AN INTEGRAL INVARIANCE PRINCIPLE FOR DIFFERENTIAL INCLUSIONS WITH APPLICATIONS IN ADAPTIVE CONTROL

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Abstract. The Byrnes-Martin integral invariance principle for ordinary differential equations is extended to differential inclusions on $\mathbb{R}^N$. The extended result is applied in demonstrating the existence of adaptive stabilizers and servomechanisms for a variety of nonlinear system classes.

Key words. adaptive control; differential inclusions; invariance principles; nonlinear systems; universal servomechanisms.

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1. Introduction. Suppose that $\dot{x} = f(x)$ generates a semidynamical system on $\mathbb{R}^N$ with semiflow φ and so, for each $x^0 \in \mathbb{R}^N$, $x(\cdot) = \varphi(\cdot, x^0)$ is the unique maximal forwards-time solution of the initial-value problem $\dot{x} = f(x)$, $x(0) = x^0$. In [2], Byrnes & Martin prove the following integral invariance principle: if $\varphi(\cdot, x^0)$ is bounded and $\int_0^\infty l(\varphi(t, x^0))dt < \infty$ for some continuous function $l : \mathbb{R}^N \to \mathbb{R}_+ := [0, \infty)$, then $\varphi(t, x^0)$ tends, as $t \to \infty$, to the largest invariant (with respect to the differential equation) set in $l^{-1}(0)$, the zero level set of l. This result has ramifications in adaptive control, some of which are highlighted in the present paper. However, we wish to consider the (adaptive) control problem in a fairly general setting that allows time-variation in the underlying differential equations, possible non-uniqueness of solutions, and discontinuous feedback strategies: each of these features places the problem outside the scope of [2]. For this reason we develop, in Theorem 2.10, an integral invariance principle for initial-value problems of the form $\dot{x} \in X(x)$, $x(0) = x^0$, where the set-valued map X is defined on some open domain $G \subset \mathbb{R}^N$ and is assumed to be upper semicontinuous with non-empty, convex and compact values. In the case $G = \mathbb{R}^N$, Theorem 2.10 contains the following generalization of the Byrnes-Martin result: if $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^N$ is a bounded solution and $\int_0^\infty l(x(s))ds < \infty$ for some lower semicontinuous $l : \mathbb{R}^N \to \mathbb{R}_+$, then $x(t)$ tends, as $t \to \infty$, to the largest weakly-invariant (with respect to the differential inclusion) set in $l^{-1}(0)$. One particular consequence of Theorem 2.10 is to facilitate the derivation of a nonsmooth extension, to differential inclusions, of LaSalle's invariance principle for differential equations: this extension may be of independent interest and is presented in Theorem 2.11. The remainder of the paper is devoted to the application (in a collection of five lemmas) of the generalized integral invariance principle to demonstrate, by construction and for a variety of nonlinear system classes, the existence of a single adaptive controller that achieves (without system identification, parameter estimation or injection of probing signals) some prescribed objective for every system in the underlying class.

2. Differential inclusions. Some known facts (tailored[1] to our immediate purpose) pertaining to differential inclusions are first assembled.

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[1] Variants of Propositions 2.2, 2.4 and 2.8 can be found in, for example, [8], [19]: for general treatments of differential inclusions and related topics in set-valued analysis, nonsmooth control and optimization see [1], [4], [6], [7], [8] and [12].

2.1. Maximal solutions. Consider the non-autonomous initial-value problem

$$\dot{x}(t) \in X(t, x(t)), \; x(t) \in G, \quad x(t_0) = x^0, \tag{1}$$

where $G \neq \emptyset$ is an open subset of $\mathbb{R}^N$. The set-valued map $(t, x) \mapsto X(t, x) \subset \mathbb{R}^N$ in (1) is assumed to be upper semicontinuous[2] on $\mathbb{R} \times G$, with non-empty, convex and compact values. This is sufficient (see, for example, [1, Chapter 2, Theorem 3]) to ensure that, for each $(t_0, x^0) \in \mathbb{R} \times G$, (1) admits a solution: an X-arc[3] $x \in AC([t_0, \omega); G)$, with $x(t_0) = x^0$.

Definition 2.1. *A solution x of (1) is said to be* maximal, *if it does not have a proper right extension which is also a solution of (1).*

Proposition 2.2. *Every solution of (1) can be extended to a maximal solution.*

Definition 2.3. *A solution $x \in AC([t_0, \omega); G)$ of (1) is* precompact *if it is maximal and the closure* $\mathrm{cl}(x([t_0, \omega)))$ *of its trajectory is a compact subset of G.*

Proposition 2.4. *If $x \in AC([t_0, \omega); G)$ is a precompact solution of (1), then $\omega = \infty$.*

2.2. Limit sets. Here, we specialize to the autonomous case of (1), rewritten as

$$\dot{x}(t) \in X(x(t)), \quad x(t) \in G, \quad x(0) = x^0 \tag{2}$$

where, without loss of generality, $t_0 = 0$ is assumed. The map $x \mapsto X(x) \subset \mathbb{R}^N$ (with domain G) is upper semicontinuous with non-empty, convex and compact values.

Definition 2.5. *Let $x \in AC([0, \omega); G)$ be a maximal solution of (2). A point $\bar{x} \in \mathbb{R}^N$ is an* ω-limit point of x *if there exists an increasing sequence $(t_n) \subset [0, \omega)$ such that $t_n \to \omega$ and $x(t_n) \to \bar{x}$ as $n \to \infty$. The set $\Omega(x)$ of all ω-limit points of x is the* ω-limit set of x.

Definition 2.6. *Let $C \subset \mathbb{R}^N$ be non-empty. A function $x \in AC([0, \omega); G)$ is said to* approach C *if $d_C(x(t)) \to 0$ as $t \to \omega$, where d_C is the (Euclidean) distance function for C defined (on $\mathbb{R}^N$) by $d_C(v) := \inf\{\|v - c\| \mid c \in C\}$.*

Definition 2.7. *Relative to (2), $S \subset \mathbb{R}^N$ is said to be a* weakly-invariant set *if, for each $x^0 \in S \cap G$, there exists at least one maximal solution $x \in AC([0, \omega); G)$ of (2) with $\omega = \infty$ and with trajectory $x([0, \omega))$ in S.*

Proposition 2.8. *If x is a precompact solution of (2), then $\Omega(x)$ is a non-empty, compact, connected subset of G. Moreover, $\Omega(x)$ is the smallest closed set approached by x and is weakly invariant.*

2.3. Invariance principles. For later use, the following fact (a specialization of a more general result [4, Theorem 3.1.7]) is first recorded.

Proposition 2.9. *Let $I = [a, b]$, let non-empty $K \subset G$ be compact. If $(x_n) \subset AC(I; K)$ is a sequence of X-arcs and there exists a scalar Φ such that, for all n, $\|\dot{x}_n(t)\| \leq \Phi$ for almost all $t \in I$, then (x_n) has a subsequence converging uniformly to an X-arc $x \in AC(I; K)$.*

We now arrive at the main result, which generalizes [2, Theorem 1.2].

[2] The set-valued map X is upper semicontinuous if it is upper semicontinuous at every point $\bar{\xi}$ of its domain in the sense that, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $X(\xi) \subset X(\bar{\xi}) + \varepsilon\mathbb{B}$ for all ξ with $\|\xi - \bar{\xi}\| < \delta$, where $\mathbb{B}$ (with closure $\bar{\mathbb{B}}$) denotes the open unit ball centred at 0 in $\mathbb{R}^N$.

[3] For an interval $I \subset \mathbb{R}$ and $S \subset \mathbb{R}^N$, $AC(I; S)$ denotes the space of functions $I \to S$ that are absolutely continuous on compact subintervals of I. For simplicity, we write $AC(I)$ in place of $AC(I; I)$; the same notational convention applies to other function spaces. A function $x \in AC(I; G)$ is said to be an X-arc if it satisfies the differential inclusion in (1) almost everywhere.

THEOREM 2.10. *Let $l : G \to \mathbb{R}$ be lower semicontinuous. Suppose that $U \subset G$ is non-empty and that $l(z) \geq 0$ for all $z \in U$.*

If x is a precompact solution of (2) with trajectory in U and $l \circ x \in L^1(\mathbb{R}_+)$, then x approaches the largest weakly-invariant set in $\Sigma := \{z \in \mathrm{cl}(U) \cap G |\ l(z) \leq 0\}$.

Proof. By Proposition 2.4, x has maximal interval of existence $\mathbb{R}_+ = [0, \infty)$ and, by Proposition 2.8, has non-empty ω-limit set $\Omega(x)$. Clearly $\Omega(x) \subset \mathrm{cl}(U) \cap G$. Let $\bar{x} \in \Omega(x)$ and so there exists $(t_n) \subset \mathbb{R}_+$ with $t_n \to \infty$ and $x(t_n) \to \bar{x}$ as $n \to \infty$. Define $K := \mathrm{cl}(x(\mathbb{R}_+))$. By upper semicontinuity of X together with compactness of its values, $X(K)$ is compact and so $\dot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^N)$, with norm $\|\dot{x}\|_\infty$. Write $I = [0, 1]$ and define a sequence (x_n) of X-arcs $x_n \in AC(I; K)$ by $x_n(s) := x(s+t_n)$. Evidently, $x_n(0) \to \bar{x}$ as $n \to \infty$. By Proposition 2.9 (with $\Phi = \|\dot{x}\|_\infty$), (x_n) has a subsequence – which we do not relabel – converging uniformly to an X-arc $x^* \in AC(I; K)$, with $x^*(0) = \bar{x}$.

Define $\lambda^* \in L^1(I; \mathbb{R})$ by $\lambda^*(s) := l(x^*(s))$ and the sequence $(\lambda_n) \subset L^1(I; \mathbb{R})$ by $\lambda_n(s) := l(x_n(s))$. By lower semicontinuity of l, together with continuity of x, x^* and (uniform) convergence of (x_n) to x^*, it follows that λ^* and λ_n, $n \in \mathbb{N}$, are lower semicontinuous with $\liminf_{n\to\infty} \lambda_n(s) \geq \lambda^*(s)$ for all $s \in I$. By Fatou's Lemma,

$$\int_0^t \lambda^*(s)ds \leq \int_0^t \liminf_{n\to\infty} \lambda_n(s)\, ds \leq \liminf_{n\to\infty} \int_0^t \lambda_n(s)\, ds \quad \forall\ t \in I.$$

By the hypotheses, $l(x(t)) \geq 0$ for all t and $\int_0^\infty l(x(t))dt < \infty$. Therefore,

$$W(t) := \int_t^\infty l(x(s))ds \quad \forall\ t \geq 0$$

defines a monotone function $W \in AC(\mathbb{R}_+)$ with $W(t) \downarrow 0$ as $t \to \infty$. Hence,

$$\begin{aligned} 0 = \lim_{n\to\infty} W(t_n) - \lim_{n\to\infty} W(t+t_n) &= \lim_{n\to\infty} \int_{t_n}^{t+t_n} l(x(s))ds \\ &= \lim_{n\to\infty} \int_0^t l(x(s+t_n))ds = \lim_{n\to\infty} \int_0^t \lambda_n(s)ds \geq \int_0^t \lambda^*(s)ds \quad \forall\ t \in I. \end{aligned}$$

Seeking a contradiction, suppose $\epsilon := \lambda^*(0) > 0$. Then, by lower semicontinuity of λ^*, there exists $t \in (0, 1]$ such that $\lambda^*(s) > \lambda^*(0) - \frac{1}{2}\epsilon = \frac{1}{2}\epsilon$ for all $s \in (0, t)$, whence the contradiction

$$0 \geq \int_0^t \lambda^*(s)ds > \tfrac{1}{2}\epsilon t > 0.$$

Therefore, $0 \geq \lambda^*(0) = l(x^*(0)) = l(\bar{x})$ and so $\bar{x} \in \Sigma$. Since $\bar{x} \in \Omega(x)$ is arbitrary, we have $\Omega(x) \subset \Sigma$. By Proposition 2.8, x approaches $\Omega(x)$ and the latter is a weakly-invariant set. Therefore, x approaches the largest weakly-invariant set in Σ. ☐

The next result is a nonsmooth extension, to differential inclusions, of LaSalle's theorem [11, Chapter 2, Theorem 6.4]: a smooth version (that is, restricted to smooth functions V) is given in [19, Theorem 1] and a nonsmooth version is proved in [23, Theorem 3]; the alternative proof given below is considerably simpler, by virtue of its use of the integral invariance principle. First, we give Clarke's [4] definition of a generalized directional derivative $V^o(z; \phi)$ of a locally Lipschitz function $V : G \to \mathbb{R}$ at z in direction ϕ:

$$V^o(z; \phi) := \limsup_{\substack{y\to z \\ h\downarrow 0}} \frac{V(y + h\phi) - V(y)}{h}\ .$$

The map $(z,\phi)\mapsto V^o(z;\phi)$ is upper semicontinuous (in the sense of real-valued functions) and, for each z, the map $\phi\mapsto V^o(z;\phi)$ is Lipschitz continuous.

THEOREM 2.11. *Let $V: G\to\mathbb{R}$ be locally Lipschitz. Define*

$$u: G\to\mathbb{R},\ z\mapsto u(z):=\max\{V^o(z;\phi)|\ \phi\in X(z)\}.$$

Suppose that $U\subset G$ is non-empty and that $u(z)\le 0$ for all $z\in U$.

If x is a precompact solution of (2) with trajectory in U, then, for some constant $c\in V(\mathrm{cl}(U)\cap G)$, x approaches the largest weakly-invariant set in $\Sigma\cap V^{-1}(c)$, where

$$\Sigma=\{z\in\mathrm{cl}(U)\cap G|\ u(z)\ge 0\}.$$

Proof. This result is essentially a corollary to Theorem 2.10 insofar as the essence of the proof is to show that the hypotheses of Theorem 2.10 hold with $l:=-u$.

We first show that u is upper semicontinuous (and so $l\equiv -u$ is lower semicontinuous). Let $z\in G$ be arbitrary and let $(z_n)\subset G$ be such that $z_n\to z$ as $n\to\infty$. From $(u(z_n))$ we extract a subsequence $(u(z_{n_k}))$ with $u(z_{n_k})\to\limsup_{n\to\infty}u(z_n)$ as $k\to\infty$. For each k, let ϕ_k be a maximizer of continuous $V^o(z_{n_k};\cdot)$ over compact $X(z_{n_k})$, and so $u(z_{n_k})=V^o(z_{n_k};\phi_k)$. Let $\varepsilon>0$. By upper semicontinuity of X, we have $X(z_{n_k})\subset X(z)+\varepsilon\mathbb{B}$ for all k sufficiently large. Since $\phi_k\in X(z_{n_k})$ and $X(z)$ is compact, (ϕ_k) contains a subsequence converging to $\phi^*\in\mathrm{cl}(X(z)+\varepsilon\mathbb{B})$. Since $\varepsilon>0$ is arbitrary and $X(z)$ is compact, $\phi^*\in X(z)$. Thus, invoking upper semicontinuity of $V^o(\cdot;\cdot)$, we may conclude that $\limsup_{n\to\infty}u(z_n)\le V^o(z;\phi^*)\le u(z)$, whence upper semicontinuity of u.

Observe that, for all $z\in U$,

$$V_+(z;\phi):=\liminf_{h\downarrow 0}\frac{V(z+h\phi)-V(z)}{h}\le V^o(z;\phi)\le u(z)\le 0\quad\forall\ \phi\in X(z). \tag{3}$$

By Proposition 2.4, x has interval of existence $\mathbb{R}_+$. Let $\mathcal{O}\subset\mathbb{R}_+$ denote the set of measure zero on which the derivative $\dot{x}(t)$ fails to exist. Since V is locally Lipschitz, for each $t\in\mathbb{R}_+\backslash\mathcal{O}$ there exists a constant L_t such that, for all $h>0$ sufficiently small,

$$\begin{aligned}V(x(t+h))-V(x(t))&\le V(x(t)+h\dot{x}(t))-V(x(t))+L_t|x(t+h)-x(t)-h\dot{x}(t)|\\&\le V(x(t+h))-V(x(t))+2L_t|x(t+h)-x(t)-h\dot{x}(t)|.\end{aligned}$$

Therefore,

$$\liminf_{h\downarrow 0}\frac{V(x(t+h))-V(x(t))}{h}=V_+(x(t);\dot{x}(t))\quad\forall\ t\in\mathbb{R}_+\backslash\mathcal{O}. \tag{4}$$

Next, we prove that $V\circ x:\ t\mapsto V(x(t))$ is nonincreasing on $\mathbb{R}_+$. This we do by showing that $V\circ x$ is nonincreasing on every compact subinterval. Let $[\alpha,\beta]\subset\mathbb{R}_+$, and let $K\subset G$ be compact and such that $x([\alpha,\beta])\subset K$. Since V is locally Lipschitz on G, it is Lipschitz on K. Thus, the restriction of $V\circ x$ to $[\alpha,\beta]$ is a composition of a Lipschitz function and an absolutely continuous function and so is itself absolutely continuous. It now follows from (3) and (4) that

$$V(x(t))-V(x(\tau))\ =\ \int_\tau^t(V\circ x)'\ \le\ \int_\tau^t u(x(s))ds\le 0\quad\forall\ t,\tau\in[\alpha,\beta],\ t\ge\tau.$$

Therefore, $t \mapsto V(x(t))$ is nonincreasing on $[\alpha, \beta]$. Since $[\alpha, \beta] \subset \mathbb{R}_+$ is arbitrary, we conclude that $V \circ x$ is nonincreasing on $\mathbb{R}_+$ with

$$V(x(t)) - V(x(\tau)) \le \int_\tau^t u(x(s))ds \le 0 \quad \forall\, t, \tau \in \mathbb{R}_+,\ t \ge \tau. \tag{5}$$

By continuity of V and precompactness of x, we conclude that $V(x(\cdot))$ is bounded. Therefore, $V(x(t)) \downarrow c := \inf_{t \in \mathbb{R}_+} V(x(t)) \in \mathbb{R}$ as $t \to \infty$. It follows that (i) $\Omega(x) \subset V^{-1}(c)$ and (ii) for all $t \ge 0$,

$$\int_0^t l(x(s))ds \equiv -\int_0^t u(x(s))ds \le V(x(0)) - V(x(t)) \le V(x(0)) - c < \infty.$$

An application of Theorem 2.10 completes the proof. □

3. Adaptive control. Approaches to adaptive control may be classified into methods that – either implicitly or explicitly – exhibit some aspect of identification of the process to be controlled, and methods that seek only to control. The latter approach, to be adopted here and sometimes referred to as universal control, has its origins in the work of Byrnes and Willems [3, 27], Mårtensson [13, 14, 15], Morse [16], Nussbaum [17] and others (see [9] for a survey and comprehensive bibliography). In common with its above-cited precursors, this section of the present paper is concerned with demonstrating the existence – under relatively weak assumptions – of a single controller that achieves some prescribed objective for every system in the underlying class. In contrast with its above-cited precursors (which deal mainly with classes of linear systems, possibly subject to “mild” nonlinear perturbations, in a context of adaptive linear feedback), the present paper considers strongly nonlinear systems and nonsmooth feedback (for an overview of adaptive control of nonlinear systems in the context of *smooth* feedback, see [18]). In essence, the ensuing two subsections provide a unified analysis – unified through its use of the integral invariance principle – of various problems in nonlinear adaptive control (some closely-related problems have been individually investigated, via alternative analyses, in [5], [10], [20] and [21]).

3.1. Scalar systems. First, consider scalar systems of the form

$$\dot y(t) = f(p(t), y(t)) + bu(t), \quad y(t), u(t) \in \mathbb{R},\ p(t) \in \mathbb{R}^P, \quad y(t_0) = y^0 \tag{6}$$

where parameters $b \in \mathbb{R}$, $P \in \mathbb{N}$ and functions f, p are unknown. The state $y(t)$ is available for feedback. We will identify (6) with the quadruple (b, f, p, P).

For any function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that is both continuous and positive definite ($\phi(s) > 0$ for all $s \ne 0$), we denote, by $\mathcal{N}_\phi$, the set of system quadruples (b, f, p, P) satisfying the following three assumptions.

Assumption A. $b \ne 0$.

Assumption B. $(p, y) \mapsto f(p, y)$, $\mathbb{R}^P \times \mathbb{R} \to \mathbb{R}$, is a continuous function and is ϕ-bounded uniformly with respect to p in compact sets: precisely, for every compact $K \subset \mathbb{R}^P$, there exists scalar μ_K such that $|f(p, y)| \le \mu_K \phi(|y|)$ for all $(p, y) \in K \times \mathbb{R}$.

Assumption C. $p(\cdot) \in L^\infty(\mathbb{R}; \mathbb{R}^P)$.

By virtue of Assumption C, without loss of generality $t_0 = 0$ may be assumed in (6): this we will do, without further comment, throughout this subsection.

Examples 3.1.

(a) Let $\phi : |y| \mapsto \exp(|y|)$. Then all polynomial systems, of arbitrary degree, of the form $\dot y(t) = p_1(t) + p_2(t)y(t) + \cdots p_P(t)y^{P-1}(t) + bu(t)$, with $b \ne 0$ and coefficient functions $p_i(\cdot) \in L^\infty(\mathbb{R})$, are of class $\mathcal{N}_\phi$.

(b) Suppose that Assumptions A, C hold and that the only *a priori* information on continuous f is its behaviour "at infinity", captured in the following manner: for some known continuous $\hat{\phi}:\mathbb{R}_+\to\mathbb{R}_+$, $(p,y)\mapsto f(p,y)$ is $O(\hat{\phi}(|y|))$, as $|y|\to\infty$, uniformly with respect to p in compact sets in the sense that, for every compact $K\subset\mathbb{R}^P$, there exist scalars c_K and C_K such that, for all $p\in K$, $|f(p,y)|\le c_K\hat{\phi}(|y|)$ for all $|y|>C_K$. Then Assumption B holds with $\phi:=1+\hat{\phi}$, and so $(b,f,p,P)\in\mathcal{N}_\phi$. $\diamond$

3.1.1. Adaptive stabilizer. Let $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous, positive-definite function. Assuming only that the function ϕ and the instantaneous state $y(t)$ are available for control purposes, we will show that the following adaptive feedback strategy (appropriately interpreted) is a $\mathcal{N}_\phi$-universal stabilizer in the sense that it assures that the state of (6) approaches $\{0\}$ for all quadruples $(b,f,p,P)\in\mathcal{N}_\phi$ whilst maintaining boundedness of the controller function $\lambda(\cdot)$:

$$u(t)=\nu(\lambda(t))\phi(|y(t)|)\mathrm{sgn}(y(t)),\quad \dot{\lambda}(t)=\phi(|y(t)|)|y(t)|,\quad \lambda(0)=\lambda^0, \tag{7}$$

where ν is any continuous function $\mathbb{R}\to\mathbb{R}$ with the properties:

$$\text{(a)}\quad \limsup_{\eta\to\infty}\frac{1}{\eta}\int_0^\eta \nu=+\infty,\qquad \text{(b)}\quad \liminf_{\eta\to\infty}\frac{1}{\eta}\int_0^\eta \nu=-\infty. \tag{8}$$

For example, $\nu:\theta\mapsto\theta^2\cos\theta$ suffices.

In view of the discontinuous nature of the feedback (however, note that, if $\phi(0)=0$, then the feedback is continuous), we interpret the strategy (7) in the set-valued sense

$$u(t)\in\nu(\lambda(t))\phi(|y(t)|)\sigma(y(t)),\quad \dot{\lambda}(t)=\phi(|y(t)|)|y(t)|,\ \lambda(0)=\lambda^0 \tag{9}$$

with $y\mapsto\sigma(y)\subset\mathbb{R}$ given by

$$\sigma(y):=\begin{cases}\{\mathrm{sgn}(y)\}, & y\neq 0\\ [-1,+1], & y=0\,.\end{cases} \tag{10}$$

Let $(b,f,p,P)\in\mathcal{N}_\phi$. By properties of $f(\cdot,\cdot)$ and boundedness of $p(\cdot)$, there exists a scalar μ such that $|f(p(t),y)|\le\mu\phi(|y|)$ for all (t,y).

We embed the feedback-controlled system in a differential inclusion on $\mathbb{R}^2$:

$$\dot{x}(t)\in X(x(t)),\quad x(t)=(y(t),\lambda(t))\in G:=\mathbb{R}^2,\quad x(0)=x^0=(y^0,\lambda^0) \tag{11}$$

where $x\mapsto X(x)\subset\mathbb{R}^2$ is given by

$$X(x)\equiv X(y,\lambda):=\{v+bu|\ |v|\le\mu\phi(|y|),\ u\in\nu(\lambda)\phi(|y|)\sigma(y)\}\times\{\phi(|y|)|y|\}.$$

X is upper semicontinuous on $\mathbb{R}^2$ with non-empty, convex and compact values. Therefore, for each $x^0\in\mathbb{R}^2$, the initial-value problem (11) has a solution and, by Proposition 2.2, every solution can be extended to a maximal solution.

LEMMA 3.2. *Let $x^0\in\mathbb{R}^2$ be arbitrary and let $x(\cdot)=(y(\cdot),\lambda(\cdot))$ be a maximal solution of (11), defined on its maximal interval of existence $[0,\omega)$. Then (i) $\omega=\infty$; (ii) $\lim_{t\to\infty}\lambda(t)$ exists and is finite; (iii) $y(t)\to 0$ as $t\to\infty$.*

Proof. The essence of the proof is to establish boundedness of $x(\cdot)$: whence, by Proposition 2.4, assertion (i) and, by monotonicity, assertion (ii): state convergence to zero (assertion (iii)) is then an immediate consequence of Theorem 2.10.

For almost all $t \in [0, \omega)$, we have

$$y(t)\dot{y}(t) \leq [\mu + b\nu(\lambda(t))]\, \phi(|y(t)|)|y(t)| = [\mu + b\nu(\lambda(t))]\dot{\lambda}(t) \tag{12}$$

which, on integration, yields

$$0 \leq y^2(t) \leq y^2(\tau) + 2\mu[\lambda(t) - \lambda(\tau)] + 2b \int_{\lambda(\tau)}^{\lambda(t)} \nu, \quad \forall\, t, \tau \in [0, \omega),\ t \geq \tau. \tag{13}$$

Seeking a contradiction, suppose that solution component $\lambda(\cdot)$ (monotone increasing) is unbounded. Fix τ such that $\lambda(\tau) \geq 1$. Dividing by $\lambda(t) \geq \lambda(\tau) \geq 1$ in (13), gives

$$0 \leq \text{constant} + \frac{b}{\lambda(t)} \int_{\lambda(\tau)}^{\lambda(t)} \nu, \quad \forall\, t \in [\tau, \omega)$$

Recalling that $b \neq 0$ and taking limit inferior as $t \uparrow \omega$ ($\lambda(t) \uparrow \infty$) leads to a contradiction of one or the other of properties (8). Hence, $\lambda(\cdot)$ is bounded and so, by (13), $y(\cdot)$ is also bounded. Therefore, $x(\cdot) = (y(\cdot), \lambda(\cdot)) \in AC([0, \omega); \mathbb{R}^2)$ is a precompact solution of (11) and so, by Proposition 2.4, $\omega = \infty$. Defining $l : \mathbb{R}^2 \to \mathbb{R}_+$, $x \equiv (y, \lambda) \mapsto \phi(|y|)|y|$, for which $l^{-1}(0) = \{0\} \times \mathbb{R}$ and $\dot{\lambda} = l \circ x$, we may conclude, by boundedness of $\lambda(\cdot)$, that $l \circ x \in L^1(\mathbb{R}_+)$ and so, by Theorem 2.10 (with $U = \mathbb{R}^2$), $x(\cdot) = (y(\cdot), \lambda(\cdot))$ approaches the set $\{0\} \times \mathbb{R}$. In particular, $y(t) \to 0$ as $t \to \infty$ and, by boundedness and monotonicity of $\lambda(\cdot)$, $\lim_{t\to\infty} \lambda(t)$ exists and is finite. □

3.1.2. Adaptive servomechanism. We now turn attention to the servomechanism problem for scalar systems (6): that is, the construction of controls that cause the state to track, asymptotically, reference signals $r(\cdot)$ of some given class in the sense that $|y(t) - r(t)| \to 0$ as $t \to \infty$. For the class of reference signals (previously adopted in [20], [10], [21]) we take the (Sobolev) space $\mathcal{R} = W^{1,\infty}(\mathbb{R})$ of functions $r \in (AC \cap L^\infty)(\mathbb{R})$ with essentially bounded derivative $\dot{r} \in L^\infty(\mathbb{R})$, equipped with the norm $\|r\|_{1,\infty} = \|r\|_\infty + \|\dot{r}\|_\infty$ where $\|\ \|_\infty$ denotes the norm on $L^\infty(\mathbb{R})$.

We impose a stronger assumption on the function f by requiring that Assumption B should hold for some known, continuous, positive-definite, nondecreasing function ϕ with the additional property that, for each $R \geq 0$, there exists a scalar ρ_R such that

$$\phi(|e + r|) \leq \rho_R \phi(|e|) \quad \forall\ (e, r) \in \mathbb{R} \times [-R, R]. \tag{14}$$

Note that, by positive definiteness of ϕ together with property (14), $\phi(0) > 0$.

EXAMPLE 3.3. Let $\phi : |y| \mapsto \exp(|y|)$, which has the property (14), and so all polynomial systems (of arbitrary degree and with coefficients in $L^\infty(\mathbb{R})$), as cited earlier in Example 3.1(a), remain admissible. $\diamond$

Let ϕ be a continuous, positive-definite, nondecreasing function with property (14). We claim that, in order to assure that the tracking error $e(\cdot) = y(\cdot) - r(\cdot)$ approaches $\{0\}$ for all reference signals $r \in \mathcal{R}$ and all quadruples $(b, f, p, P) \in \mathcal{N}_\phi$, it suffices to replace every occurrence of $y(t)$ in (9) by $e(t)$. Proof of this claim follows.

Let $(b, f, p, P) \in \mathcal{N}_\phi$. Write $\tilde{P} = P + 2$ and define the continuous function

$$\tilde{f} : \mathbb{R}^{\tilde{P}} \times \mathbb{R} \to \mathbb{R}, \quad (\tilde{p}, e) \equiv (p, r, s, e) \mapsto f(p, e + r) - s.$$

Let $\tilde{K} \subset \mathbb{R}^{\tilde{P}}$ be compact and so there exist compact $K \subset \mathbb{R}^P$ and $R > 0$ such that $\tilde{K} \subset K \times [-R, R]^2$. By properties of f and ϕ, there exist constants μ_K and ρ_R such that, for all $(p, r, s) \equiv \tilde{p} \in \tilde{K} \subset K \times [-R, R]^2$,

$$|\tilde{f}(\tilde{p}, e)| \leq |f(p, e + r)| + |s| \leq \mu_K \phi(|e + r|) + R \leq \mu_K \rho_R \phi(|e|) + R \leq \tilde{\mu}_{\tilde{K}} \phi(|e|)$$

with $\tilde{\mu}_{\tilde{K}} := \mu_K \rho_R + R/\phi(0)$. Thus, $\tilde{f}$ is ϕ-bounded uniformly with respect to $\tilde{p}$ in compact sets.

Let $r \in \mathcal{R}$ be arbitrary. Then $\tilde{p}(\cdot) := (p(\cdot), r(\cdot), \dot{r}(\cdot)) \in L^\infty(\mathbb{R}; \mathbb{R}^{\tilde{P}})$ and so $(b, \tilde{f}, \tilde{p}, \tilde{P}) \in \mathcal{N}_\phi$. Expressed in terms of the tracking error $e(t) = y(t) - r(t)$, the system dynamics have the form

$$\dot{e}(t) = \tilde{f}(\tilde{p}(t), e(t)) + bu(t), \quad \tilde{p}(t) = (p(t), r(t), \dot{r}(t)) \in \mathbb{R}^{\tilde{P}}, \quad e(0) = e^0.$$

We are now in precisely the same context, modulo notation, as in the case of an adaptive stabilizer and so, replacing all occurrences of $y(t)$ in (9) by $e(t)$ to yield

$$u(t) \in \nu(\lambda(t))\phi(|e(t)|)\sigma(e(t)), \quad \dot{\lambda}(t) = \phi(|e(t)|)|e(t)|, \ \lambda(0) = \lambda^0, \tag{15}$$

then the same argument (as used to establish Lemma 3.2) applies *mutatis mutandis* to conclude that (15) is an $(\mathcal{R}, \mathcal{N}_\phi)$-universal servomechanism: for each $r(\cdot) \in \mathcal{R}$ and $(b, f, p, P) \in \mathcal{N}_\phi$, every solution $(e(\cdot), \lambda(\cdot))$ of the controlled system has maximal interval of existence $\mathbb{R}_+$ with $e(t) \to 0$ as $t \to \infty$, and $\lim_{t\to\infty} \lambda(t)$ exists and is finite.

3.1.3. Practical stabilization and tracking by continuous feedback. The adaptive strategies outlined above are (generically) of a discontinuous feedback nature. From a viewpoint of practical utility, this feature might be regarded as unpleasant. Here, we investigate the possibility of adopting smooth approximations to the discontinuous feedbacks. Of course, in so doing, one would expect to pay a price. It will be shown that, if the objective of attractivity of the zero state (in the stabilization case) or asymptotic tracking (in the case of a servomechanism) is weakened to requiring global attractivity of any (arbitrarily small) prescribed neighbourhood of zero or, for the servomechanism problem, tracking to within any prescribed (arbitrarily small but non-zero) error margin, then such approximations are feasible. We will present this result only in the context of the stabilization problem (imposing the additional property (14), the corresponding result for the servomechanism problem is readily inferred by analogy with Section 3.1.2). The naïve idea, as developed in [10] and [21], is to inhibit the adaption whenever the state lies within the prescribed neighbourhood of zero.

Let $\epsilon > 0$ be arbitrary and let d_ϵ denote the distance function for the set $[-\epsilon, \epsilon]$: thus, $d_\epsilon(x) := |x| - \epsilon$ if $|x| \geq \epsilon$; $d_\epsilon(x) = 0$ if $|x| < \epsilon$.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and positive-definite. Let $\mathrm{sat}_\epsilon : \mathbb{R} \to \mathbb{R}$ be any continuous function (arbitrarily smooth) such that (i) $|\mathrm{sat}_\epsilon(x)| \leq 1$ for all x and (ii) $\mathrm{sat}_\epsilon(x) = \mathrm{sgn}(x)$ whenever $|x| \geq \epsilon$. We will show that the following strategy assures that the state of (6) approaches the interval $[-\epsilon, \epsilon]$ for all quadruples $(b, f, p, P) \in \mathcal{N}_\phi$:

$$u(t) = \nu(\lambda(t))\phi(|y(t)|)\mathrm{sat}_\epsilon(y(t)), \quad \dot{\lambda}(t) = \phi(|y(t)|)d_\epsilon(y(t)), \quad \lambda(0) = \lambda^0$$

where, as before, ν is any continuous function with properties (8).

Let $(b, f, p, P) \in \mathcal{N}_\phi$ and so there exists constant μ such that $|f(p(t), y)| \leq \mu\phi(|y|)$ for all (t, y). Define the set-valued map $y \mapsto \sigma_\epsilon(y)$ by

$$\sigma_\epsilon(y) := \begin{cases} \{\mathrm{sgn}(y)\}, & |y| \geq \epsilon \\ [-1, +1], & |y| < \epsilon \ . \end{cases}$$

Evidently, sat_ϵ is a continuous selection from σ_ϵ. We now embed the smooth-feedback-controlled system in the following differential inclusion on $\mathbb{R}^2$:

$$\dot{x}(t) \in X_\epsilon(x(t)), \quad x(t) = (y(t), \lambda(t)) \in \mathbb{R}^2, \quad x(0) = x^0 = (y^0, \lambda^0) \tag{16}$$

where $x \mapsto X_\epsilon(x) \subset \mathbb{R}^2$ is given by

$$X_\epsilon(x) \equiv X_\epsilon(y,\lambda) := \{v + bu \,|\, |v| \le \mu\phi(|y|),\ u \in \nu(\lambda)\phi(|y|)\sigma_\epsilon(y)\} \times \{\phi(|y|)d_\epsilon(y)\}.$$

X_ϵ is upper semicontinuous on $\mathbb{R}^2$ with non-empty, convex and compact values. Therefore, for each $x^0 \in \mathbb{R}^2$, the initial-value problem (16) has a solution and every solution has a maximal extension.

LEMMA 3.4. *Let $x^0 \in \mathbb{R}^2$ be arbitrary and let $x(\cdot) = (y(\cdot), \lambda(\cdot))$ be a maximal solution of (16), defined on its maximal interval of existence $[0,\omega)$. Then (i) $\omega = \infty$; (ii) $\lim_{t\to\infty}\lambda(t)$ exists and is finite; (iii) $d_\epsilon(y(t)) \to 0$ as $t \to \infty$.*

Proof. For almost all $t \in [0,\omega)$,

$$\begin{aligned}(d/dt)(\tfrac{1}{2}d_\epsilon^2(y(t))) &= d_\epsilon(y(t))\mathrm{sat}_\epsilon(y(t))\dot{y}(t)\\ &\le [\mu + b\nu(\lambda(t))]\,\phi(|y(t)|)d_\epsilon(y(t)) = [\mu + b\nu(\lambda(t))]\dot{\lambda}(t)\end{aligned}$$

which, on integration, yields

$$0 \le d_\epsilon^2(y(t)) \le d_\epsilon^2(y(\tau)) + 2\mu[\lambda(t) - \lambda(\tau)] + 2b\int_{\lambda(\tau)}^{\lambda(t)} \nu \tag{17}$$

valid for all $t, \tau \in [0,\omega)$, with $t \ge \tau$. By precisely the same contradiction argument as employed previously in the case of the discontinuous stabilizer, we may deduce that $x(\cdot) = (y(\cdot),\lambda(\cdot)) \in AC([0,\omega);\mathbb{R}^2)$ is a precompact solution of (16) and so $\omega = \infty$. Defining $l : \mathbb{R}^2 \to \mathbb{R}$, $x \equiv (y,\lambda) \mapsto \phi(|y|)d_\epsilon(y)$ (and so $\dot{\lambda} = l \circ x$), we conclude, by boundedness of $\lambda(\cdot)$, that $l \circ x \in L^1(\mathbb{R}_+)$). Therefore, by Theorem 2.10 (with $U = \mathbb{R}^2$), $x(\cdot) = (y(\cdot),\lambda(\cdot))$ approaches the set $l^{-1}(0) \equiv \{(y,\lambda) |\ d_\epsilon(y) = 0\}$. In particular, $d_\epsilon(y(t)) \to 0$ as $t \to \infty$ and, by monotonicity of bounded $\lambda(\cdot)$, $\lim_{t\to\infty}\lambda(t)$ exists and is finite. ∎

3.1.4. Dynamically perturbed scalar systems. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and positive definite. Let $\Sigma_1 = (b,f,p,P) \in \mathcal{N}_\phi$. We wish to consider the situation wherein Σ_1 is subject to perturbations generated through its interconnection with a dynamical system Σ_2.

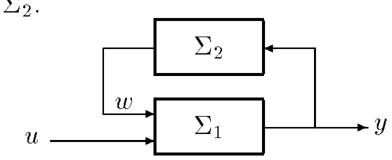

Figure 1

System Σ_2 is assumed to correspond to a differential equation (driven by the state of the scalar system Σ_1) on $\mathbb{R}^N$ of the form

$$\Sigma_2 : \quad \dot{\zeta}(t) = g(y(t),\zeta(t)), \quad w(t) = h(\zeta(t)), \quad \zeta(0) = \zeta^0 \in \mathbb{R}^N \tag{18}$$

with input $y(t)$, and scalar output $w(t)$ perturbing Σ_1. Notationally, we identify the system Σ_2 with the triple (g,h,N). The overall system has representation (on $\mathbb{R}\times\mathbb{R}^N$)

$$\begin{cases} \dot{y}(t) = f(p(t),y(t)) + h(\zeta(t)) + bu(t) \\ \dot{\zeta}(t) = g(y(t),\zeta(t)), \quad (y(0),\zeta(0)) = (y^0,\zeta^0). \end{cases} \tag{19}$$

We will define, via Assumption D below, a class $\mathcal{P}_\psi$ of admissible systems $\Sigma_2 = (g,h,N)$, such that the $\mathcal{N}_\phi$-universal stabilizer of Section 3.1.1 is readily modified to yield $(\mathcal{N}_\phi, \mathcal{P}_\psi)$-universal stabilizer. Before stating Assumption D, we cite Sontag's concept of input-to-state stability [24, 25] (see also [26]) in the context of (18) with g assumed to be locally Lipschitz and with $y(\cdot)$ regarded as an independent input of class $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$. System (18) is *input-to-state stable (ISS)* if there exist a continuous, strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, with $\gamma(0) = 0$, and a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, with $\beta(0, t) = 0$ for all t and having the properties that, for each $t \geq 0$, $\beta(\cdot, t)$ is strictly increasing and, for each $s \geq 0$, $\beta(s, t) \downarrow 0$ as $t \to \infty$, such that, for every $\zeta^0 \in \mathbb{R}^N$ and every $y(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, the (unique) maximal solution $\zeta(\cdot)$ of the initial-value problem (18) satisfies $\|\zeta(t)\| \leq \beta(\|\zeta^0\|, t) + \gamma(\|y_t\|_\infty)$ for all $t \geq 0$, where y_t denotes the truncation of y at t, that is, $y_t(s) = y(s)$ if $s \leq t$ and $y_t(s) = 0$ if $s > t$. If (18) is ISS, then it is forward complete and has the convergent-input, convergent-state property: for each $\zeta^0 \in \mathbb{R}^N$ and $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, the unique solution $\zeta(\cdot)$ of the initial-value problem has maximal interval of existence $\mathbb{R}_+$ and, if $y(t) \to 0$ as $t \to \infty$, then $\zeta(t) \to 0$ as $t \to \infty$.

For continuous $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we denote, by $\mathcal{P}_\psi$ the set of system triples $\Sigma_2 = (g, h, N)$ satisfying the following:

Assumption D. (i) $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz; (ii) system (18) is input-to-state stable (ISS); (iii) $h : \mathbb{R}^N \to \mathbb{R}$ is continuous; (iv) there exist a function $\alpha_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and scalar $\alpha_1 > 0$, such that, for each $(\zeta^0, y(\cdot)) \in \mathbb{R}^N \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}))$, the (unique) solution $\zeta(\cdot)$ of (18) satisfies

$$\int_0^t h(\zeta(s))y(s)ds \leq \alpha_0(\|\zeta^0\|) + \alpha_1 \int_0^t \psi(|y(s)|)|y(s)|ds \quad \forall\, t \geq 0.$$

While Assumption D is rather restrictive, it is not difficult to identify non-trivial classes of systems for which the assumption holds. Three such examples follow, the first of which is easily seen, the second and third can be verified by arguments invoking [22, Theorem 2].

Examples 3.5.

(a) Let $\psi : |y| \mapsto |y|$ and suppose (g, h, N) defines a linear system with $g : (y, \zeta) \mapsto A\zeta + By$, $h : \zeta \mapsto C\zeta$. If A has spectrum, $\mathrm{spec}(A)$, in the open left half complex plane $\mathbb{C}_-$, then $(g, h, N) \in \mathcal{P}_\psi$.

(b) More generally, let $\psi : |y| \mapsto |y|^k$, $k \geq 1$. Assume $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz and $h : \mathbb{R}^N \to \mathbb{R}$ is continuous. In addition, assume g and h are positively homogeneous of degree k. If $\{0\}$ is an asymptotically stable equilibrium of the unforced system $\dot\zeta(t) = g(0, \zeta(t))$, then $(g, h, N) \in \mathcal{P}_\psi$. For example, with $k = 3$ and $\psi : |y| \mapsto |y|^3$, systems (with $N = 1$ and with unknown real parameters a_i) of the form $\dot\zeta = a_1\zeta^3 + a_2\zeta^2 y + a_3\zeta y^2 + a_4 y^3$, with output $w = a_5\zeta^3$, are of class $\mathcal{P}_\psi$, provided that $a_1 < 0$.

(c) Let $k > 0$, $k_h \geq 1$ and $\psi : |y| \mapsto \frac{1}{2}\left(|y|^{(k+1)k_h - 1} + |y|^{1/k}\right)$. Assume $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz and that $h : \mathbb{R}^N \to \mathbb{R}$ is continuous and positively homogeneous of degree k_h. If, in addition, g is positively homogeneous of degree k_g, with $1 \leq k_g \leq (k + 1)k_h$, and $\{0\}$ is an asymptotically stable equilibrium of the unforced system $\dot\zeta(t) = g(0, \zeta(t))$, then $(g, h, N) \in \mathcal{P}_\psi$. For example, systems (with $N = 1$ and unknown parameters ρ, $a_i \in \mathbb{R}$) of the form $\dot\zeta = |\zeta|^\rho[a_1\zeta + a_2 y]$, with linear output $y = a_3\zeta$ (in which case $k_h = 1$), are of class $\mathcal{P}_\psi$, provided that $a_1 < 0$ and $0 \leq \rho \leq k$. $\diamond$

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and positive definite. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. We will show that, for $(\mathcal{N}_\phi, \mathcal{P}_\psi)$-universal stabilization, it suffices to replace both occurrences of ϕ in (9) by $\phi+\psi$, to yield

(20) $u(t) \in \nu(\lambda(t))(\phi+\psi)(|y(t)|)\sigma(y(t)),\ \ \dot\lambda(t) = (\phi+\psi)(|y(t)|)|y(t)|,\ \ \lambda(0)=\lambda^0.$

Let $(b,f,p,P) \in \mathcal{N}_\phi$ and $(g,h,N) \in \mathcal{P}_\psi$. Then there exists constant μ such that $|f(p(t),y)| \le \mu\phi(|y|)$ for all (t,y). The feedback-controlled system (19-20) can be embedded in a differential inclusion on $\mathbb{R}^{N+2}$:

$$\left\{\begin{array}{ll} \dot x(t) \in X(x(t)), & x(t) = (y(t),\zeta(t),\lambda(t)) \in G := \mathbb{R}^{N+2} \\ x(0) = x^0 = (y^0,\zeta^0,\lambda^0) & \end{array}\right. \tag{21}$$

where $x = (y,\zeta,\lambda) \mapsto X(x) \subset \mathbb{R}^{N+2}$ is given by

$$\begin{aligned} X(x) \equiv X(y,\zeta,\lambda) := \{h(\zeta)+v+bu \mid\ |v| \le \mu\phi(|y|),\ u \in \nu(\lambda)(\phi+\psi)(|y|)\sigma(y)\} \\ \times \{g(y,\zeta)\} \times \{(\phi+\psi)(|y|)|y|\}. \end{aligned}$$

X is upper semicontinuous on $\mathbb{R}^{N+2}$ with non-empty, convex and compact values. Therefore, for each $x^0 \in \mathbb{R}^{N+2}$, the initial-value problem (21) has a solution and every solution can be maximally extended.

Lemma 3.6. *Let $x^0 \in \mathbb{R}^{N+2}$ be arbitrary and let $x(\cdot) = (y(\cdot),\zeta(\cdot),\lambda(\cdot))$ be a maximal solution of (21), defined on its maximal interval of existence $[0,\omega)$. Then (i) $\omega=\infty$; (ii) $\lim_{t\to\infty}\lambda(t)$ exists and is finite; (iii) $(y(t),\zeta(t)) \to (0,0)$ as $t\to\infty$.*

Proof. For almost all $t \in [0,\omega)$, we have

$$\begin{aligned} y(t)\dot y(t) &\le h(\zeta(t))y(t) + \mu\phi(|y(t)|)|y(t)| + b\nu(\lambda(t))(\phi+\psi)(|y(t)|)|y(t)| \\ &\le h(\zeta(t))y(t) + [\mu + b\nu(\lambda(t))]\dot\lambda(t) \end{aligned}$$

which, on integration, yields

$$0 \le y^2(t) \le y^2(\tau) + 2\int_\tau^t h(\zeta(s))y(s)ds + 2\mu[\lambda(t)-\lambda(\tau)] + 2b\int_{\lambda(\tau)}^{\lambda(t)} \nu$$

valid for all $t,\tau \in [0,\omega)$, with $t \ge \tau$. Invoking Assumption D, we have

$$0 \le y^2(t) \le y^2(\tau) + 2\alpha_0(\|\zeta(\tau)\|) + 2[\alpha_1+\mu][\lambda(t)-\lambda(\tau)] + 2b\int_{\lambda(\tau)}^{\lambda(t)} \nu. \tag{22}$$

By the same contradiction argument as that employed in the proof of Lemma 3.2, we may deduce that $\lambda(\cdot)$ is bounded. Boundedness of $y(\cdot)$ then follows from (22). That $\zeta(\cdot)$ is bounded is a consequence of the ISS property of $\Sigma_2 = (g,h,N)$. Therefore, $x(\cdot) = (y(\cdot),\zeta(\cdot),\lambda(\cdot)) \in AC([0,\omega);\mathbb{R}^{N+2})$ is a precompact solution of (21) and so $\omega=\infty$. Defining $l : \mathbb{R}^{N+2} \to \mathbb{R}$, $\ x \equiv (y,\zeta,\lambda) \mapsto (\phi+\psi)(|y|)|y|$ (and so $\dot\lambda = l \circ x$), we may conclude, by boundedness of $\lambda(\cdot)$, that $l \circ x \in L^1(\mathbb{R}_+)$. Therefore, by Theorem 2.10 (with $U = \mathbb{R}^{N+2}$), $x(\cdot)$ approaches the set $\{(y,\zeta,\lambda) \mid\ y = 0\}$. In particular, $y(t) \to 0$ as $t\to\infty$ and so, by the convergent-input, convergent-state property of the ISS system $\Sigma_2 = (g,h,N)$, we may also conclude that $\zeta(t) \to 0$ as $t\to\infty$. Finally, by boundedness and monotonicity of $\lambda(\cdot)$, $\lim_{t\to\infty}\lambda(t)$ exists and is finite. □

Example 3.7. *Linear, minimum-phase systems of relative degree one.* This class has played a central rôle in the development of universal adaptive control. In appropriate coordinates, such systems have state space representations of the form

$$\dot y = A_1 y + A_2\zeta + CBu, \quad \dot\zeta = A_3 y + A_4\zeta. \tag{23}$$

In the single-input, single-output case, we identify (23) and (19) with

$$f:(p,y)\mapsto A_1y,\quad h:\zeta\mapsto A_2\zeta,\quad b=CB,\quad g:(y,\zeta)\mapsto A_3y+A_4\zeta.$$

By the relative-degree-one assumption, $b = CB \neq 0$ and, by the minimum-phase assumption, $\mathrm{spec}(A_4)\subset\mathbb{C}_-$. Defining $\phi\equiv\psi:|y|\mapsto\frac{1}{2}|y|$, we see that every single-input, single-output, linear, minimum-phase system of relative degree one is of class $(\mathcal{N}_\phi,\mathcal{P}_\psi)$ and the control (20) reduces to the ubiquitous Byrnes-Willems strategy: $u(t)=\nu(\lambda(t))y(t)$, $\dot\lambda(t)=y^2(t)$, $\lambda(0)=\lambda^0$. $\diamond$

REMARKS 3.8. In considering the case of dynamically perturbed scalar systems, we treated only the problem of adaptive stabilization. The adaptive servomechanism of Section 3.1.2 can also be modified to incorporate dynamically perturbed systems, when the dynamic perturbations are generated by linear systems $\dot\zeta = A\zeta + By$, $w = C\zeta$, $\mathrm{spec}(A)\subset\mathbb{C}_-$, as already described in Example 3.5(a) above. For such perturbations, the (modified) servomechanism assures convergence to zero of the tracking error, convergence to a finite limit of the adapting parameter, and boundedness of the evolution $t\mapsto\zeta(t)$ of the perturbing system. We omit full details here.

3.2. Planar systems. In all applications of the integral invariance principle in Section 3.1 above, the conclusion that $x(t)$ tends, as $t\to\infty$, to the zero level set $l^{-1}(0)$ proved sufficient for our purposes: the additional property that $x(\cdot)$ approaches the largest weakly-invariant subset of $l^{-1}(0)$ was redundant. Here, we treat a class of systems for which the latter property can be fruitfully exploited. We consider planar systems (with scalar control u) described by a second-order differential equation:

$$(24)\quad\begin{cases}\ddot y(t)=d(t)\dot y(t)+f(p(t),y(t))+bu(t), & y(t),u(t),d(t)\in\mathbb{R},\quad p(t)\in\mathbb{R}^P,\\ (y(t_0),\dot y(t_0))=(y^0,v^0),\end{cases}$$

where the parameters $b\in\mathbb{R}$, $P\in\mathbb{N}$ and functions d, f, p are unknown. The variable $y(t)$, but not its derivative $\dot y(t)$, is available for feedback. We identify (24) with the quintuple (b,d,f,p,P). For $\delta>0$ and continuous, positive-definite, nondecreasing function $\phi:\mathbb{R}_+\to\mathbb{R}_+$, we denote, by $\mathcal{N}_{\delta,\phi}$, the set of system quintuples (b,d,f,p,P) for which Assumption A (that is, $b\neq 0$) holds, together with the following three assumptions.

ASSUMPTION E. $d\in(C\cap L^\infty)(\mathbb{R})$ and, for some $\epsilon>0$, $d(t)\le-\delta-2\epsilon$ for all t.

ASSUMPTION F. $(p,y)\mapsto f(p,y)$, $\mathbb{R}^P\times\mathbb{R}\to\mathbb{R}$ is continuous and is continuously differentiable in its first argument. Both f and D_1f ($\equiv\partial f/\partial p$) are ϕ-bounded uniformly with respect to p in compact sets: precisely, for every compact $K\subset\mathbb{R}^P$, there exists scalar μ_K such that $|f(p,y)|+\|D_1f(p,y)\|\le\mu_K\phi(|y|)$ for all $(p,y)\in K\times\mathbb{R}$.

ASSUMPTION G. $p(\cdot)\in W^{1,\infty}(\mathbb{R};\mathbb{R}^P)$.

By virtue of Assumptions E and G, without loss of generality $t_0=0$ may be assumed in (24): this we will do, without further comment, throughout.

In Section 3.2.1 below, we will show that $(b,d,f,p,P)\in\mathcal{N}_{\delta,\phi}$, for some known $\delta>0$ and known continuous positive-definite nondecreasing function ϕ, is sufficient *a priori* information for adaptive stabilizability of (24) by feedback of the variable $y(t)$ alone: in essence, Assumption E compensates for the inaccessibility of the velocity variable $\dot y(t)$ by requiring that the system should exhibit dissipitive dependence (loosely quantifiable by the known constant δ) on that variable.

EXAMPLE 3.9. As motivation for (24), consider a single-degree-of-freedom mechanical system with position, but not velocity, available for feedback and with some constant (but unknown) natural damping d quantified by a known parameter δ in the sense that $d < -\delta < 0$. If we suppose that Assumption F holds with $\phi : |y| \mapsto 1+|y|^3$, then, for example, the following particular realizations are admissible.

Nonlinear pendulum with disturbed support (disturbance $p(\cdot) \in W^{1,\infty}(\mathbb{R})$):-

$$\ddot{y}(t) = d\dot{y}(t) + (a + p(t))\sin(y(t)) + bu(t), \quad a, b \in \mathbb{R},\ b \neq 0,$$

Duffing equation with extraneous disturbance ($p(\cdot) \in W^{1,\infty}(\mathbb{R})$):-

$$\ddot{y}(t) = d\dot{y}(t) + a_1 y(t) + a_2 y^3(t) + p(t) + bu(t), \quad a_1, a_2, b \in \mathbb{R},\ b \neq 0.$$

In the absence of control, such systems are potentially "chaotic". $\diamond$

3.2.1. Adaptive stabilizer. Let $\delta > 0$ and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, positive-definite, nondecreasing function. Assuming only that δ, ϕ and the instantaneous state $y(t)$ are available for control purposes, our goal is to demonstrate the existence of an adaptive feedback strategy that provides $\mathcal{N}_{\delta,\phi}$-universal stabilization in the sense that it assures that the state of (24) approaches $\{0\}$ for all quintuples $(b, d, f, p, P) \in \mathcal{N}_{\delta,\phi}$ whilst maintaining boundedness of the controller function $\lambda(\cdot)$.

Define the continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by $\gamma(\xi) := \xi + \phi(\xi)$ and let Γ denote its indefinite integral: $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi \mapsto \int_0^\xi \gamma$. We claim that the following (formal) strategy is a $\mathcal{N}_{\delta,\phi}$-universal stabilizer:

$$\left\{ \begin{array}{l} u(t) = \nu(\eta(t))\gamma(|y(t)|)\mathrm{sgn}(y(t)), \quad \eta(t) := \delta\lambda(t) + \Gamma(|y(t)|) \\ \dot{\lambda}(t) = \gamma(|y(t)|)|y(t)|, \quad \lambda(0) = \lambda^0 \in \mathbb{R}. \end{array} \right. \tag{25}$$

where ν is any continuous function with properties (8).

Let $(b, d, f, p, P) \in \mathcal{N}_{\delta,\phi}$. Introducing the coordinate transformation $z(t) = \dot{y}(t) + \delta y(t)$, we may express (24) in the form

$$\left\{ \begin{array}{l} \dot{y}(t) = -\delta y(t) + z(t) \\ \dot{z}(t) = (\delta + d(t))(z(t) - \delta y(t)) + f(p(t), y(t)) + bu(t) \\ (y(0), z(0)) = (y^0, z^0). \end{array} \right. \tag{26}$$

The feedback (25) is interpreted in the set-valued sense:

$$\left\{ \begin{array}{l} u(t) \in \nu(\eta(t))\gamma(|y(t)|)\sigma(y(t)), \quad \eta(t) = \delta\lambda(t) + \Gamma(|y(t)|) \\ \dot{\lambda}(t) = \gamma(|y(t)|)|y(t)|, \quad \lambda(0) = \lambda^0 \end{array} \right. \tag{27}$$

with the map $y \mapsto \sigma(y)$ defined as before in (10).

Writing $x(t) = (y(t), z(t), \lambda(t))$, the overall adaptive feedback controlled system may be embedded in the following differential inclusion on $\mathbb{R}^3$

$$\dot{x}(t) \in X(t, x(t)), \quad x(0) = x^0 = (y^0, z^0, \lambda^0) \tag{28}$$

where $X : (t, x) \equiv (t, y, z, \lambda) \mapsto \{-\delta y + z\} \times X_2(t, x) \times \{\gamma(|y|)|y|\} \subset \mathbb{R}^3$, with

$$X_2(t, x) := \{(\delta + d(t))(z - \delta y) + f(p(t), y) + bu \mid u \in \nu(\delta\lambda + \Gamma(|y|))\gamma(|y|)\sigma(y)\}.$$

X is upper semicontinuous on $\mathbb{R}\times\mathbb{R}^3$ and takes non-empty, convex and compact values in $\mathbb{R}^3$. Therefore, for each $x^0 \in \mathbb{R}^3$, the initial-value problem (28) has a solution and every solution can be extended into a maximal solution.

LEMMA 3.10. *Let $x^0 \in \mathbb{R}^3$ be arbitrary and let $x(\cdot) = (y(\cdot), z(\cdot), \lambda(\cdot))$ be a maximal solution of (28) defined on its maximal interval of existence $[0,\omega)$. Then (i) $\omega = \infty$; (ii) $\lim_{t\to\infty} \lambda(t)$ exists and is finite; (iii) $(y(t), z(t)) \to (0,0)$ as $t \to \infty$.*

Proof. On $\mathbb{R}$, define the locally Lipschitz function $\Phi : r \mapsto \Gamma(|r|)$, with directional derivative at r in direction s given by

$$\Phi^\dagger(r; s) = \lim_{h\downarrow 0} \frac{\Phi(r + hs) - \Phi(r)}{h} = \begin{cases} \gamma(|r|)\mathrm{sgn}(r)s, & r \neq 0 \\ \gamma(0)|s|, & r = 0. \end{cases} \tag{29}$$

Let $F_1 \in AC([0,\omega); \mathbb{R})$ denote the composition $\Phi \circ y$ and let $\mathcal{O}_1 \subset [0,\omega)$ be the set (of measure zero) of points t at which the derivative $\dot{F}_1(t)$ fails to exist. A straightforward argument (analogous to that yielding (4)) gives

$$\dot{F}_1(t) = \Phi^\dagger(y(t); \dot{y}(t)) \quad \forall\ t \in [0,\omega)\backslash\mathcal{O}_1. \tag{30}$$

By properties of f and p (Assumptions F and G), the function

$$F_2 : t \mapsto \int_0^{y(t)} f(p(t), \xi)\, d\xi.$$

is of class $AC([0,\omega); \mathbb{R})$. Let $\mathcal{O}_2$ denote the set (of measure zero) of points t at which $\dot{p}(t)$ fails to exist. Again by properties of f and p, there exists $\mu > 0$ such that

$$\begin{aligned} \dot{F}_2(t) &= \int_0^{y(t)} \langle D_1 f(p(t), \xi), \dot{p}(t)\rangle\, d\xi + f(p(t), y(t))\dot{y}(t) \\ &\geq -\mu\phi(|y(t)|)|y(t)| + f(p(t), y(t))z(t) \quad \forall\ t \in [0,\omega)\backslash\mathcal{O}_2. \end{aligned} \tag{31}$$

Define $F_c \in AC([0,\omega); \mathbb{R})$, parameterized by $c > 0$, as follows $F_c : t \mapsto cF_1(t) - F_2(t)$. By Assumptions F, G and definition of Γ, F_c is such that, for all c sufficiently large,

$$F_c(t) \geq \tfrac{1}{2}cy^2(t) \quad \forall\ t. \tag{32}$$

Moreover, by (30) and (31),

$$\begin{aligned} \dot{F}_c(t) &= c\dot{F}_1(t) - \dot{F}_2(t) = c[\dot{\eta}(t) - \delta\gamma(|y(t)|)|y(t)|] - \dot{F}_2(t) \\ &\leq c\dot{\eta}(t) + (\mu - c\delta)\gamma(|y(t)|)|y(t)| - z(t)f(p(t), y(t)). \quad \forall\ t \in [0,\omega)\backslash(\mathcal{O}_1 \cup \mathcal{O}_2). \end{aligned} \tag{33}$$

Let $Y := \{t \in [0,\omega) \mid (y(t), \dot{y}(t)) \neq (0,0)\}$ be the set of points t at which $y(t)$ and $\dot{y}(t)$ are not both zero. Observe that (i) every point t of the subset $\mathcal{O}_0 := \{t \in Y \mid y(t) = 0\}$ is an isolated point implying that $\mathcal{O}_0$ is countable and so has measure zero, and (ii) $y(t) = \dot{y}(t) = z(t) = 0$ for all $t \in [0,\omega)\backslash Y$. From these observations together with (29), (30) and writing $\mathcal{O} := \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$ (of measure zero), we deduce that

$$\begin{aligned} \forall\ t \in [0,\omega)\backslash\mathcal{O}, \quad uz(t) &= \nu(\eta(t))[\delta\gamma(|y(t)|)|y(t)| + \Phi^\dagger(y(t); \dot{y}(t))] \\ &= \nu(\eta(t))\dot{\eta}(t) \quad \forall\ u \in \nu(\eta(t))\gamma(|y(t)|)\sigma(y(t)). \end{aligned} \tag{34}$$

Define $V_c \in AC([0,\omega); \mathbb{R})$ by $V_c(t) := F_c(t) + \frac{1}{2}z^2(t)$. Invoking (33), (34) and Assumption E,

$$\begin{aligned} \dot{V}_c(t) &\leq \dot{F}_c(t) - 2\epsilon z^2(t) - (d(t) + \delta)\delta y(t)z(t) \\ &\qquad + z(t)f(p(t), y(t)) + b\nu(\eta(t))\dot{\eta}(t) \\ &\leq [\Delta^2/(4\epsilon) + \mu - c\delta]\gamma(|y(t)|)|y(t)| - \epsilon z^2(t) + [c + b\nu(\eta(t))]\dot{\eta}(t) \end{aligned} \tag{35}$$

for almost all $t \in [0,\omega)$, wherein we have used the fact that

$$-(d(t)+\delta)\delta y(t)z(t) \le \Delta|y(t)z(t)| \le \epsilon z^2(t) + \frac{\Delta^2}{4\epsilon}y^2(t) \le \epsilon z^2(t) + \frac{\Delta^2}{4\epsilon}\gamma(|y(t)|)|y(t)|.$$

with $\Delta := (\|d(\cdot)\|_\infty + \delta)\delta$. Now fix c sufficiently large so that $\Delta^2(4\epsilon)^{-1} + \mu - c\delta \le 0$ and (32) holds, in which case, $V_c(t) \ge \frac{1}{2}\left[cy^2(t) + z^2(t)\right]$ for all t and

$$\dot{V}_c(t) \le [c + b\nu(\eta(t))]\dot{\eta}(t) \quad \text{for a.a. } t \in [0,\omega) \tag{36}$$

which, on integration, yields

$$0 \le \tfrac{1}{2}\left[cy^2(t) + z^2(t)\right] \le V_c(t) \le V_c(\tau) + c[\eta(t) - \eta(\tau)] + b\int_{\eta(\tau)}^{\eta(t)} \nu \tag{37}$$

for all $t, \tau \in [0,\omega)$, with $t \ge \tau$.

We first show that the function $\eta(\cdot)$ (and hence $\lambda(\cdot)$) is bounded. By properties (8) of ν, there exist increasing sequences $(\hat{\eta}_n)_{n\in\mathbb{N}}$ and $(\tilde{\eta}_n)_{n\in\mathbb{N}}$, with $\hat{\eta}_n \to \infty$ and $\tilde{\eta}_n \to \infty$ as $n \to \infty$, such that

$$\text{(a)}\quad \frac{1}{\hat{\eta}_n}\int_{\hat{\eta}_1}^{\hat{\eta}_n} \nu(\theta)d\theta \;\to +\infty \qquad \text{(b)}\quad \frac{1}{\tilde{\eta}_n}\int_{\tilde{\eta}_1}^{\tilde{\eta}_n} \nu(\theta)d\theta \;\to -\infty \tag{38}$$

as $n \to \infty$. Without loss of generality, we may assume $\hat{\eta}_1, \tilde{\eta}_1 \ge 1$. Seeking a contradiction, suppose that $\eta(\cdot)$ is unbounded. Now, $\eta(\cdot)$ is bounded from below (in particular, $\eta(t) \ge \delta\lambda^0$ for all $t \ge 0$) and so, by the supposition, $\eta(\cdot)$ is unbounded from above. Therefore, there exist increasing sequences $(\hat{t}_n), (\tilde{t}_n) \subset [0,\omega)$ such that, for all n,

$$\eta(\hat{t}_n) = \hat{\eta}_n \quad \text{and} \quad \eta(\tilde{t}_n) = \tilde{\eta}_n.$$

Now, either $b > 0$ or $b < 0$. If $b > 0$, then (37) and (38b) combine to yield the contradiction

$$0 \le \text{constant} + \frac{b}{\tilde{\eta}_n}\int_{\tilde{\eta}_1}^{\tilde{\eta}_n} \nu \;\to\; -\infty \quad \text{as } n \to \infty.$$

If $b < 0$, then (37) and (38a) combine to yield the contradiction

$$0 \le \text{constant} - \frac{|b|}{\hat{\eta}_n}\int_{\hat{\eta}_1}^{\hat{\eta}_n} \nu \;\to\; -\infty \quad \text{as } n \to \infty.$$

Therefore, $\eta(\cdot)$ (and hence $\lambda(\cdot)$) is bounded. Boundedness of $\eta(\cdot)$ and $\lambda(\cdot)$, together with (37), imply boundedness of $y(\cdot)$ and $z(\cdot)$. This establishes assertion (i) and assertion (ii) follows by monotonicity of $\lambda(\cdot)$. It remains to prove assertion (iii).

By boundedness of $d(\cdot)$, $p(\cdot)$ and $x(\cdot)$, there exists $\rho > 0$ such that $X_2(t, x(t)) \subset \rho\bar{\mathbb{B}}$ for all $t \in [0,\infty)$ and so $x(\cdot) = (y(\cdot), z(\cdot), \lambda(\cdot))$ is a precompact solution of the autonomous initial-value problem

$$\dot{x}(t) \in \{-\delta y(t) + z(t)\} \times \rho\bar{\mathbb{B}} \times \{\gamma(|y(t)|)|y(t)|\}, \quad x(0) = x^0. \tag{39}$$

Moreover, by boundedness of $\lambda(\cdot)$,

$$\int_0^\infty y^2(s)ds \le \int_0^\infty \dot{\lambda}(s)ds < \infty.$$

Therefore, by Theorem 2.10, $x(\cdot)$ approaches the largest weakly-invariant (relative to the autonomous differential inclusion (39)) set W in $\{(y,z,\lambda)|\ y=0\}$. Let $\bar{w}=(0,\bar{z},\bar{\lambda})\in W$. By definition of weak invariance, the initial-value problem

$$\dot{w}(t)=(\dot{w}_1(t),\dot{w}_2(t),\dot{w}_3(t))\in\{-\delta w_1(t)+w_2(t)\}\times\rho\bar{\mathbb{B}}\times\{\gamma(|w_1(t)|)|w_1(t)|\},\quad w(0)=\bar{w}$$

has at least one solution $w(\cdot)=(w_1(\cdot),w_2(\cdot),w_3(\cdot))$ with maximal interval of existence $\mathbb{R}_+$ and with trajectory in $W\subset\{(y,z,\lambda)|\ y=0\}$. Since $\dot{w}_1(t)=-\delta w_1(t)+w_2(t)$ and $w_1(\cdot)\equiv 0$ in W, it follows that $w_2(\cdot)\equiv 0$ and so $\bar{z}=w_2(0)=0$. Therefore, we conclude that the largest weakly-invariant set in W is contained in the set $\{(y,z,\lambda)|\ y=0=z\}$ and so the solution $x(\cdot)$ approaches the set $\{(0,0)\}\times\mathbb{R}$. In particular, $(y(t),z(t))\to(0,0)$ as $t\to\infty$. □

3.2.2. Adaptive servomechanism. We now turn attention to the servomechanism problem for planar systems (26): that is, the construction of controls that cause the system to track, asymptotically, any reference signal $r(\cdot)$ of some given class, in the sense that both the tracking error $e(t)=y(t)-r(t)$ and its derivative $\dot{e}(t)=\dot{y}(t)-\dot{r}(t)$ tend to zero as $t\to\infty$. For the class of reference signals we take the (Sobolev) space $\mathcal{R}=W^{3,\infty}(\mathbb{R})$ of functions $r\in\left(C^2\cap L^\infty\right)(\mathbb{R})$ with $\dot{r}\in\left(C^1\cap L^\infty\right)(\mathbb{R})$ and $\ddot{r}\in W^{1,\infty}(\mathbb{R})$, equipped with the norm

$$\|r\|_{3,\infty}=\|r\|_\infty+\|\dot{r}\|_\infty+\|\ddot{r}\|_\infty+\|\dddot{r}\|_\infty.$$

For the servomechanism problem, we restrict the underlying class of systems by imposing a stronger assumption on the function f: Assumption F* below should hold for some known, continuous, positive-definite, nondecreasing function ϕ having the additional property (14). Specifically, for real $\delta>0$ and continuous, positive-definite, nondecreasing function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ with property (14), we denote, by $\mathcal{N}^*_{\delta,\phi}$, the set of quintuples (b,d,f,p,P) satisfying Assumptions A, E and F* following.

Assumption F*. $(p,y)\mapsto f(p,y)$, $\mathbb{R}^P\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Both f and its gradient function $Df=(D_1f,D_2f)$ ($\equiv(\partial f/\partial p,\partial f/\partial y)$) are ϕ-bounded uniformly with respect to p in compact sets: precisely, for every compact $K\subset\mathbb{R}^P$, there exists scalar μ_K such that $|f(p,y)|+\|Df(p,y)\|\le\mu_K\phi(|y|)$ for all $(p,y)\in K\times\mathbb{R}$.

Example 3.11. The function $\phi:|y|\mapsto 1+|y|^3$ has property (14) and the systems described in Example 3.9 are admissible. ◊

Let $\delta>0$ and let ϕ be a continuous, positive-definite, nondecreasing function with property (14). We claim that, in order to assure convergence to zero of both the tracking error $e(t)=y(t)-r(t)$ and its derivative $\dot{e}(t)$ for all reference signals $r\in\mathcal{R}$ and all quintuples $(b,d,f,p,P)\in\mathcal{N}^*_{\delta,\phi}$, it suffices to replace every occurrence of $y(t)$ in (25) by $e(t)$. Proof of this claim follows.

Let $(b,d,f,p,P)\in\mathcal{N}^*_{\delta,\phi}$. Write $\tilde{P}=P+3$ and define the continuous function

$$\tilde{f}:\mathbb{R}^{\tilde{P}}\times\mathbb{R}\to\mathbb{R},\quad(\tilde{p},e)\equiv(p,r,v,w,e)\mapsto f(p,e+r)-\delta v-w.$$

By properties of f, $\tilde{f}$ is continuously differentiable with respect to its first argument ($\tilde{p}$), with $D_1\tilde{f}$ given by

$$D_1\tilde{f}(\tilde{p},e)\equiv(\partial\tilde{f}/\partial\tilde{p})(\tilde{p},e)=(D_1f(p,e+r),D_2f(p,e+r),-\delta,-1).$$

Let $\tilde{K}\subset\mathbb{R}^{\tilde{P}}$ be compact and so there exist compact $K\subset\mathbb{R}^P$ and $R>0$ such that $\tilde{K}\subset K\times[-R,R]^3$. By properties of f and ϕ, there exist constants μ_K and ρ_R such that, for all $(p,r,v,w) \equiv \tilde{p} \in \tilde{K} \subset K \times [-R,R]^3$,

$$\begin{aligned}|\tilde{f}(\tilde{p},e)| + \|D_1\tilde{f}(\tilde{p},e)\| &\le |f(p,e+r)| + \delta|v| + |w| + \|Df(p,e+r)\| + \delta + 1\\ &\le \mu_K\phi(|e+r|) + (1+\delta)(1+R)\\ &\le \mu_K\rho_R\phi(|e|) + (1+\delta)(1+R) \le \tilde{\mu}_{\tilde{K}}\phi(|e|)\end{aligned}$$

with $\tilde{\mu}_{\tilde{K}} := \mu_K\rho_R + ((1+\delta)(1+R))/\phi(0)$. Therefore, $\tilde{f}$ satisfies Assumption F.

Let $r \in \mathcal{R} \equiv W^{3,\infty}(\mathbb{R})$. Then $t \mapsto \tilde{p}(t) := (p(t), r(t), \dot{r}(t), \ddot{r}(t)) \in \mathbb{R}^{\tilde{P}}$ is of class $W^{1,\infty}(\mathbb{R};\mathbb{R}^{\tilde{P}})$ and so $(b,d,\tilde{f},\tilde{p},\tilde{P}) \in \mathcal{N}_{\delta,\phi}$. Expressed in terms of the tracking error $e(t) = y(t) - r(t)$ and adopting the coordinate transformation $z(t) = \dot{e}(t) + \delta e(t)$, the underlying dynamics have the form

$$\left\{\begin{aligned}&\dot{e}(t) = -\delta e(t) + z(t)\\ &\dot{z}(t) = (\delta + d(t))(z(t) - \delta e(t)) + \tilde{f}(\tilde{p}(t), e(t)) + bu(t)\\ &(e(0), z(0)) = (e^0, z^0)\end{aligned}\right. \tag{40}$$

We are now in precisely the same context, modulo notation, as in the case of an adaptive stabilizer and so, replacing all occurrences of $y(t)$ in (9) by $e(t)$, viz.

$$\left\{\begin{aligned}&u(t) \in \nu(\eta(t))\gamma(|e(t)|)\sigma(e(t)), \quad \eta(t) = \delta\lambda(t) + \Gamma(|e(t)|)\\ &\dot{\lambda}(t) = \gamma(|e(t)|)|e(t)|, \ \lambda(0) = \lambda^0,\end{aligned}\right. \tag{41}$$

then the same argument, as used to establish Lemma 3.10, applies *mutatis mutandis* to conclude that (41) is an $(\mathcal{R}, \mathcal{N}^*_{\delta,\phi})$-universal servomechanism: for each $r(\cdot) \in \mathcal{R}$ and $(b,d,f,p,P) \in \mathcal{N}^*_{\delta,\phi}$, every solution $(e(\cdot), z(\cdot), \lambda(\cdot))$ of the feedback controlled system has maximal interval of existence $\mathbb{R}_+$ with $(e(t), z(t)) \to (0,0)$ as $t \to \infty$ and, moreover, $\lim_{t\to\infty}\lambda(t)$ exists and is finite.

3.2.3. Dynamically perturbed planar systems. Let $\delta > 0$ and let the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, positive definite and nondecreasing. Let $\Sigma_1 = (b,d,f,p,P) \in \mathcal{N}_{\delta,\phi}$. Here, we consider the case where Σ_1 is subject to perturbations generated through its interconnection with a dynamical system Σ_2 (as depicted in Figure 1).

The system Σ_2 is assumed to correspond to a differential equation (driven by the variable $y(t)$ of system Σ_1) on $\mathbb{R}^N$ of the form (18) with input $y(t)$, and scalar output $w(t)$ perturbing Σ_1. As before, we identify the system Σ_2 with the triple (g,h,N). Writing $z(t) = \dot{y}(t) + \delta y(t)$, the overall system has representation (on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$)

$$\left\{\begin{aligned}&\dot{y}(t) = -\delta y(t) + z(t)\\ &\dot{z}(t) = (\delta + d(t))z(t) - (\delta + d(t))\delta y(t) + f(p(t), y(t)) + h(\zeta(t)) + bu(t)\\ &\dot{\zeta}(t) = g(y(t), \zeta(t)), \quad (y(0), z(0), \zeta(0)) = (y^0, z^0, \zeta^0)\end{aligned}\right. \tag{42}$$

We will define, via Assumption H below, a class $\mathcal{P}_\psi$ of admissible systems $\Sigma_2 = (g,h,N)$, such that the $\mathcal{N}_{\delta,\phi}$-universal stabilizer of Section 3.2.1 is readily modified to yield a $(\mathcal{N}_{\delta,\phi}, \mathcal{P}_\psi)$-universal stabilizer.

For continuous $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we denote, by $\mathcal{P}_\psi$ the set of system triples $\Sigma_2 = (g,h,N)$ satisfying the following:

ASSUMPTION H. (i) $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz; (ii) system (18) is input-to-state stable (ISS); (iii) $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous; (iv) there exist a function $\alpha_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and a scalar $\alpha_1 > 0$, such that, for each $(\zeta^0, y(\cdot)) \in \mathbb{R}^N \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, the (unique) solution $\zeta(\cdot)$ of (18) satisfies

$$\int_0^t h^2(\zeta(s))ds \le \alpha_0(\|\zeta^0\|) + \alpha_1 \int_0^t \psi(|y(s)|)|y(s)|ds.$$

EXAMPLES 3.12.

(a) Assumption H holds for the class of linear systems in Example 3.5(a).

(b) More generally, assume $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz and $h : \mathbb{R}^N \to \mathbb{R}$ is continuous. Assume further that h is positively homogeneous of degree $k \ge 1$ and that g is positively homogeneous of degree k_g, $1 \le k_g \le 2k$. Let $\psi : |y| \mapsto |y|^{2k-1}$. If $\{0\}$ is an asymptotically stable equilibrium of the unforced system $\dot\zeta(t) = g(0, \zeta(t))$, then it can be shown (by an argument invoking [22, Theorem 2]) that $(g, h, N) \in \mathcal{P}_\psi$. For example, if $\psi : |y| \mapsto |y|$, then systems (with $N = 1$ and with unknown real parameters a_i) of the form $\dot\zeta = a_1\zeta|\zeta| + a_2 y^2$, $w = a_3\zeta$, are of class $\mathcal{P}_\psi$, provided that $a_1 < 0$. $\diamond$

Let $\delta > 0$ and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, positive definite and nondecreasing. As before, let $\gamma : \xi \mapsto \xi + \phi(\xi)$ and $\Gamma : \xi \mapsto \int_0^\xi \gamma$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continous with indefinite integral $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi \mapsto \int_0^\xi \psi$.

We will show that, for $(\mathcal{N}_{\delta,\phi}, \mathcal{P}_\psi)$-universal stabilization, it suffices to replace both occurrences of γ in (25) by $\gamma + \psi$ and to replace the single occurrence of Γ by $\Gamma + \Psi$, to yield

$$(43)\quad \begin{cases} u(t) \in \nu(\eta(t))(\gamma + \psi)(|y(t)|)\sigma(y(t)), \quad \eta(t) = \delta\lambda(t) + (\Gamma + \Psi)(|y(t)|) \\ \dot\lambda(t) = (\gamma + \psi)(|y(t)|)|y(t)|, \ \lambda(0) = \lambda^0 \end{cases}$$

Let $(b, d, f, p, P) \in \mathcal{N}_{\delta,\phi}$ and $(g, h, N) \in \mathcal{P}_\psi$. The feedback-controlled system (42-43) can be embedded in a differential inclusion on $\mathbb{R}^{N+3}$:

$$(44)\quad \begin{cases} \dot x(t) \in X(t, x(t)) \\ x(t) = (y(t), z(t), \zeta(t), \lambda(t)) \in G := \mathbb{R}^{N+3}, \quad x(0) = x^0 = (y^0, z^0, \zeta^0, \lambda^0) \end{cases}$$

where the set-valued map $(t, x) \equiv (t, y, z, \zeta, \lambda) \mapsto X(t, x) \subset \mathbb{R}^{N+3}$ is given by

$$X(t, x) = \{-\delta y + z\} \times X_2(t, x) \times \{g(y, \zeta)\} \times \{(\gamma + \psi)(|y|)|y|\}$$

$$X_2(t, x) := \{(\delta + d(t))z - (\delta + d(t))\delta y + f(p(t), y) + h(\zeta) + bu| \\ u \in \nu(\delta\lambda + (\Gamma + \Psi)(|y|)\sigma(y)\}.$$

X is upper semicontinuous on $\mathbb{R} \times \mathbb{R}^{N+3}$ and takes non-empty, convex and compact values in $\mathbb{R}^{N+3}$. Therefore, for each $x^0 \in \mathbb{R}^{N+3}$, the initial-value problem (44) has a solution and every solution can be maximally extended.

LEMMA 3.13. *Let $x^0 \in \mathbb{R}^{N+3}$ be arbitrary and let $x(\cdot) = (y(\cdot), z(\cdot), \zeta(\cdot), \lambda(\cdot))$ be a maximal solution of (44) defined on its maximal interval of existence $[0, \omega)$. Then (i) $\omega = \infty$; (ii) $\lim_{t\to\infty} \lambda(t)$ exists and is finite; (iii) $(y(t), z(t), \zeta(t)) \to (0, 0, 0)$ as $t \to \infty$.*

Proof. Let F_c and V_c, parameterized by $c > 0$, be defined as in the proof of Lemma 3.10. By an argument essentially the same as that adopted in the proof of Lemma 3.10 and choosing c sufficiently large, we arrive at a counterpart to (35):

$$\dot{V}_c(t, y(t), z(t)) \leq -\epsilon z^2(t) + |h(\zeta(t))||z(t)| + [c + b\nu(\eta(t))]\dot{\eta}(t) \tag{45}$$

for almost all $t \in [0, \omega)$. Invoking the inequality $|h(\zeta)||z| \leq \frac{1}{2}\epsilon z^2 + \frac{1}{2}\epsilon^{-1}h^2(\zeta)$, then integrating and invoking Assumption H, we have (for c sufficiently large)

$$\begin{aligned}\tfrac{1}{2}[cy^2(t) + z^2(t)] \leq V_c(t, y(t), z(t)) &\leq V_c(\tau, y(\tau), z(\tau)) \\ &+ \tfrac{1}{2}\epsilon^{-1}\alpha_0(\|\zeta(\tau)\|) + \tfrac{1}{2}\epsilon^{-1}\alpha_1[\lambda(t) - \lambda(\tau)] \\ &+ c[\eta(t) - \eta(\tau)] + b\int_{\eta(\tau)}^{\eta(t)} \nu\end{aligned} \tag{46}$$

for all $t, \tau \in [0, \omega)$, with $t \geq \tau$. A straightforward modification of the contradiction argument previously used in the proof of Lemma 3.10, establishes boundedness of $\eta(\cdot)$ (and hence of $\lambda(\cdot)$). Boundedness of $\eta(\cdot)$ and $\lambda(\cdot)$, together with (46), imply boundedness of $y(\cdot)$ and $z(\cdot)$. That $\zeta(\cdot)$ is bounded is a consequence of the ISS property of $\Sigma_2 = (g, h, N)$. This establishes assertion (i) and assertion (ii) follows by monotonicity of $\lambda(\cdot)$. It remains to prove assertion (iii). With minor modification, the argument used in the proof of Lemma 3.10 applies to conclude that $x(\cdot)$ approaches the set $\{(y, z, \zeta, \lambda) |\ y = 0 = z\}$. In particular, $(y(t), z(t)) \to (0, 0)$ as $t \to \infty$ and so, by the convergent-input, convergent-state property of the ISS system $\Sigma_2 = (g, h, N)$, we may also conclude that $\zeta(t) \to 0$ as $t \to \infty$. □

EXAMPLE 3.14. *Linear Minimum-Phase Systems of Relative Degree Two.* Let (A, B, C) define a linear, single-input u, single-output y minimum-phase system on $\mathbb{R}^{N+2}$ of relative degree two. Denoting its Markov parameters by $m_k := CA^{k-1}B$, then $m_1 = 0$, $m_2 \neq 0$ and the system has a representation (on $\mathbb{R}^2 \times \mathbb{R}^N$) of the form

$$\ddot{y} = A_1 y + (m_3/m_2)\dot{y} + A_2\zeta + m_2 u, \quad \dot{\zeta} = A_3 y + A_4\zeta. \tag{47}$$

If we assume that $-m_3/m_2 > \delta > 0$ (that is, the system exhibits natural damping quantified by known δ), then we may identify (47) and (42) by setting

$$f : (p, y) \mapsto A_1 y,\ g : (y, \zeta) \mapsto A_3 y + A_4\zeta,\ h : \zeta \mapsto A_2\zeta, \quad b = m_2,\ d(\cdot) \equiv m_3/m_2.$$

By the relative-degree-two assumption, $b = m_2 \neq 0$ and, by the minimum-phase assumption, $\mathrm{spec}(A_4) \subset \mathbb{C}_-$. Defining $\phi \equiv \psi : |y| \mapsto \frac{1}{2}|y|$, we see that every relative-degree-two, minimum-phase system with $m_3/m_2 < -\delta$ is of class $(\mathcal{N}_{\delta,\phi}, \mathcal{P}_\psi)$ and we recover (modulo notation) the adaptive stabilizer proposed previously in [5]:

$$u(t) = \nu(\eta(t))y(t), \quad \eta(t) = \delta\lambda(t) + y^2(t), \quad \dot{\lambda}(t) = 2y^2(t),\ \lambda(0) = \lambda^0. \qquad \diamond$$

REMARKS 3.15. We conclude with some observations on the servomechanism problem for dynamically perturbed planar systems. Akin to Remarks 3.8, the adaptive servomechanism of Section 3.2.2 can also be modified to incorporate dynamically perturbed systems, when the dynamic perturbations are generated by linear systems $\dot{\zeta} = A\zeta + By$, $w = C\zeta$, $\mathrm{spec}(A) \subset \mathbb{C}_-$. For such perturbations, the (modified) servomechanism assures convergence to zero of the tracking error and its derivative, convergence to a finite limit of the adapting parameter, and boundedness of the evolution $t \mapsto \zeta(t)$ of the perturbing system. For brevity, we omit full details here.

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