x) \in U$, $\partial_x F(\lambda, x)$ is a Fredholm operator of index 0 whose spectrum intersects $(-\infty, 0]$ at a finite number of isolated eigenvalues of finite multiplicity and that, for (λ, x) in compact subsets of $\bar{S} \cap U$, the spectra of $\partial_x F(\lambda, x)$ are uniformly bounded below.

Definition 7.6. *If (G1) and (G2) hold, for $(\lambda, x) \in U$ let $i(\lambda, x) \in \mathbb{N}_0$ denote the Morse index of $\mathcal{J}(\lambda, .)$ at x, which is the number of negative eigenvalues (counted according to multiplicity) of $\partial_x F(\lambda, x)$.*

The following result is an immediate corollary of [31].

Proposition 7.7. *Suppose (G1) and (G2) hold and let* $\tilde{h}_1 \in C^1([-1,1],\mathbb{R})$ *and* $\tilde{h}_2 \in C^1([-1,1],X)$ *be such that* $\tilde{h}_1 : [-1,1] \to \mathbb{R}$ *is injective and, when* $\tilde{h} = (\tilde{h}_1,\tilde{h}_2)$, $\tilde{h}(\sigma) \in U$, $F(\tilde{h}(\sigma)) = 0$ *for all* $|\sigma| \leq 1$ *and the partial Fréchet derivative,* $\partial_x F(\tilde{h}(\sigma))$, *is invertible for* $\sigma \neq 0$.

If, in addition, $i(\tilde{h}(1)) \neq i(\tilde{h}(-1))$, *then* $\tilde{h}(0)$ *is a bifurcation point, which means that every neighbourhood of* $\tilde{h}(0)$ *in* $\mathbb{R}\times X$ *contains a solution of* $F = 0$ *that is not in the range of* $\tilde{h}$.

Theorem 7.8. *Suppose that (C1)-(C8) and (G1)-(G2) hold. Let* $h = (h_1, h_2)$ *be given by Theorem 7.4. If* $\limsup_{s\to\infty} i(h(s)) = \infty$, *then there are infinitely many values* $s > 0$ *at which* $h(s)$ *is a turning point or a bifurcation point.*

Proof. If the number of turning points is finite, there are infinitely many values of $s > 0$ such that $h(s)$ is not a turning point and the index $i \circ h$ jumps at s. By Proposition 7.7, $h(s)$ is a bifurcation point. □

Remark. This result illustrates the advantage that global bifurcation theory based on real-analyticity has over its counterpart based on topological degree theory, when real-analytic problems are under consideration. The former allows a global curve to be parametrised beyond the first bifurcation point, whereas the latter does not. This limitation of the topological approach curtailed the effectiveness of Plotnikov's Lemma (Section 8.3) to showing the existence of *at least one* bifurcation point or turning point on a global continuum of solitary waves. The current methods applied to his problem would lead to an analogue of Theorem 8.2 for the solitary wave problem. □

8 Stokes Waves

In this section the abstract theory of Section 7 is used to obtain the global bifurcation and secondary bifurcation of symmetric Stokes waves. (Symmetry here means that w is even in (5.1), or that u is odd and v is even in (2.1).) The results of Section 6 are then used to obtain the existence of sub-harmonic symmetric Stokes-wave bifurcations.

Before this, we need to examine more closely the relation between our equation (5.1) and the water-wave problem described in Section 2.1. From this examination further properties of solutions of (5.1) can be extracted using existing water-wave theory [61, Section 5]. Also relevant here is the connection between (5.1) and Nekrasov's equation, in which the independent variable is the angle θ between the free surface, suitably parametrised, and the horizontal, which is the basis of much previous work on Stokes waves (see [61] for a survey).

8.1 Stokes Waves Preliminaries

Henceforth we regard (5.1) in the following setting, which is the restriction to even functions of the problem treated in Sections 3 and 5. To re-cap, in notation relevant to Section 7, the problem is

$$F(\lambda, w) = \mathcal{C}w' - \lambda\{w + w\mathcal{C}w' + \mathcal{C}(ww')\} = 0 \tag{8.1}$$

for $\lambda \in \mathbb{R}$ and $w \in X$, where X is the real Banach space of all 2π-periodic even functions that are in $H^1_{loc}(\mathbb{R})$. Let Y be the real Banach space of all 2π-periodic even functions in $L^2_{loc}(\mathbb{R})$, where the norms on X and Y are the usual H^1 and L^2 norms over a period $[-\pi, \pi]$, and the inner-product on Y is denoted by $\langle \cdot, \cdot \rangle$.

Let (λ, w) be a solution of (8.1) with $(2\lambda)^{-1} > w \not\equiv 0$. (Note that w is not constant.) We inferred in Sections 2.4 and 3 that w and $\mathcal{C}w$ are real-analytic functions which satisfy the identity

$$(1 - 2\lambda w)\{{w'}^2 + (1 + \mathcal{C}w')^2\} = 1 \text{ on } \mathbb{R}, \tag{8.2}$$

${w'}^2 + (1 + \mathcal{C}w')^2$ is nowhere zero and (8.8) is a parametrisation of the free boundary of a Stokes wave of period 2π (in dimensionless coordinates). If, in addition $w' \leq 0$ on $[0, \pi]$, it follows by a result of Spielvogel [55] that $\inf_{s\in\mathbb{R}}(1 + \mathcal{C}w'(s)) > 0$. From this it follows that the odd real-analytic function

$$\theta := -\arctan\{w'/(1 + \mathcal{C}w')\} \tag{8.3}$$

has $-\pi/2 < \theta < \pi/2$ and satisfies Nekrasov's equation

$$\theta(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{|\sin((t+s)/2)|}{|\sin((t-s)/2)|}\frac{\sin\theta(t)}{3\left\{\nu^{-1} + \int_0^t \sin\theta(s)ds\right\}},\ t \in [-\pi, \pi]. \tag{8.4}$$

(See [61, Sections 5 and 6] for a proof of Spielvogel's result and the subsequent derivation of Nekrasov's equation.)

In this formulation the parameter ν, which we call Nekrasov's parameter, is given in terms of the variables in (2.2) with $\sigma = 0$ by $\nu = 3g\Lambda c/\pi|\nabla\phi(0)|^3$ (see [61, Section 6]), and it is clear from (8.3) that θ is the angle of inclination of the free surface to the horizontal. Therefore, since the parameter λ in (8.1) is given in terms of μ in (2.22) with $\sigma = 0$ by $\lambda = \mu^{-1} = (\pi c^2/\Lambda g)^{-1}$ (see (2.10)), we find that

$$\begin{aligned} \nu &= \frac{3g\Lambda c}{\pi|\nabla\phi(0)|^3} = \frac{3g\Lambda}{\pi c^2}\ \frac{c^3}{|\nabla\phi(0)|^3} = 3\lambda\ \frac{c^3}{|\nabla\phi(0)|^3} \\ &= 3\lambda\left(w'(0)^2 + (1 + \mathcal{C}w'(0))^2\right)^{3/2}\ \text{(see (2.20))} \\ &= 3\lambda(1 - 2\lambda w(0))^{-3/2}\ \text{ by (8.2).} \end{aligned}$$

Let

$$\nu(\lambda, w) = 3\lambda\eta(\lambda, w) \text{ where } \eta(\lambda, w) = (1 - 2w(0))^{-3/2}. \tag{8.5}$$

That $\nu(\lambda, w) \geq 3$ follows as in [60] by multiplying equation (8.4) by $\sin t$ and integrating, using the positivity of the odd functions θ, $\sin\theta$ and $\sin t$ on $(0, \pi)$.

Also the value of λ in (8.1) can be recovered from w and the value of $\nu(\lambda, w)$ by the formula [4, formula (A2), page 665]

$$(3\lambda)^{1/3} = \frac{1}{\pi}\int_0^\pi \frac{\cos\theta(t)}{\left\{\nu(\lambda, w)^{-1} + \int_0^t \sin\theta(s)ds\right\}^{1/3}} dt < \nu(\lambda, w)^{1/3}. \tag{8.6}$$

Hence $\eta(\lambda, w) > 1$, $w(0) > 0$ and

$$\lambda \geq \min\left\{\frac{1}{4w(0)}, \frac{\nu(\lambda, w)}{3} 2^{-3/2}\right\} \geq \min\left\{\frac{1}{4w(0)}, 2^{-3/2}\right\}. \tag{8.7}$$

8.2 Global Analytic Bifurcation of Stokes Waves

We now check the hypotheses of Section 7.

From the boundedness and linearity of the conjugate operator $\mathcal{C}$, it follows that $F : \mathbb{R} \times X \to Y$ is real-analytic.

Let $U \subset \mathbb{R} \times X$ be the set of all (λ, w) such that $\lambda > 0$, $w \neq 0$, w is non-increasing on $[0, \pi]$, $\max w = w(0) < (2\lambda)^{-1}$ and the map

$$s \mapsto (s + \mathcal{C}w(s), w(s)) \in \mathbb{R}^2,\ \ s \in \mathbb{R},\ \text{is injective}\ . \tag{8.8}$$

For $(\lambda, w) \in U$, let

$$H(\lambda, w)(t) = \frac{\sin\theta(t)}{3\left\{\nu(\lambda, w)^{-1} + \int_0^t \sin\theta(s)ds\right\}},\quad t \in [-\pi, \pi], \tag{8.9}$$

where $\nu(\lambda, w)$ and $\theta(t) \in (-\pi/2, \pi/2)$ for $t \in [-\pi, \pi]$, are given by (8.3) and (8.5), respectively.

First we note that $\{(\lambda, w) \in U : F(\lambda, w) = 0\}$ is relatively open in $\{(\lambda, w) \in \mathbb{R} \times X : F(\lambda, w) = 0\}$. To see this, note that with θ and $H(\lambda, \theta)$ given by (8.3) and (8.9), respectively, when $(\lambda, w) \in U$ is a solution of equation (8.1), $H(\lambda, w) \geq 0$ on $[0, \pi]$. Since (8.4) can be re-written in the form

$$\theta(t) = \frac{1}{\pi}\int_0^\pi \log\frac{|\sin((t+s)/2)|}{|\sin((t-s)/2)|} H(\lambda, w)ds \tag{8.10}$$

for $t \in [0, \pi]$, we deduce that θ and $H(\lambda, w)$ are strictly positive on $(0, \pi)$. Differentiating (8.10) at $t = 0$ and π, (recall that w, and hence θ, is smooth) gives

$$\theta'(0) = \frac{1}{\pi}\int_0^\pi \frac{H(\lambda, w)}{\tan(s/2)} ds > 0$$

and

$$\theta'(\pi) = -\frac{1}{\pi}\int_0^\pi H(\lambda, w)\tan(s/2)ds < 0.$$

Therefore, from (8.3), $w' < 0$ on $(0, \pi)$, $w''(0) < 0$ and $w''(\pi) > 0$.

Now let $\{(\lambda_n, w_n), n \in \mathbb{N}\}$ be a sequence of solutions of equation (8.1) in $\mathbb{R} \times X$ that converges to $(\lambda, w) \in U$. From the regularity theory for (8.1), we infer that $w_n \to w$ in $C^k(\mathbb{R})$ for all $k \in \mathbb{N}$ and hence that $(\lambda_n, w_n) \in U$ for large $n \in \mathbb{N}$.

For hypotheses (C3) to (C5) to hold, it is sufficient that $\partial_w F(\lambda, w)$ is a Fredholm operator of index 0 at every $(\lambda, w) \in U$. Observe that

$$F(\lambda, w) = (1 - 2\lambda w)(\mathcal{C}w' + w) + (2\pi)^{-1}\lambda\mathcal{G}(w') - (1 - 2\lambda w + \lambda)w, \qquad (8.11)$$

where (see (3.2))

$$\frac{1}{2\pi}\mathcal{G}(w') = w\mathcal{C}w' - \mathcal{C}(ww') = \frac{1}{8\pi}\int_{-\pi}^{\pi}\left(\frac{w(x) - w(y)}{\sin((x-y)/2)}\right)^2 dy$$

and $\mathcal{G}$ is sequentially continuous from the weak $L^2(-\pi, \pi)$-topology to the strong $L^2(-\pi, \pi)$-topology by Theorem 3.4. Hence $\partial_w F(\lambda, w)$ is a Fredholm operator of index 0 as the sum of an invertible linear operator and a compact operator.

Let us check condition (C6) with

$$S = \{(\lambda, w) \in U :\ F(\lambda, w) = 0, \partial_w F(\lambda, w) \text{ invertible with bounded inverse}\}.$$

Let D be a subset of $\bar{S} \cap U$ on which ν is bounded. For $(\lambda, w) \in D$, $w(0) > 0$ and $\int_{-\pi}^{\pi} w(t)dt \leq 0$ by integrating equation (8.1) on $(-\pi, \pi)$. Hence $w(t) = 0$ at some $t \in (0, \pi)$. The corresponding θ given by (8.3) is known to be uniformly bounded by $\pi/5$ [1] and therefore $w(0) \leq \pi$. This gives on D a positive lower-bound for λ and finite upper-bounds for $\eta(\lambda, w)$, $\max|w'|$ (see equation(8.2)) and $||w||_X$. The structure of F given by (8.11), the compactness of $\mathcal{G}$ and inequality (8.6) show that D is relatively compact in $\mathbb{R} \times X$.

A combination of local (Section 5.1) and global (Section 7) bifurcation theory for equation (8.1) gives that $\bar{\lambda} = 1$ is a value of λ at which a d-arc $\mathcal{A}_0 \subset U$ of solutions bifurcates from the line of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. This is what is required in Condition (C7).

Finally, let a sequence $\{(\lambda_n, w_n)\} \subset \bar{S} \cap U$ converge to $(\lambda, w) \notin U$ and $\sup \nu(\lambda_n, w_n) < \infty$. Clearly $F(\lambda, w) = 0$, w is non-increasing on $[0, \pi]$, $\lambda > 0$, $w(0) < (2\lambda)^{-1}$, $\inf_{s \in \mathbb{R}} \mathcal{C}w'(s) \geq -1$, ${w'}^2 + (1 + \mathcal{C}w')^2 > 0$ and the map (8.8) is injective. As $(\lambda, w) \notin U$, we conclude that $w = 0$. As $\bar{\lambda} = 1$ is the only positive value of λ at which $\mathcal{A}_0 \subset U$ can bifurcate from the line of trivial solutions $\{(\lambda, 0); \lambda \in \mathbb{R}\}$ of (8.1), , it follows that $\lambda = 1$ and $(\lambda_n, w_n) \in \mathcal{A}_0$ for all large n. In conclusion, Theorem 7.4 holds for the classical water-wave problem described by equation (8.1).

Let $\{(\lambda_n, w_n); n \in N\} \subset U$ be a sequence of solutions of equation (8.1) such that $\nu(\lambda_n, w_n) \to \infty$. Passing to a subsequence if necessary, the corresponding sequence of functions θ_n, defined in terms of (λ_n, θ_n) by (8.3), converge in C^k for all $k \in \mathbb{N}_0$ on every compact subinterval of $(0, \pi]$ to some even and periodic function θ that describes a wave of greatest height. An important feature is that θ is discontinuous at $s = 0$ and

$$\sup\{|\theta(s)| :\ |s| < \epsilon\} \geq \pi/6, \quad \text{for all } \epsilon > 0 \qquad (8.12)$$

(see [43, 60, 61]). Moreover θ is positive on $(0, \pi)$.

8.3 Turning Points and Bifurcation Points

The aim of this section is to show that the conclusion of Theorem 7.8 holds for the water-wave problem.

We have observed in Section 3 that the critical points with respect to $w \in X$ of the functional

$$\mathcal{J}(\lambda, w) = \frac{1}{2}\int_{-\pi}^{\pi} \{w\mathcal{C}w' - \lambda w^2(1 + \mathcal{C}w')\}dt$$

are smooth and satisfy equation (8.1).

First, recall from Theorem 5.1 that for all $(\lambda, w) \in U$ such that $F(\lambda, w) = 0$, there exists a linear homeomorphism $A(\lambda, w)$ on Y defined in terms of (λ, w) by (5.14), such that X is an invariant subspace for $A(\lambda, w)$ and

$$A(\lambda, w)^*\partial_w F(\lambda, w)A(\lambda, w)u = \mathcal{C}u' + Q(\lambda, w)u \; = T(\lambda, w)u, \text{ say,} \tag{8.13}$$

for all $u \in X$, where by (8.3)

$$\begin{aligned} Q(\lambda, w)(x) = &- \ \text{Imag}\frac{W'}{W} - \lambda|W|^2(1 + \mathcal{C}w') \\ &= \theta'(x) - \frac{\cos\theta(x)}{3\left\{\nu(\lambda, w)^{-1} + \int_0^x \sin\theta(s)ds\right\}}, \end{aligned} \tag{8.14}$$

by the definition of θ, since $\lambda^{-1}|W|^{-3} = 3\left\{\nu(\lambda, w)^{-1} + \int_0^x \sin\theta(s)ds\right\}$ (see [61, Section 6]).

From the fact that the Morse index $i(\lambda, w)$ of the critical point w of $\mathcal{J}(\lambda, \cdot)$ is the maximal dimension of all the linear subspaces $V \subset X$ such that the quadratic form $u \mapsto \langle u, \partial_x F(\lambda, w)u\rangle$ is negative definite, we infer that $i(\lambda, w)$ is the number of negative eigenvalues of $T(\lambda, w)$ (counted according to multiplicity). Let the non-decreasing sequence of eigenvalues of $T(\lambda, w)$ be denoted by $\{\alpha_n(\lambda, w); n \in \mathbb{N}\}$.

The next result is due to Plotnikov [49] in the case of solitary waves.

Plotnikov's Lemma. *If* $(\lambda_n, w_n) \in U$ *satisfies* $F(\lambda_n, w_n) = 0$ *for each* $n \in \mathbb{N}$ *and if* $\nu(\lambda_n, w_n) \to \infty$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} \alpha_k(\lambda_n, w_n) = -\infty$ *for all* $k \in \mathbb{N}$. *In particular* $\lim_{n\to\infty} i(\lambda_n, w_n) = \infty$.

Proof. For $k \in \mathbb{N}$, let $v_k(\lambda_n, w_n)$ be a normalised eigenvector of $T(\lambda_n, w_n)$ corresponding to the eigenvalue $\alpha_k(\lambda_n, w_n)$. For convenience, we adopt the convention $\alpha_0(\lambda_n, w_n) = -\infty$ and $v_0(\lambda_n, w_n) = 0$. For $\phi \in X$, $n \in \mathbb{N}$, $R \geq 0$

and $s \in \mathbb{N}_0$ such that $\alpha_s(\lambda_n, w_n) < -R \le \alpha_{s+1}(\lambda_n, w_n)$, we get

$$\begin{aligned}
\langle T(\lambda_n, w_n)\phi, \phi\rangle &\ge \sum_{j=1}^{s} \alpha_j(\lambda_n, w_n)\langle \phi, v_j(\lambda_n, w_n)\rangle^2 - R||\phi||_Y^2 \\
&= \sum_{j=1}^{s} \alpha_j(\lambda_n, w_n)^{-1}\langle \phi, T(\lambda_n, w_n)v_j(\lambda_n, w_n)\rangle^2 - R||\phi||_Y^2 \\
&\ge \sum_{j=1}^{s} \alpha_s(\lambda_n, w_n)^{-1}\langle T(\lambda_n, w_n)\phi, v_j(\lambda_n, w_n)\rangle^2 - R||\phi||_Y^2 \\
&\ge \alpha_s(\lambda_n, w_n)^{-1}||T(\lambda_n, w_n)\phi||_Y^2 - R||\phi||_Y^2.
\end{aligned}$$

Passing to a subsequence, we can assume that $\{\theta_n\}$ defined by formula (8.3) converges in the sense explained at the end of the previous section to $\theta \neq 0$, and that $\lim_{n\to\infty} \alpha_k(\lambda_n, w_n)$ exists in $[-\infty, +\infty]$ for all $k \in \mathbb{N}$.

Seeking a contradiction, suppose that for some $s \in \mathbb{N}_0$

$$\lim_{n\to\infty} \alpha_s(\lambda_n, w_n) = -\infty \text{ and } \lim_{n\to\infty} \alpha_{s+1}(\lambda_n, w_n) = r > -\infty.$$

Now, by choosing $R \ge 0$ so that $-R$ is smaller than r, and $\phi \in X$ so that

$$\sup\{||T(\lambda_n, w_n)\phi||_Y : n \in \mathbb{N}\} < \infty, \tag{8.15}$$

we can deduce that

$$\liminf_{n\to\infty}\langle T(\lambda_n, w_n)\phi, \phi\rangle \ge -R||\phi||_Y^2. \tag{8.16}$$

Note from (8.12) that (8.15) is satisfied if $\phi \in X$ vanishes on a neighbourhood of 0.

Let $g \in C^\infty(\mathbb{R}, [0,1])$ be even, have compact support in $[-1,1]$ and be equal to 1 on $[-1/2, 1/2]$. For $\epsilon \in (0, 1/4)$ and $|t| \le \pi$, set $g_\epsilon(t) = g(t/\epsilon)$, $\phi_\epsilon(t) = g(t) - g_\epsilon(t)$ and extend ϕ_ϵ on $\mathbb{R}$ by periodicity. Since ϕ_ϵ vanishes near 0, inequality (8.15) holds (with an upper-bound depending on ϵ), and so does property (8.16). Clearly

$$\begin{aligned}
&|\int_{-\pi}^{\pi} \theta_n'(t)\phi_\epsilon^2(t)dt| = 2|\int_{-\pi}^{\pi} \theta_n(t)\phi_\epsilon(t)\phi_\epsilon'(t)dt| \\
&\qquad \le 2\max\{|\theta_n|\}\int_{-\infty}^{+\infty} \left(|g'(t)| + \epsilon^{-1}|g'(t/\epsilon)|\right) dt \le \pi\int_{-\infty}^{+\infty} |g'(t)|dt.
\end{aligned}$$

From the inequalities $\sup \epsilon^{-1/2}||g_\epsilon||_{L^2(-\pi,\pi)} < \infty$, and $\sup \epsilon^{1/2}||g_\epsilon'||_{L^2(-\pi,\pi)} < \infty$, we infer that

$$\sup_{0<\epsilon<1/4} \int_{-\pi}^{\pi} \phi_\epsilon \mathcal{C}\phi_\epsilon' dt < \infty \tag{8.17}$$

and

$$\limsup_{n\to\infty}\langle T(\lambda_n, w_n)\phi_\epsilon, \phi_\epsilon\rangle \leq C - \liminf_{n\to\infty}\int_{-\pi}^{\pi}\frac{\cos\theta_n(t)\phi_\epsilon^2(t)}{3\left\{\nu(\lambda_n, w_n)^{-1} + \int_0^t \sin\theta_n(s)ds\right\}}dt$$
$$\leq C - \int_\epsilon^{1/2}\frac{\cos(\pi/4)}{3\int_0^t\sin(\pi/4)ds}dt = C - (1/3)\ln(2^{-1}/\epsilon)$$

for some constant $C > 0$ that is independent of ϵ. For small positive ϵ, this contradicts inequality (8.16), in which $R \geq 0$ is also independent of ϵ. □

Remark. The above estimates are somewhat sharper than at first glance might appear. To see this, in the proof let ψ_ϵ be the *odd* periodic function defined by $\psi_\epsilon = \phi_\epsilon$ on $[0, \pi]$. Recall from Section 3 that

$$\int_{-\pi}^{\pi}\psi_\epsilon\mathcal{C}\psi_\epsilon' dt = \int_{-\pi}^{\pi}\{\psi_\epsilon\mathcal{C}\psi_\epsilon' - \mathcal{C}(\psi_\epsilon\psi_\epsilon')\}dt = \frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{\psi_\epsilon(x)-\psi_\epsilon(y)}{\sin((x-y/2))}\right\}^2 dxdy.$$

Then, instead of crucial estimate (8.17), we get

$$\begin{aligned}
\int_{-\pi}^{\pi}\psi_\epsilon\mathcal{C}\psi_\epsilon' dt &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{\psi_\epsilon(x)-\psi_\epsilon(y)}{\sin((x-y/2))}\right\}^2 dxdy \\
&\geq \frac{1}{\pi}\int_0^{\pi}\int_0^{\pi}\left\{\frac{\psi_\epsilon(x)+\psi_\epsilon(y)}{x+y}\right\}^2 dxdy \\
&\geq \frac{4}{\pi}\int_\epsilon^{1/2}\int_\epsilon^{1/2}(x+y)^{-2}dxdy \\
&\geq -\frac{4}{\pi}\ln(\epsilon) - \text{Const.}
\end{aligned}$$

Therefore one cannot draw the conclusion that, as the Stokes highest wave is approached, there are large numbers of negative eigenvalues of the linearised problem with *odd* eigenfunctions. In fact this result is false as we will see in the next Theorem, which says that in the space of odd functions, zero is always a simple eigenvalue the eigenfunction of which is a result of the problem's translation invariance. □

Note that when $(\lambda, w) \in U$ is a solution of (8.1), then $w \neq 0$, w' is odd and differentiation of (8.1) gives that $v = w'$ satisfies the linear eigenvalue problem

$$\partial_w F(\lambda, w)v = \mathcal{C}v' - \lambda\{v + v\mathcal{C}w' + w\mathcal{C}v' + \mathcal{C}((vw)')\} = \gamma v, \quad v \in H_0, \tag{8.18}$$

with $w' \leq 0$ on $[0, \pi]$ and $\gamma = 0$. Also note that, for general γ, any solution v of (8.18) can be written as a sum of its even and odd parts, each of which is also a solution of (8.18) with the same value of γ, because w is even. We will show that $\gamma = 0$ is always a simple eigenvalue of (8.18) in the subspace of H_0 comprised of odd functions. The following observation is central.

Suppose that $v \in H_0$ is odd. Then $v\mathcal{C}v' - \mathcal{C}(vv')$ is odd and it is shown in Section 3 that, for $x \in [-\pi, \pi]$,

$$
\begin{aligned}
&(v\mathcal{C}v' - \mathcal{C}(vv'))(x) = \frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{v(x)-v(y)}{\sin(\frac{x-y}{2})}\right\}^2 dy \\
&= \frac{1}{8\pi}\int_0^{\pi}\left[\left\{\frac{v(x)-v(y)}{\sin(\frac{x-y}{2})}\right\}^2 + \left\{\frac{v(x)+v(y)}{\sin(\frac{x+y}{2})}\right\}^2\right] dy \\
&= \frac{1}{8\pi}\int_0^{\pi}\{v(x)^2+v(y)^2\}\left\{\frac{1}{\sin^2(\frac{x+y}{2})} + \frac{1}{\sin^2(\frac{x-y}{2})}\right\} dy \\
&+ \frac{1}{4\pi}\int_0^{\pi} v(x)v(y)\left\{\frac{1}{\sin^2(\frac{x+y}{2})} - \frac{1}{\sin^2(\frac{x-y}{2})}\right\} \\
&= \frac{1}{8\pi}\int_0^{\pi}\{v(x)^2+v(y)^2\}\left\{\frac{1}{\sin^2(\frac{x+y}{2})} + \frac{1}{\sin^2(\frac{x-y}{2})}\right\} dy \\
&- \frac{1}{4\pi}\int_0^{\pi}\frac{v(x)v(y)\sin x \sin y}{\sin^2(\frac{x+y}{2})\sin^2(\frac{x-y}{2})} dy.
\end{aligned} \tag{8.19}
$$

Let $\tilde{v}$ denote the odd function on $[-\pi, \pi]$ which coincides with $|v|$ on $[0, \pi]$. Then (8.19) shows that

$$
(v\mathcal{C}v' - \mathcal{C}(vv'))(x) \geq (\tilde{v}\mathcal{C}\tilde{v}' - \mathcal{C}(\tilde{v}\tilde{v}'))(x) \text{ for all } x \in [0, \pi], \tag{8.20}
$$

with equality only if $\tilde{v} \equiv \pm v$. This leads to the required result.

Theorem 8.1. *For every solution $(\lambda, w) \in U$ of (8.1), $v = w'$ is the only odd solution (up to normalisation) of equation (8.18) with $\gamma = 0$.*

Proof. Let H denote the set of odd functions in H_0. Then the odd eigenfunctions of (8.18) are the critical points of $Q(v)$ under the constraint $||v||_{L_2[-\pi,\pi]} = 1$, where $Q : H \to \mathbb{R}$ is defined by

$$
\begin{aligned}
Q(v) &= \int_{-\pi}^{\pi}\left\{v\mathcal{C}v' - \lambda(v^2 + v^2\mathcal{C}w' + 2wv\mathcal{C}v')\right\} dx \\
&= \int_{-\pi}^{\pi}\left\{(1-2\lambda w)(v\mathcal{C}v' - \mathcal{C}(vv')) - \lambda v^2(1+\mathcal{C}w') + (1-2\lambda w)\mathcal{C}(vv')\right\} dx \\
&= \int_{-\pi}^{\pi}\left\{(1-2\lambda w)(v\mathcal{C}v' - \mathcal{C}(vv')) - \lambda v^2(1+2\mathcal{C}w')\right\} dx \\
&= 2\int_0^{\pi}(1-2\lambda w)\left\{\frac{1}{8\pi}\int_0^{\pi}\{v(x)^2+v(y)^2\}\left(\frac{1}{\sin^2(\frac{x+y}{2})} + \frac{1}{\sin^2(\frac{x-y}{2})}\right) dy\right. \\
&\left. - \frac{1}{4\pi}\int_0^{\pi}\frac{v(x)v(y)\sin x \sin y}{\sin^2(\frac{x+y}{2})\sin^2(\frac{x-y}{2})} dy\right\} dx - 2\int_0^{\pi}\lambda\{v^2 + 2v^2\mathcal{C}w'\} dx
\end{aligned}
$$

Since $\min(1-2\lambda w) > 0$ because $(\lambda, w) \in U$, it follows by (8.20) that

$$Q(\tilde{v}) \leq Q(v) \quad \text{and} \quad ||\tilde{v}||_{L_2[-\pi,\pi]} = ||v||_{L_2[-\pi,\pi]}. \tag{8.21}$$

As there exists a constant $M > 0$ such that Q is coercive in the sense that

$$Q(v) \geq M^{-1}||v||_H^2 - M||v||^2_{L^2(-\pi,\pi)},$$

it follows easily that there exists $\overline{v} \in H$ such that

$$Q(\overline{v}) = \min\{Q(v) : v \in H, ||v||_{L_2[-\pi,\pi]} = 1\}. \tag{8.22}$$

From (8.20) it follows that there is no loss of generality in assuming that $\overline{v} \geq 0$ on $[0, \pi]$, and $\overline{v}$ is an eigenfunction of (8.18) for some $\gamma = \overline{\gamma}$, say. But $w' \in H$ is an eigenfunction of (8.18), which is also non-negative on $[0, \pi]$. Since eigenfunctions corresponding to distinct eigenvectors must be orthogonal in $L_2[-\pi, \pi]$, we conclude that $\overline{v}$ is an eigenvector of (8.18) with $\gamma = 0$ and hence that $Q \geq 0$ on H.

For contradiction, let $u \in H$ be an eigenfunction of (8.18) with $\gamma = 0$ that is linearly independent of $\overline{v}$. Regularity theory for (8.18) implies that u and $\overline{v}$ are continuous. Hence there exists a linear combination $\overline{u} \in H$ of u and $\overline{v}$ such that $||\overline{u}||_{L^2(-\pi,\pi)} = 1$ and $\overline{u}$ is both negative and positive on $(0, \pi)$. Letting $\tilde{u} \in H$ be such that $\tilde{u} = |\overline{u}|$ on $(0, \pi)$, equations (8.22) and (8.20) lead to the contradiction

$$Q(\overline{v}) \leq Q(\tilde{u}) < Q(\overline{u}) = Q(\overline{v}).$$

□

From Theorems 7.4 and 7.8, we deduce the following result on the global bifurcation of Stokes waves, which shows in particular (part (vii)) that the maximal admissible trail of symmetric Stokes waves has an infinity of points which are either turning points or bifurcation points. In the next section we associate these points with sub-harmonic bifurcations of symmetric Stokes waves.

Theorem 8.2. *For equation* (8.1) *there exist a continuous function* $h = (h_1, h_2) : (0, \infty) \to \mathbb{R} \times X$ *such that*

(i) $h : (0, \infty) \to \bar{S} \cap U$;

(ii) $h((0, 1)) \subset \mathcal{A}_0$ *and* $\lim_{s\to 0} h(s) = (1, 0)$;

(iii) *if* ν *denotes Nekrasov's parameter then* $\lim_{s\to\infty} \nu(h(s)) = \infty$ *and* h *is injective on* $h^{-1}(S)$;

(iv) *at all points* $s \in h^{-1}(S)$, h *is real-analytic with* $h_1'(s) \neq 0$;

(v) *the set* $h^{-1}(\bar{S}\backslash S) \subset (0, \infty)$ *consists of isolated values;*

(vi) *locally near every point* $s_0 \in h^{-1}(\bar{S}\backslash S)$, *there exists an injective and continuous re-parametrisation of the set originally parametrised by* h, *such that* $s = \gamma(\sigma)$, $|\sigma| \leq 1$, $s_0 = \gamma(0)$ *and* $h \circ \gamma$ *is a real-analytic function (whose derivative may vanish only at* 0*).*

(vii) There are infinitely many values of $s > 0$ at which $h(s) \in \bar{S} \setminus S$ is a turning point or a bifurcation point. The set of $h(s)$ in $\bar{S} \setminus S$ for all such s is also infinite.

8.4 Sub-Harmonic Bifurcations

Levi-Civita [35] conjectured that if 2Λ is the *minimal* period of a Stokes wave solution of (2.2), then the *minimal Froude number* $\pi c^2/g\Lambda$ is not less than 1.

We will show that Levi-Civita's conjecture is false. In fact, the Froude number calculated using the minimal period of a Stokes wave can be arbitrarily small because there are sub-harmonic bifurcations of solutions of minimal period $2q\Lambda$, for arbitrarily large prime numbers q, from curves of solutions with minimal period 2Λ. Hence even in a *connected* set of Stokes waves with constant wavelength (here connected means with respect to the speeds and any reasonable definition of the distances between wave profiles) solutions may have different minimal wavelengths. This is a period-multiplying phenomenon and the minimal Froude number is not a continuous function on the set of Stokes waves.

To be more precise we need some notation. With $\mathcal{E}_q$, $q \in \mathbb{N}$, the continua of even solutions of (8.1) which were introduced in Section 5.1 and h the function whose properties are given by Theorem 8.2, let

$$\mathcal{H}_1 = \{h(s) : s \in (0, \infty)\} \subset \mathcal{E}_1,$$
$$\mathcal{H}_q = \{(q\lambda, w_{(q)}) : (\lambda, w) \in \mathcal{H}_1\} \subset \mathcal{E}_q,$$

and

$$\mathcal{A}_q = \{(q\lambda, w_{(q)}) : (\lambda, w) \in \mathcal{A}_1\} \subset \mathcal{H}_q.$$

When $(\lambda, w) \in \mathcal{H}_q$, w has period 2π, but its minimal period is $2\pi/q$. We will prove that for q sufficiently large there bifurcates from $\mathcal{H}_q$, 2π-periodic solutions with minimal period strictly larger that $2\pi/q$ which, if q is a prime number, are of minimal period 2π.

From (8.7) there exists A and B such that

$$0 < A \leq \lambda \leq B \text{ for all } (\lambda, w) \in \mathcal{H}_1, \tag{8.23}$$

and hence the *minimal* Froude number (see (2.10)) is bounded below and above by B^{-1} and A^{-1}, respectively, on $\cup_{q \in \mathbb{N}} \mathcal{H}_q$. However, if a solution of minimal period 2π bifurcates from $\mathcal{H}_q$ for large q, its minimal Froude number near to the point of bifurcation lies in an interval $[(qB)^{-1}, (qA)^{-1}]$, and so is small as $q \to \infty$. These bifurcation points are in $\mathcal{H}_q \setminus \Upsilon_q(\delta)$, for some $\delta > 0$ by Theorem 5.2.

To prove all this, we now introduce a discrete family of Banach spaces of even functions of different periods, and a corresponding discrete family of equations that are each equivalent to the equation (8.1).

For every natural number $q \geq 2$, let $X_q \subset X^q$ (X^q is defined at the end of Section 5.2) be the real Banach space of all $2q\pi$-periodic *even, real-valued,* functions that are in $H^1_{loc}(\mathbb{R})$, and let $Y_q \subset Y^q$ be the real Banach space of all $2q\pi$-periodic even functions in $L^2_{loc}(\mathbb{R})$. The conjugate operator acting on Y_q defined (as in Section 5.2) in terms of Fourier series by $\mathcal{C}_q(e^{ikt/q}) = -i\mathrm{sgn}(k)e^{ikt/q}$, $t \in \mathbb{R}$, is a bounded linear operator from Y_q into itself. Let $\mathcal{J}_q(\lambda, w) : \mathbb{R} \times X_q \to \mathbb{R}$ be defined by

$$\mathcal{J}_q(\lambda, w) = \frac{1}{2}\int_{-q\pi}^{q\pi} \{w\mathcal{C}_q w' - \lambda w^2(1 + \mathcal{C}_q w')\}dt$$

and let $\partial_w \mathcal{J}_q(\lambda, w)u = \langle F_q(\lambda, w), u\rangle_q$, for all $u \in X_q$, where

$$F_q(\lambda, w) = \mathcal{C}_q w' - \lambda\{w + w\mathcal{C}_q w' + \mathcal{C}_q(ww')\}, \quad w \in X_q. \tag{8.24}$$

Then, for all $w \in X = X_1 \subset X_q$, $q \in \mathbb{N}$,

$$\mathcal{J}_q(\lambda, w) = q\mathcal{J}(\lambda, w)$$

and

$$F_q(w, \lambda) = 0 \text{ if and only if } F(\lambda, w) = 0.$$

The sets $\mathcal{H}_q$, when re-scaled onto the interval $(-q\pi, q\pi)$ by $\mathcal{H}_q \ni (\lambda, w) \mapsto (\lambda/q, qw(t/q)) \in \mathbb{R} \times X_q$, gives rise to the same set $\mathcal{H} \subset \mathbb{R} \times X_q$ for every q. The set $\mathcal{H}$ comprises 2π-periodic solutions of (8.24) for all q and is parametrised by $\{h(s) : s \in (0, \infty)\}$, as in Theorem 8.2. The only difference between $\mathcal{H}_1$ and $\mathcal{H}$ is that the former is in $\mathbb{R} \times X$ while the latter is in $\mathbb{R} \times X_q$.

Moreover if $F(\lambda, w) = 0$ and $(\lambda, w) \in U \subset \mathbb{R} \times X$, then $\partial_w F_q(\lambda, w)$ is a self-adjoint operator on $X_q \subset Y_q$ whose spectrum intersects $(-\infty, 0]$ at isolated eigenvalues of finite multiplicity. Now let $i_q(\lambda, w)$ be the number of these eigenvalues (counted according to multiplicity). Then clearly $i_q(\lambda, w) \geq i(\lambda, w)$.

Recall from Theorem 5.1 and (8.13) that the homeomorphism $A_q(\lambda, w)$: $Y_q \to Y_q$ defined by $A_q(\lambda, w)u = (1 + \mathcal{C}_q w')u + w'\mathcal{C}_q u$ for all $u \in Y_q$ leaves X_q invariant and, for all $u \in X_q$,

$$A_q(\lambda, w)^*\partial_w F_q(\lambda, w)A_q(\lambda, w)u(x) = \mathcal{C}_q u'(x) + Q(\lambda, w)u(x) = T_q(\lambda, w)u, \text{ say,}$$

where θ and $Q(\lambda, w)$ are given by the formulae (8.3) and (8.14).

Therefore, $i_q(\lambda, w)$ is the number of negative eigenvalues of $T_q(\lambda, w)$ counted according to multiplicity.

Since $\mathcal{H} = \{h(0, \infty)\} \subset \mathbb{R} \times X_q$, for every $q \in \mathbb{N}$, a change in the Morse index $i_q(h(s))$ of $\mathcal{J}_q(h(s))$ as s tends to infinity may cause secondary bifurcations of solutions in $\mathbb{R} \times X_q$ (see Proposition 7.7). In the remains of this section, it is established that the Morse index of $\mathcal{J}_q \circ h$ goes to infinity much faster than that of $\mathcal{J} \circ h$, as $s \to \infty$, provided that q is sufficiently large. This implies that the Morse index of $\mathcal{J}_q \circ h(s)$ jumps at infinitely many values of $s > 0$ that are in $h^{-1}(S)$. Such jumps create secondary bifurcations of solutions in $\mathbb{R} \times X_q$ that are *not* in $\mathbb{R}\times X$. If, moreover, q is a prime number, the bifurcating solutions are of *minimal* period $2q\pi$.

The first step is to establish how the Morse indices of $\mathcal{J}_q$, for different q, are related by introducing a continuous family of operators $\{\mathcal{T}_\tau, \tau\in[0,1]\}$. Recall, from the opening paragraph of Section 6, the definitions of the operator C_τ and its domain D_τ, and for $(\lambda,w)\in\mathcal{H}\subset\mathbb{R}\times X$, let $\mathcal{T}_\tau(\lambda,w)$ be the self-adjoint operator defined on $D_\tau\subset\tilde{L}^2$ by

$$\mathcal{T}_\tau(\lambda,w)u(x)=C_\tau u'(x)+Q(\lambda,w)u(x). \tag{8.25}$$

From Theorem 6.3 and (6.6) we find that the spectrum of $\mathcal{T}_\tau(\lambda,w)$ is of the form $\gamma_0^\tau(\lambda,w)\le\gamma_1^\tau(\lambda,w)\le\gamma_2^\tau(\lambda,w)\le\dots$ with

$$|\gamma_n^\tau(\lambda,w)-\gamma_n^0(\lambda,w)|\le 1/2 \text{ for all } n\in\mathbb{N}_0 \text{ and all } \tau\in[0,1],$$

and

$$\lim_{n\to\infty}\inf_{\tau\in[0,1]}\gamma_n^\tau(\lambda,w)=\infty. \tag{8.26}$$

Also, for all $n\in\mathbb{N}_0$, the function $\tau\to\gamma_n^\tau(\lambda,w)$ is continuous on $[0,1]$ and takes any given value at most a finite number of times.

It is useful to record here the relations between T_q in (8.4) and $\mathcal{T}_\tau$ in (8.25). First, all eigenvalues of $T(\lambda,w)$ defined in (8.13) are eigenvalues of $\mathcal{T}_0(\lambda,w)$, but the converse statement is only partially true. Namely, if $\mathcal{T}_0(\lambda,w)u=\gamma_n^0u$ for some $u\in D_0$ the real part of which is not identically zero, then the real part extended periodically to $\mathbb{R}$ is an *even* function and therefore an eigenfunction of $T(\lambda,w)$. The imaginary part of such a function u is *odd* and does not belong to X if it is non-zero. Thus the eigenvalues of $T(\lambda,w)$ form a subsequence of the eigenvalues of $\mathcal{T}_0(\lambda,w)$, and all eigenfunctions of $T(\lambda,w)$ restricted to $(-\pi,\pi)$ are eigenfunctions of $\mathcal{T}_0(\lambda,w)$.

Analogously, if $u\in D_1$ satisfies $\mathcal{T}_1(\lambda,w)u=\gamma_n^1u$ and the real part of u does not vanish identically, then the real part of u extended periodically to $\mathbb{R}$ is an eigenfunction of $T_2(\lambda,w)$, and if $u\in D_\tau$ satisfies $\mathcal{T}_\tau(\lambda,w)u=\gamma_n^\tau u$ with $u\not\equiv 0$ and $\tau=2m/q$ for some integer $0<2m<q$, then the real part of u does not vanish identically and is an eigenfunction of $T_q(\lambda,w)$ once it has been extended periodically to $\mathbb{R}$.

From this last remark, we obtain that, for $q\ge 3$,

$$i_q(\lambda,w)\ge \operatorname{Card}\{(n,m)\in\mathbb{N}_0\times\mathbb{N}:\gamma_n^{2m/q}(\lambda,w)<0,\ 2m<q\}.$$

Conversely, suppose that $u\in X_q$ satisfy $T_q(\lambda,w)u=\gamma u$. Then it follows from Lemma 6.1 that $\gamma=\gamma_n^{2m/q}$ for some $n\in\mathbb{N}_0$ and q with $-q<2m\le q$. But, by the remark following (6.6), $\gamma_n^{-2m/q}=\gamma_n^{2m/q}$ and hence we may infer that for $q\ge 2$,

$$i_q(\lambda,w)\le \operatorname{Card}\{(n,m)\in\mathbb{N}_0^2:\gamma_n^{2m/q}(\lambda,w)<0,\ m\le q/2\}.$$

We can now state our main result on sub-harmonic bifurcation.

Theorem 8.3. *Let $h = (h_1, h_2) : (0, \infty) \to \mathbb{R} \times X$ be given by Theorem 8.2 and let $n \in \mathbb{N}_0$. Let s_1 and s_2 be elements of $h^{-1}(S)$, chosen using Plotnikov's Lemma and Theorem 8.2, such that*

$$s_2 > s_1 > 0, \quad \gamma_n^0(h(s_1)) - 1/2 > 0 \text{ and } \gamma_n^0(h(s_2)) + 1/2 < 0.$$

Then, for all natural numbers q sufficiently large, there exists $s(q) \in (s_1, s_2)$ such that every neighbourhood of $h(s(q))$ in $\mathbb{R} \times X_q$ contains a solution of $F_q = 0$ that does not belong to $\mathbb{R} \times X$. Moreover, for all sufficiently large $Q \in \mathbb{N}$, the set $\{s(q) : q \in \mathbb{N},\ q \geq Q\} \subset (s_1, s_2)$ is infinite.

Proof. On one hand, for $q \geq 3$,

$$i_q(h(s_2)) \geq \mathrm{Card}\{(j, m) \in \mathbb{N}_0 \times \mathbb{N} : \gamma_j^{2m/q}(h(s_2)) < 0,\ j \leq n,\ 2m < q\} \geq (n+1)(q/2-1).$$

On the other hand, $i_q(h(s_1)) \leq n(q/2 + 1)$. Hence, for q large enough, $i_q(h(s_2)) - i_q(h(s_1))$ is strictly greater than

$$\mathrm{Card}\{(s, j, \tau) : \gamma_j^\tau(h(s)) = 0, s \in h^{-1}(\bar{S} \backslash S), s_1 < s < s_2, j \in \mathbb{N}_0, 0 \leq \tau \leq 1\}.$$

Note that, by (8.26) and the assertion that follows it, this cardinality is finite. We deduce that the jumps of $i_q \circ h$ in (s_1, s_2) cannot all occur at points of $h^{-1}(\bar{S} \backslash S)$ and, as a consequence, there exists $s(q) \in (s_1, s_2) \cap h^{-1}(S)$ so that $i_q \circ h$ has a jump-discontinuity at $s(q)$ (By analyticity, this discontinuity is isolated). We conclude by considering $h : (0, \infty) \to \mathbb{R} \times X \subset \mathbb{R} \times X_q$ as a map taking its values in $\mathbb{R} \times X_q$ and by applying Theorem 7.7. The final assertion of the Theorem is a consequence of the fact that, for every $s \in \mathbb{R}$, the set $\{(j, \tau) : \gamma_j^\tau(h(s)) = 0,\ j \in \mathbb{N}_0,\ 0 \leq \tau \leq 1\}$ is finite. □

Remarks.

(a) The preceding Theorem relates to Levi-Civita conjecture as follows. For a large prime number q the bifurcation point $h(s(q)) \in \mathbb{R} \times X_q$ is a point near which there are solutions of $F_q = 0$ with minimal period $2q\pi$. When re-scaled, they correspond to solutions of (8.1) in $\mathbb{R} \times X$ with minimal period 2π bifurcating from a point of $\mathcal{H}_q$. As explained earlier, this shows that the minimal Froude number of a Stokes wave can be arbitrarily small and the conjecture of Levi-Civita is false. (Levi-Civita's conjecture was studied in [19]. However the discussion in the first paragraph of [19, page 1049] *excludes from consideration* solutions of the type obtained in our theorem).

(b) In general, it is not clear that the solutions in X_q so obtained are of minimal period $2q\pi$. However, this is true when q is prime.

(c) For two distinct prime integers q and q', the corresponding bifurcating solutions are distinct, and $s_1 = s_1(n)$ can be chosen such that $\lim_{n\to\infty} s_1(n) = \infty$. This follows from Plotnikov's lemma (in section 8.3) and the fact that the sequence $\{\alpha_n(\lambda, w) : n \in \mathbb{N}\}$ is a subsequence of $\{\gamma_n^0(\lambda, w) : n \in \mathbb{N}_0\}$.

(d) It would be interesting to know that $\lambda < 1$ on $\mathcal{H}_1$ or $\mathcal{E}_1$, since this would mean that the Froude numbers of the corresponding waves exceed 1.

□

9 Appendix

9.1 Analyticity

This section begins with a brief reminder about analytic functions between finite or infinite dimensional Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We follow [24, Section 3.1.24], [27, Chapter 26] and [63].

Let X and Y be Banach spaces over $\mathbb{K}$ and for $k \in \mathbb{N}$ let X^k denote the Banach space product of X with itself k times. For $k \in \mathbb{N}$, a k-linear operator $u_k : X^k \to Y$ is bounded if for some constant M

$$||u_k(x_1, \dots, x_k)|| \leq M||x_1|| \dots ||x_k|| \text{ for all } x_1, \dots, x_k \in X.$$

The infimum of all M with this property is denoted by $||u_k||$, the norm of u_k. If, moreover, $u_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = u_k(x_1, \dots, x_k)$ for all $(x_1, \dots, x_k) \in X^k$ and all permutations σ of $\{1, \dots, k\}$, then the bounded k-linear operator u_k is called a *k-form*. In that case $u_k(x, \dots, x)$ is abbreviated as $u_k x^k$. By convention, the 0-form u_0 is an element of Y.

A function $\phi : X \to Y$ is said to be *analytic* at $x_0 \in X$ if and only if it is expressible as a series

$$\phi(x) = \sum_{k \in \mathbb{N}_0} u_k(x - x_0)^k, \tag{9.1}$$

where $u_0 = \phi(0) \in Y$, u_k is a k-form for all $k \in \mathbb{N}$, and the series is uniformly convergent for x in a neighbourhood of x_0. In this case, ϕ is infinitely Fréchet differentiable in a neighbourhood of x_0 and $u_k = (1/k!)\phi^{(k)}(x_0)$ for $k \in \mathbb{N}$, where $\phi^{(k)}(x_0)$ is the k^{th} derivative of ϕ at x_0.

The series (9.1) is uniformly convergent in a neighbourhood of x_0 if and only if there exists $\rho > 0$ such that

$$\tilde{M} := \sup\{||u_k(x - x_0)^k|| : \ ||x - x_0|| \leq \rho, k \in \mathbb{N}_0\} < \infty,$$

and by the polarisation identity for k-forms ([24, Sections 1.9.3 and 3.1.24] $||u_k|| \leq e^k \rho^{-k} \tilde{M}$ for all $k \in \mathbb{N}_0$. If ϕ is analytic at x_0 it is analytic at every point in an open neighbourhood of x_0.

Let Z be another Banach space over $\mathbb{K}$. For $k, l \in \mathbb{N}_0$ and any $(k + l)$-form $u_{k+l} : (X \times Y)^{k+l} \to Z$ let

$$u_{k,l} x^k y^l = u_{k,l} x^k y^l = u_{k+l}(x, 0)^k (0, y)^l \in Z, \quad (x, y) \in X \times Y.$$

(When $k = 0$ (similarly when $l = 0$) $(x, 0)^k$ is absent from the formula.) The function $u_{k,l} : X \times Y \to Z$ is clearly continuous, k-homogeneous with respect to x and l-homogeneous with respect to y. A function $\phi : X \times Y \to Z$ is then analytic at (x_0, y_0) if and only if it is expressible as

$$\phi(x, y) = \sum_{k,l \in \mathbb{N}_0} u_{k,l}(x - x_0)^k (y - y_0)^l$$

with, for some $\rho > 0$,

$$\sup\{||u_{k,l}(x - x_0)^k (y - y_0)^l|| : \ ||x - x_0|| \leq \rho, ||y - y_0|| \leq \rho, k, l \in \mathbb{N}_0\} < \infty.$$

Again, ϕ is infinitely differentiable near (x_0, y_0) and if $k + l > 0$,

$$u_{k,l} = \frac{1}{k!l!} \frac{\partial^k}{\partial x^k} \frac{\partial^l \phi}{\partial y^l}(x_0, y_0).$$

Analytic Implicit Function Theorem. *Let $U \subset X \times Y$ be an open set, $F : U \to Z$ be analytic at $(x_0, y_0) \in U$, $F(x_0, y_0) = 0$ and let the partial Fréchet derivative of F with respect to y at (x_0, y_0), $F_y[x_0, y_0] : Y \to Z$, be a linear homeomorphism.*

Then for some open neighbourhood $V \subset X$ of x_0, there exists a function $f : V \to Y$ which is analytic at x_0 such that $f(x_0) = y_0$, $F(x, f(x)) = 0$ on V, and $f(x) = y$ whenever $F(x, y) = 0$ with (x, y) close enough to (x_0, y_0).

Proof. See [63], for example. □

Let $B_n \subset \mathbb{K}^n$ be the open unit ball centred at the origin 0. A subset $\mathcal{N} \neq \emptyset$ of X is said to be an *embedded analytic submanifold of X of dimension $n \in \mathbb{N}_0$* if, for each $x \in \mathcal{N}$, there exists an analytic function $\eta_x : B_n \to X$ such that $\eta_x(0) = x$, η_x maps open subsets of B_n into relatively open subsets of $\mathcal{N}$, and the null space of the derivative, $d\eta_x(0) : \mathbb{K}^n \to X$, is trivial. (A zero-dimensional embedded analytic submanifold is a collection of isolated points of X.)

Locally near $0 \in \mathbb{K}^n$, each such function η_x has an analytic left inverse defined in a neighbourhood in X of x, by the analytic implicit function theorem.

9.2 Germs of Analytic Varieties

If U is open in $\mathbb{K}^n$, $n \in \mathbb{N}$, and F is a finite collection of $\mathbb{K}$-valued analytic functions on U, then

$$\text{var}(U, F) = \{x \in U : f(x) = 0 \text{ for all } f \in F\} \subset \mathbb{K}^n$$

is called the *analytic subvariety of U generated by F.* A point $x \in \text{var}(U, F)$ is said to be *m-regular* for $m \in \mathbb{N}_0$ if, for some neighbourhood $\hat{U}$ of x, $\hat{U} \cap \text{var}(U, F)$ is a analytic submanifold of dimension m.

Let $a \in \mathbb{K}^n$. Two subsets $S, T \subset \mathbb{K}^n$ are said to be *equivalent at a* if $S \cap U = T \cap U$ for some open neighbourhood U of a. The corresponding equivalence class of S, denoted by $\gamma_a(S)$, is called the *germ of S at a.* The germ $\gamma_a(S)$ admits S as *a representative.* Finite unions, finite intersections and complementation of germs are defined via representatives. We shall use the notations $\gamma_a(\emptyset) = \emptyset$ and $\gamma_a(\{a\}) = \{a\}$. (Note that $a \notin \gamma_a(S)$ is a possibility.)

An analytic variety germ at a is the germ at a of some analytic subvariety $\mathrm{var}(U, F)$. The set of all analytic variety germs at a is denoted by $\mathcal{V}_a(\mathbb{K}^n)$. Analytic variety germs are referred to as real- or complex- depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ and $\alpha \neq \emptyset$, then $a \in \alpha$, and if $\alpha, \beta \in \mathcal{V}_a(\mathbb{K}^n)$, then $\alpha \cap \beta$ and $\alpha \cup \beta$ are in $\mathcal{V}_a(\mathbb{K}^n)$, but not $\alpha \backslash \beta$ in general.

The *dimension*, $\dim \alpha \leq n$, of $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ is the largest $m \in \mathbb{N}_0$ such that all representatives $\mathrm{var}(U, F)$ of α have an m-regular point. If no such number exists, $\dim \alpha = -1$ by convention. (This definition of dimension is standard for complex analytic varieties [64, Definition 9A and Remark 9B, Chapter 2, Section 9].)

A germ $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ is said to be *irreducible* if $\alpha = \alpha_1 \cup \alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{V}_a(\mathbb{K}^n)$ implies $\alpha = \alpha_1$ or $\alpha = \alpha_2$.

Recall how the difference between analyticity when $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ is simply illustrated by the example in which $X = Y = \mathbb{C}$ and the function is complex conjugation. When $\mathbb{K} = \mathbb{R}$ it is analytic, but when $\mathbb{K} = \mathbb{C}$ it is not (since complex conjugation is not linear over $\mathbb{C}$). This distinction between real and complex analytic functions notwithstanding, $\mathbb{R}^n \subset \mathbb{C}^n$ and there is a nice relationship between $\mathcal{V}_a(\mathbb{R}^n)$ and $\mathcal{V}_a(\mathbb{C}^n)$ when $a \in \mathbb{R}^n$.

Proposition 9.1. *If* $a \in \mathbb{R}^n$, *for all* $\alpha \in \mathcal{V}_a(\mathbb{R}^n)$, *there exists* $\tilde{\alpha} \in \mathcal{V}_a(\mathbb{C}^n)$ *such that* $\alpha \subset \tilde{\alpha}$ *and* $\beta \in \mathcal{V}_a(\mathbb{C}^n)$ *contains* $\tilde{\alpha}$ *whenever* $\alpha \subset \beta$. *Moreover* $\alpha = \tilde{\alpha} \cap \gamma_a(\mathbb{R}^n)$, $\dim_{\mathbb{R}} \alpha = \dim_{\mathbb{C}} \tilde{\alpha}$, *and* $\tilde{\alpha}$ *is* irreducible *if and only if* α *is irreducible. Such an* $\tilde{\alpha}$ *is unique and is called the* complexification *of* α.

Proof. See [14, Section 8] and [46, Chapter V]. □

Proposition 9.2.

(a) *For* $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$, $\dim \alpha = -1$ *if and only if* $\alpha = \emptyset$,

(b) *Every non-empty family* H *of germs in* $\mathcal{V}_a(\mathbb{K}^n)$ *has an element that contains no other element of* H.

(c) *Every germ* $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ *can be written as a union* $\cup_{i=1}^{\nu} \alpha_i$ *of a finite family of irreducible germs* $\alpha_1, \dots, \alpha_\nu \in \mathcal{V}_a(\mathbb{K}^n)$, *such that* $\alpha_i \subset \alpha_j$ *only if* $i = j$. *Up to permutations, this decomposition is unique and* $\alpha_1, \dots, \alpha_\nu$, *are called the* irreducible components of α.

(d) For $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$, $\dim \alpha = \sup_{1 \leq i \leq \nu} \dim \alpha_i$, where α_i, $1 \leq i \leq \nu$, are the irreducible components of α.

(e) If $\alpha, \beta \in \mathcal{V}_a(\mathbb{K}^n)$, α is irreducible and β is strictly contained in α (that is, $\beta \backslash \alpha = \emptyset$ and $\alpha \backslash \beta \neq \emptyset$), then $\dim \beta < \dim \alpha$.

(f) If $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ is an irreducible germ of dimension $m \in \mathbb{N}_0$, there exists $\beta \in \mathcal{V}_a(\mathbb{K}^n)$ such that $\beta \subset \alpha$, $\dim \beta < \dim \alpha$ and the set germ $\alpha \backslash \beta$ admits a representative that is an m-dimensional analytic submanifold.

(g) If $\alpha, \beta \in \mathcal{V}_a(\mathbb{K}^n)$, the germ $\alpha \backslash \beta$ admits a sequence of representatives $\{S_n : n \in \mathbb{N}\}$ whose diameters converge to 0 and such that every S_n is a union of finitely many connected analytic submanifolds of various dimensions.

(h) $\dim \alpha = 0$ if and only if $\alpha = \{a\}$.

(i) The germ $\gamma_a(\mathbb{K}^n) \in \mathcal{V}_a(\mathbb{K}^n)$ is irreducible.

Proof. (a) Since dim $\emptyset = -1$, and since a non-empty complex analytic variety has non-negative dimension (see [64, Chapter 2, Section 2, page 44]), (a) follows from Proposition 9.1.

(b) In the case when $\mathbb{K} = \mathbb{R}$ this result is given in [24, Section 3.4.5, page 321]; when $\mathbb{K} = \mathbb{C}$, see [24, Section 3.4.12, page 338]. (c) See [24, Section 3.4.5, page 321] for the case $\mathbb{K} = \mathbb{R}$ and [24, Section 3.4.12] for $\mathbb{K} = \mathbb{C}$. (d) This is [24, Theorem 3.4.8 (14), page 328]. (e) See [24, Theorem 3.4.8 (15), page 328 and Section 3.4.12].

(f) This is stated in [14, Proposition 10, page 93]. (g) See [24, Theorem 3.4.8 (11), page 326 and Section 3.4.12].

(h) If $\alpha = \{a\}$, then clearly $\dim \alpha = 0$. If $\dim \alpha = 0$ then each of its non-empty irreducible components has dimension 0, by part (a). Thus each is an isolated point by part (f) and since $a \in \alpha$, $\alpha = \{a\}$.

Part (i) is immediate from (c) and (d). □

Corollary 9.3.

(a) If $\alpha, \beta \in \mathcal{V}_a(\mathbb{K}^n)$, then $\dim(\alpha \cup \beta) = \max(\dim \alpha, \dim \beta)$.

(b) If α_1, α_2 are two distinct irreducible components of $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$, then $\dim(\alpha_1 \cap \alpha_2) < \dim \alpha$.

(c) Theorem 9.2(f) holds if α is reducible.

(d) If $\alpha, \beta \in \mathcal{V}_a(\mathbb{K}^n)$, α is irreducible with $\dim \alpha = 1$ and $\alpha \not\subset \beta \neq \emptyset$, then $\alpha \cap \beta = \{a\}$.

Proof. (a) This follows from parts (c) and (e) of the preceding theorem.

(b) As $\alpha_1 \cap \alpha_2$ is strictly contained in α_1, part (e) of the theorem gives

$$\dim(\alpha_1 \cap \alpha_2) < \dim \alpha_1 \leq \dim \alpha.$$

(c) Let $\alpha_1, \dots, \alpha_\nu$ be the irreducible components of α and let β_i be given by part (f) of the theorem applied to α_i $(1 \leq i \leq \nu)$. Set

$$\beta = \cup\{\alpha_i : \dim(\alpha_i) < \dim(\alpha), 1 \leq i \leq \nu\} \cup \{\cup_{i=1}^{\nu} \beta_i\} \;\cup\; \{\cup_{j \neq k}(\alpha_j \cap \alpha_k)\} \subset \alpha.$$

By parts (a) and (b) of the theorem we obtain that $\dim \beta < \dim \alpha$.

(d) As $\alpha \cap \beta \notin \{\alpha, \emptyset\}$ and α is irreducible, we obtain $\dim(\alpha \cap \beta) = 0$ by part (f) of the theorem; part (h) of the theorem gives $\alpha \cap \beta = \{a\}$. □

Proposition 9.4. *Let $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ admit the representative var(U, F). For $l \in \mathbb{N}$, let $M \subset U$ be an l-dimensional analytic submanifold which is a subset of the set of l-regular points of var(U, F) and $\gamma_a(M) \neq \emptyset$. Then there exists $\mu \in \mathcal{V}_a(\mathbb{K}^n)$ such that* *(a)* $\gamma_a(M) \subset \mu \subset \alpha$;

(b) If $\tilde{\mu}$ *satisfies* $\gamma_a(M) \subset \tilde{\mu} \subset \alpha$, *then* $\mu \subset \tilde{\mu}$;

(c) $\dim \mu = l$.

Proof. Let $H \subset \mathcal{V}_a(\mathbb{K}^n)$ be the family of germs that contain $\gamma_a(M)$. By Theorem 9.2, H contains an element $\mu \subset \alpha$ which contains no other member of H. Clearly, μ satisfies (a). As H is closed under finite intersection, (b) holds also and clearly $\dim \mu \geq l$.

Suppose, for contradiction, that $\dim \mu > l$. By part (f) of Theorem 9.2 applied to μ and part (c) of Corollary 9.3, there exists $\beta \in \mathcal{V}_a(\mathbb{K}^n)$ such that $\beta \subset \mu$, $\dim \beta < \dim \mu$ and $\gamma_a(M) \subset \beta$, since $\dim \mu > l$. This contradicts the minimality of μ and the proof is complete. □

We now recall some elementary structure theory for 1-dimensional irreducible germs of analytic varieties.

Proposition 9.5. *For* $n \geq 2$, *let* $\alpha \in \mathcal{V}_a(\mathbb{K}^n)$ *be an irreducible germ at* $a = 0 \in \mathbb{K}^n$ *of dimension* 1 *such that*

$$\alpha \not\subset \gamma_a(\{e\} \times \mathbb{K}^{n-1}) \tag{9.2}$$

where $e = 0 \in \mathbb{K}$. *Then there exists* $\delta > 0$, $p \in \mathbb{N}$ *and a holomorphic mapping* $h : \{t \in \mathbb{C} : |t| < \delta\} \to \mathbb{C}^{n-1}$ *such that* $h(0) = 0$ *and*

$$V = \{(t^p, h(t)) : \ t \in \mathbb{C}, |t| < \delta, t^p \in \mathbb{K}, h(t) \in \mathbb{K}^{n-1}\},$$

is a representative of α. *Moreover the mapping* $\mathbb{C} \ni t \to (t^p, h(t)) \in \mathbb{C}^n$ *can be assumed to be injective for* $|t|$ *small.*

Proof. When $\mathbb{K} = \mathbb{C}$, this is proved in [37, Section II.6.2, page 173] under the assumption of 1-regularity, which is ensured here by (9.2) (see [37, Section III.4.4, Corollary, page 200]).

The case $\mathbb{K} = \mathbb{R}$ follows from Proposition 9.1. □

Corollary 9.6. *Let* n, α, h *and* V *be as in the proposition, with* $\mathbb{K} = \mathbb{R}$. *Then there exists* $q \in \{0, \dots, p-1\}$ *such that the map* $s \mapsto (e^{iq\pi} s^p, h(e^{iq\pi/p} s))$ *is a bijection from* $\{s \in \mathbb{R} : |s| < \delta\}$ *onto* $V \subset \mathbb{R}^n$ *if* $\delta > 0$ *is chosen small enough.*

Proof. First observe that t^p, $t \neq 0$, is real if and only if $t = e^{iq\pi/p} s$ for some $s > 0$ and $q \in \{0, \dots, 2p-1\}$. Moreover, since $\dim \alpha > 0$, there exists $q \in \{0, \dots, 2p-1\}$ such that the function $[0, \delta) \ni s \mapsto h(e^{iq\pi/p} s)$ is real-valued.

Let $q' \in \{0, \dots, 2p-1\}$ be such that the function $[0, \delta) \ni s \mapsto h(e^{iq'\pi/p} s)$ is also real-valued and let us show that $q - q' \in p\mathbb{Z}$. Since

$$\frac{d^k}{ds^k} h(e^{iq\pi/p} s)|_{s=0} \text{ and } \frac{d^k}{ds^k} h(e^{iq'\pi/p} s)|_{s=0}$$

are real and

$$\frac{d^k}{ds^k} h(e^{iq'\pi/p} s)|_{s=0} = e^{i\pi k(q'-q)/p} \frac{d^k}{ds^k} h(e^{iq\pi/p} s)|_{s=0}$$

for all $k \in \mathbb{N}_0$, we obtain that

$$\frac{d^k}{ds^k} h(e^{iq\pi/p} s)|_{s=0} = 0$$

for $k(q' - q) \notin p\mathbb{Z}$. Letting $k_0 \in \mathbb{N}$ be the generator of the ideal $\{k \in \mathbb{Z} : k(q' - q) \in p\mathbb{Z}\}$, we get that k_0 divides p and $(t^p, h(t)) = ((t^{k_0})^{p/k_0}, \tilde{h}(t^{k_0}))$ on $\{t \in \mathbb{C} : |t| < \delta\}$ for some holomorphic map $\tilde{h}$. From the injectivity of $\mathbb{C} \ni t \mapsto (t^p, h(t))$, we deduce that $k_0 = 1$ and $q' - q \in p\mathbb{Z}$.

On the other hand, choosing $q' \in \{q - p, q + p\} \cap \{0, \dots, 2p - 1\}$, the map $\mathbb{R} \ni s \mapsto h(-e^{iq\pi/p} s) = h(e^{iq'\pi/p} s)$ is real-valued for $s > 0$ small. Hence the map $s \mapsto (e^{iq\pi} s^p, h(e^{iq\pi/p} s))$ is a bijection from $\{s \in \mathbb{R} : |s| < \delta\}$ onto V if $\delta > 0$ is small enough. Inverting the roles of q and q' if necessary, we can assume that $q < p$. $\square$

Remark. After permuting coordinates, we can always assume that relation (9.2) holds. $\square$

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