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Degenerations of $(1,3)$ abelian surfaces and Kummer surfaces

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Abstract. We study degenerations of Kummer surfaces associated to certain divisors in Nieto's quintic threefold and show how they arise from boundary components of a suitable toroidal compactification of the corresponding Siegel modular threefold.

The aim of this paper is to study the degenerations of an interesting class of Kummer surfaces in $\mathbb{P}^3$ in terms of degenerations of the corresponding abelian surfaces. The quintic threefold in $\mathbb{P}^4$ given by

$$N = \left\{ \sum_{i=0}^{5} u_i = \sum_{i=0}^{5} \frac{1}{u_i} = 0 \right\} \subset \mathbb{P}^5.$$

is the closure of the locus parametrizing H_{22}-invariant quartics with 16 skew lines. The smooth surfaces of this type are parametrized by a non-empty open set N^s of N. These surfaces are Kummer surfaces associated to abelian surfaces with a $(1,3)$–polarization. The action of the Heisenberg group on the Kummer surface corresponds to a level-2 structure on the abelian surface. If A is an abelian surface and $\mathcal{L}$ is a symmetric line bundle representing this polarization then $\mathcal{L}^{\otimes 2}$ is the unique totally symmetric line bundle representing the $(2,6)$–polarization. Under the involution $\iota : x \mapsto -x$ the space $H^0(\mathcal{L}^{\otimes 2})$ decomposes into eigenspaces $H^0(\mathcal{L}^{\otimes 2})^+$ and $H^0(\mathcal{L}^{\otimes 2})^-$ of dimensions 8 and 4 respectively. The linear system given by $H^0(\mathcal{L}^{\otimes 2})^-$ defines a rational map $A - - \to \mathbb{P}^3$ which induces a map from the smooth Kummer surface $\widetilde{\mathrm{Km}}(A)$ to $\mathbb{P}^3$. In fact for general A this defines an embedding of $\widetilde{\mathrm{Km}}(A)$ into $\mathbb{P}^3$. With respect to a suitable basis of the linear system which defines the map to $\mathbb{P}^3$ the image surface is H_{22}-invariant.

Let $\mathcal{A}_{1,3}(2)$ be the moduli space of $(1,3)$–polarized abelian surfaces with a level-2 structure. Then the map which associates to an abelian surface its Kummer surface defines a $2:1$ map $\mathcal{A}_{1,3}(2) \to N$ (see also the paper [**HNS**]). It was already shown by one of the authors [**Ni**] that the complement of N^s in N consists of two sets of 15 planes called the V- and S-planes respectively (see also [**BN**]). In fact this author studied in [**Ni**] the configuration of V- and S-planes and computed the

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structure of the singular Kummer surface which belongs to a general point on these planes showing in particular that the generic point on a V-plane corresponds to a quotient of a product of elliptic curves by a $\mathbb{Z}/2 \times \mathbb{Z}/2$ action. In [**HNS**] we proved this result by a different method and showed that the Kummer surfaces which are parametrized by the V-planes are exactly the bielliptic surfaces in the sense of [**HW**]. In this paper we consider the quartic surfaces which are parametrized by the 15 S-planes and we will show that they correspond to Kummer surfaces of degenerate abelian surfaces. More precisely we construct an explicit degeneration of abelian surfaces which gives rise to these singular Kummer surfaces.

1. Theta functions

In this section we will give an explicit description of a basis of the space $H^0(\mathcal{L}^{\otimes 2})^-$ which defines the map from A to $\mathbb{P}^3$, factoring through the Kummer surface, in terms of theta functions. Our standard reference for theta functions is Igusa's book [**I**]. We shall denote points of the Siegel upper half plane $\mathbb{H}_2$ by $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$, and $z = (z_1, z_2)$ will denote the coordinates on $\mathbb{C}^2$. For every pair $(m', m'') \in \mathbb{R}^2 \times \mathbb{R}^2$ we define the theta function

$$\Theta_{m'm''}(\tau, z) = \sum_{q \in \mathbb{Z}^2} e^{2\pi i[\frac{1}{2}(q+m')\tau^t(q+m')+(q+m')^t(z+m'')]}.$$

Given a point $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$ we associate to it a period matrix

$$\Omega_\tau = \begin{pmatrix} 2\tau_1 & 2\tau_2 & 2 & 0 \\ 2\tau_2 & 2\tau_3 & 0 & 6 \end{pmatrix}$$

and the lattice

$$L_\tau = \mathbb{Z}^4\Omega_\tau = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$$

generated by the columns e_i of the period matrix Ω_τ. The abelian surface

$$A_\tau = \mathbb{C}^2/L_\tau$$

has a $(1,3)$–(and hence also a $(2,6)$–)polarization. Normally $0 \in A_\tau$ is chosen as the origin and the involution given by taking the inverse is $\iota : x \mapsto -x$. For reasons which will become apparent later we shall want to define the origin of A_τ as the image of the point

$$\omega = \frac{1}{2}(1,1)\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \frac{1}{2}(\tau_1 + \tau_2, \tau_2 + \tau_3).$$

Note that with respect to 0 this is a 4-torsion point. Then the involution with respect to ω is given by

$$\iota_\omega(z) = -z + 2\omega.$$

Finally we set

$$\tau' = \begin{pmatrix} \tau_1/2 & \tau_2/6 \\ \tau_2/6 & \tau_3/18 \end{pmatrix}, \quad z' = (z_1/2, z_2/6).$$

The main objects of this section are the functions

$$\widehat{\Theta}_{\alpha\beta}(\tau, z) := \Theta_{00\frac{\alpha}{2}\frac{\beta}{6}}(\tau', z' - \omega'), \quad \alpha = 0, 1;\ \beta = 0, \ldots, 5.$$

Note that $\widehat{\Theta}_{\alpha+2,\beta} = \widehat{\Theta}_{\alpha\beta}$ and $\widehat{\Theta}_{\alpha,\beta+6} = \widehat{\Theta}_{\alpha\beta}$ so that we can read the indices cyclically.

LEMMA 1.1. (i) *The functions* $\widehat{\Theta}_{\alpha\beta}$ *are all sections of the same line bundle* $\mathcal{L}_\tau$ *on* A_τ.
(ii) $\mathcal{L}_\tau$ *represents a polarization of type* $(2,6)$.

PROOF. (i) We must prove that the automorphy factor of the functions $\widehat{\Theta}_{\alpha\beta}$ with respect to $z \mapsto z + e_i$ does not depend on (α, β). This follows immediately from the formulae $(\Theta 1) - (\Theta 5)$ of [**I**, pp. 49, 50].
(ii) Since the type of a polarization is constant in families it is enough to prove the statement for $\tau_2 = 0$ where

$$A_\tau = E(\tau_1) \times E(\tau_3)$$

with

$$E(\tau_1) = \mathbb{C}/(\mathbb{Z}2\tau_1 + \mathbb{Z}2),\ E(\tau_3) = \mathbb{C}/(\mathbb{Z}2\tau_3 + \mathbb{Z}6).$$

In this case

$$\widehat{\Theta}_{\alpha\beta}(\tau, z) = \vartheta_{0\frac{\alpha}{2}}(\tau_1/2, z_1/2 - \omega_1/2)\ \vartheta_{0\frac{\beta}{6}}(\tau_3/18, z_2/6 - \omega_2/6)$$

where we use ϑ to denote theta functions in one variable. We claim that the degree on $E(\tau_1)$ is 2 and that the degree on $E(\tau_3)$ is 6. Indeed the first claim follows since

$$\begin{aligned} \vartheta_{0\frac{\alpha}{2}}(\tau_1/2, z_1/2 - \omega_1/2) = 0 \quad &\Leftrightarrow \quad z_1/2 \in (\mathbb{Z}\tau_1/2 + \mathbb{Z}) - \alpha/2 + \omega_1/2 \\ &\Leftrightarrow \quad z_1 \in (\mathbb{Z}\tau_1 + \mathbb{Z}2) - \alpha + \omega_1. \end{aligned}$$

This means that $\vartheta_{0\frac{\alpha}{2}}(\tau_1/2, z_1/2 - \omega_1/2)$ has two zeroes on $E(\tau_1)$. The other claim follows in exactly the same way. □

We shall denote the sections of $\mathcal{L}_\tau$ defined by $\widehat{\Theta}_{\alpha\beta}(\tau, z)$ by $\widehat{s}_{\alpha\beta}$. By general theory the twelve sections $\widehat{s}_{\alpha\beta};\quad \alpha = 0, 1,\ \beta = 0, \ldots, 5$ form a *basis* of $H^0(\mathcal{L}_\tau)$. (Cf. [**I**, p.75] for an analogous statement.)

We now want to describe the symmetry properties of the line bundle $\mathcal{L}_\tau$ and the sections $\widehat{s}_{\alpha\beta}$. The kernel of the map

$$\begin{aligned} \lambda : \ A_\tau \ &\to \ \mathrm{Pic}^0 A_\tau \\ x \ &\mapsto \ t_x^* \mathcal{L}_\tau \otimes \mathcal{L}_\tau^{-1} \end{aligned}$$

where t_x is translation by x is equal to

$$\ker \lambda = (\mathbb{Z}\frac{e_1}{2} + \mathbb{Z}\frac{e_2}{6} + \mathbb{Z}\frac{e_3}{2} + \mathbb{Z}\frac{e_4}{6})L_\tau \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z}/6)^2.$$

We set $\rho_6 := e^{2\pi i/6}$.

PROPOSITION 1.2. (i) The group $\ker \lambda$ acts on the sections $\widehat{s}_{\alpha\beta}$ as follows

$$\begin{array}{llllllll} e_1/2 & : & \widehat{s}_{\alpha\beta} & \mapsto & (-1)^\alpha\ \widehat{s}_{\alpha\beta}\ , & e_2/6 & : \widehat{s}_{\alpha\beta} & \mapsto \ \rho_6^{-\beta}\ \widehat{s}_{\alpha\beta} \\ e_3/2 & : & \widehat{s}_{\alpha\beta} & \mapsto & \widehat{s}_{\alpha+1,\beta}\ , & e_4/6 & : \widehat{s}_{\alpha\beta} & \mapsto \ \widehat{s}_{\alpha,\beta+1}. \end{array}$$

(ii) The involution ι_ω acts on the sections $\widehat{s}_{\alpha\beta}$ by

$$\iota_\omega : \widehat{s}_{\alpha\beta} \mapsto \widehat{s}_{-\alpha,-\beta}.$$

PROOF. (i) We shall prove this for $e_1/2$, the other cases being similar. Again using [**I**, pp. 49, 50] we find

$$\begin{array}{rcl}
\widehat{\Theta}_{\alpha\beta}(\tau, z+\frac{e_1}{2}) & = & \Theta_{00\frac{\alpha}{2}\frac{\beta}{6}}(\tau', z'-\omega'+(\tau_1/2, \tau_2/6)) \\
& \overset{(\Theta 3)}{\sim} & e^{2\pi i(-(1,0)\binom{\alpha/2}{\beta/6})} \Theta_{10\frac{\alpha}{2}\frac{\beta}{6}}(\tau', z'-\omega') \\
& \overset{(\Theta 1)}{=} & (-1)^{\alpha}\Theta_{00\frac{\alpha}{2}\frac{\beta}{6}}(\tau', z'-\omega') \\
& = & (-1)^{\alpha}\widehat{\Theta}_{\alpha\beta}(\tau, z).
\end{array}$$

Here $\sim$ denotes equality up to a nowhere vanishing function which is independent of α and β.
(ii) Here we have that

$$\begin{array}{rcl}
\widehat{\Theta}_{\alpha\beta}(\tau, -z+2\omega) & = & \Theta_{00\frac{\alpha}{2}\frac{\beta}{6}}(\tau', -z'+\omega') \\
& \overset{(\Theta 1)}{=} & \Theta_{00-\frac{\alpha}{2}-\frac{\beta}{6}}(\tau', z'-\omega') \\
& = & \widehat{\Theta}_{-\alpha-\beta}(\tau, z)
\end{array}$$

where indices are to be read cyclically. □

REMARK 1.3. (i) Part (i) of the above proposition gives an explicit description of the lifting of the group $(\mathbb{Z}/2)^2 \times (\mathbb{Z}/6)^2$ to the Heisenberg group H_{26}.
(ii) Note that part (ii) of the above proposition is true for any choice of the point ω and hence in particular also for the involution ι itself.

We can now describe a basis of the eigenspaces $H^0(\mathcal{L}_\tau)^+$ and $H^0(\mathcal{L}_\tau)^-$ as follows:

$$\begin{array}{ll}
\widehat{u}_{\alpha\beta} = \widehat{s}_{\alpha\beta} + \widehat{s}_{-\alpha,-\beta} \in H^0(\mathcal{L}_\tau)^+; & \alpha \in \{0,1\}, \beta \in \{0,1,2,3\} \\
\widehat{t}_{\alpha\beta} = \widehat{s}_{\alpha\beta} - \widehat{s}_{-\alpha,-\beta} \in H^0(\mathcal{L}_\tau)^-; & (\alpha,\beta) = (0,1),(0,2),(1,1),(1,2).
\end{array}$$

For our purposes it is, however, better to work with a different basis of $H^0(\mathcal{L}_\tau)^-$.

$$\begin{array}{rcccccccc}
\widehat{g}_0 & := & \widehat{t}_{01} & + & \widehat{t}_{11} & - & \widehat{t}_{02} & - & \widehat{t}_{12} \\
\widehat{g}_1 & := & -\widehat{t}_{01} & - & \widehat{t}_{11} & - & \widehat{t}_{02} & - & \widehat{t}_{12} \\
\widehat{g}_2 & := & \widehat{t}_{01} & - & \widehat{t}_{11} & - & \widehat{t}_{02} & + & \widehat{t}_{12} \\
\widehat{g}_3 & := & -\widehat{t}_{01} & + & \widehat{t}_{11} & - & \widehat{t}_{02} & + & \widehat{t}_{12}.
\end{array}$$

Recall the Heisenberg group H_{22} from [**BN**]. The group H_{22} has order 32 and

$$H_{22}/\text{ centre } \cong (\mathbb{Z}/2)^4$$

is the group generated by the elements

$$\begin{array}{rcccl}
\sigma_1 & : & (z_0 : z_1 : z_2 : z_3) & \mapsto & (z_2 : z_3 : z_0 : z_1) \\
\sigma_2 & : & (z_0 : z_1 : z_2 : z_3) & \mapsto & (z_1 : z_0 : z_3 : z_2) \\
\tau_1 & : & (z_0 : z_1 : z_2 : z_3) & \mapsto & (z_0 : z_1 : -z_2 : -z_3) \\
\tau_2 & : & (z_0 : z_1 : z_2 : z_3) & \mapsto & (z_0 : -z_1 : z_2 : -z_3).
\end{array}$$

The group ${}_2A_\tau$ of 2-torsion points of A_τ is contained in $\ker\lambda$. Here we identify ${}_2A_\tau$ with translations of A_τ of order 2. Using the translations $x \mapsto x + e_i/2$ as generators we obtain an identification of ${}_2A_\tau$ with $(\mathbb{Z}/2)^4$. A straightforward calculation using

Proposition 1.2 and the definition of the basis $\widehat{g}_0, \dots, \widehat{g}_3$ shows that

$$\begin{array}{lllll}
e_1/2: & (\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) & \mapsto & (\widehat{g}_2 : \widehat{g}_3 : \widehat{g}_0 : \widehat{g}_1) & = \sigma_1(\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) \\
e_2/2: & (\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) & \mapsto & (\widehat{g}_1 : \widehat{g}_0 : \widehat{g}_3 : \widehat{g}_2) & = \sigma_2(\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) \\
e_3/2: & (\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) & \mapsto & (\widehat{g}_0 : \widehat{g}_1 : -\widehat{g}_2 : -\widehat{g}_3) & = \tau_1(\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) \\
e_4/2: & (\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3) & \mapsto & (\widehat{g}_0 : -\widehat{g}_1 : \widehat{g}_2 : -\widehat{g}_3) & = \tau_2(\widehat{g}_0 : \widehat{g}_1 : \widehat{g}_2 : \widehat{g}_3)
\end{array}$$

We can, therefore, summarize our results as follows:

THEOREM 1.4. (i) *The basis $\widehat{g}_0, \dots, \widehat{g}_3 \in H^0(\mathcal{L}_\tau)^-$ defines a rational map from A_τ to $\mathbb{P}^3$ which factors through $\widetilde{\mathrm{Km}}(A_\tau)$. This map is equivariant with respect to the action of ${}_2A_\tau \cong (\mathbb{Z}/2)^4$ on A_τ and of $H_{22}/\text{centre} \cong (\mathbb{Z}/2)^4$ on $\mathbb{P}^3$. In particular the image is H_{22}-invariant.*
(ii) *The Kummer surface $\widetilde{\mathrm{Km}}(A_\tau)$ is embedded as a smooth quartic surface if and only if A_τ is neither a product nor a bielliptic abelian surface. If A_τ is bielliptic then $\widetilde{\mathrm{Km}}(A_\tau)$ is mapped to a quartic with four nodes; if A_τ is a product, then $\widetilde{\mathrm{Km}}(A_\tau)$ is mapped $2:1$ onto a quadric.*

PROOF. (i) Follows immediately from our above calculations.
(ii) This was shown in [**HNS**]. □

REMARK 1.5. $\mathcal{L}_\tau$ is the unique totally symmetric line bundle with respect to the involution ι_ω.

2. Degenerations

In this section we construct degenerations of $(1,3)$–polarized abelian surfaces which correspond to points on the S-planes. The construction of degenerating families of abelian varieties is in general technically complicated (see e.g. [**FC**], [**AN**]). Although we cannot avoid these technicalities entirely, we have tried to present our construction in a way which uses only a minimum of technical steps. These, however, cannot be avoided.

We consider the group

$$P = \left\{ \left(\begin{array}{c|cc} \mathbf{1} & 2\mathbb{Z} & 6\mathbb{Z} \\ & 6\mathbb{Z} & 18\mathbb{Z} \\ \hline 0 & \multicolumn{2}{c}{\mathbf{1}} \end{array} \right) \right\} \subset \mathrm{Sp}(4, \mathbb{Z}).$$

Note that this is the lattice contained in the parabolic subgroup of $\Gamma_{1,3}(2) \cap \Gamma_{1,3}^{\mathrm{lev}}$ which fixes the isotropic plane $h = (0,0,1,0) \wedge (0,0,0,1)$. Here $\Gamma_{1,3}(2)$ is the group which defines the moduli space of abelian surfaces with a $(1,3)$–polarization and a level-2 structure, whereas $\Gamma_{1,3}^{\mathrm{lev}}$ belongs to the moduli space of $(1,3)$–polarized abelian surfaces with a canonical level structure (cf. [**HKW**, I.1]). There are two reasons for considering this group. One is that we can then make use of the constructions in [**HKW**] which from our point of view is the most economical way to construct the degenerations which we are interested in; the second reason is that, at least with the known constructions of degenerations of abelian surfaces, the presence of a canonical level structure is necessary. We could also use the method of Alexeev and Nakamura [**AN**] which likewise goes back to Mumford's construction [**M**], and [**Nak**], [**Nam**]. For the surfaces which we are interested in it makes, however, little difference which of these methods we choose.

The group P acts on $\mathbb{H}_2$ by

$$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 + 2\mathbb{Z} & \tau_2 + 6\mathbb{Z} \\ \tau_2 + 6\mathbb{Z} & \tau_3 + 18\mathbb{Z} \end{pmatrix}.$$

The partial quotient of $\mathbb{H}_2$ by P is given by

$$\begin{array}{ccc} \mathbb{H}_2 & \to & \mathbb{H}_2/P \subset (\mathbb{C}^*)^3 \\ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} & \mapsto & (e^{2\pi i\tau_1/2}, e^{2\pi i\tau_2/6}, e^{2\pi i\tau_3/18}) = (t_1, t_2, t_3). \end{array}$$

Recall that

$$A_\tau = \mathbb{C}^2/L_\tau$$

where

$$L_\tau = \mathbb{Z}\begin{pmatrix} 2\tau_1 \\ 2\tau_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2\tau_2 \\ 2\tau_3 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 0 \\ 6 \end{pmatrix} = L'_\tau + L''.$$

Here L'_τ is spanned by the first two columns of Ω_τ and L'' by the last two. Obviously L'' does not depend on τ and

$$\mathbb{C}^2/L'' = (\mathbb{C}^*)^2.$$

We shall use the coordinates

$$w_1 = z_1/2, \quad w_2 = z_2/6$$

on $(\mathbb{C}^*)^2$. The lattice L'_τ acts on the trivial torus bundle $\mathbb{H}_2/P \times (\mathbb{C}^*)^2$ by

$$(m,n) : (t_1, t_2, t_3; w_1, w_2) \mapsto (t_1, t_2, t_3;\ t_1^{2m}\ t_2^{6n} w_1, t_2^{2m}\ t_3^{6n} w_2).$$

We have to extend the trivial bundle $\mathbb{H}_2/P \times (\mathbb{C}^*)^2$ to the boundary in such a way that the action of L'_τ also extends. The general theory of toroidal compactifications of moduli spaces of abelian surfaces (the material which is relevant in our situation can be found in [**HKW**]) leads us to consider first the map

$$\begin{array}{ccc} (\mathbb{C}^*)^3 & \to & \mathbb{C}^3 \\ (t_1, t_2, t_3) & \mapsto & (t_1t_2, t_2t_3, t_2^{-1}) = (T_1, T_2, T_3). \end{array}$$

Let

$$B := \overset{\circ}{(\overline{\mathbb{H}_2/P})}$$

be the interior of the closure of $\mathbb{H}_2/P$ in $\mathbb{C}^3$ in the $\mathbb{C}$–topology. (What we have considered here is an open part of the partial compactification in the direction of the cusp corresponding to $h = (0,0,1,0) \wedge (0,0,0,1)$. The surfaces $B \cap \{T_i = 0\}$ are mapped to boundary surfaces in the Igusa compactification of the moduli space $\mathcal{A}^{\text{lev}}_{1,3}(2)$ of $(1,3)$–polarized abelian surfaces with both a level-2 and a canonical level structure.) In terms of the coordinates T_i the action of L'_τ is now given by

$$(m,n) : (T_1, T_2, T_3; w_1, w_2) \mapsto (T_1, T_2, T_3;\ T_1^{2m} T_3^{2m-6n} w_1, T_2^{6n} T_3^{6n-2m} w_2).$$

Here we are particularly interested in degenerations which are given by $\tau_1 \to i\infty$. This is equivalent to $t_1 = 0$ and hence corresponds to points on the surface $T_1 = 0$.

We now consider the space

$$\tilde{\mathcal{P}} = \text{Proj } R_{\Phi,\Sigma} \to \text{Spec } \mathbb{C}[T_1, T_2, T_3] \cong \mathbb{C}^3$$

which was defined in [**HKW**, p.210]. Let

$$\mathcal{P} := \tilde{\mathcal{P}}|_B.$$

Then $\mathcal{P}$ is a partial compactification of the trivial torus bundle $\mathbb{H}_2/P \times (\mathbb{C}^*)^2$ over $\mathbb{H}_2/P \subset B$. Moreover the action of L_τ on the trivial torus bundle extends to an action on $\mathcal{P}$. The construction of $\tilde{\mathcal{P}}$ is originally due to Mumford [**M**, final example]. Let

$$\bar{A} := \mathcal{P}/L_\tau.$$

Then we have a diagram

$$\begin{array}{ccc} A & \subset & \bar{A} \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{H}_2/P & \subset & B \end{array}$$

where $A = (\mathbb{H}_2/P \times (\mathbb{C}^*)^2)/L_\tau$ is the universal family. In particular $\bar{A}$ extends the universal family A to the boundary. The fibres

$$\bar{A}_u = \pi^{-1}(u)$$

over "boundary points" $u \in B\backslash(\mathbb{H}_2/P)$ are degenerate abelian surfaces. We are interested in the fibres $\bar{A}_u$ over points $u = (0, T_2, T_3)$ with $T_2T_3 \neq 0$. These are the corank 1 degenerations associated to the boundary component given by $\tau_1 \to i\infty$. Note that if $T_2T_3 \neq 0$ then this gives $t_2 = T_3^{-1}$ and $t_3 = T_2T_3$. In particular the point u determines a point $(\tau_2, \tau_3) \in \mathbb{C} \times \mathbb{H}_1$ where τ_2 and τ_3 are uniquely defined up to $6\mathbb{Z}$ and $18\mathbb{Z}$ respectively.

We can now formulate the main result of this section.

THEOREM 2.1. *Let $u = (0, T_2, T_3) \in B$. Then $\bar{A}_u$ is a degenerate abelian surface with the following properties:*
(i) *$\bar{A}_u$ is a corank 1 degeneration. More precisely $\bar{A}_u$ is a chain of two elliptic ruled surfaces $A_{u,1}, A_{u,2}$ i.e. there exists an elliptic curve E and a line bundle $\mathcal{M}_u \in \mathrm{Pic}^0(E)$ such that $A_{u,i} = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{M}_u)$, $i = 1, 2$. The surfaces $A_{u,i}$ are glued with a glueing parameter e as shown below in Figure 1.*
(ii) *The base curve $E \cong E(\tau_3) = \mathbb{C}/(\mathbb{Z}2\tau_3 + \mathbb{Z}6)$.*
(iii) *The line bundle $\mathcal{M}_u = \mathcal{O}_{E(\tau_3)}(6[\tau_2] - 6[0])$ where $[\tau_2], [0]$ are the points of $E(\tau_3)$ given by τ_2 and 0.*
(iv) *The glueing parameter $e = [2\tau_2] \in E(\tau_3)$.*

PROOF. We can derive this from [**HKW**, part II]. There the quotient $\hat{A} = \mathcal{P}/\hat{L}$ was considered where $\hat{L} \cong \mathbb{Z}^2$ acts on the trivial torus bundle $\mathbb{H}_2/P \times (\mathbb{C}^*)^2$ by

$$(m, n) : (T_1, T_2, T_3; w_1, w_2) \mapsto (T_1, T_2, T_3; T_1^m T_3^{m-n} w_1, T_2^n T_3^{n-m} w_2).$$

Hence L'_τ is a subgroup of $\hat{L}$ with $\hat{L}/L'_\tau \cong (\mathbb{Z}/2) \times (\mathbb{Z}/6)$. This means that we can use the description of $\hat{A}$ given in [**HKW**, part II] to give a description of $\bar{A}$. In the terminology of [**HKW**] the group $L'_\tau =< s^{-2}, r^{-6} >$. The statements (i) and (ii) now follow exactly as in the proof of [**HKW**, Theorem II.3.10]. In particular, the fact that $s^2 \in L'_\tau$, but $s \notin L'_\tau$ implies that $\bar{A}_u$ has two irreducible components. The statement about the base curve E follows from diagram [**HKW**, II.3.13]. Statements (iii) and (iv) are an immediate consequence of the proof of [**HKW**, Proposition (II.3.20)]. □

Our next task is to study the involutions ι and ι_ω on A and their extensions to $\bar{A}$. If we choose $0 \in A_\tau = \mathbb{C}^2/L_\tau$ as the origin, then this defines a section of A which extends to a section of $\bar{A}$. Moreover, the involution $\iota : z \mapsto -z$ defines an involution of A which extends to $\bar{A}$ (this is the involution given by [**HKW**, Lemma (II.2.9)(ii)].

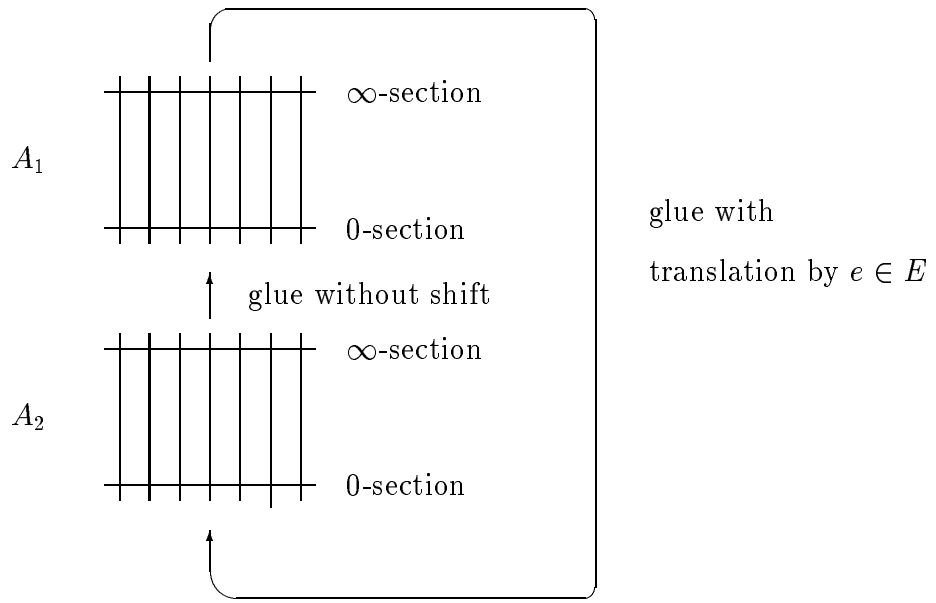


FIGURE 1. Glueing of the surface $\bar{A}_u$

But we said in section 1 that we wanted to choose $\omega = [(\tau_1 + \tau_2)/2, (\tau_2 + \tau_3)/2]$ as the origin. This point is a 4-torsion point of A_τ if we choose 0 as the origin. This choice of origin will be necessary for what follows, but at this point it has the disadvantage that it only defines a multisection of A, not a section. Nevertheless this multisection extends to $\bar{A}$. We also claim that the involution $\iota_\omega(z) = -z + 2\omega$ extends to $\bar{A}$. Since $z \mapsto -z$ is defined on $\bar{A}$ it is enough to show that the translation $z \mapsto z + 2\omega$ is defined on A and extends to $\bar{A}$. This is easy to see, since $z \mapsto z + 2\omega$ in terms of the coordinates w_1, w_2 is given by

$$(w_1, w_2) \mapsto (t_1 t_2^3 w_1, t_2 t_3^3 w_2) = (T_1 T_3^{-2} w_1, T_2^3 T_3^2 w_2).$$

This is the element $s^{-1}r^{-3} \in \hat{L}$ and hence acts on $\mathcal{P}$ and on $\bar{A} = \mathcal{P}/L'_\tau$. Since $s^{-2}r^{-6} \in L'_\tau$, this is an involution. In particular ι_ω defines an involution on the fibres $\bar{A}_u$ of $\bar{A}$. Recall that for $u = (0, T_2, T_3)$ with $T_2 T_3 \neq 0$ the surface $\bar{A}_u$ has two irreducible components $A_{u,i}$, $i = 1, 2$ and that the singular locus of $\bar{A}_u$ consists of two disjoint elliptic curves E_1 and E_2 with $E_i \cong E(\tau_3) = \mathbb{C}/(\mathbb{Z}2\tau_3 + \mathbb{Z}6)$.

PROPOSITION 2.2. The involution ι_ω interchanges the two components $A_{u,1}$ and $A_{u,2}$ of $\bar{A}_u$ and induces an involution on each of the two curves E_1 and E_2 with four fixed points on each of these curves.

PROOF. The involution ι fixes each of the surfaces $A_{u,1}$ and $A_{u,2}$ and interchanges E_1 and E_2. Addition by the 2-torsion point 2ω also interchanges $A_{u,1}$ and $A_{u,2}$ as well as E_1 and E_2. Hence ι_ω interchanges $A_{u,1}$ and $A_{u,2}$ but induces non-trivial involutions on E_1 and E_2. In order to determine the fixed points of ι_ω it is sufficient to compute the limit of the fixed points of ι_ω in A_τ as $\tau_1 \to i\infty$. The

sixteen fixed points of ι_ω on A_τ are given by

$$[(\tau_1+\tau_2, \tau_2+\tau_3)/2 + \varepsilon_1(\tau_1,\tau_2) + \varepsilon_2(\tau_2,\tau_3) + \varepsilon_3(1,0) + \varepsilon_4(0,3)] \in A_\tau$$

where $\varepsilon_i = 0$ or 1. As $\tau_1 \to i\infty$ these 16 points come together in pairs; more precisely any two points which only differ by ε_3 have the same limit. This gives us eight points of which four lie on each of the curves E_i (depending on whether $\varepsilon_1 = 0$ or 1). These points are given by

$$[(\tau_2+\tau_3)/2 + \varepsilon_2\tau_3 + \varepsilon_4 3] \in E(\tau_3) \qquad (\varepsilon_1 = 0)$$
$$[(\tau_2+\tau_3)/2 + \tau_2 + \varepsilon_2\tau_3 + \varepsilon_4 3] \in E(\tau_3) \quad (\varepsilon_1 = 1).$$

□

Figure 2 indicates the action of ι_ω and the position of the eight fixed points on $\bar{A}_u$.

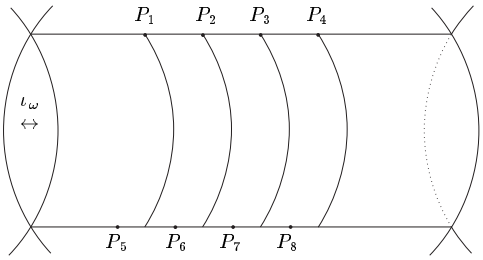


FIGURE 2. The involution ι_ω

Note that two fixed points lie on one ruling if and only if $[\tau_2]$ is a 2-torsion point on $E(\tau_3)$, i.e. if and only if the glueing parameter $e = [2\tau_2] = 0$.

The next step is to extend the polarization to the degenerate abelian surfaces. Ideally we would like to glue the line bundle $\mathcal{L}_\tau$ on A_τ to a line bundle $\mathcal{L}$ on A and to extend this line bundle to $\bar{A}$ in such a way that the sections $\widehat{s}_{\alpha\beta}$ as well as the action of the symmetry group (see Proposition 1.2) extend. At this point, however, we encounter a fundamental difficulty. We have seen that it is possible to extend A to $\bar{A}$ in such a way that the symmetries, and here in particular the involution ι_ω, extend. It is also possible to define a suitable line bundle $\mathcal{L}$ and extend it to a line bundle $\bar{\mathcal{L}}$ on $\bar{A}$ (see [**HKW**, II.5]). But it is not possible to do this in such a way that the action of the symmetry group also extends to $\bar{\mathcal{L}}$. (This leads in particular to a numerical contradiction on the fibre over the origin $0 \in B$.) For this reason we shall now restrict ourselves to taking the partial quotient with respect to the group

$$P' = \left\{ \left(\begin{array}{c|cc} \mathbf{1} & 2\mathbb{Z} & 0 \\ & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\mathbf{1}} \end{array} \right) \right\} \subset \mathrm{Sp}(4,\mathbb{Z}).$$

This is the lattice contained in the stabilizer of the line generated by $l_0 = (0,0,1,0)$ in the group $\Gamma_{1,3}(2) \cap \Gamma_{1,3}^{\mathrm{lev}}$. The partial quotient defined by this group is given by the map

$$\begin{array}{rcl} \mathbb{H}_2 & \to & \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{H}_1 \\ \left(\begin{array}{cc} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{array} \right) & \mapsto & (t_1 = e^{2\pi i \tau_1/2}, \tau_2, \tau_3). \end{array}$$

Partial compactification of $\mathbb{H}_2/P'$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{H}_1$ is given by

$$B' := \overset{\circ}{(\overline{\mathbb{H}_2/P'})}$$

The two partial quotients with respect to P and P' are related by the glueing map

$$\begin{array}{rccc} \varphi: & B' & \to & B \\ & (t_1,\tau_2,\tau_3) & \mapsto & (t_1t_2, t_2t_3, t_2^{-1}) \end{array}$$

where $t_2 = e^{2\pi i\tau_2/6}$ and $t_3 = e^{2\pi i\tau_3/18}$. The image of φ is $B\backslash(B\cap\{T_2T_3=0\})$ and the map φ is unramified onto its image. We can pull the family $\bar{A}$ over B back to B' via φ and we shall denote the resulting family by $\bar{A}'$. This family extends the universal family A' over $\mathbb{H}_2/P'$. We shall denote the projection from $\bar{A}'$ to B' by π'.

PROPOSITION 2.3. (i) The line bundles $\mathcal{L}_\tau$ on A_τ glue to a line bundle $\mathcal{L}'$ on A'. (ii) The line bundle $\mathcal{L}'$ can be extended to a line bundle $\bar{\mathcal{L}}'$ on $\bar{A}'$ in such a way that the sections $\widehat{s}_{\alpha\beta}$ as well as the action of the Heisenberg group H_{26} and the involution ι_ω extend.

PROOF. A straightforward computation shows that with respect to the coordinates $t_1 = e^{2\pi i\tau_1/2}$ and $w_1 = e^{2\pi iz_1/2}, w_2 = e^{2\pi iz_2/6}$ the functions $\widehat{\Theta}_{\alpha\beta}(\tau,z)$ are given by:

$$\begin{aligned} \widehat{\Theta}_{\alpha\beta}(\tau,z) &= \Theta_{00\,\frac{\alpha}{2}\frac{\beta}{6}}(\tau', z'-\omega') \\ &= \sum_{q\in\mathbb{Z}^2} t_1^{\frac{1}{2}q_1(q_1-1)}\exp\{9\pi i q_2(q_2-3)\tau_3\} \\ &\qquad \cdot\exp\{6\pi i(2q_1q_2-3q_1-q_2)\tau_2\} w_1^{q_1}w_2^{q_2}(-1)^{\alpha q_1}\rho_6^{\beta q_2}. \end{aligned}$$

In particular this shows that we can consider these functions as functions on $(\mathbb{H}_2/P')\times(\mathbb{C}^*)^2$. Similarly we find that with respect to the lattice L_τ the functions $\widehat{\Theta}_{\alpha\beta}(\tau,z)$ have the following transformation behaviour. For $(k,l)\in\mathbb{Z}^2$:

$$\widehat{\Theta}_{\alpha\beta}(\tau, z+(2k,6l)) = \widehat{\Theta}_{\alpha\beta}(\tau,z).$$

For $(m,n)\in\mathbb{Z}^2$:

$$\begin{aligned} &\widehat{\Theta}_{\alpha\beta}\left(\tau, z+(m,n)\begin{pmatrix} 2\tau_1 & 2\tau_2 \\ 2\tau_2 & 2\tau_3\end{pmatrix}\right) \\ &\quad = \Theta_{00\,\frac{\alpha}{2}\,\frac{\beta}{6}}\left(\tau', z'-\omega'+(2m,6n)\begin{pmatrix}\tau_1' & \tau_2' \\ \tau_2' & \tau_3'\end{pmatrix}\right) \\ &\quad = \exp\left\{2\pi i\left[-\tfrac{1}{2}(2m,6n)\begin{pmatrix}\tau_1' & \tau_2' \\ \tau_2' & \tau_3'\end{pmatrix}\begin{pmatrix}2m \\ 6n\end{pmatrix} - (2m,6n)(z'-\omega')\right]\right\} \\ &\qquad \cdot\Theta_{2m,6n,\frac{\alpha}{2},\frac{\beta}{6}}(\tau', z'-\omega') \\ &\quad = t_1^{-2m^2+m}w_1^{-2m}w_2^{-6n}e^{2\pi i[(-2mn+\frac{m^2}{2})\tau_2+(-n^2+\frac{n}{2})\tau_3]}\widehat{\Theta}_{\alpha\beta}(\tau,z). \end{aligned}$$

These calculations show claim (i). To prove (ii) we have to consider the limit as $t_1\to 0$. Here we find

$$\begin{aligned} \lim_{t_1\to 0}\widehat{\Theta}_{\alpha\beta}(\tau,z) &= \sum_{q_2\in\mathbb{Z}} e^{9\pi i q_2(q_2-3)}e^{6\pi i(-q_2\tau_2)}w_2^{q_2}\rho_6^{\beta q_2} \\ &\qquad +(-1)^\alpha w_1\sum_{q_2\in\mathbb{Z}} e^{9\pi i q_2(q_2-3)}e^{2\pi i(q_2-3)\tau_2}w_2^{q_2}\rho_6^{\beta q_2} \\ &= \vartheta_{0\,\frac{k}{6}}(\tau_3/6, (z_2-\tau_3/2-\tau_2/2)/6) \\ &\qquad +(-1)^\alpha w_1 e^{2\pi i\left(-\frac{\tau_2}{4}\right)}\vartheta_{0\,\frac{k}{6}}(\tau_3/6, (z_2-\tau_3/2+\tau_2/2)/6)\,. \end{aligned}$$

To prove that $\mathcal{L}'$ can be extended to a line bundle $\bar{\mathcal{L}}'$ on $\bar{A}'$ we can argue exactly as in the proof of [**HKW**, Proposition (II.5.13)], the only difference being that we took the partial quotient with respect to a smaller group. (Note that if we take the quotient with respect to P we no longer obtain integer exponents of t_2.) The extension of the action of the symmetry group follows as in the proof of [**HKW**, Proposition (II.5.41)]. □

3. The map to $\mathbb{P}^3$

We consider boundary points $u' = (0, \tau_2, \tau_3) \in B'$ and $u = (0, t_2 t_3, t_2^{-1}) \in B$ (so $u = \varphi(u')$) and the associated degenerate abelian surfaces

$$\bar{A}'_{u'} = \bar{A}_u = A_{u,1} \cup A_{u,2}$$

where $A_{u,1} = A_{u,2}$ is an elliptic ruled surface. We gave a precise description of the surfaces $A_{u,i}$ and the way they are glued in Theorem 2.1. Recall that $A_{u,i}$ is an elliptic ruled surface with two disjoint sections $E_i, i = 1, 2$ of self-intersection number $E_i^2 = 0$. We shall denote the fibre over a point P of the base curve by f_P. Recall also the line bundle $\bar{\mathcal{L}}'$ on $\bar{A}'$. We set

$$\bar{\mathcal{L}}_u := \bar{\mathcal{L}}'|_{\bar{A}'_u} = \bar{\mathcal{L}}'|_{\bar{A}_u}, \quad \mathcal{L}_{u,i} := \bar{\mathcal{L}}'|_{A_{u,i}}.$$

In the proof of Proposition 2.3 we computed that

$$\begin{aligned}\lim_{t_1 \to 0} \widehat{\Theta}_{\alpha\beta}(\tau, z) = \quad & \vartheta_{0\frac{k}{6}}(\tau_3/6, (z_2 - \tau_3/2 - \tau_2/2)/6) \\ & + w^{-1} \vartheta_{0\frac{k}{6}}(\tau_3/6, (z_2 - \tau_3/2 + \tau_2/2)/6).\end{aligned}$$

The theta function $\vartheta_{0\frac{k}{6}}(\tau_3/6, (z_2 - \tau_3/2 + \tau_2/2)/6)$ has six zeroes on the elliptic curve $E(\tau_3) = \mathbb{C}/(\mathbb{Z}2\tau_3 + \mathbb{Z}6)$. Hence

$$\deg \mathcal{L}_{u,i}|_{E_i} = 6.$$

Since the exponent of w is -1 it follows that

$$\deg \mathcal{L}_{u,i}|_{f_P} = 1.$$

(See [**HKW**, Proposition(II.5.35)] for similar considerations in the $(1, p)$ case.) Hence

$$\mathcal{L}_{u,i} = \mathcal{O}_{A_{u,i}}(E_1 + 6 f_{P_i})$$

for a suitable point $P_i \in E(\tau_3)$. (The point P_i can be computed from $\lim_{t_1 \to 0} \widehat{\Theta}_{\alpha\beta}(t, z)$ and the normal bundle of E_1 in $A_{u,i}$, but we shall not need this later.) Standard arguments using Riemann-Roch show that

$$h^0(A_{u,i}, \mathcal{L}_{u,i}) = h^0(A_{u,i}, \mathcal{O}_{A_{u,i}}(E_1 + 6 f_{P_i})) = 12.$$

PROPOSITION 3.1. The restriction $\bar{\mathcal{L}}_u = \bar{\mathcal{L}}'|_{\bar{A}'_{u'}} = \bar{\mathcal{L}}'|_{\bar{A}_u}$ of $\bar{\mathcal{L}}'$ to the degenerate abelian surface $\bar{A}_u$ has the following properties:
(i) $h^0(\bar{A}_u, \bar{\mathcal{L}}_u) = 12$,
(ii) The restriction map rest : $H^0(\bar{A}', \bar{\mathcal{L}}') \to H^0(\bar{A}_u, \bar{\mathcal{L}}_u)$ is surjective.
(iii) The restriction map rest : $H^0(\bar{A}_u, \bar{\mathcal{L}}_u) \to H^0(A_{u,i} \mathcal{L}_{u,i})$ is an isomorphism.

PROOF. We consider the space $V \subset H^0(\bar{A}', \bar{\mathcal{L}}')$ which is spanned by the twelve sections $\widehat{s}_{\alpha\beta}$. Since these sections are a basis of $H^0(A_\tau, \mathcal{L}_\tau)$ for every τ the space V has dimension 12. We claim that the restriction map

$$\text{rest} : V \to H^0(\bar{A}_u, \bar{\mathcal{L}}_u)$$

is injective. By our computation of $\lim_{t\to 0}\widehat{\Theta}_{\alpha\beta}(\tau,z)$ it follows that this map is not identically zero. It is also H_{26}-equivariant and hence our claim follows if we can show that V is irreducible as an H_{26}–module. But this is easy to see: As an H_6-module $V = V_0 \oplus V_1$ where $V_i = \text{span } (\widehat{s}_{i\beta}, \beta = 0,\dots,5)$. The H_6–modules V_0 and V_1 are irreducible, Moreover addition by $e_3/2$ interchanges V_0 and V_1. Hence $h^0(\bar{A}_u, \bar{\mathcal{L}}_u) \geq 12$.

We have already remarked that $h^0(A_{u,i}, \mathcal{O}_{A_{u,i}}(E_1 + 6f_{P_i})) = 12$. Our next claim is that the map

$$\text{rest } : H^0(A_{u,i}, \mathcal{L}_{u,i}) \to H^0(E_1, \mathcal{L}_{u,i}|_{E_1}) \oplus H^0(E_2, \mathcal{L}_{u,i}|_{E_2})$$

is an isomorphism. Since the vector spaces on both side have the same dimension, namely 12, it is enough to prove injectivity. This follows from $h^0(A_{u,i}, \mathcal{O}_{A_{u,i}}(-E_2 + 6f_{P_i})) = 0$. But now this implies that glueing sections on $A_{u,1}$ and $A_{u,2}$ along E_1 and E_2 gives at least $2 \times 6 = 12$ conditions. Hence $h^0(\bar{A}_u, \bar{\mathcal{L}}_u) \leq 12$. With our previous argument this shows that $h^0(\bar{A}_u, \bar{\mathcal{L}}_u) = 12$ and hence both (i) and (ii) are proved. This also shows that

$$\text{rest} : H^0(\bar{A}_u, \bar{\mathcal{L}}_u) \to H^0(A_{u,i}, \mathcal{L}_{u,i})$$

is an isomorphism and hence we have shown (iii). □

Since we are interested in the map to $\mathbb{P}^3$ given by $H^0(\mathcal{L})^-$ we consider the subspace

$$V^- = \langle \widehat{g}_0, \widehat{g}_1, \widehat{g}_2, \widehat{g}_3 \rangle \subset V \subset H^0(\bar{A}', \bar{\mathcal{L}}')$$

and

$$\begin{aligned} V_u^- &= \text{rest } (V^- \to H^0(\bar{A}_u, \bar{\mathcal{L}}_u)), \\ V_{u,i}^- &= \text{rest } (V^- \to H^0(A_{u,i}, \mathcal{L}_{u,i})). \end{aligned}$$

The spaces V_u^- and $V_{u,i}^-$ are 4–dimensional. We want to study the map

$$\varphi_{V_u^-} : \bar{A}_u - - \to \mathbb{P}^3.$$

The sections $\widehat{g}_i$ vanish at the eight "2-torsion" points $P_1, \dots, P_8$ on $\bar{A}_u$ and hence

$$V_u^- \subset H^0(A_{u,i}, \mathcal{O}_{A_{u,i}}(E_1 + 6f_{P_i} - \sum_{j=1}^{8} P_j)).$$

Again using restriction to E_1 and E_2 it follows that the vector space on the right hand side has dimension 4 and hence

$$V_u^- = H^0(A_{u,i}, \mathcal{O}_{A_{u,i}}(E_1 + 6f_{P_i} - \sum_{j=1}^{8} P_j)).$$

We shall first consider the *product case*, i.e. $e = [\tau_2] = 0$. In this case $A_{u,i} = E(\tau_3) \times \mathbb{P}^1$ and there are four rulings which contain two of the points P_j each. These four rulings are, therefore, in the base locus of the linear system $|V_u^-|$. Removing this base locus we obtain the complete linear system of a line bundle on $A_{u,i}$ which has degree 2 on the sections E_i and degree 1 on the fibres. This maps $A_{u,i}$ $2:1$ onto a quadric. Since the map $\bar{A}_u - - \to \mathbb{P}^3$ factors through $A_{u,i}$ this shows that the "Kummer surface" $\bar{A}_u/\iota_\omega$ is mapped $2:1$ onto a quadric. It should be noted that double quadrics arise not only from degenerations of abelian surfaces, but also from special abelian surfaces, namely products (cf. Theorem 1.4). In fact what happens is that the map from the moduli space $\mathcal{A}_{1,3}(2)$ (or its extension to a

toroidal compactification) contracts each Humbert surface parametrizing product surfaces to a double point in N corresponding to a quadric.

From now on we shall assume $e \neq 0$. Then no fibre of the ruling contains two of the points P_i. We have to recall the notion of an *elementary transformation* of a ruled surface S at a point P. This consists of first blowing up S in P and then blowing down the strict transform of the fibre through P. The result is again a ruled surface $\mathrm{elm}_P S$. Let

$$\hat{A}_u := \mathrm{elm}_{P_1,\dots,P_8}(A_{u,i}).$$

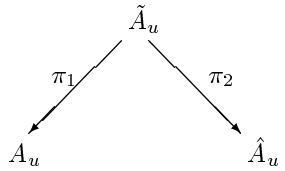
Then $\hat{A}_u$ has again two disjoint sections E_1 and E_2 with $E_i^2 = 0$. In particular

$$\hat{A}_u = \mathbb{P}(\mathcal{O}_{E(\tau_3)} \oplus \widehat{\mathcal{M}}_u) \text{ for some } \widehat{\mathcal{M}}_u \in \mathrm{Pic}^0(E(\tau_3)).$$

Note that $\widehat{\mathcal{M}}_u$ and $\widehat{\mathcal{M}}_u^{-1}$ define the same $\mathbb{P}_1$-bundle. It is straightforward to compute the normal bundle of the sections E_1 and E_2 in $A_{u,i}$ (cf. [**HKW**, p. 229]. Since we can control the self-intersection of a section under blowing up and blowing down and since we know the points P_j it is straightforward to show that $\widehat{\mathcal{M}}_u^{\pm 1} = \mathcal{O}_{E(\tau_3)}(2[\tau_2] - 2[0])$, and hence

$$\hat{A}_u = \mathbb{P}(\mathcal{O}_{E(\tau_3)} \oplus \mathcal{O}_{E(\tau_3)}(2[\tau_2] - 2[0])).$$

Consider the diagram



where $A_u = A_{u,i}$ and $\tilde{A}_u$ is A_u blown up in $P_1, \dots, P_8$. We denote the exceptional divisors over $P_1, \dots, P_8$ by $E^1, \dots, E^8$. The line bundle $\pi_1^* \mathcal{L}_{u,i} \otimes \mathcal{O}_{\tilde{A}_u}(-E^1 - \dots - E^8)$ has degree 0 on the strict transforms of the fibres through the points P_j. Hence

$$\hat{\mathcal{L}}_u := \pi_{2*}(\pi_1^* \mathcal{L}_{u,i} \otimes \mathcal{O}_{\tilde{A}_u}(-E^1 - \dots - E^8)) \in \mathrm{Pic}\ \hat{A}_u$$

is a line bundle on $\hat{A}_u$. The degree of $\hat{\mathcal{L}}_u$ is 1 on a ruling and 2 on the sections E_i. Hence

$$\hat{\mathcal{L}}_u = \mathcal{O}_{\hat{A}_u}(E_1 + 2 f_Q)$$

for a suitable point Q on the base curve. (Clearly Q can be computed explicitly, but this is immaterial for our purposes.) By the usual arguments

$$h^0(\hat{A}_u, \hat{\mathcal{L}}_u) = 4$$

and the rational map from A_u to $\hat{A}_u$ defines an isormorphism

$$\pi_{2*}\pi_1^* : V^- \cong H^0(\hat{A}_u, \hat{\mathcal{L}}_u).$$

PROPOSITION 3.2. Let $e \neq 0$. The linear system $|V^-|$ on $\hat{A}_u$ has the following properties:
(i) $|V^-|$ is base point free.
(ii) $|V^-|$ maps the two sections E_i each $2:1$ onto two skew lines.
(iii) $|V^-|$ is very ample outside the sections E_i. More precisely, if a cluster ζ of length 2 (i.e. two points or a point and a tangent direction) is not embedded, then ζ is contained in E_1 or E_2.

PROOF. This follows easily from Reider's theorem. We write

$$|V^-| = |E_1 + 2f_Q| = |K + L|$$

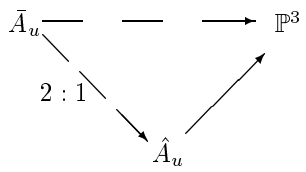
where the canonical divisor $K = -E_1 - E_2$ and $L = 2E_1 + E_2 + 2f_Q$. Then $L^2 = 12$. If $|V^-|$ is not base point free, then there exists a curve D with $L.D = 0$ or 1. Clearly such a curve cannot exist. If $|V^-|$ fails to embed a cluster ζ then there exists a curve $D \supset \zeta$ with $L.D = 2$ and $D^2 = 0$. Then D must be a section with $D.E_i = 0$. Since $\widehat{\mathcal{M}}_u \neq \mathcal{O}_{E(\tau_3)}$ (here we use $e = 2[\tau_2] \neq 0$) it follows that $D = E_1$ or $D = E_2$. Finally note that the restriction of $|V^-|$ to the elliptic curves E_i gives a complete linear system of degree 2. Hence these curves are mapped $2:1$ to lines. Since pairs (x, y) with $x \in E_1$ and $y \in E_2$ are separated, there lines are skew. □

We can now summarize our results as follows.

THEOREM 3.3. *Let $\bar{A}_u$ be a corank 1 degenerate abelian surface over a point $u = (0, T_2, T_3) \in B$ with $T_2T_3 \neq 0$ and consider the Kummer map given by the linear system $|V^-|$:*

$$\varphi_{|V^-|} : \bar{A}_u - - \to \mathbb{P}^3.$$

(i) *If $e = 0$ then $\bar{A}_u$ is mapped $4:1$ onto a smooth quadric.*
(ii) *If $e \neq 0$ then there is a commutative diagram.*



The image of $\bar{A}_u$ is an elliptic ruled surface which has double points along two skew lines but no other singularities.

PROOF. By construction the Kummer map $\bar{A}_u - - \to \mathbb{P}^3$ factors through $\bar{A}_u/\iota_\omega = A_{u,i}$. All other statements follow from our above discussion of the map $\varphi_{|V^-|} : \hat{A}_u \to \mathbb{P}^3$ given by the linear system $|V^-|$. □

This theorem explains the irreducible singular quartic surfaces which are parametrized by the S-planes, appearing already in [**Ni**, Section 5-2].

We want to conclude this paper with some remarks. We have already seen that the degenerate surfaces with $e = 0$ correspond to products. The limits of *bielliptic* abelian surfaces are characterised by $2e = 0, e \neq 0$. Geometrically this means that the surfaces $\bar{A}_u$ contain degenerate elliptic curves which are 4-gons, i.e. cycles consisting of 4 rulings. The degenerate abelian surfaces $\bar{A}_u$ where u is on another boundary component or where more than one of the T_i vanishes can also be described. If $T_3 = 0$ and $T_1T_2 \neq 0$ then $\bar{A}_u$ is a chain of 6 elliptic ruled surfaces. If two of the T_i are zero, then $\bar{A}_u$ consists of 12 quadrics, whereas $\bar{A}_0$ has 36 components, of which 24 are $\mathbb{P}^2$ and 12 are $\mathbb{P}^2$ blown up in three points. Limits of polarizations of type $(2, 6)$ exist on these surfaces, but as we pointed out before, there is no possibility of defining the Kummer map globally over B. On the other hand it is easy to construct degenerations of the ruled surfaces $A_{u,i}$ which lead to a union of two quadrics intersecting along a quadrangle or to a tetrahedron.

Finally we want to comment on the boundary components of the Igusa compactification of the moduli space $\mathcal{A}_{1,3}(2)$. These are enumerated by the *Tits building* of the group $\Gamma_{1,3}(2)$, i.e. by the equivalence classes modulo $\Gamma_{1,3}(2)$ of the lines and

isotropic planes in $\mathbb{Q}^4$. The Tits building was calculated by Friedland in [**F**]: for details, and for some other cases, see [**FS**]. There are 30 equivalence classes of lines. These correspond to the 15 equivalence classes of short, respectively long vectors. Each set of 15 lines is naturally parametrized by $(\mathbb{Z}/2)^4 \setminus \{0\} = \mathbb{P}^3(\mathbb{F}_2)$. The 15 planes are parametrized by the 15 isotropic planes in $\mathrm{Gr}(1, \mathbb{P}^3(\mathbb{F}_2))$. The isotropic planes are a hyperplane section of $\mathrm{Gr}(1, \mathbb{P}^3(\mathbb{F}_2))$ embedded as a quadric via the Plücker embedding. The 15 short and the 15 long vectors as well as the 15 planes are identified under the group $\Gamma_{1,3}/\Gamma_{1,3}(2) \cong \mathrm{Sp}(4, \mathbb{F}_2) \cong S_6$. That is, there are two equivalence classes of lines modulo $\Gamma_{1,3}$ and one plane (see also [**HKW**, Theorem(I.3.40)]). Finally the involution V_3 of the maximal arithmetic subgroup $\Gamma^*_{1,3}$ identifies short and long vectors (see [**G**, Folgerung 3.7] and [**HNS**, Section 2]). In our computations above the boundary component given by $T_1 = 0$ corresponds to a short vector, whereas the boundary component given by $T_3 = 0$ corresponds to a long vector. We described the degenerate abelian surfaces associated to points on a boundary component corresponding to a short vector. The matrix V_3 (and similarly any involution gV_3 where g is an element of $\Gamma_{1,3}$ – cf. [**HNS**, Theorem 2.4]) interchanges boundary components associated to short vectors with boundary components associated to long vectors. It should, however, be pointed out that the induced action of V_3 on the Igusa compactification $\mathcal{A}^*_{1,3}$ is only a rational map, not a morphism. This follows since the boundary components associated to long, and short vectors are not isomorphic: although their open parts (i.e. away from the corank-2 boundary components) are isomorphic (namely to the open Kummer modular surface $K^0(1)$), they contain different configurations of rational curves in the corank-2 boundary components. This follows from [**B**, Satz III.5.19] and [**W**, Theorem 4.13]. The degenerate abelian surfaces belonging to points on $T_3 = 0$ are different from those associated to points on $T_1 = 0$: they are a cycle of six elliptic ruled surfaces rather than two. At first this looks like a contradiction to [**HNS**, Theorem 2.4], but this is not the case. The polarization on each of the six components of the surface A_P where $P \in \{T_3 = 0\}$ is of the form $\mathcal{O}(E_1 + 2f_P)$. On four of the six components the linear system $|V^-|$ has a base locus consisting of a section. These four components are contracted. The other two components are identified under $|V^-|$ and are mapped to a quartic which is an elliptic ruled surface singular along two skew lines. In this way we find the same images in $\mathbb{P}^3$ as in the case $T_1 = 0$.

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