# Meagre Functions and Asymptotic Behaviour of Dynamical Systems

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**Abstract.** A measurable function $x : J \subset \mathbb{R} \to X$ ($X$ a metric space) is said to be $C$-meagre if $C \subset X$ is non-empty and, for every closed set $K \subset X$ with $K \cap C = \emptyset$, $x^{-1}(K)$ has finite Lebesgue measure. This concept of meagreness is shown to provide a unifying framework which facilitates a variety of characterizations, extensions or generalizations of diverse facts pertaining to asymptotic behaviour of dynamical systems.

## 1 Introduction

By way of motivation, consider the initial-value problem $\dot{x} = f(x)$, $x(0) = \xi \in \mathbb{R}^N$, with $f : \mathbb{R}^N \to \mathbb{R}^N$ locally Lipschitz, and denote its unique solution by $x : t \mapsto \phi(t, \xi)$ defined on its maximal interval of existence $I_\xi = [0, \tau_\xi)$. Let $g : \mathbb{R}^N \to \mathbb{R}$ be continuous. Assume that $x$ is a bounded solution of the initial-value problem (in which case, $I_\xi = [0, \infty)$), then $g \circ x$ is uniformly continuous. If, in addition, $g \circ x \in L^1$, then we may conclude (by an elementary observation frequently referred to as Barbălat's lemma [5] (see, also, [18, §21, Lemma 1])) that $g(x(t)) \to 0$ as $t \to 0$. Since, by boundedness, $x(t)$ approaches, as $t \to \infty$, its non-empty, compact, invariant $\omega$-limit set $\Omega$, we also conclude that $\Omega \subset g^{-1}(0)$. It follows that $x(t)$ tends, as $t \to \infty$, to the largest invariant subset of $g^{-1}(0)$. This is a short proof of Theorem 1.2 of [6] (the proof in [6] is based on arguments involving properties of the flow generated by the differential equation and does not make use of Barbălat's lemma). A simple corollary is a version of LaSalle's invariance principle [14, Chap.2, Theorem 6.4]: if $\phi(t, \xi) : [0, \infty) \to \mathbb{R}^N$ is a bounded solution of the initial-value problem, $V : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, and $g(x) := \langle \nabla V(x), f(x) \rangle \leq 0$ for all $x$ in some neighbourhood of the orbit $\phi([0, \infty), \xi)$, then $\phi(t, \xi)$ tends, as $t \to \infty$, to the largest invariant subset in $g^{-1}(0)$. In this paper, we present variants of the above observations (and other related results) in contexts of systems defined by abstract semiflows, finite-dimensional differential inclusions, and non-autonomous semilinear infinite-dimensional systems. For example, in the context of a semiflow $\phi$, we identify a class of functions $g$ for which convergence (as $t \to \infty$) of $x(t) = \phi(t, \xi)$ to the largest invariant set in $g^{-1}(0)$ is a consequence of the hypothesis $g \circ x \in L^1$ without positing *a priori* that $\phi([0, \infty), \xi)$ be bounded or relatively compact; as a corollary, we obtain that $L^p$-trajectories ($1 \leq p < \infty$) of semiflows converge to 0 as $t \to \infty$. These and other results are shown to follow – in an entirely elementary manner – from three technical lemmas pertaining to the concept of meagre functions. In addition, via the concept of meagreness, some sufficient conditions for asymptotic stability of compacta in the context of semiflows on metric spaces are presented which provide characterizations for asymptotic stability in the context of semiflows on metric spaces with the property that every ball of finite radius is relatively compact. Furthermore, some characterizations of exponential stability for linear semigroups are provided.

Whilst a subset of the results on asymptotic behaviour presented in Section 5 below are (partially) known, we emphasize that an important aspect of the paper is the simplicity of the approach by which the results are obtained in a natural and direct way. Finally we remark that, whilst the technical lemmas on meagre functions and the results on abstract semiflows are derived under the assumption that the underlying space $X$ is a metric space, most of these results (in a suitably modified form) remain true if $X$ is a topological space or, even more generally, a sequential-convergence space or a limit space (as adopted in [3]).

## 2 Notation and terminology

Throughout, $X$ denotes a metric space, with metric $\rho$, $\mathbb{R}_+ := [0, \infty)$ and $J$ denotes a non-empty, measurable subset of $\mathbb{R}_+$. For non-empty $C \subset X$, $d_C : X \to \mathbb{R}_+$ denotes its *distance function* given by $d_C(v) := \inf\{\rho(c, v) \mid c \in C\}$. The function $d_C$ is globally Lipschitz of rank 1: $|d_C(v) - d_C(w)| \le \rho(v, w)$ for all $v, w \in X$. For $\epsilon > 0$, $\mathbb{B}_\epsilon(C)$ denotes the $\epsilon$-neighbourhood of $C$ given by $\{v \in X \mid d_C(v) < \epsilon\}$; if $C = \{c\}$ (singleton) then we simply write $\mathbb{B}_\epsilon(c)$. A function $x : J \to X$ is said *to approach* $C$ if $d_C(x(t)) \to 0$ as $t \to \sup J$. A function $x : J \to X$ is *relatively compact* if $\mathrm{cl}(x(J))$ (that is, the closure of its trajectory) is compact. A point $l \in X$ is an $\omega$-limit point of $x : J \to X$ if there exists a sequence $(t_n) \subset J$ with $t_n \to \sup J$ and $x(t_n) \to l$ as $n \to \infty$; the set $\Omega(x)$ of all such $\omega$-limit points is the $\omega$-limit set of $x$. The $\omega$-limit set $\Omega(x)$ is always closed, and, if $x$ is relatively compact, then $\Omega(x)$ is non-empty, compact, is approached by $x$ and is the smallest closed set so approached. The restriction of a function $x : J \to X$ to $I \subset J$ is denoted by $x|_I$. Lebesgue measure on $\mathbb{R}$ is denoted by $\mu$.
A function $y : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{K}$-function if it continuous and strictly increasing with $y(0) = 0$; $y$ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function with $y(s) \to \infty$ as $s \to \infty$. A function $(s, t) \mapsto z(s, t) \in \mathbb{R}_+$ with domain $\mathbb{R}_+ \times (0, \infty)$ is a $\mathcal{KL}$-function if, for each $t$, $z(\cdot, t)$ is a $\mathcal{K}$-function and, for each $s$, $z(s, \cdot)$ is a decreasing function.

## 3 Meagre functions

**Definition.** Let $C \subset X$ be non-empty. A measurable function $x : J \to X$ is *$C$-meagre* if for every closed set $K \subset X$ with $K \cap C = \emptyset$, the set $x^{-1}(K) := \{t \in J \mid x(t) \in K\}$ has finite measure $\mu(x^{-1}(K)) < \infty$. If $X$ is a vector space and $C = \{0\}$, then we use the simple term *meagre* in place of the more cumbersome *$\{0\}$-meagre.*

For measurable $x : J \to X$ and non-empty $C \subset X$, it is clear that

$$x \text{ is } C\text{-meagre} \quad \Rightarrow \quad d_C \circ x \text{ is meagre.}$$

The converse is false in general, but is true in the case of *compact* $C$. We also remark that the concept of a $C$-meagre function, with $C$ compact, is intimately related to the concept of *convergence in measure*: a sequence $(f_n)$ of measurable functions $J \subset \mathbb{R} \to \mathbb{R}$ is said to converge in measure to $f : J \to \mathbb{R}$ if, for each $\epsilon > 0$, $\lim_{n\to\infty} \mu(\{t \in J \mid |f_n(t) - f(t)| \ge \epsilon\}) \to 0$ as $n \to \infty$. The relationship is described in the following proposition.

**Proposition 1.** Let $C \subset X$ be non-empty and compact. Let $x : J \to X$ be a measurable function and define $\gamma := d_C \circ x$. The following statements are equivalent:-

(i) $x$ is $C$-meagre;

(ii) $\gamma$ is meagre;

(iii) for every $\epsilon > 0$, $\lim_{\tau\to\infty} \mu(\{t \in J\backslash[0,\tau] \mid \gamma(t) \geq \epsilon\}) = 0$;

(iv) for every sequence $(\tau_n) \subset \mathbb{R}_+$ with $\tau_n \to \infty$ as $n \to \infty$, the sequence $(\gamma_n)$ of functions $J \to \mathbb{R}_+$ given by

$$\gamma_n(t) := \begin{cases} 0, & t \in J \cap [0,\tau_n] \\ \gamma(t), & t \in J\backslash[0,\tau_n] \end{cases}$$

converges in measure to the zero function.

**Proof.** That (i) implies (ii) is clear. That (ii) implies (i) is a consequence of the observation that, for every closed set $K$ with $K \cap C = \emptyset$,

$$\mu(x^{-1}(K)) \leq \mu(\{t \in J \mid \gamma(t) \geq \epsilon\}),$$

where $\epsilon := \inf_{k\in K} d_C(k) > 0$ (positive by virtue of closedness of $K$ and compactness of $C$). Equivalence of (ii) and (iii) is a consequence of the observation that, for every $\epsilon > 0$,

$$\begin{aligned}\mu(\{t \in J \mid \gamma(t) \geq \epsilon\}) &= \mu(\{t \in J \cap [0,\tau] \mid \gamma(t) \geq \epsilon\}) + \mu(\{t \in J\backslash[0,\tau] \mid \gamma(t) \geq \epsilon\}) \\ &\leq \tau + \mu(\{t \in J\backslash[0,\tau] \mid \gamma(t) \geq \epsilon\}) \quad \forall\, \tau \in \mathbb{R}_+.\end{aligned}$$

Finally, equivalence of (iii) and (iv) is a consequence of the observation that, for every $\epsilon > 0$ and every sequence $(\tau_n) \subset \mathbb{R}_+$,

$$\mu(\{t \in J\backslash[0,\tau_n] \mid \gamma(t) \geq \epsilon\}) = \mu(\{t \in J \mid \gamma_n(t) \geq \epsilon\}) \quad \forall\, n.$$

■

As in [24], a measurable function $x : J \to X$, $X = (X, \|\cdot\|)$ a Banach space, is said to be of (Marcinkiewicz) *weak-$L^p$ class*, $1 \leq p < \infty$, written $x \in wk\text{-}L^p(J;X)$, if there exists a constant $c > 0$ such that

$$\mu(\{t \in J \mid \|x(t)\| \geq \lambda\}) < \frac{c}{\lambda^p} \quad \forall\, \lambda > 0.$$

It is straightforward to verify that $x \in L^p(J;X)$ implies $x \in wk\text{-}L^p(J;X)$.

We record a further simple fact.

**Proposition 2.** Let $X$ be a Banach space. If $x \in wk\text{-}L^p(J;X)$ for some $p \in [1,\infty)$, then $x$ is meagre.

The next result asserts that essentially-bounded functions of weak-$L^p$ class are necessarily of class $L^q$ for all $q > p$: this will prove useful in a later study of exponential stability of linear semigroups.

**Proposition 3.** Let $X$ be a Banach space. If $x \in L^\infty(\mathbb{R}_+;X)$ and $x \in wk\text{-}L^p(\mathbb{R}_+;X)$ for some $p \in [1,\infty)$, then $x \in L^q(\mathbb{R}_+;X)$ for all $q > p$.

**Proof.** By hypothesis, there exists $\Lambda > 0$ such that $\|x(t)\| \le \Lambda$ almost everywhere. For each $n \in \mathbb{N}$, define

$$S_n := \left\{ s \mid \frac{\Lambda}{2^n} \le \|x(t)\| \le \frac{\Lambda}{2^{n-1}} \right\}.$$

Since $x \in wk\text{-}L^p(\mathbb{R}_+; X)$, $\mu(S_n) \le c\Lambda^{-p}2^{np}$. Therefore, for all $q \in (p, \infty)$,

$$\int_0^\infty \|x(t)\|^q dt = \sum_{n\in\mathbb{N}} \int_{S_n} \|x(t)\|^q dt \le \sum_{n\in\mathbb{N}} \Lambda^q 2^{-nq+q}\mu(S_n) \le 2^q\Lambda^{q-p} \sum_{n\in\mathbb{N}} \left(\frac{1}{2^{q-p}}\right)^n < \infty.$$

■

The final result of this section provides another connection between meagre functions and convergence of integrals: this will play a central rôle in a later study of global asymptotic stability of compacta in a context of semiflows.

**Proposition 4.** Let $\mathcal{M}$ be a non-empty set of measurable functions $\mathbb{R}_+ \to \mathbb{R}_+$. The following statements are equivalent.

(i) There exists a $\mathcal{KL}$-function $\kappa : \mathbb{R}_+ \times (0,\infty) \to \mathbb{R}_+$ such that

$$\mu(\{t \mid \gamma(t) \ge \lambda\}) \le \kappa(\gamma(0), \lambda) \quad \forall\ (\gamma, \lambda) \in \mathcal{M} \times (0,\infty). \tag{1}$$

(ii) There exist $\mathcal{K}$-functions $\alpha$ and $\beta$ such that

$$\int_0^\infty \alpha(\gamma(t))dt \le \beta(\gamma(0)) \quad \forall\ \gamma \in \mathcal{M}. \tag{2}$$

**Proof.** (i) $\Rightarrow$ (ii): Assume that (i) holds. Let $(\lambda_n)_{n\in\mathbb{Z}} \subset (0,\infty)$ be a bi-sequence with $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$ and such that

$$\lim_{n\to-\infty} \lambda_n = 0\,, \quad \lim_{n\to\infty} \lambda_n = \infty\,.$$

Choose a bi-sequence $(\eta_n)_{n\in\mathbb{Z}} \subset (0,\infty)$ with the properties

$$\sum_{n\in\mathbb{Z}} \eta_n < \infty \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \eta_n \kappa(|n|, \lambda_n) < \infty\,.$$

For example, the bi-sequence given by

$$\eta_0 = 1\,, \quad \eta_n = \min\{1\,,\, 1/\kappa(|n|, \lambda_n)\}/n^2 \ \text{ if } n \ne 0$$

suffices. Define $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\beta(\delta) := \sum_{n\in\mathbb{Z}} \eta_n \kappa(\delta, \lambda_n).$$

By properties of $\kappa$ and $(\eta_n)_{n\in\mathbb{Z}}$, this sum converges uniformly for $\delta$ in compact intervals and so $\beta$ is continuous. Moreover, by properties of $\kappa$, $\beta$ is strictly increasing and $\beta(0)=0$. Therefore, $\beta$ is a $\mathcal{K}$-function. Define $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ by

$$\alpha(0)=0\,,\qquad \alpha(\lambda_{n+1})=\sum_{m=-\infty}^{n}\eta_m\quad \forall\, n\in\mathbb{Z}\,,$$

$$\alpha(\lambda)=\frac{1}{\lambda_{n+1}-\lambda_n}[(\alpha(\lambda_{n+1})-\alpha(\lambda_n))\lambda+\lambda_{n+1}\alpha(\lambda_n)-\lambda_n\alpha(\lambda_{n+1})]\quad \forall\,\lambda\in[\lambda_n,\lambda_{n+1}]\,.$$

The last condition means that $\alpha$ interpolates linearly on $[\lambda_n,\lambda_{n+1}]$. Evidently, $\alpha$ is a $\mathcal{K}$-function.

Let $\gamma\in\mathcal{M}$ be arbitrary and, for each $n\in\mathbb{N}$, define $M_n:=\{t\,|\,\gamma(t)\in[\lambda_n,\lambda_{n+1})\}$. Then,

$$\begin{aligned}
\beta(\gamma(0)) &= \sum_{n\in\mathbb{Z}}\eta_n\kappa(\gamma(0),\lambda_n)\geq\sum_{n\in\mathbb{Z}}\eta_n\mu(\{t\,|\,\gamma(t)\geq\lambda_n\})\\
&= \sum_{n\in\mathbb{Z}}\eta_n\sum_{m=n}^{\infty}\mu(M_m)=\sum_{m\in\mathbb{Z}}\mu(M_m)\sum_{n=-\infty}^{m}\eta_n=\sum_{m\in\mathbb{Z}}\mu(M_m)\alpha(\lambda_{m+1})\\
&\geq \sum_{m\in\mathbb{Z}}\int_{M_m}\alpha(\gamma(t))dt=\int_0^\infty\alpha(\gamma(t))dt\,.
\end{aligned}$$

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Then

$$\mu(\{t\,|\,\gamma(t)\geq\lambda\})\alpha(\lambda)\leq\int_0^\infty\alpha(\gamma(t))dt\leq\beta(\gamma(0))$$

and so (a) holds with $\kappa$ given by $\kappa(\delta,\lambda):=\beta(\delta)/\alpha(\lambda)$. ∎

## 4 Three key lemmas

The main purpose of this section is to present three lemmas, Lemmas 6, 7 and 9 below, which form the basis for later results. These lemmas, which are unremarkable in themselves, collectively provide a unifying framework that facilitates the derivation of diverse results pertaining to asymptotic behaviour of dynamical systems. Loosely speaking, Lemma 6 identifies conditions under which meagreness implies convergence, Lemma 7 provides a technical convenience, and Lemma 9 establishes a connection between meagreness and $\omega$-limit sets.

In the sequel we shall repeatedly extract subsequences of sequences. In order to avoid multiple subscripts, it is useful to note the trivial fact that a sequence $(\tilde{x}_n)\subset X$ is a subsequence of a given sequence $(x_n)\subset X$ if, and only if, there exists a strictly increasing map $\sigma:\mathbb{N}\to\mathbb{N}$ such that $\tilde{x}_n=x_{\sigma(n)}$ for all $n$. Throughout, the symbols $\sigma$ and $\sigma_1$ will be used to denote such strictly increasing mappings from $\mathbb{N}$ to $\mathbb{N}$.

In the context of measurable functions $\gamma:\mathbb{R}_+\to\mathbb{R}$, the following hypothesis (H) will be widely invoked.

$$\text{(a) } \gamma \text{ is meagre, or (b) } \lim_{t\to\infty}\int_t^{t+h} |\gamma(\tau)|\, d\tau = 0 \text{ for some } h > 0\,. \tag{H}$$

**Proposition 5.** Let $\gamma : \mathbb{R}_+ \to \mathbb{R}$ be measurable and satisfy (H). Then, for all sequences $(\epsilon_n) \subset (0,\infty)$, $(b_n) \subset \{-1,1\}$ and $(t_n) \subset \mathbb{R}_+$ with $t_{n+1} - t_n \geq 1$ for all $n$, there exists a sequence $(s_n) \subset \mathbb{R}_+$ and a subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that, for all $n$, $|\gamma(s_n)| < \epsilon_n$ and $b_n(t_{\sigma(n)} - s_n) \geq 0$ with $t_{\sigma(n)} - s_n \to 0$ as $n \to \infty$.

**Proof.** For $k, n \in \mathbb{N}$, set

$$U_k^n := \{t \in [t_k - (b_n+1)/4n\,,\, t_k - (b_n-1)/4n] \mid |\gamma(t)| < \epsilon_n\}\,.$$

For each fixed $n$, the sets $U_k^n$, $k \in \mathbb{N}$, do not overlap since $t_{k+1} - t_k \geq 1$. By (H), there exists a sequence $(k_n) \subset \mathbb{N}$ such that, for all $n$, $k_n < k_{n+1}$ and $U_{k_n}^n \neq \emptyset$. The result follows by setting $\sigma(n) = k_n$ and choosing $s_n \in U_{\sigma(n)}^n$ for all $n \in \mathbb{N}$. ∎

**Remark.** Conditions (a) and (b) in (H) are independent (in the sense that any one does not imply the other) as the following examples illustrate: let $X = \mathbb{R}$ and, on $\mathbb{R}_+$, define functions $\gamma_a$, $\gamma_b$: $\mathbb{R}_+ \to \mathbb{R}$ by

$$\gamma_a(t) := \begin{cases} n^2, & t \in \cup_n [n, n+(1/n^2)] \\ 0, & \text{otherwise} \end{cases} \qquad \gamma_b(t) := \begin{cases} 1, & t \in \cup_n [n, n+(1/n)] \\ 0, & \text{otherwise} \end{cases}$$

then $\gamma_a$ satisfies (a) but not (b), whilst $\gamma_b$ satisfies (b) but not (a). However, for a large class of functions, the conditions (a) and (b) in (H) are equivalent, as will be shown in the corollary to Lemma 6 below.

**Definitions.** Let $x : J \to X$ be measurable with non-empty $\omega$-limit set $\Omega(x)$. An $\omega$-limit point $l \in \Omega(x)$ is said to be robust if, for all sequences $(s_n), (t_n) \subset J$ with (i) $\lim_{n\to\infty} s_n = \sup J$, (ii) $t_n \geq s_n$ and (iii) $\lim_{n\to\infty}(t_n - s_n) = 0$,

$$\lim_{n\to\infty} x(s_n) = l \quad \Rightarrow \quad \lim_{n\to\infty} x(t_n) = l.$$

If every $l \in \Omega(x)$ is robust, then the $\omega$-limit set $\Omega(x)$ is said to be robust. For each $l \in X$, define

$$\mathcal{R}_l(J;X) := \{x : J \to X \mid x \text{ measurable},\ l \in \Omega(x),\ l \text{ robust}\}$$

and

$$\mathcal{R}(J;X) := \{x : J \to X \mid x \text{ measurable},\ \Omega(x) \text{ non-empty and robust}\}.$$

Whilst the class $\mathcal{R}(J;X)$ may appear somewhat obscure, it is a convenient artefact for our purposes: in the following remarks, two concrete subclasses of $\mathcal{R}(J;X)$ are identified.

**Remarks.** 1. If $x$ corresponds to the orbit of a local semiflow and has non-empty $\omega$-limit set, then $x \in \mathcal{R}(\mathbb{R}_+; X)$ (see the proof of Theorem 10 below).

2. If $x : J \to X$ is uniformly continuous and has non-empty $\omega$-limit set, then $x \in \mathcal{R}(J; X)$.

3. Evidently, the converse of remark 2 is not true, since there are trivial examples of discontinuous functions belonging to $\mathcal{R}(J; X)$. If $x \in \mathcal{R}(J; X)$ is continuous and relatively compact, then $x$ is uniformly continuous, as can be easily shown; however, as the following example shows, it is not difficult to construct continuous functions $x \in \mathcal{R}(J; X)$ such that $x$ is not relatively compact and not uniformly continuous. Set $k_n := 3^{n-1}$, $n \in \mathbb{N}$, $b_1 := 1$ and $b_n := (k_n^2 - k_n)/(k_n^2 - 1)$ for all $n \geq 2$. Define $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\gamma(t) := \begin{cases} 0\,, & 0 \leq t \leq 1 \\ t - k_n\,, & k_n \leq t \leq 2k_n \\ k_n - k_n(t - 2k_n)\,, & 2k_n \leq t \leq 2k_n + 1/k_n \\ k_n - 1 - b_n(t - 2k_n - 1/k_n)\,, & 2k_n + 1/k_n \leq t \leq 3k_n\,. \end{cases}$$

Then $\gamma$ is a continuous and unbounded function. Since $\gamma(2k_n) - \gamma(2k_n + 1/k_n) = 1$ for all $n$, it is clear that $\gamma$ is not uniformly continuous. To show that $\gamma \in \mathcal{R}(\mathbb{R}_+; \mathbb{R})$, let $l \in \Omega(\gamma) = \mathbb{R}_+$ and let $(s_n) \subset \mathbb{R}_+$ be a sequence with $\lim_{n\to\infty} s_n = \infty$ and such that $\lim_{n\to\infty} \gamma(s_n) = l$. Let $\epsilon \in (0,1)$ and set $\Delta_n = [2k_n - \epsilon, 2k_n + \epsilon]$. Then, since the sequence $(\gamma(s_n))$ is bounded, elementary considerations yield the existence of $N \in \mathbb{N}$ such that $s_n \notin \Delta_m$ for all $n \geq N$ and all $m \in \mathbb{N}$ and a routine argument then gives $\gamma \in \mathcal{R}(\mathbb{R}_+; \mathbb{R})$.

4. To provide a further link between robustness and uniform continuity, let $x : \mathbb{R}_+ \to X$ be a continuous function with $\Omega(x) \neq \emptyset$ and consider the following three statements

(i) for each $z \in X$, there exists a neighbourhood $U \subset X$ of $z$ such that either $x^{-1}(U) = \emptyset$ or $x$ is uniformly continuous on $x^{-1}(U)$;

(ii) $x \in \mathcal{R}(\mathbb{R}_+; X)$;

(iii) if $U \subset X$ is compact, then either $x^{-1}(U) = \emptyset$ or $x$ is uniformly continuous on $x^{-1}(U)$;

It is not difficult to show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and that, if $X$ is locally compact, then all three statements are equivalent.

We now present the first of the three key lemmas.

**Lemma 6.** Let $\gamma : \mathbb{R}_+ \to \mathbb{R}$ be measurable and, for $\epsilon > 0$, define $J_\epsilon := \{t \in \mathbb{R}_+ \mid |\gamma(t)| < \epsilon\}$. Then the following statements hold:-

(i) $\gamma$ satisfies (H) and $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R}) \quad \Leftrightarrow \quad \lim_{t\to\infty} \gamma(t) = 0$;

(ii) if $\gamma$ satisfies (H) and is continuous and there exists some $\epsilon > 0$ such that $\gamma|_{J_\epsilon} \in \mathcal{R}_0(J_\epsilon;\mathbb{R})$, then $\lim_{t\to\infty} \gamma(t) = 0$.

**Proof.** (i) "$\Rightarrow$" Assume $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$ satisfies (H). Seeking a contradiction, suppose that $\lim_{t\to\infty} \gamma(t) \neq 0$. Then there exist $\epsilon > 0$ and $(t_n) \subset \mathbb{R}_+$, with $t_{n+1} - t_n \geq 1$ for all $n$, such that $|\gamma(t_n)| \geq \epsilon$. By Proposition 5, there exists a sequence $(s_n) \subset \mathbb{R}_+$ and a subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that $\lim_{n\to\infty} \gamma(s_n) = 0$ and, for all $n$, $t_{\sigma(n)} - s_n \geq 0$ with $t_{\sigma(n)} - s_n \to 0$ as $n \to \infty$. Since $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$, $\lim_{n\to\infty} \gamma(t_{\sigma(n)}) = 0$, yielding the contradiction

$$0 < \epsilon \leq \lim_{n\to\infty} |\gamma(t_{\sigma(n)})| = 0 .$$

"$\Leftarrow$" Now assume $\lim_{t\to\infty} \gamma(t) = 0$. Then $\Omega(\gamma) = \{0\}$ and $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$. Moreover, for all $\lambda > 0$, $\mu\{t \in \mathbb{R}_+ \mid |\gamma(t)| \geq \lambda\}$ is finite and so $\gamma$ is meagre.

(ii) Let $\epsilon > 0$ be such that $\gamma|_{J_\epsilon} \in \mathcal{R}_0(J_\epsilon;\mathbb{R})$. Again, seeking a contradiction, suppose that the claim is not true. Then $\limsup_{t\to\infty} |\gamma(t)| > \delta$ for some $\delta \in (0,\epsilon)$. Using (H) and the continuity of $\gamma$ it is clear that there exists $(t_n) \subset \mathbb{R}_+$ with $t_{n+1} - t_n \geq 1$ for all $n$ and such that $|\gamma(t_n)| = \delta$. By Proposition 5, there exist a sequence $(s_n) \subset \mathbb{R}_+$ and a subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that, for all $n$, $|\gamma(s_n)| < \delta/n$ and $t_{\sigma(n)} - s_n \geq 0$ with $t_{\sigma(n)} - s_n \to 0$ as $n \to \infty$. By construction, $t_{\sigma(n)}, s_n \in J_\epsilon$ for all $n$. Since $\gamma|_{J_\epsilon} \in \mathcal{R}_0(J_\epsilon;\mathbb{R})$, it follows that $\lim_{n\to\infty} \gamma(t_{\sigma(n)}) = 0$, yielding the contradiction

$$0 < \delta = |\gamma(t_{\sigma(n)})| \to 0 , \quad \text{as } n \to \infty .$$

∎

The following corollary is a consequence of part (i) of the above lemma.

**Corollary.** Let $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$. Condition (a) in (H) holds if, and only if, condition (b) in (H) holds.

**Remark.** As previously mentioned in the Introduction, an elementary result, frequently referred to as Barbălat's lemma and widely invoked in stability analyses of (controlled) dynamical systems, is the following: let $\gamma : \mathbb{R}_+ \to \mathbb{R}$ be uniformly continuous; if

$$\lim_{t\to\infty} \int_0^t \gamma(\tau)\, d\tau \quad \text{exists and is finite,}$$

then $\lim_{t\to\infty} \gamma(t) = 0$. This result is subsumed by Lemma 6.

**Definition.** For $C \subset G \subset X$, $C$ non-empty, let $\mathcal{D}_C(G)$ be the set of all Borel functions $g : G \to \mathbb{R}$ such that $g^{-1}(0) = C$, $g$ is continuous at $c$ for all $c \in C$ and $\inf\{|g(z)| \mid z \in K \cap G\} > 0$ for any closed set $K \subset X$ with $K \cap C = \emptyset$ and $K \cap G \neq \emptyset$.

For example, if $C \subset X$ is non-empty and compact, then $d_C \in \mathcal{D}_C(X)$.

Next, we present the second of the three key lemmas.

**Lemma 7.** Let $C \subset G \subset X$, $C$ non-empty, $g \in \mathcal{D}_C(G)$ and $(z_n) \subset G$. If $g(z_n) \to 0$ as $n \to \infty$, then $d_C(z_n) \to 0$ as $n \to \infty$. If, in addition, $C$ is compact, then there exists $c \in C$ and a subsequence $(z_{\sigma(n)})$ of $(z_n)$ such that $z_{\sigma(n)} \to c$ as $n \to \infty$.

**Proof.** Assume $g(z_n) \to 0$ as $n \to \infty$ and, for contradiction, suppose that $d_C(z_n) \not\to 0$ as $n \to \infty$. Then there exists $\epsilon > 0$ and a subsequence $(z_{\sigma(n)})$ of $(z_n)$ such that $d_C(z_{\sigma(n)}) \geq \epsilon$ for all $n$. Let $K$ be the closed set

$$K := \{z \in X \mid d_C(z) \geq \epsilon\}.$$

Then $K \cap C = \emptyset$, $z_{\sigma(n)} \in K \cap G$ for all $n$, and, since $g \in \mathcal{D}_C(G)$, we have a contradiction: $\inf_n |g(z_{\sigma(n)})| > 0$. Hence, $d_C(z_n) \to 0$ as $n \to 0$. Now assume, in addition, that $C$ is compact. Then there exists a sequence $(c_n) \subset C$ such that $d_C(z_n) = \rho(z_n, c_n) \to 0$ as $n \to \infty$ and, for some subsequence $(c_{\sigma(n)})$, $c_{\sigma(n)} \to c \in C$ as $n \to \infty$. Therefore, $z_{\sigma(n)} \to c \in C$ as $n \to \infty$. ■

Before presenting the last of the three key lemmas, we prove a preliminary result.

**Proposition 8.** Let $x : J \to X$ be measurable, let $G \subset X$ with $G \supset \mathrm{cl}(x(J))$, let $C \subset G$ be non-empty and compact and let $g \in \mathcal{D}_C(G)$. If $x \in \mathcal{R}(J; X)$ and $\Omega(x) \cap C \neq \emptyset$, then $\gamma := g \circ x \in \mathcal{R}_0(J; \mathbb{R})$.

**Proof.** Since $\Omega(x) \cap C \neq \emptyset$ it follows that $0 \in \Omega(\gamma)$. Let $(s_n), (t_n) \subset J$ be sequences such that $\lim_{n\to\infty} s_n = \sup J$, $\lim_{n\to\infty} \gamma(s_n) = 0$, $t_n \geq s_n$ and $\lim_{n\to\infty}(t_n - s_n) = 0$. Seeking a contradiction, suppose the claim is not true, i.e. $\gamma \notin \mathcal{R}_0(J; \mathbb{R})$. Then there exists a subsequence $(t_{\sigma(n)})$ of $(t_n)$ and $\epsilon > 0$ such that $|\gamma(t_{\sigma(n)})| \geq \epsilon$ for all $n$. By Lemma 7, there exists $c \in C$ and a subsequence $(s_{\sigma_1(n)})$ of $(s_{\sigma(n)})$ such that $\lim_{n\to\infty} x(s_{\sigma_1(n)}) = c$. Since $x \in \mathcal{R}(J; X)$, it follows that $\lim_{n\to\infty} x(t_{\sigma_1(n)}) = c$. Thus we arrive at the contradiction

$$0 < \epsilon \leq \|\gamma(t_{\sigma_1(n)})\| \to 0\,, \quad \text{as } n \to \infty\,.$$ ■

**Corollary.** Let $C \subset X$ be non-empty, let $x : \mathbb{R}_+ \to X$ be measurable, set $\gamma := d_C \circ x$ and define $J_\epsilon := \{t \in \mathbb{R}_+ \mid \gamma(t) < \epsilon\}$. Then the following statements hold:-

(i) if (a) $C$ is compact, (b) $x$ is $C$-meagre, and (c) $x \in \mathcal{R}(\mathbb{R}_+; X)$, then $x$ approaches $C$;

(ii) if (a) $C$ is compact, (b) $x$ is $C$-meagre and continuous, and (c) there exists some $\epsilon > 0$ such that and $x|_{J_\epsilon} \in \mathcal{R}(J_\epsilon; \mathbb{R})$, then $x$ approaches $C$;

(iii) if (a) $x$ is $C$-meagre and continuous, and (b) there exists some $\epsilon > 0$ such that $x|_{J_\epsilon}$ is uniformly continuous, then $x$ approaches $C$.

**Proof.** In order to prove statement (i), we first note that, by $C$-meagreness of $x$, there exist $(c_n) \subset C$ and $(t_n) \subset \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that $\rho(x(t_n), c_n) \to 0$ as $n \to \infty$. Invoking compactness of $C$, it follows that $\Omega(x) \cap C \neq \emptyset$.

Since $d_C \in \mathcal{D}_C(X)$, Proposition 8 shows that $\gamma := d_C \circ x \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$. By $C$-meagreness of $x$, $\gamma$ is meagre, and the claim follows from Lemma 6, part (i). To prove statement (ii), note that $\Omega(x) \cap C = \Omega(x|_{J_\epsilon}) \cap C \neq \emptyset$. By using part (ii) (instead of part (i)) of Lemma 6, statement (ii) can now be proved in exactly the same manner as in the proof of statement (i). Finally, to prove statement (iii), observe that the assumptions on $x$ imply that $\gamma := d_C \circ x$ is meagre and continuous and $\gamma|_{J_\epsilon} \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$ for some $\epsilon > 0$. Hence, by part (ii) of Lemma 6, $\lim_{t\to\infty}\gamma(t) = 0$, and thus $x$ approaches $C$. ∎

**Lemma 9.** Let $x \in \mathcal{R}(\mathbb{R}_+;X)$, let $G \subset X$ with $G \supset \mathrm{cl}(x(\mathbb{R}_+))$ and let $C \subset G$ be non-empty and compact. Then the following statements hold:-

(i) if $g \in \mathcal{D}_C(G)$ and $\gamma = g \circ x$ satisfies (H), then $x$ approaches $C$ and $\Omega(x) \subset C$;

(ii) if $g : G \to \mathbb{R}$ is such that $|g|$ (that is, the map $z \mapsto |g(z)|$) is lower semicontinuous and $\gamma = g \circ x$ satisfies (H), then $\Omega(x) \subset g^{-1}(0)$.

**Proof.** (i) Using (H) we may conclude that there exists a sequence $(t_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$ and such that $\lim_{n\to\infty}\gamma(t_n) = 0$. By Lemma 7, there exists $c \in C$ and a subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that $\lim_{n\to\infty} x(t_{\sigma(n)}) = c$, showing that $\Omega(x) \cap C \neq \emptyset$. Therefore, by Proposition 8, $\gamma \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$. By Lemma 6, part (i) it follows that $\lim_{t\to\infty}\gamma(t) = 0$. Again using Lemma 7, it follows that $x$ approaches $C$ which implies, by compactness of $C$, that $\Omega(x) \subset C$.

(ii) For contradiction, suppose $\Omega(x) \not\subset g^{-1}(0)$. Then there exists $l \in \Omega(x)$ such that $\epsilon := |g(l)|/2 > 0$. Since $l$ is an $\omega$-limit point, there exists $(t_n) \subset \mathbb{R}_+$ with $t_n \to \infty$ and $x(t_n) \to l$ as $n \to \infty$. We may assume that $t_{n+1} - t_n \geq 1$ for all $n$. By Proposition 5, there exists $(s_n) \subset \mathbb{R}_+$ and some subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that $|g(x(s_n))| < \epsilon$ with $s_n - t_{\sigma(n)} \geq 0$ and $s_n - t_{\sigma(n)} \to 0$ as $n \to \infty$. Since $x \in \mathcal{R}(\mathbb{R}_+;X)$, $x(s_n) \to l$ as $n \to \infty$ which, together with lower semicontinuity of $|g|$, yields the contradiction:

$$\epsilon \geq \liminf_{n\to\infty} |g(x(s_n))| \geq |g(l)| = 2\epsilon > 0.$$

∎

# 5 Asymptotic behaviour of dynamical systems

We now present a variety of ramifications of Lemmas 6, 7 and 9 in qualitative analyses of dynamical systems.

## 5.1 Semiflows

Let $X$ be a metric space. A map $\phi : D \subset (\mathbb{R}_+ \times X) \to X$ is a *local semiflow* if

(i) $D$ is open (in the relative topology in $\mathbb{R}_+ \times X$) and for each $\xi \in X$ there exists $\tau_\xi \in (0,\infty]$ such that $D = \cup_{\xi\in X}([0,\tau_\xi) \times \{\xi\})$;

(ii) $\phi : D \to X$ is continuous;

(iii) $\phi(0,\xi) = \xi$ for all $\xi \in X$;

(iv) if $\xi \in X$, $t_1 \in [0,\tau_\xi)$ and $t_2 \in [0,\tau_{\phi(t_1,\xi)})$, then $t_1+t_2 \in [0,\tau_\xi)$, and $\phi(t_2,\phi(t_1,\xi)) = \phi(t_1+t_2,\xi)$.

If $D = \mathbb{R}_+ \times X$, then $\phi$ is called a *global semiflow*.

In the above definition of semiflow, we have postulated joint continuity of the map $(t,\xi) \mapsto \phi(t,\xi)$. For each $t > 0$, joint continuity at $(t,\xi) \in D$ is implied by separate continuity (that is, continuity of the map $s \mapsto \phi(s,\xi)$ at $t$ and continuity of the map $\theta \mapsto \phi(t,\theta)$ at $\xi$); however, this implication may fail at points $(0,\xi) \in D$ (if $\phi$ is a (local) *flow*, then joint continuity is equivalent to separate continuity), see, for example, [4, 7]).

Let $\phi$ be a (local or global) semiflow, let $\xi \in X$ and write $x(\cdot) = \phi(\cdot,\xi)$. The ensuing facts are standard ([10, 1, 25]): if $\tau_\xi < \infty$, then $\Omega(x) = \emptyset$; $\Omega(x)$ is positively-invariant; if $x$ is relatively compact, then $\tau_\xi = \infty$ and $\Omega(x)$ is non-empty, compact, connected, invariant, is approached by $x$ and is the smallest closed set so approached.

### 5.1.1 Asymptotic behaviour of local semiflows

The second part of the following result gives a generalization of [6, Theorem 1.2] applicable in the context of local semiflows.

**Theorem 10.** Let $\phi : D \to X$ be a local semiflow, let $\xi \in X$ and assume that $\tau_\xi = \infty$. Write $x(\cdot) := \phi(\cdot,\xi) : \mathbb{R}_+ \to X$, let $G \subset X$ with $G \supset \mathrm{cl}(x(\mathbb{R}_+))$ and let $C \subset G$ be non-empty and compact. Then the following statements hold:-

(i) if $g \in \mathcal{D}_C(G)$ and $\gamma = g \circ x$ satisfies (H), then $x$ is relatively compact and approaches the largest invariant set in $C$;

(ii) if $g : G \to \mathbb{R}$ is such that $|g|$ is lower semicontinuous and $\gamma = g \circ x$ satisfies (H), then $\Omega(x)$ is contained in the largest positively-invariant set in $g^{-1}(0)$. If, in addition, $x$ is relatively compact, then $x$ approaches the largest invariant set in $g^{-1}(0)$.

**Proof.** We first show that if $\Omega(x) \neq \emptyset$, then $x \in \mathcal{R}(\mathbb{R}_+;X)$. To this end let $l \in \Omega(x)$ and let $(s_n),(t_n) \subset \mathbb{R}_+$ be sequences with $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} x(s_n) = l$, $t_n \geq s_n$ and $\lim_{n\to\infty}(t_n - s_n) = 0$. Using the properties of the semiflow we have that

$$x(t_n) = \phi(t_n,\xi) = \phi(t_n - s_n, \phi(s_n,\xi)) = \phi(t_n - s_n, x(s_n)) \to l\,, \quad \text{as } n \to \infty\,,$$

and so $l$ is robust. Since $l \in \Omega(x)$ is arbitrary, we may conclude that $x \in \mathcal{R}(\mathbb{R}_+;X)$. To prove statement (i) we note that, by (H), there exist $(t_n) \subset \mathbb{R}_+$ such that $t_n \to \infty$ and $g(x(t_n)) \to 0$ as $n \to \infty$. By compactness of $C$ and Lemma 7, there exists $c \in C$ and a subsequence $(t_{\sigma(n)})$ of $(t_n)$ such that $x(t_{\sigma(n)}) \to c$ as $n \to \infty$. Thus, $\Omega(x) \cap C \neq \emptyset$ and so, in particular, $\Omega(x) \neq \emptyset$. Therefore, by the above argument, $x \in \mathcal{R}(\mathbb{R}_+;X)$. Invoking part (i) of Lemma 9 shows that $x$ approaches $C$. Since $C$ is compact and $x$ is continuous it follows that $x$ is relatively compact.

Therefore $x$ approaches $\Omega(x)$. Now $\Omega(x)$ is the smallest closed set so approached and hence $\Omega(x) \subset C$. Statement (i) now follows since $\Omega(x)$ is invariant.
To prove the first part of statement (ii), assume that $\Omega(x) \neq \emptyset$ (if $\Omega(x) = \emptyset$ there is nothing to prove). Then $x \in \mathcal{R}(\mathbb{R}_+; X)$ and part (ii) of Lemma 9 yields $\Omega(x) \subset g^{-1}(0)$. Since $\Omega(x)$ is positively-invariant, it follows that $\Omega(x)$ is contained in the largest positively-invariant set in $g^{-1}(0)$. Under the additional assumption that $x$ is relatively compact, $x$ approaches its non-empty, compact, invariant $\omega$-limit set $\Omega(x) \subset g^{-1}(0)$ and so must approach the largest invariant set in $g^{-1}(0)$. ■

**Remark.** An alternative derivation of the second part of statement (ii) (wherein $x$ is assumed to be relatively compact) is possible, based on invariance principle due to Ball [3, Theorem 2.3, part (ii)]. This alternative derivation (not given here) is, however, nontrivial; the above approach seems to be more natural and direct.

The following corollary (which has the flavour of the $L^p$ stability considerations in [23]) is a consequence of part (i) of Theorem 10.

**Corollary.** Let $X$ be a Banach space, let $\phi : D \to X$ be a local semiflow, let $\xi \in X$ with $\tau_\xi = \infty$ and let $\alpha \in \mathcal{D}_{\{0\}}(\mathbb{R}_+)$ be such that $\alpha(v) > 0$ for all $v > 0$. If

$$\int_0^\infty \alpha(\|\phi(t,\xi)\|)\, dt < \infty\,,$$

then $\lim_{t\to\infty} \phi(t,\xi) = 0$.

In particular, if $\phi$ is a global semiflow and is $L^p$ stable for some $p \in [1,\infty)$ (in the sense that $\phi(\cdot,\xi) \in L^p(\mathbb{R}_+; X)$ for all $\xi$), then $\{0\}$ is a global attractor.

### 5.1.2 Semiflows and asymptotic stability of compacta

Let $X$ be a metric space and let $\phi : \mathbb{R}_+ \times X \to X$ be a global semiflow. Relative to $\phi$, a non-empty set $C \subset X$ is said to be *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for all $\xi \in \mathbb{B}_\delta(C)$, $\phi(t,\xi) \in \mathbb{B}_\epsilon(C)$ for all $t \in \mathbb{R}_+$; $C \subset X$ is said to be *globally asymptotically stable* if it is both stable and a global attractor (that is, for all $\xi \in X$, $\phi(\cdot,\xi)$ approaches $C$). Consider the following three statements

(i) relative to the semiflow $\phi$, $C$ is globally asymptotically stable;

(ii) there exists a $\mathcal{KL}$-function $\kappa$ such that

$$\mu(\{t \mid d_C(\phi(t,\xi)) \geq \lambda\}) \leq \kappa(d_C(\xi), \lambda) \quad \forall\ (\xi,\lambda) \in X \times (0,\infty); \tag{3}$$

(iii) there exist $\mathcal{K}$-functions $\alpha$ and $\beta$ such that

$$\int_0^\infty \alpha(d_C(\phi(t,\xi)))dt \leq \beta(d_C(\xi)) \quad \forall\ \xi \in X. \tag{4}$$

**Theorem 11.** Let non-empty $C \subset X$ be compact.

(a) (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i), but (i) $\not\Rightarrow$ (iii).

(b) If $X$ satisfies the compactness condition

$$\mathbb{B}_r(z) \text{ is relatively compact for all } (r,z) \in (0,\infty) \times X\,, \tag{K}$$

then (i), (ii) and (iii) are equivalent.

**Proof.** (iii) $\Leftrightarrow$ (ii): On setting $\mathcal{M} = \{\phi(\cdot,\xi) |\ \xi \in X\}$, the equivalence of (ii) and (iii) follows by Proposition 4.

(ii) $\Rightarrow$ (i): Let (ii) hold. We will deduce (i) by establishing stability of $C$ and global attractivity of $C$.

*Stability of $C$:* Seeking a contradiction, suppose that $C$ is not stable. Then there exists $\epsilon > 0$ and sequences $(\xi'_k) \subset X$, $(t_k) \subset \mathbb{R}_+$ such that

$$d_C(\xi'_k) < \frac{1}{k}, \quad d_C(\phi(t_k,\xi'_k)) > \epsilon \quad \forall\, k \in \mathbb{N}. \tag{5}$$

Choose sequences $(k_n), (l_n) \subset \mathbb{N}$ such that

$$l_{n+1} > k_n > l_n\,, \quad \forall\, n \quad \text{and} \quad \lim_{n\to\infty} \kappa(1/k_n, 1/l_n) = 0\,. \tag{6}$$

Using the fact that $\kappa$ is a $\mathcal{KL}$-function, sequences $(k_n)$ and $(l_n)$ with the above properties can be defined recursively by setting $l_1 := 1$ and

$$k_n := l_n + \min\{j \in \mathbb{N} \,|\, \kappa(1/j, 1/l_n) < 1/n\}\,, \quad l_{n+1} := k_n + 1\,.$$

Setting $\xi_n := \xi'_{k_n}$, and using (5), (6) and continuity of $d_C(\phi(\cdot,\xi_n))$, it follows that there exist $r_n, s_n$ with $0 < r_n < s_n < t_{k_n}$, such that for all sufficiently large $n$

$$d_C(\phi(r_n,\xi_n)) = \frac{1}{l_n}, \quad d_C(\phi(s_n,\xi_n)) = \epsilon, \quad \frac{1}{l_n} < d_C(\phi(s,\xi_n)) < \epsilon \quad \forall\, s \in (r_n, s_n).$$

Therefore, for all sufficiently large $n$

$$s_n - r_n \le \mu\{s \mid d_C(\phi(s,\xi_n)) \ge 1/l_n\} \le \kappa(d_C(\xi_n), 1/l_n) < \kappa(1/k_n, 1/l_n),$$

and so, by (6),

$$\lim_{n\to\infty} (s_n - r_n) = 0\,. \tag{7}$$

By construction, $\lim_{n\to\infty} d_C(\phi(r_n,\xi_n)) = 0$. By compactness of $C$, there exist $c \in C$ and a subsequence $(\phi(r_{\sigma(n)}, \xi_{\sigma(n)}))$ of $(\phi(r_n,\xi_n))$ such that $\lim_{n\to\infty} \phi(r_{\sigma(n)}, \xi_{\sigma(n)}) = c$. Hence, by invoking (7), we obtain a contradiction

$$0 < \epsilon = d_C(\phi(s_{\sigma(n)}, \xi_{\sigma(n)})) = d_C(\phi(s_{\sigma(n)} - r_{\sigma(n)}, \phi(r_{\sigma(n)}, \xi_{\sigma(n)}))) \to 0\,, \quad \text{as } n \to \infty\,.$$

Therefore $C$ is stable.

*Global attractivity of $C$:* This is an immediate consequence of hypothesis (ii) and part (i) of Theorem 10.

(i) $\Rightarrow$ (ii) *if $X$ satisfies* (K): Assume that $X$ satisfies (K) and that (i) holds. By global attractivity of $C$, the following is well-defined for each $(\xi,\lambda)\in X\times(0,\infty)$:

$$\theta_0(\xi,\lambda):=\inf\{\tau\mid d_C(\phi(t,\xi))<\lambda\ \forall\, t>\tau\}\geq 0.$$

For each $(r,\lambda)\in\mathbb{R}_+\times(0,\infty)$, write

$$\theta(r,\lambda)=\sup\{\theta_0(\xi,\lambda)|\ d_C(\xi)\leq r\}\geq 0.$$

We claim that $\theta(r,\lambda)<\infty$ for all $(r,\lambda)\in\mathbb{R}_+\times(0,\infty)$. For contradiction, suppose otherwise. Then there exist $(r,\lambda)\in\mathbb{R}_+\times(0,\infty)$ and a sequence $(\xi_n)\subset X$ such that

$$\theta_0(\xi_n,\lambda)\to\infty\quad\text{as } n\to\infty,\qquad d_C(\xi_n)\leq r,\quad d_C(\phi(\theta_0(\xi_n,\lambda),\xi_n))=\lambda\quad\forall\, n\in\mathbb{N}. \tag{8}$$

By stability of $C$, there exists $\delta>0$ such that

$$d_C(\xi)<\delta\quad\Rightarrow\quad d_C(\phi(t,\xi))<\lambda/2\quad\forall\, t\geq 0.$$

By compactness of $C$ and condition (K), $\mathrm{cl}(\mathbb{B}_r(C))$ is compact and so we may assume (passing to a subsequence which we do not re-label) that $(\xi_n)$ $(\subset \mathrm{cl}(\mathbb{B}_r(C)))$ converges to $\xi^*$ with $d_C(\xi^*)\leq r$. By global attractivity of $C$, there exists $t^*\in\mathbb{R}_+$ such that $d_C(\phi(t^*,\xi^*))<\delta/2$. Fix $N\in\mathbb{N}$ sufficiently large so that $\rho(\phi(t^*,\xi_n),\phi(t^*,\xi^*))<\delta/2$ for all $n>N$. It follows that $\phi(t^*,\xi_n)\in\mathbb{B}_\delta(C)$ for all $n>N$, and so

$$d_C(\phi(t+t^*,\xi_n))=d_C(\phi(t,\phi(t^*,\xi_n))<\lambda/2\quad\forall\, t\geq 0\quad\forall\, n>N,$$

contradicting (8). Therefore, $\theta(r,\lambda)\in\mathbb{R}_+$ for every $(r,\lambda)\in\mathbb{R}_+\times(0,\infty)$.

On $\mathbb{R}_+\times(0,\infty)$, define $\kappa_0$ by

$$\kappa_0(r,\lambda):=\sup\{\mu(\{t\,|\,d_C(\phi(t,\xi))\geq\lambda\})\,|\,d_C(\xi)\leq r\}\leq\theta(r,\lambda).$$

Clearly, for each $\lambda\in(0,\infty)$, $\kappa_0(\cdot,\lambda)$ is increasing and, for each $r\in\mathbb{R}_+$, $\kappa_0(r,\cdot)$ is decreasing. Define $\kappa_1:\mathbb{R}_+\times(0,\infty)\to\mathbb{R}_+$ by setting for each $\lambda\in(0,\infty)$

$$\kappa_1(0,\lambda):=0\,,\qquad \kappa_1(r,\lambda):=2r\int_r^{2r}s^{-2}\kappa_0(s,\lambda)\,ds\,,\quad\text{if } r>0.$$

For each $\lambda\in(0,\infty)$, the function $\kappa_1(\cdot,\lambda)$ is continuous on $\mathbb{R}_+$ and it is easy to check that

$$\kappa_1(r,\lambda)\geq\kappa_0(r,\lambda)\qquad\forall\,(r,\lambda)\in\mathbb{R}_+\times(0,\infty)\,. \tag{9}$$

A routine calculation yields that for all $r,\lambda>0$

$$\frac{\partial\kappa_1}{\partial r}(r,\lambda)=\frac{1}{r}[2(\kappa_1(r,\lambda)-\kappa_0(r,\lambda))+\kappa_0(2r,\lambda)]\,,$$

which shows in combination with (9) that the function $\kappa_1(\cdot,\lambda)$ is increasing for every $\lambda\in(0,\infty)$, It follows that

$$\kappa:(r,\lambda)\mapsto r+\kappa_1(r,\lambda)$$

is a $\mathcal{KL}$-function and, for all $(\xi,\lambda)\in X\times(0,\infty)$,

$$\mu(\{t\,|\,d_C(\phi(t,\xi))\geq\lambda\})\leq\kappa_0(d_C(\xi),\lambda)\leq\kappa(d_C(\xi),\lambda)).$$

It remains only to show that, if $X$ does not satisfy condition (K), then (i) does not imply (iii).

(i) $\not\Rightarrow$ (iii) *if $X$ does not satisfy condition* (K): Observe that, if $X$ is a Banach space, if $C=\{0\}$ and if the flow is linear (i.e. $\phi(t,\xi)=S(t)\xi$, where $S=(S(t),t\in\mathbb{R}_+)$ is a strongly continuous semigroup of linear bounded operators), then condition (iii) implies uniform exponential stability of $S$ by Rolewicz's theorem [20, Theorem 2] (see, also, [16, Theorem 3.2.2]). Since it is well known that, for linear strongly continuous semigroups of bounded linear operators on an infinite-dimensional Banach space $X$, global asymptotic stability does not imply uniform exponential stability, it follows that (i) cannot imply (iii). ∎

**Remarks.** 1. If $X$ is a normed vector space, then condition (K) is equivalent to local compactness of $X$. Therefore part (b) of the theorem implies that statements (i), (ii) and (iii) are equivalent in the case of a locally compact normed vector space $X$. Hence these equivalences hold in particular for global semiflows generated by ordinary differential equations on $X=\mathbb{R}^N$. In this context of ordinary differential equations, the equivalence of (i) and (iii) in the theorem is not new: it is implicit in the integral variant of Sontag's concept of input-to-state stability for finite-dimensional controlled systems given in [22, Theorem 1]. Moreover, in the context of ordinary differential equations on $X=\mathbb{R}^N$, the proof of the implication (i) $\Rightarrow$ (iii) (equivalently, (i) $\Rightarrow$ (ii)) follows easily by converse Lyapunov theory [15, Theorem 2.9] and, furthermore, if statement (iii) is modified by postulating $\mathcal{K}_\infty$ functions $\alpha$ and $\beta$, then the equivalence of (i), (ii) and (iii) remains true.

2. In proving that (i) $\not\Rightarrow$ (iii) in the case of a metric space $X$ which does not satisfy condition (K), we appealed to Rolewicz's theorem. However, there is also a direct argument based on a simple counterexample. Let $X=\ell^2(\mathbb{N})$, let $(\epsilon_n)\subset(0,\infty)$ with $\lim_{n\to\infty}\epsilon_n=0$ and set, for $t\in\mathbb{R}_+$,

$$S(t)=\mathrm{diag}_{n\in\mathbb{N}}(e^{-\epsilon_n t})\,.$$

Clearly, $S$ is a strongly continuous semigroup of linear operators on $X$, and it is easy to show that $\{0\}$ is globally asymptotically stable. Let $\alpha$ be an arbitrary $\mathcal{K}$-function and for $m\in\mathbb{N}$, let $\xi^{(m)}=(\xi_n^{(m)})$ be the $m$-th unit vector in $X=\ell^2(\mathbb{N})$, i.e. $\xi_n^{(m)}=0$ if $n\neq m$ and $\xi_n^{(m)}=1$ if $n=m$. Then

$$\int_0^\infty\alpha(\|S(t)\xi^{(m)}\|)\,dt=\sum_{k=1}^\infty\int_{k-1}^k\alpha(\|S(t)\xi^{(m)}\|)\,dt\geq\sum_{k=1}^\infty\alpha(e^{-\epsilon_m k})\to\infty\,,\quad\text{as } m\to\infty\,,$$

showing that (iii) does not hold.

### 5.1.3 Characterizations of exponential stability for linear semigroups

In the proof of Theorem 11 above, reference has been made to a result by Rolewicz on uniform exponential stability of linear semigroups. Here, we discuss in more detail this and similar results, and provide two new characterizations of uniform exponential stability.

Let $X$ be a Banach space and let $S = (S(t))_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup of bounded linear operators on $X$. The semigroup $S$ is *uniformly exponentially stable* if there exist constants $a > 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{-at} \quad \forall\, t \in \mathbb{R}_+.$$

Let $p \in [1, \infty)$. The semigroup $S$ is *$L^p$-stable* if $S(\cdot)\xi \in L^p(\mathbb{R}_+; X)$ for all $\xi \in X$. The semigroup $S$ is *wk-$L^p$-stable* if $S(\cdot)\xi \in wk\text{-}L^p(\mathbb{R}_+; X)$ for all $\xi \in X$. The equivalence of uniform exponential stability and $L^p$-stability is well known (originally proved by Datko [8] for a Hilbert space $X$ and $p = 2$; the general result later established by Pazy [17]). Here, we establish the equivalence of uniform exponential stability and *wk*-$L^p$-stability, and also provide an equivalent to Rolewicz's characterization [20], [16, Section 3.2] (for related results, see also, [19, 13, 27]) of uniform exponential stability. We remark that the concept of "weak $L^p$ stability" considered in [27] differs fundamentally from the concept of *wk*-$L^p$-stability considered here.

**Theorem 12.** The following statements are equivalent:-

(i) $S$ is exponentially stable;

(ii) $S$ is $L^p$-stable for some $p \in [1, \infty)$;

(iii) $S$ is *wk*-$L^p$-stable for some $p \in [1, \infty)$;

(iv) there exists a non-decreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0 with $\alpha(0) = 0$ and $\alpha(v) > 0$ for all $v > 0$, such that $\alpha(\|S(\cdot)\xi\|) \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ for all $\xi \in X$;

(v) there exists a non-decreasing function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0 with $\beta(0) = 0$ and $\beta(v) > 0$ for all $v > 0$, such that $\beta(\|S(\cdot)\xi\|) \in wk\text{-}L^1(\mathbb{R}_+; \mathbb{R}_+)$ for all $\xi \in X$;

**Proof.** As already mentioned, the equivalence of (i) and (ii) is well known. That (ii) implies (iii) is evident. We now show that (iii) implies (ii). Assume (iii) holds. Let $\xi \in X$. By Proposition 2, $\|S(\cdot)\xi\|$ is meagre. By part (i) of Theorem 10 applied in the context of the global semiflow $\phi : (t, \xi) \mapsto S(t)\xi$ and the function $g \in \mathcal{D}_{\{0\}}(X)$, $g : z \mapsto \|z\|^p$, we may conclude that $S(\cdot)\xi$ approaches $\{0\}$. In particular, $S(\cdot)\xi$ is bounded and so, by Proposition 3, $S(\cdot)\xi \in L^q(\mathbb{R}_+; X)$ for all $q > p$.
The equivalence of (i) and (iv) is a consequence of [16, Theorem 3.2.2] (a generalization of Rolewicz's result [20, Theorem 2]). That (iv) implies (v) is evident. It remains only to prove that (v) implies (iv). Assume (v) holds and let $\xi \in X$. By part (i) of Theorem 10, $S(\cdot)\xi$ approaches $\{0\}$. Therefore, $\beta(\|S(\cdot)\xi\|)$ is bounded and so, by Proposition 3 and defining $\alpha := \beta^2$, $\alpha(\|S(\cdot)\xi\|) \in L^1(\mathbb{R}_+; X)$. ∎

### 5.2 Non-autonomous ordinary differential equations and differential inclusions

Until otherwise indicated, $X = \mathbb{R}^N$. Our next goal is to highlight some ramifications of Lemmas 6, 7 and 9 in the study of asymptotic behaviour of solutions of initial-value problems for autonomous differential inclusions (for background see, for example [2, 11, 9]),

$$\dot{x}(t) \in F(x(t)), \quad x(0) = \xi, \quad x(t) \in X \tag{10}$$

where $x \mapsto F(x) \subset X$ is a non-empty-set-valued map defined on $X$. By a solution of (10), we mean an absolutely continuous function $x : I \to X$, defined on some interval $I \subset \mathbb{R}_+$ containing 0 and of non-zero length $|I|$, such that $x(0) = \xi$ and the differential inclusion in (10) is satisfied for almost all $t \in I$; $x$ is a maximal solution if it does not have a proper right extension (i.e. to an interval $J$, with $I \subset J \subset [0, \infty)$ and $|J| > |I|$) which is also a solution. We note that, by Zorn's lemma, every solution can be extended to a maximal solution (that is, a solution on a maximal interval of existence $I$). If $F$ is upper semicontinuous and convex-compact-valued, and $x : I \to G$ is a maximal *bounded* solution, then $\sup I = \infty$ and $x$ approaches its non-empty, compact, connected, weakly-invariant $\omega$-limit set $\Omega(x)$ (and $\Omega(x)$ is the smallest closed set approached by $x$). By *weak invariance* of $\Omega(x)$ we mean that, for each $l \in \Omega(x)$, there exists at least one maximal solution of the initial-value problem $\dot{z} \in F(z)$, $z(0) = l$, with trajectory in $\Omega(x)$.
Note that (10) subsumes not only autonomous differential equations but also classes of non-autonomous differential equations. Let $D \subset \mathbb{R}^P$ and $f : D \times \mathbb{R}^N \to \mathbb{R}^N$, and consider, for example, an equation of the form $\dot{x}(t) = f(d(t), x(t))$, with $d(\cdot)$ measurable and taking values in $D$: this equation is embedded in (10) on setting $F(x) := f(D, x)$, the embedding being in the sense that any solution of the differential equation is *a fortiori* a solution of the differential inclusion (of course, if $f$ is continuous and $D$ is compact, then, by Filippov's lemma [12], we have equivalence of the solution sets in the sense that $x$ is a solution of the inclusion if, and only if, there exists measurable $t \mapsto d(t) \in D$ such that $x$ is a solution of the equation).

**Theorem 13.** Assume that $F$ maps bounded sets to bounded sets. Let $x : \mathbb{R}_+ \to X$ be a bounded solution of (10). Let $G \subset X$ with $G \supset \mathrm{cl}(x(\mathbb{R}_+))$. If $g : G \to \mathbb{R}$ is such that $|g|$ is lower semicontinuous and $\gamma := g \circ x$ satisfies (H), then $x$ approaches $g^{-1}(0)$.

**Proof.** By (10), together with boundedness of $x$ and the assumption that $F$ maps bounded sets to bounded sets, it follows that $\dot{x} \in L^\infty(\mathbb{R}_+; X)$ and so $x$ is uniformly continuous. Therefore, $x \in \mathcal{R}(\mathbb{R}_+; X)$ and, by part (ii) of Lemma 9, $\Omega(x) \subset g^{-1}(0)$. By boundedness of $x$, $\Omega(x)$ is non-empty and is approached by $x$. ∎

A corollary of this result provides another generalization of [6, Theorem 1.2]. This generalization is proved (by other arguments) in [21], wherein it is also shown to have applications in adaptive control of uncertain dynamical systems.

**Corollary.** Assume that $F$ is upper semicontinuous with convex, compact values. Let $x : \mathbb{R}_+ \to X$ be a bounded solution of (10). Let $G \subset X$ with $G \supset \mathrm{cl}(x(\mathbb{R}_+))$. If $g : G \to \mathbb{R}$ is such that $|g|$ is lower semicontinuous and $\gamma := g \circ x$ satisfies (H), then $x$ approaches the largest weakly-invariant set in $g^{-1}(0)$.

**Proof.** By properties of $F$, it maps bounded sets to bounded sets and so, by Theorem 13, $\Omega(x) \subset g^{-1}(0)$. By boundedness of $x$ and properties of $F$, $\Omega(x) \neq \emptyset$ is weakly-invariant and is approached by $x$, and hence $x(t)$ tends to the largest weakly-invariant set in $g^{-1}(0)$ as $t \to \infty$. ∎

As shown in [21, Theorem 2.11], a (nonsmooth) version of LaSalle's invariance principle follows readily from the Corollary to Theorem 13.

**Theorem 14.** Let $G \subset X$ be open and assume that $F$ is upper semicontinuous on $G$ with convex, compact values. Let $V : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz and let $V^o(z; v)$ denote the Clarke directional derivative of $V$ at $z$ in direction $v$. Define

$$g : G \to \mathbb{R}, \quad g : z \mapsto g(z) := - \max_{v \in F(z)} V^o(z; v).$$

If (i) $g(z) \geq 0$ for all $z \in G$ and (ii) $x : \mathbb{R}_+ \to X$ is a bounded solution of (10) with $\mathrm{cl}(x(\mathbb{R}_+)) \subset G$, then $x$ approaches the largest weakly-invariant set in $g^{-1}(0)$.

**Proof.** The proof of this result can be found in [21]: here we simply sketch the argument.
By upper semicontinuity of $V^o(\cdot;\cdot)$, together with properties of $F$, the function $g$ is lower semicontinuous. By boundedness of $x$ and continuity of $V$, $V \circ x$ is bounded. Moreover,

$$V(x(\tau)) - V(x(t)) \geq \int_\tau^t g(x(s)) ds \geq 0 \quad \forall\, t, \tau \in \mathbb{R}_+,\ t \geq \tau,$$

and so the bounded function $V \circ x$ is monotone non-increasing. It follows that

$$\int_0^\infty g(x(s))\, ds \leq V(x(0)) - \lim_{t \to \infty} V(x(t)) < \infty.$$

The assertion now follows by the Corollary to Theorem 13. ∎

The next result establishes that, under a weak condition on $F$, solutions of (10) on $\mathbb{R}_+$ which are meagre with respect to $C$ necessarily approach $C$.

**Proposition 15.** Let $C \subset X$ be non-empty. Assume that $F$ is bounded on a neighbourhood of $C$. If $x : \mathbb{R}_+ \to X$ is a solution of (10) and is $C$-meagre, then $x$ approaches $C$.

**Proof.** Let $\epsilon > 0$ be sufficiently small so that $F(\mathbb{B}_\epsilon(C))$ is bounded. Define $\gamma(\cdot) := d_C(x(\cdot))$ and $J_\epsilon := \{t \in \mathbb{R}_+ \,|\, \gamma(t) < \epsilon\}$. Since $x$ is $C$-meagre, $\gamma$ is meagre. By meagreness of $\gamma$, 0 is an $\omega$-limit point of $\gamma|_{J_\epsilon}$. If $0 \in \Omega(\gamma|_{J_\epsilon})$ is robust (that is, if

$\gamma|_{J_\epsilon} \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$), then the result follows by part (ii) of Lemma 6. To show that $\gamma|_{J_\epsilon} \in \mathcal{R}_0(\mathbb{R}_+;\mathbb{R})$, it is sufficient to prove that $\gamma|_{J_\epsilon}$ is uniformly continuous. To this end note that by the boundedness of $F(\mathbb{B}_\epsilon(C))$, $\dot{x}|_{J_\epsilon} \in L^\infty(J_\epsilon;X)$. Evidently, $J_\epsilon$ is non-empty and open (in the relative topology for $\mathbb{R}_+$) and so is the union of a countable disjoint class of (relatively) open intervals:

$$J_\epsilon = \cup_{n\in\mathbb{N}}(s_n,t_n)\cap\mathbb{R}_+.$$

Let $\delta > 0$ be arbitrary and set $\lambda := \delta/2\alpha$, where $\alpha := 1 + \|\dot{x}|_{J_\epsilon}\|_\infty$. Let $s,t \in J_\epsilon$. Without loss of generality, we assume $s \le t$. Then, for some $m,n \in \mathbb{N}$, $s \in (s_m,t_m)$ and $t \in (s_n,t_n)$ with either (i) $m = n$ or (ii) $0 < t_m \le s_n < \infty$.

*Case (i):* If $m = n$, then

$$\begin{aligned} t,s \in (s_m,t_m),\ |t-s| \le \lambda \ \Rightarrow\ |\gamma(t)-\gamma(s)| &= |d_C(x(t)) - d_C(x(s))| \\ &\le \|x(t)-x(s)\| \le \int_s^t \|\dot{x}\| \le \alpha\lambda < \delta, \end{aligned}$$

wherein the Lipschitz property of $d_C$ has been invoked.

*Case (ii):* Assume $0 < t_m \le s_n < \infty$. Clearly,

$$s \in (s_m,t_m),\ t \in (s_n,t_n),\ |t-s| \le \lambda \ \Rightarrow\ |t-s_n| \le \lambda \quad \text{and} \quad |t_m - s| \le \lambda. \tag{11}$$

Moreover, by definition of $J_\epsilon$ and continuity of $\gamma(\cdot) = d_C(x(\cdot))$,

$$\gamma(t_m) = \gamma(s_n) = \epsilon$$

and so, by the Lipschitz property of $d_C$,

$$|\gamma(t)-\gamma(s)| \le |\gamma(t)-\gamma(s_n)| + |\gamma(t_m)-\gamma(s)| \le \|x(t)-x(s_n)\| + \|x(t_m)-x(s)\|, \tag{12}$$

Therefore, invoking (11) and (12), we conclude

$$\begin{aligned} s \in (s_m,t_m),\ t \in (s_n,t_n),\ |t-s| \le \lambda \ \Rightarrow\ |\gamma(t)-\gamma(s)| &\le \int_{s_n}^t \|\dot{x}\| + \int_s^{t_m} \|\dot{x}\| \\ &\le 2\lambda\alpha = \delta \end{aligned}$$

The above arguments show that $|\gamma(t)-\gamma(s)| \le \delta$ whenever $t,s \in J_\epsilon$ and $|t-s| \le \lambda$, whence uniform continuity of $\gamma|_{J_\epsilon}$. ∎

The following corollary is immediate.

**Corollary.** Assume that $F$ is bounded on a neighbourhood of 0. If $x : \mathbb{R}_+ \to X$ is a solution of (10) and $x \in L^p(\mathbb{R}_+;X)$ for some $1 \le p < \infty$, then $x$ approaches $\{0\}$.

### 5.3 Non-autonomous semilinear infinite-dimensional systems

Let $X$ be a Banach space and let $A : \mathrm{dom}(A) \to X$ generate a strongly continuous semigroup $S = (S(t))_{t\in\mathbb{R}_+}$ of linear bounded operators on $X$. Let $\|\cdot\|$ denote the norm in $X$. Moreover, let $B \in \mathcal{L}(U, X_{-1})$. Here $U$ is a Banach space (with norm $\|\cdot\|_U$) and $X_{-1}$ is the completion of $X$ with respect to the norm $\|x\|_{-1} = \|(sI-A)^{-1}x\|$, where $s$ is in the resolvent set of $A$. Different choices for $s$ lead to equivalent norms. The semigroup $S$ extends to a strongly continuous semigroup on $X_{-1}$ which will be denoted by the same symbol. We assume that $B$ is an admissible input operator for $S$ in the sense of Weiss [26], i.e. for each $T \geq 0$ there exists $a_T > 0$ such that

$$\|\int_0^T S(t)Bu(t)\,dt\|^2 \leq a_T \int_0^T \|u(t)\|_U^2\,dt\,, \quad \text{for all } u \in L^2_{loc}(\mathbb{R}_+; U)\,.$$

Finally, let $N : \mathbb{R}_+ \times X \to U$, $(t, y) \mapsto N(t, y)$ be continuous in $t$ and locally Lipschitz in $y$, uniformly in $t$ on bounded intervals. The latter condition means, that for all $r, t \geq 0$ there exists $L(r, t) > 0$ such that

$$\|N(s,y) - N(s,z)\|_U \leq L(r,t)\|y - z\|\,, \quad \text{for all } s \in [0,t],\, y, z \in \mathbb{B}_r(0)\,.$$

Under these conditions, for given $\xi \in X$, the integral equation

$$x(t) = S(t)\xi + \int_0^t S(t-\tau)BN(\tau, x(\tau))\,d\tau\,, \quad t \geq 0 \tag{13}$$

has a unique continuous $X$-valued solution defined on a maximal interval of existence; (13) can be thought of as the evolution equation of the system obtained by applying the nonlinear feedback $u(t) = N(t, x(t))$ to the linear controlled system given formally by $\dot{x} = Ax + Bu$.

Moreover, we assume that $S$, $B$ and $N$ satisfy the following assumption.

Assumption (A): For every bounded $K \subset X$, there exists $h_K > 0$ and a function $\theta_K : [0, h_K] \to \mathbb{R}_+$, continuous at 0 with $\theta_K(0) = 0$ and such that, whenever $h \in [0, h_K]$, $t \geq 0$ and $x : [t, t+h] \to K$ is continuous,

$$\|\int_t^{t+h} S(t+h-\tau)BN(\tau, x(\tau))\,d\tau\| \leq \theta_K(h)\,.$$

A condition which only involves the nonlinearity $N$ and which guarantees that Assumption (A) is satisfied is given by the following assumption.

Assumption (B): For every bounded $K \subset X$, there exists $h_K > 0$ and a function $\theta_K : [0, h_K] \to \mathbb{R}_+$, continuous at 0 with $\theta_K(0) = 0$ and such that, whenever $h \in [0, h_K]$, $t \geq 0$ and $x : [t, t+h] \to K$ is continuous,

$$\int_t^{t+h} \|N(\tau, x(\tau))\|_U^2\,d\tau \leq \theta_K(h)\,.$$

If $B \in \mathcal{L}(U,X)$, we can absorb $B$ into $N$, and hence may assume, without loss of generality, that $U = X$ and $B = I$. If in addition we have $A = 0$, then Assumption (A) is just the condition introduced by Artstein in [14, Appendix A] and the integral equation (13) is equivalent to the initial-value problem

$$\dot{x}(t) = N(t, x(t)), \quad x(0) = \xi .$$

**Proposition 16.** Suppose that Assumption (A) holds. Let $\omega \in \mathbb{R}$ and, for $t \in \mathbb{R}_+$, set $S_\omega := (e^{\omega t} S(t))_{t \in \mathbb{R}_+}$. Then Assumption (A) remains valid if $S$ is replaced by $S_\omega$.

**Proof.** For all $x \in X_{-1}$ we have

$$S_\omega(t)x = S(t)x + \omega \int_0^t S(t-\sigma) S_\omega(\sigma) x \, d\sigma . \tag{14}$$

Let $K \subset X$ be bounded, and let $h_K$, $\theta_K$ be as in Assumption (A). Let $h \in [0, h_K]$, $t \geq 0$ and let $x : [t, t+h] \to K$ be continuous. It suffices to show that there exist a constant $\hat{h}_K \in (0, h_K]$ and a function $\hat{\theta}_K : [0, \hat{h}_K] \to \mathbb{R}_+$, continuous at 0 with $\hat{\theta}_K(0) = 0$, such that

$$\| \int_t^{t+h} S_\omega(t+h-\tau) B N(\tau, x(\tau)) \, d\tau \| \leq \hat{\theta}_K(h) \quad \forall \ h \in [0, \hat{h}_K] .$$

Setting $\tilde{N}(\tau) := N(\tau, x(\tau))$, we have

$$\begin{aligned} &\| \int_t^{t+h} \int_0^{t+h-\tau} S(t+h-\tau-\eta) S_\omega(\eta) B \tilde{N}(\tau) \, d\eta \, d\tau \| \\ &\qquad\qquad = \| \int_0^h S_\omega(\eta) \left( \int_t^{t+h-\eta} S(t+h-\eta-\tau) B \tilde{N}(\tau) \, d\tau \right) d\eta \| . \end{aligned}$$

Let $a > 0$ be such that $\|S_\omega(t)\| \leq a$ for all $t \in [0, h_K]$. By continuity of $\theta_K$ at zero, there exists $\hat{h}_K \in (0, h_K]$ such that

$$b := \sup\{\theta_K(\eta) \,|\, 0 \leq \eta \leq \hat{h}_K\} < \infty .$$

Therefore, for all $h \in [0, \hat{h}_K]$,

$$\| \int_t^{t+h} \int_0^{t+h-\tau} S(t+h-\tau-\eta) S_\omega(\eta) B \tilde{N}(\tau) \, d\eta \, d\tau \| \leq abh .$$

Invoking (14), we obtain for all $h \in [0, \hat{h}_K]$

$$\| \int_t^{t+h} S_\omega(t+h-\tau) B N(\tau, x(\tau)) \, d\tau \| \leq \theta_K(h) + ab|\omega|h =: \hat{\theta}_K(h) .$$

■

**Proposition 17.** Suppose that Assumption (A) holds and that $x : \mathbb{R}_+ \to X$ is a solution of (13). If $x$ is bounded and $\Omega(x) \neq \emptyset$, then $x \in \mathcal{R}(\mathbb{R}_+; X)$.

**Proof.** Choose $\omega \in \mathbb{R}$ such that the semigroup $S_\omega := (e^{\omega t}S(t))_{t\in\mathbb{R}_+}$ is exponentially stable. Clearly

$$x(t) = S_\omega(t)\xi + \int_0^t S_\omega(t-\tau)(BN(\tau, x(\tau)) - \omega x(\tau))\,d\tau\,.$$

To show that $x \in \mathcal{R}(\mathbb{R}_+; X)$, let $l \in \Omega(x)$ and let $(s_n), (t_n) \subset \mathbb{R}_+$ be sequences with $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} x(s_n) = l$, $t_n \geq s_n$ for all $n$ and $\lim_{n\to\infty}(t_n - s_n) = 0$. Setting

$$v(t) := \int_0^t S_\omega(t-\tau)(BN(\tau, x(\tau)) - \omega x(\tau))\,d\tau\,,$$

it follows that $\lim_{n\to\infty} v(s_n) = l$. It is sufficient to show that $\lim_{n\to\infty} v(t_n) = l$. To this end note that

$$v(t_n) = S_\omega(t_n - s_n)v(s_n) + w_n\,, \tag{15}$$

where

$$w_n := \int_{s_n}^{t_n} S_\omega(t_n-\tau)(BN(\tau, x(\tau)) - \omega x(\tau))\,d\tau\,.$$

Since the operator $B$ is an admissible control operator for $S$, it is also admissible for $S_\omega$, and hence for each $T \geq 0$ there exists $\hat{a}_T > 0$ such that

$$\|\int_0^T S_\omega(t)Bu(t)\,dt\|^2 \leq \hat{a}_T \int_0^T \|u(t)\|_U^2\,dt\,, \quad \text{for all } u \in L^2_{loc}(\mathbb{R}_+; U)\,.$$

Setting $K := \mathrm{cl}(x(\mathbb{R}_+))$, it follows from the boundedness of $x$ that $K$ is a bounded set, and hence $k := \sup\{\|z\| \mid z \in K\} < \infty$. Using Assumption (A) and Lemma 16, we may conclude that there exists $\hat{h}_K > 0$ and a function $\hat{\theta}_K : [0, \hat{h}_K] \to \mathbb{R}_+$, continuous at 0 with $\hat{\theta}_K(0) = 0$, such that, for all sufficiently large $n$,

$$\|w_n\| \leq \hat{\theta}_K(t_n - s_n) + \hat{a}_{\hat{h}_K} k|\omega|(t_n - s_n)\,.$$

Therefore, $\lim_{n\to\infty} w_n = 0$, and combining this with the fact that $\lim_{n\to\infty} v(s_n) = l$, we obtain from (15) that $\lim_{n\to\infty} v(t_n) = l$. ■

### 5.3.1 Asymptotic behaviour of solutions to (13)

The final result is, in a sense, an infinite-dimensional, non-autonomous, version of Theorem 10.

**Theorem 18.** Suppose that Assumption (A) holds. Let $x : \mathbb{R}_+ \to X$ be a *bounded* solution of (13). Let $G \subset X$ with $G \supset \mathrm{cl}(x(\mathbb{R}_+)$ and let $C \subset G$ be non-empty and compact. Then the following statements hold:-

(i) if $g \in \mathcal{D}_C(G)$ and $\gamma = g \circ x$ satisfies (H), then $x$ is relatively compact and $\Omega(x) \subset C$ (hence, in particular, $x$ approaches $C$);

(ii) if $g : G \to \mathbb{R}$ is such that $|g|$ is lower semicontinuous and (H) holds for $\gamma = g \circ x$, then $\Omega(x) \subset g^{-1}(0)$. If, in addition, $x$ is relatively compact, then $x$ approaches $g^{-1}(0)$.

**Proof.** To prove (i), we first conclude that $\Omega(x) \cap C \neq \emptyset$ by the same argument as previously used in the proof of statement (i) of Theorem 10. Therefore, $\Omega(x) \neq \emptyset$ and so, by Proposition 17, $x \in \mathcal{R}(\mathbb{R}_+; X)$. By Lemma 9(i), $x$ approaches $C$. Relative compactness of $x$ now follows by continuity of $x$ and compactness of $C$, and so $x$ approaches $\Omega(x)$. Since $\Omega(x)$ is the smallest closed set so approached, $\Omega(x) \subset C$.

To prove (ii), assume $\Omega(x) \neq \emptyset$ (if $\Omega(x) = \emptyset$, then (ii) holds trivially). By Proposition 17, $x \in \mathcal{R}(\mathbb{R}_+; X)$ and Lemma 9(ii) then yields $\Omega(x) \subset g^{-1}(0)$. To complete the proof, simply note that, if $x$ is relatively compact, then $x$ approaches $\Omega(x)$. ∎

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## REFERENCES

[1] H. Amann, "Ordinary Differential Equations: An Introduction to Nonlinear Analysis", Walter de Gruyter, Berlin - New York, 1990.

[2] J.P. Aubin and A. Cellina, "Differential Inclusions", Springer-Verlag, Berlin, 1984.

[3] J.M. Ball, On the asymptotic behaviour of generalized processes, with applications to nonlinear evolution equations, *J. Diff. Eq.* **27** (1978), 224–265.

[4] J.M. Ball, Continuity of nonlinear semigroups, *J. Funct. Anal.* **17** (1974), 91–103.

[5] I. Barbălat, Systèmes d'équations différentielles d'oscillations non linéaires, *Revue de Mathématiques Pures et Appliquées*, Bucharest, **IV** (1959), 267–270.

[6] C.I. Byrnes and C.F. Martin, An integral-invariance principle for nonlinear systems, *IEEE Trans. Autom. Control* **AC-40** (1995), 983–994.

[7] P.R. Chernoff and J.E. Marsden, "Properties of Infinite-Dimensional Hamiltonian Systems", Springer-Verlag, Berlin, 1974.

[8] R. Datko, Extending a theorem of A.M. Lyapunov to Hilbert space, *J. Math. Anal. Appl.* **32** (1970), 610–616.

[9] K. Deimling, "Multivalued Differential Equations", Walter de Gruyter, Berlin - New York, 1992.

[10] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel and H.-O. Walther, "Delay Equations: Functional-, Complex-, and Nonlinear Analysis", Springer-Verlag, New York, 1995.

[11] A.F. Filippov, "Differential Equations with Discontinuous Righthand Sides", Kluwer, Dordrecht, 1988.

[12] A.F. Filippov, On certain questions in the theory of optimal control, *SIAM J. Control* **1** (1962), 76–84.

[13] A. Ichikawa, Equivalence of $L_p$ stability and exponential stability for a class of nonlinear semigroups, *Nonlin. Anal., Theory, Methods & Applics.* **8** (1984), 805–815.

[14] J.P. LaSalle, "The Stability of Dynamical Systems", SIAM Reg. Conf. Ser. in App. Math. 25, SIAM, Philadelphia, 1976.

[15] Y. Lin, E.D. Sontag and Y. Wang, A smooth converse Lyapunov theorem for robust stability, *SIAM J. Control & Opt.* **34** (1996), 124–160.

[16] J. van Neerven, "The Asymptotic Behaviour of Semigroups of Linear Operators", Birkhäuser Verlag, Basel, 1996.

[17] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, New York, 1983.

[18] V.M. Popov, "Hyperstability of Control Systems", Springer-Verlag, Berlin, 1973.

[19] A.J. Pritchard and J. Zabczyk, Stability and stabilizability of infinite-dimensional systems, *SIAM Rev* **23** (1981), 25–52.

[20] S. Rolewicz, On uniform $N$-equistability, *J. Math. Anal. Appl.* **115** (1986), 434–441.

[21] E.P. Ryan, An integral invariance principle for differential inclusions with application in adaptive control, *SIAM J. Control & Opt.*, to appear (1998).

[22] E.D. Sontag, Comments on integral variants of ISS, preprint (1997).

[23] A. Strauss, Liapunov functions and $L^p$ solutions of differential equations, *Trans. AMS* **119** (1965), 37–50.

[24] A. Torchinsky, "Real-Variable Methods in Harmonic Analysis", Academic Press, New York, 1986.

[25] J.A. Walker, "Dynamical Systems and Evolution Equations: Theory and Applications", Plenum Press, New York, 1980.

[26] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control & Opt.* **27** (1989), 527–545.

[27] G. Weiss, Weak $L^p$-stability of a linear semigroup on a Hilbert space implies exponential stability, *J. Diff. Eq.* **76** (1988), 269–285.