Title: Optimal L_p estimates for the solutions of elliptic equations in bounded domains

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Abbreviated title: Optimal elliptic estimates.

Abstract: Linear second-order elliptic Dirichlet problems are considered in a bounded domain $\Omega \subset \mathbb{R}^N$. Isoperimetric inequalities are used to obtain sharp estimates of the solution operator from $L_p(\Omega)$ to $L_q(\Omega)$, p, $q \in [1, \infty]$ and $1/p < 2/N + 1/q$. The methodology used is an extension of techniques developed by Talenti. The main result removes an exponential weighting factor present in previous estimates due to Alvino, Trombetti and Lions.

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1 Introduction

For Ω, a bounded open subset of $\mathbb{R}^N$, we consider the boundary value problem

$$\mathcal{L}[u] \equiv -\sum_{i,j=1}^{N}(a_{ij}(x)u_{x_i})_{x_j} - \sum_{i=1}^{N}(b_i(x)u)_{x_i} + c(x)u = f(x), \qquad x \in \Omega, \quad (1.1)$$

for u zero on $\partial\Omega$. It is assumed that $a_{ij} \in L_\infty(\Omega)$ and that, for some $\nu > 0$,

$$\sum_{ij=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \qquad \xi \in \mathbb{R}^N, \ \ x \in \overline{\Omega}. \quad (1.2)$$

Additionally, it is assumed that b, $c \in L_\infty(\Omega)$ and that

$$\text{ess.sup.} \sum_{i=1}^{N} b_i^2(x) \leq M^2. \quad (1.3)$$

The norm $\|u\|_{L_q(\Omega)} = \|u^\sharp\|_{L_q(B_R)}$, where $u^\sharp$ is a spherically symmetric decreasing rearrangement of u over a ball $B_R \equiv B(0,\, R)$ with the same measure as Ω. Using isperimetric inequalities, $u^\sharp(r)$ may be compared with $v(r)$, the solution of a symmetrized problem over B_R. Since $v(r)$ is the solution of a one-dimensional ODE, it may calculated or numerically computed with relative ease to yield upper bounds for $\|u\|_{L_q(\Omega)}$.

The Schwarz symmetrization of a function $\phi \in L_1(\Omega)$ is defined as follows. Let

$$E(k) = \{x \in \Omega \,:\, |\phi(x)| > k\}\,, \qquad \mu(k) = |E(k)|, \qquad k \geq 0, \quad (1.4)$$

where $|E|$ means Meas(E). The decreasing rearrangement of u as a function of measure, is given by

$$\phi^\star(s) = \sup\{k \geq 0 \,:\, \mu(k) \geq s\}\,, \qquad s \in [0,\, |\Omega|]. \quad (1.5)$$

The Schwarz rearrangement of ϕ is now defined by

$$\phi^\sharp(x) = \phi^\star(C_N|x|^N), \qquad x \in B_R \equiv B_R, \quad (1.6)$$

where $C_N = 2\pi^{N/2}/\Gamma(N/2)$ is the spherical constant and $R = (|\Omega|/C_N)^{1/N}$. The corresponding increasing rearrangements are given by

$$\phi_\star(s) = \phi^\star(|\Omega| - s), \quad s \in [0,\, |\Omega|]; \qquad \phi_\sharp(x) = \phi_\star(C_N|x|^N), \quad x \in B_R.$$

We consider the following symmetrized comparison equation for (1.1):

$$\mathcal{L}^\sharp[v] \equiv -\nu\Delta v - \nabla.\left(\frac{Mxv}{|x|}\right) + \tilde{c}(x)v = f^\sharp(x), \tag{1.7}$$

for $x \in B_R$, subject to $v(x) = 0$, $|x| = R$. Here, for $x \in \Omega$,

$$\tilde{c}(x) = [(c^+)_\sharp - (c^-)^\sharp](x),\ c^+(x) = \max(c(x), 0),\ c^-(x) = \max(-c(x), 0). \tag{1.8}$$

Provided that the solution of (1.7) is unique, it must also be spherically symmetric, so that

$$L[\overline{v}] \equiv -\nu\frac{d^2\overline{v}}{dr^2} - \left(\frac{\nu(N-1)}{r} + M\right)\frac{d\overline{v}}{dr} - \left(\frac{M(N-1)}{r} - \overline{c}\right)\overline{v} = \overline{f}, \tag{1.9}$$

where $\overline{v}(r) = v(x)$, $\overline{c}(r) = \tilde{c}(x)$ and $\overline{f}(r) = f^\sharp(x)$ for $|x| = r$, $r \in [0, R]$.

The main result of this paper is to show that if (1.7) possesses a positive solution then

$$\int_0^s u^\star(\eta)\,d\eta \le \int_0^s v^\star(\eta)\,d\eta, \quad s \in [0, |\Omega|]. \tag{1.10}$$

If Φ is any non-negative, non-decreasing, weakly convex function, then, according to [13], (1.10) implies that

$$\begin{aligned}\int_{E(u^\star(s))} \Phi(|u(x)|)\,dx &= \int_0^s \Phi(u^\star(\eta))\,d\eta \\ &\le \int_0^s \Phi(v^\star(\eta))\,d\eta = \int_{B_r} \Phi(v^\sharp(x))\,dx, \end{aligned}\tag{1.11}$$

for $s \in [0, |\Omega|]$ and $r = (s/C_N)^{1/N}$. Consequently the $L_p(\Omega)$ and, more generally, Orlicz norms of u are bounded in terms of those of $v^\sharp$.

The history of isoperimetric inequalities is discussed in the book of Kawohl [16]. Work directly relevant to this paper was initiated by Bandle [6] and Talenti [17], who considered the Schwarz symmetrization of Poisson's equation. Extensions by Talenti [18] and Alvino and Trombetti [2, 3] for (1.1) in the case $c \equiv 0$ established that

$$u^\star(s) \le v^\star(s), \qquad s \in (0, |\Omega|]. \tag{1.12}$$

However, when $\mathcal{L} = -\Delta + \lambda$, (1.12) can fail when λ is a sufficiently large positive constant. Nevertheless, Alvino, Trombetti and Lions [4] proved that the weaker condition (1.10) holds in this case. It was also shown in [4] that even (1.10) fails in general when u satisfies the problem

$$\mathcal{L}^*[u] \equiv -\sum_{i,j=1}^N (a_{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^N b_i(x)u_{x_i} + c(x)u = f(x), \qquad x \in \Omega, \tag{1.13}$$

with zero Dirichlet boundary conditions, and v satisfies its symmetrization

$$(\mathcal{L}^\sharp)^*[v] \equiv -\nu\Delta v + M\frac{x}{|x|}.\nabla v + \tilde{c}(x)v = f^\sharp. \tag{1.14}$$

(Here $*$ denotes the formal adjoint.) Despite this, bounds in the case of (1.1), combined with a duality argument, yield sharp estimates of the norm of the mapping $(\mathcal{L}^*)^{-1} : L_p(\Omega) \to L_q(\Omega)$. However, for (1.13), as distinct from (1.1), a sharp bound on $\|u\|_q$ for each $f \in L_p(\Omega)$ is not available using this method.

2 Main Results

In order to guarantee a solution for (1.1) and (1.7) it is necessary to make further assumptions. A well-known sufficient condition is $c(x) \geq 0$. We work under the following weaker assumption:

(P) There is a strictly positive $H \in C^1[0,\, R]$ satisfying

$$\begin{aligned} L^*[H] &\equiv -\nu\frac{d^2H}{dr^2} - \left(\frac{\nu(N-1)}{r} - M\right)\frac{dH}{dr} + \overline{c}H = 0, \\ H(0) &= 1, \qquad H'(0) = 0. \end{aligned} \tag{2.1}$$

In order to obtain estimates in all $L_p(\Omega)$ spaces, $p \in [1,\, \infty]$, we consider the weakest one, $p = 1$. We define

$$\begin{aligned} D(\mathcal{A}) &= \{u \in W_0^{1,1}(\Omega) \,|\, \mathcal{L}[u] \in L_1(\Omega)\}, \\ \mathcal{A} &: D(\mathcal{A}) \longrightarrow L_1(\Omega), \qquad \mathcal{A}[u] = \mathcal{L}[u], \quad u \in D(\mathcal{A}). \end{aligned}$$

The main result of the paper is stated below.

Theorem 1 *Suppose that, for a bounded domain $\Omega \subset \mathbb{R}^N$, $\mathcal{L}$ is of the form (1.1), that its coefficients satisfy the associated conditions, including (1.2) and (1.3), and that assumption (P) is satisfied. Then, $\mathcal{A}$ has a bounded inverse, and for all $f \in L_1(\Omega)$,*

$$\int_{E(u^*(s))} |u(x)|\,dx = \int_{B_r} u^\sharp(x)\,dx \leq \int_{B_r} v(x)\,dx, \quad s \in [0,\, |\Omega|]. \tag{2.2}$$

Here, $r = (s/C_N)^{1/n}$, $u = \mathcal{A}^{-1}f$ and $v(x) = \overline{v}(|x|)$ for $x \in B_R$, where

$$\overline{v}(r) = \frac{e^{-Mr/\nu}H(r)}{\nu}\int_r^R \frac{e^{My/\nu}}{H^2(y)y^{N-1}}\int_0^y H(z)z^{N-1}\overline{f}(z)\,dz\,dy. \tag{2.3}$$

Furthermore, $v = v^\sharp$, yielding the measure-coordinate form of (2.2),

$$\int_0^s u^\star(\eta)\, d\eta \le \int_0^s v^\star(\eta)\, d\eta, \qquad s \in (0,\, |\Omega|]. \tag{2.4}$$

Lemma 2.1 *Suppose that the hypotheses of Theorem 1 hold and that for some $f \in L_\infty(\Omega)$ there exists a weak solution $u \in H_0^1(\Omega) \cap L_\infty(\Omega)$ for (1.1). Then,*

$$-\nu r^{N-1}\frac{d\overline{u}}{dr} - M r^{N-1}\overline{u} + \int_0^r y^{N-1}\overline{c}(y)\overline{u}(y)\, dy \le \int_0^r \rho^{N-1}\overline{f}(\rho)\, d\rho, \tag{2.5}$$

for $r \in (0,\, R)$, where $\overline{u}(r) = u^\sharp(x)$ for $|x| = r$.

Proof: By definition, $(\mathcal{L}[u] - f,\, \psi) = 0$, for all $\psi \in H_0^1(\Omega)$. We note that $f \in L_\infty(\Omega)$ implies $u \in L_\infty(\Omega)$, see [11, Theorem 8.15]. Choosing

$$\psi(h,\, k) = \begin{cases} \operatorname{sign}[u], & |u(x)| > k + h, \\ \operatorname{sign}[u](|u(x)| - k)/h, & |u(x)| \in (k,\, k+h], \\ 0, & |u(x)| \le k, \end{cases}$$

for $h > 0$ and $k \ge 0$, cf. [18, 4], we obtain

$$\begin{aligned} &\sum_{i,j=1}^N \frac{1}{h}\int_{E(k)\setminus E(k+h)} a_{ij}u_{x_i}u_{x_j}\, dx + \sum_{i=1}^N \frac{1}{h}\int_{E(k)\setminus E(k+h)} b_i u_{x_i} u\, dx \qquad (2.6) \\ =& \int_{E(k+h)} (f - cu)\operatorname{sign}[u]\, dx + \int_{E(k)\setminus E(k+h)} (f - cu)\operatorname{sign}[u]\frac{(|u| - k)}{h}\, dx, \end{aligned}$$

where $E(k)$ is defined by (1.4).

It is shown in [17, 18] that, given $u \in H_0^1(\Omega)$, the function $\mu(k) = \operatorname{Meas} E(k)$ is always strictly decreasing for $k \in [0,\, |\Omega|]$ and that, for a.e. $k \ge 0$, $\mu(k)$ is continuous and $\mu'(k)$ exists. We assume below that k is one of these generic values. Applying the Cauchy-Schwarz inequality, (1.2) and (1.3), and letting $h \to 0+$, we obtain

$$\begin{aligned} -\nu\frac{d}{dk}\int_{E(k)} |\nabla u|^2\, dx \quad &\le \quad -Mk\frac{d}{dk}\int_{E(k)} |\nabla u|\, dx \\ &\quad + \int_{E(k)} (f - cu)\operatorname{sign}[u]\, dx. \qquad (2.7) \end{aligned}$$

Here, we have used the fact that the second term on the right-hand side of (2.6) vanishes in the limit, and also the Cauchy-Schwarz estimate

$$\lim_{h\to 0}\int_{E(k)\setminus E(k+h)} |\nabla u|\, dx \le \|u\|_{H^1(\Omega)} \lim_{h\to 0}(\mu(k+h) - \mu(k))^{1/2} = 0.$$

To rewrite the second term on the right-hand side of (2.7), we use the following properties of Schwarz symmetrizations, see [18],

$$\begin{aligned}
\int_{E(k)} |u(x)|\,dx &= \int_0^{\mu(k)} u^\star(\eta)\,d\eta,\\
\int_\omega g(x)\,dx &\le \int_0^{|\omega|} g^\star(\eta)\,d\eta,\\
\int_0^{|\omega|} g^\star(\sigma)h_\star(\eta)\,d\eta &\le \int_\omega |g(x)h(x)|\,dx \le \int_0^{|\omega|} g^\star(\eta)h^\star(\eta)\,d\eta.
\end{aligned}$$

Here, ω is an open subset of Ω, g, $h \in L_1(\Omega)$ and the inequalities in the last line hold only when the integrals exist. We deduce that

$$\begin{aligned}
\int_{E(k)} f\cdot \mathrm{sign}[u]\,dx &\le \int_0^{\mu(k)} f^\star(\eta)\,d\eta,\\
\int_{E(k)} -cu\cdot \mathrm{sign}[u]\,dx &\le \int_0^{\mu(k)} ((c^+)_\star - (c^-)^\star)(\eta)u^\star(\eta)\,d\eta.
\end{aligned}$$

We now apply the inequalities

$$\begin{aligned}
NC_N^{1/N}\mu(k)^{1-1/N} &\le -\frac{d}{dk}\int_{E(k)} |\nabla u|\,dx\\
&\le \sqrt{-\mu'(k)}\left(-\frac{d}{dk}\int_{E(k)} |\nabla u|^2\,dx\right)^{1/2}, \qquad (2.8)
\end{aligned}$$

which are derived from the classical isoperimetric inequality [12], the Fleming-Rishel co-area formula [13] and the Cauchy-Schwarz inequality. We obtain

$$\nu NC_N^{1/N}\mu^{1-1/N} \le -\mu'\left\{Mk + \mu^{-(1-1/N)}\int_0^\mu f^\star - ((c^+)_\star - (c^-)^\star)u^\star\,d\eta\right\}, \quad (2.9)$$

where $\mu = \mu(k)$.

The function $u^\star(s)$ is the inverse function of $\mu(k)$ for $s \in [0, |\Omega|]$, and the formula $(u^\star)'(\mu(k)) = 1/\mu'(k)$ is shown to be true for a.e. $k \ge 0$ for $u \in H_0^1(\Omega)$ in [18]. There, it is also shown that $u^\star$ is absolutely continuous on compact subsets of $(0, |\Omega|]$, and thus that the inverse image of a null subset of the range is a null subset of $(0, |\Omega|]$. Within appropriate ranges, therefore, we may replace 'a.e. k' by 'a.e. s' on transforming coordinates. We conclude from (2.9) that

$$\begin{aligned}
- \quad & \nu N^2C_N^2 s^{2-2/N}\frac{du^\star}{ds} - NC_N M s^{1-1/N}u^\star(s)\\
&+ \int_0^s [(c^+)_\star - (c^-)^\star](\eta)u^\star(\eta)\,d\eta \le \int_0^s f^\star(\eta)\,d\eta, \qquad \text{a.e. } s\in(0,\,|\Omega|).
\end{aligned}$$

We obtain (2.5) on changing variables to $r = (s/C_N)^{1/N}$. $\square$

Lemma 2.2 *Suppose that the hypotheses of Theorem 1 hold and that $f \in L_\infty(\Omega)$. Then, $\overline{v}(r)$, defined by (2.3), satisfies (1.9) for $r \in (0,\, R)$, $v \in H_0^1(B_R)$ is a weak solution of* (1.7) *and, for all $r \in (0,\, R)$,*

$$-\nu r^{N-1}\frac{d\overline{v}}{dr} - M r^{N-1}\overline{v} + \int_0^r y^{N-1}\overline{c}(y)\overline{v}(y)\, dy = \int_0^r y^{N-1}\overline{f}(y)\, dy. \tag{2.10}$$

In addition, there exists a constant c_1 independent of f and v such that

$$\begin{aligned} \|v\|_{L_1(B_R)} &= C_N \int_0^R r^{N-1}\overline{v}(r)\, dr \\ &\leq c_1 C_N \int_0^R r^{N-1}\overline{f}(r)\, dr = c_1 \|f^\sharp\|_{L_1(B_R)} \end{aligned} \tag{2.11}$$

Proof: As $f \in L_\infty(\Omega)$ implies that $f^\sharp \in L_\infty(B_R)$ and $\overline{f} \in L_\infty(0,\, R)$, one may differentiate (2.3) directly to verify, using the properties of $H(r)$, that the right-hand side satisfies (1.9). Simple calculation also implies that $v \in H_0^1(\Omega)$ and is thus a weak solution of (1.7). Inequality (2.10) follows from integrating (1.7), coupled with a verification that

$$\lim_{r\to 0+} r^{N-1}\left(\nu\frac{d\overline{v}}{dr}(r) + M\overline{v}(r)\right) = 0.$$

Finally, (2.11) is also the result of a straightforward calculation. $\square$

As a result of Lemmas 2.1 and 2.2, we see that, if $\overline{w} = \overline{u} - \overline{v}$,

$$-\nu r^{N-1}\frac{d\overline{w}}{dr} - M r^{N-1}\overline{w} + \int_0^r y^{N-1}\overline{c}(y)\overline{w}(y)\, dy \leq 0, \qquad r \in (0,\, R). \tag{2.12}$$

As remarked in the Introduction, it is known, even in the case $M = 0$, that this does not, in general, lead to the conclusion that $\overline{w}(r) \leq 0$. Instead, we aim for

$$\int_0^r y^{N-1}\overline{w}(y)\, dy \leq 0, \qquad r \in (0,\, R), \tag{2.13}$$

which, if we already knew that $v = v^\sharp$, would be equivalent to (1.10).

Heuristically motivating the next two lemmas, we suppose that $G(r,\, \rho)$ satisfies the Green's function equation

$$L_r[G(r,\, \rho)] = r^{N-1}\delta(r - \rho), \qquad G(R,\, \rho) = 0,$$

and that $L[\overline{w}] = \overline{g}(r)$, where $\displaystyle\int_0^r \rho^{N-1}\overline{g}(\rho)\,d\rho \le 0$ for $r \in [0,\, R]$. Then $\overline{w}(r) = \displaystyle\int_0^R \rho^{N-1} G(r,\,\rho)\overline{g}(\rho)\,d\rho$, which, after some manipulation, yields

$$\int_0^r \rho^{N-1}\overline{w}(\rho)\,d\rho = -\int_0^R \frac{\partial J}{\partial \rho}(r,\,\rho)\int_0^\rho y^{N-1}\overline{g}(y)\,dy\,d\rho. \tag{2.14}$$

where $J(r,\,\rho) \equiv \displaystyle\int_0^r y^{N-1}G(y,\,\rho)\,dy$. From the adjoint Green's function equation, we see that J satisfies

$$L^*_\rho[J](r,\,\rho) = \chi_{[0,r]}(\rho), \quad \rho \in (0,\,R), \quad \frac{\partial J}{\partial \rho}(r,\,0) = J(r,\,R) = 0, \tag{2.15}$$

where L^* is as in (2.1) and χ denotes the usual characteristic function. Since (2.12) implies that the inner integral in (2.14) is non-positive, the desired conclusion (2.13) will follow provided that $\partial J/\partial\rho$ is always non-positive. These results are rigorously established below in slightly different terms.

Lemma 2.3 *Suppose that the hypotheses of Theorem 1 hold. Then, for all $r \in (0,\, R)$,*

$$\theta_r(\rho) \equiv \frac{H(\rho)}{\nu}\int_\rho^R \frac{e^{My/\nu}}{H^2(y)y^{N-1}}\int_0^y H(z)e^{-Mz/\nu}z^{N-1}\chi_{[0,r]}(z)\,dz\,dy \tag{2.16}$$

satisfies $\theta'_r(\rho) \le 0$ for all $\rho \in [0,\, R]$.

Proof: By substitution, noting (2.1), we see that θ_r satisfies $L^*[\theta_r] = \chi_{[0,r]}$. For $h \in (0,\, R)$, let

$$\phi_h(\rho) = (\theta_r(\rho+h) - \theta_r(\rho))/h, \qquad \rho \in [0,\, R-h].$$

Calculations show that $\theta'_r(R) < 0 = \theta'_r(0)$. Hence, the continuity of θ'_r implies that, if it is somewhere positive, there are $\rho_2 < \rho_3 \in [0,\, R)$ such that $\theta'_r(\rho_2) = \theta'_r(\rho_3) = 0$ and $\theta'_r(\rho) > 0$ for $\rho \in (\rho_2,\, \rho_3)$. Now, for $\rho \in (\rho_2,\, \rho_3 - h)$, $h < \rho_3 - \rho_2$, $L^*[\phi_h] = F_h$, where

$$\begin{aligned} F_h(\rho) \;&=\; -\frac{\nu(N-1)}{\rho(\rho+h)}\theta'_r(\rho+h) + \left(\frac{\overline{c}(\rho) - \overline{c}(\rho+h)}{h}\right)\theta_r(\rho+h) \\ &\quad + \frac{\chi_{[0,r]}(\rho+h) - \chi_{[0,r]}(\rho)}{h}. \end{aligned} \tag{2.17}$$

Since $\overline{c}$ is non-decreasing and $\chi_{[0,r]}$ is non-increasing, $F_h(\rho) < 0$ for $\rho \in (\rho_2,\, \rho_3 - h)$. Setting $\zeta_h(\rho) \equiv \phi_h(\rho)/H(\rho)$, we find that

$$\left(-\nu\zeta_h'' - \left(\frac{\nu(N-1)}{\rho} - M\right)\zeta_h'\right)H - 2\nu H'\zeta_h' = F_h(\rho).$$

Multiplying by $\rho^{N-1}H^2(\rho)e^{-M\rho/\nu}$ and integrating, we obtain

$$\left[-\nu y^{N-1}H^2 e^{-My/\nu}\zeta_h'\right]_{\rho_2}^{\rho} = \int_{\rho_2}^{\rho} y^{N-1}e^{-My/\nu}H(y)F_h(y)\,dy, \qquad \rho \in (\rho_2,\, \rho_3 - h).$$

Hence, for $\rho \in (\rho_2,\, \rho_3 - h)$,

$$\begin{aligned}
[-\nu\zeta_h]_{\rho}^{\rho_3-h} &= -\int_{\rho}^{\rho_3} (\rho_2/y)^{N-1} e^{M(y-\rho_2)/\nu}\frac{H^2(\rho_2)\zeta_h'(\rho_2)}{H^2(y)}\,dy \\
&\quad + \int_{\rho}^{\rho_3} \frac{e^{My/\nu}}{H^2(y)y^{N-1}}\int_{\rho_2}^{y} z^{N-1}e^{-Mz/\nu}H(z)F_h(z)\,dz\,dy. \qquad (2.18)
\end{aligned}$$

Now, $\lim_{h\to 0+}\zeta_h(\rho_3 - h) = \theta'(\rho_3)/H(\rho_3) = 0$ and, since $\theta'(\rho_2) = 0$,

$$\begin{aligned}
\liminf_{h\to 0+}\zeta_h'(\rho_2) &= \liminf_{h\to 0+}\left(\frac{\phi_h'}{H} - \frac{\phi_h H'}{H^2}\right)(\rho_2) \\
&= \liminf_{h\to 0+}\left(\frac{\theta_r'(\rho_2+h) - \theta_r'(\rho_2)}{hH(\rho_2)}\right) - \theta_r'(\rho_2)\frac{H'(\rho_2)}{H^2(\rho_2)} \geq 0.
\end{aligned}$$

We therefore deduce, on letting $h \to 0+$ in (2.18), that $\theta_r'(\rho) \leq 0$ for $\rho \in (\rho_2,\, \rho_3)$. This contradiction proves the lemma. $\square$

Lemma 2.4 *Let the hypotheses of Lemmas 2.1 and 2.2 hold. Then, (2.2) is true.*

Proof: Multiplying (2.12), expressed in the ρ variable, by $\theta_r'(\rho)$, integrating over $(0,\, R)$ and using integrating by parts, we obtain,

$$\begin{aligned}
0 &\geq \int_0^R w(\rho)\left[-\nu\frac{d}{d\rho}\left(\rho^{N-1}\frac{d\theta_r}{d\rho}\right) + \rho^{N-1}M\frac{d\theta_r}{d\rho} + \rho^{N-1}\overline{c}(\rho)\theta_r\right]d\rho \\
&= \int_0^R \rho^{N-1}w(\rho)L^*[\theta_r](\rho)\,d\rho \\
&= \int_0^R \rho^{N-1}w(\rho)\chi_{[0,r]}(\rho)\,d\rho \\
&= \int_0^r \rho^{N-1}w(\rho)\,d\rho,
\end{aligned}$$

where we have used $\theta_r'(0) = \theta_r(R) = 0$ and $\theta_r(\rho) \geq 0 \geq \theta_r'(\rho)$, $\rho \in (0,\, R)$. $\square$

Proof of Theorem 1: Suppose that $u \in D(\mathcal{A})$, $f \in L_1(\Omega)$ and $\mathcal{A}u = f$. From the definition of a weak solution in $W_0^{1,1}(\Omega)$ of the equation $\mathcal{L}[u] = f$, there exists a sequence $\{u_n\}$ in $C_0^\infty(\Omega)$ such that $f_n \equiv \mathcal{L}[u_n] \to f$ and $u_n \to u$ in $L_1(\Omega)$. If v_n is defined for $f_n^\sharp$ via (2.3), Lemma 2.2 implies that there is a constant c_1, independent of f_n and v_n such that

$$c_1 \|\mathcal{L}[u_n]\|_{L_1(\Omega)} = c_1 \|f_n\|_{L_1(\Omega)} = c_1 \|f_n^\sharp\|_{L_1(B_R)} \geq \|v_n\|_{L_1(B_R)}.$$

On the other hand, Lemma 2.4 implies that

$$\begin{aligned} \|v_n\|_{L_1(B_R)} = C_N \int_0^R r^{N-1} \overline{v}_n(r)\, dr \quad &\geq \quad C_N \int_0^R r^{N-1} \overline{u}_n(r)\, dr \\ &= \quad \|u_n^\sharp\|_{L_1(B_R)} = \|u_n\|_{L_1(\Omega)}. \end{aligned}$$

Hence, $c_1 \|\mathcal{L}[u_n]\|_{L_1(\Omega)} \geq \|u_n\|_{L_1(\Omega)}$, which on taking the limit in n yields

$$c_1 \|\mathcal{L}[u]\|_{L_1(\Omega)} \geq \|u\|_{L_1(\Omega)}. \tag{2.19}$$

This bound implies the injectivity of $\mathcal{A} :\ D(\mathcal{A}) \to L_1(\Omega)$.

Now,

$$D(\mathcal{A}) \supset D(\mathcal{A}_2) \equiv \{u \in H_0^1(\Omega) \,|\, \mathcal{L}[u] \in L_2(\Omega)\},$$

and so $\mathcal{A}_2 : D(\mathcal{A}_2) \to L_2(\Omega)$ must also be injective. The existence of a weak solution in $D(\mathcal{A}_2)$ for each $f \in L_2(\Omega)$ now follows from the Fredholm alternative, see [11, Section 8.2]. But $L_2(\Omega)$ is dense in $L_1(\Omega)$, and so, by the closedness of $\mathcal{A}$ implied by (2.19), the range of $\mathcal{A}$ is $L_1(\Omega)$. Inequality (2.19) now implies that $\mathcal{A}$ posseses a bounded inverse $\mathcal{A}^{-1}$ such that $\|\mathcal{A}^{-1}\|_{L_1(\Omega)} \leq c_1$.

It is known, see [8] and [16, p. 23], that for all $f,\, g \in L_1(\Omega)$,

$$\|f^\sharp - g^\sharp\|_{L_1(B_R)} \leq \|f - g\|_{L_1(\Omega)}. \tag{2.20}$$

Thus, $f_n \to f$ implies $f_n^\sharp \to f^\sharp$, and we deduce from Lemma 2.2 that $v_n \to v$ in $L_1(B_R)$, where v is defined via (2.3). Since (2.2) holds for u_n and v_n, by Lemma 2.4, we deduce that (2.2) holds also for u and v.

Finally, applying (2.2) for $\Omega = B_R$ and $\mathcal{L} = \mathcal{L}^\sharp$, we obtain

$$\int_{B_r} v_n^\sharp\, dx \leq \int_{B_r} v_n\, dx \leq \int_{B_r} v_n^\sharp\, dx, \qquad r \in (0,\, R),$$

where the second inequality is a standard result of Schwarz symmetrization. Since, v_n and $v_n^\sharp$ are both spherically symmetric, and (2.3) and the proof of Lemma 2.1 imply that both are continuous, it follows that they are equal. Taking the limit, again appealing to (2.20), we deduce that $v = v^\sharp$. Inequality (2.4) now follows from (2.2). $\square$

We define

$$\begin{aligned} D(\mathcal{A}^*) &= \{u \in W_0^{1,1}(\Omega) : \mathcal{L}^*[u] \in L_1(\Omega)\}, \\ \mathcal{A}^* &: D(\mathcal{A}^*) \to L_1(\Omega), \qquad \mathcal{A}^* u = \mathcal{L}^*[u], \quad u \in D(\mathcal{A}^*). \end{aligned}$$

and put $\mathcal{A}^\sharp$ and $(\mathcal{A}^\sharp)^*$ be equal to $\mathcal{A}$ and $\mathcal{A}^*$, when $\mathcal{L} = \mathcal{L}^\sharp$ and $\mathcal{L}^* = (\mathcal{L}^\sharp)^*$, respectively. We set

$$\begin{aligned} S &= \{u \in L_1(B_R) : u(x) = \overline{u}(|x|), \ \text{ some } \overline{u} : (0,\, R) \to \mathbb{R}\}, \\ D(A) &= \{u \in W_0^{1,1}(B_R) \cap S : \mathcal{L}^\sharp[u] \in S\} \\ A &: D(A) \to S, \qquad Au = \mathcal{L}^\sharp[u], \quad u \in D(A), \\ D(A^*) &= \{u \in W_0^{1,1}(B_R) \cap S : (\mathcal{L}^\sharp)^*[u] \in S\} \\ A^* &: D(A^*) \to S, \qquad A^* u = (\mathcal{L}^\sharp)^*[u], \quad u \in D(A^*). \end{aligned}$$

Corollary 2 *Let the hypotheses of Theorem 1 hold. Then, for* $p,\, q \in [1,\, \infty]$ *such that* $1/p - 1/q < 2/N$, $\mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1} : L_p(\Omega) \longrightarrow L_q(\Omega)$, *are bounded linear operators such that*

$$\|\mathcal{A}^{-1}\|_{L_p(\Omega)\to L_q(\Omega)} \leq \|A^{-1}\|_{L_p(B_R)\to L_q(B_R)}, \tag{2.21}$$

$$\|(\mathcal{A}^*)^{-1}\|_{L_p(\Omega)\to L_q(\Omega)} \leq \|(A^*)^{-1}\|_{L_p(B_R)\to L_q(B_R)}. \tag{2.22}$$

Proof: It is apparent from Lemma 2.2, representation (2.3) and some calculations that $A^{-1} : L_p(B_R) \longrightarrow L_q(B_R)$ is a bounded linear operator, provided $1/p - 1/q < 2/N$. Given $f \in L_p(\Omega)$, $\|f^\sharp\|_{L_p(B_R)} = \|f\|_{L_p(\Omega)}$, whilst (2.4) implies that

$$\int_0^s u^\star(\eta)\, d\eta \leq \int_0^s v^\star(\eta)\, d\eta,$$

where $u = \mathcal{A}^{-1} f$ and $v = (\mathcal{A}^\sharp)^{-1} f^\sharp$. As a consequence of the argument leading to (1.11), $\|u\|_{L_q(\Omega)} \leq \|v\|_{L_q(B_R)}$ for all $q \in [1,\, \infty]$ such that $1/p - 1/q < 2/N$. Hence, (2.21) follows.

Duality arguments show that if $1/p + 1/p' = 1$, $1/q + 1/q' = 1$,

$$\begin{aligned} \|\mathcal{A}^{-1}\|_{L_p(\Omega)\to L_q(\Omega)} &= \|(\mathcal{A}^*)^{-1}\|_{L_{q'}(\Omega)\to L_{p'}(\Omega)} \\ \|A^{-1}\|_{L_p(B_R)\to L_q(B_R)} &= \|A^{-1}\|_{L_{q'}(B_R)\to L_{p'}(B_R)} \end{aligned} \tag{2.23}$$

Consequently, (2.22) follows. $\square$

Remark: Theorem 1 gives finer information than (2.21), since a specific v is available for each u and f satisfying (1.1). In the case of $\mathcal{L}^*$, however, when the first-order term is in non-conservation form, an analogue of Theorem 1 is not known — see [4] for the best available result. The estimate (2.22) is therefore more important here.

Corollary 3 *Suppose that the assumptions of Theorem 1 hold. Then, the same conclusions are valid for the operator $\mathcal{L}$, with c replaced by $d \in L_\infty(\Omega)$, such that $d(x) > c(x) - \lambda_1$, where $\lambda_1 > 0$ is an eigenvalue of A corresponding to a non-negative eigenfunction ϕ_1.*

Proof: It is apparent from (2.3) that A^{-1} is a positive operator on the cone $K = \{u \in S : u \geq 0\}$. The weak Krein-Rutman theorem, see e.g. [9, p. 191], implies that there is a $\phi_1 \in D(A^*) \cap K$, such that $(A^*)^{-1}\phi_1 = \alpha_1\phi_1$ for some $\alpha_1 > 0$. Putting $\lambda_1 = 1/\alpha_1$, we obtain

$$A^*\phi_1 = \lambda_1\phi_1.$$

Claim: $\overline{d} > \overline{c} - \lambda_1$.
If x is such that $c(x) \geq \lambda_1 > 0$, then $d(x) > 0$ and

$$\begin{aligned} d^+(x) + \lambda_1 &= d(x) + \lambda_1 > c(x) = c^+(x) \\ d^-(x) &= 0 < \lambda_1 = \lambda_1 + c^-(x). \end{aligned}$$

If x is such that $c(x) < \lambda_1$,

$$\begin{aligned} d^+(x) + \lambda_1 = \max\{d(x) + \lambda_1,\, 0\} \geq \max\{c(x),\, 0\} = c^+(x) \\ d^-(x) = \max\{-d(x),\, 0\} < \lambda_1 - c(x) \leq \lambda_1 + c^-(x). \end{aligned}$$

Thus, $d^+(x) + \lambda_1 > c^+(x)$ and $d^-(x) \leq \lambda_1 + c^-(x)$. Using standard properties of Schwarz symmetrizations, see [16, Section II.2], we deduce that

$$\begin{aligned} (d^+)_\sharp(x) &= (d^+ + \lambda_1)_\sharp(x) > (c^+)_\sharp(x) \\ (d^-)^\sharp &< (c^- + \lambda_1)^\sharp = (c^-(x))^\sharp + \lambda_1, \end{aligned}$$

from which the claim follows.

The hypotheses of Theorem 1 hold for the modified operator $\mathcal{L}$, provided that the corresponding condition (P) is true; i.e., there exists a strictly positive

$H \in C^1[0, R]$ satisfying (2.1), with $\overline{d}$ replacing $\overline{c}$. The existence of a C^1 solution H is standard — it remains to check its positivity. As may be verified by differentiation, H satisfies

$$\frac{d}{dr}\left(\frac{H}{\phi_1}\right) = \int_0^r (y/r)^{N-1} e^{M(r-y)/\nu} \frac{\phi_1(y)}{\nu\phi_1^2(r)} \left(\overline{d} - (\overline{c} - \lambda_1)\right) H(y)\, dy, \tag{2.24}$$

for $r \in (0, R)$. Because $H(0) = 1$ and H is continuous, the right-hand side is initially non-negative. A simple argument by contradiction shows that H/ϕ_1 is non-decreasing, and therefore strictly positive, for $r \in [0, R]$. Multiplying (2.24) by $\nu\phi_1^2(r)$, and letting $r \to R-$, we obtain

$$-\nu\phi_1'(R)H(R) = \int_0^R (y/R)^{N-1} e^{M(R-y)/\nu} \phi_1(y) \left(\overline{(c-\lambda)} - (\overline{c} - \lambda_1)\right) H(y)\, dy,$$

using $\phi_1(R) = 0$. Since $\overline{(c-\lambda)} > \overline{c} - \lambda_1$, the right-hand side is positive. Also, ϕ_1 decreasing at $r = R$ implies $\phi_1'(R) \leq 0$. We conclude that $\phi_1'(R) < 0$ and $H(R) > 0$. Hence, $H(r) > 0$ for $r \in [0, R]$. $\square$

3 Calculations

Given a boundary value problem of the form (1.1), we seek answers to the questions,

(a) Is condition (P) satisfied?

(b) If (P) is satisfied, can one obtain numerical values for $\overline{v}(r)$, the comparison solution of Theorem 1?

(c) If (P) is satisfied, can one obtain numerical values for $\|A^{-1}\|_{L_p(B_R)\to L_q(B_R)}$ and $\|(A^*)^{-1}\|_{L_p(B_R)\to L_q(B_R)}$, the upper bounds of Corollary 2 for the solution operator of (1.1) and its adjoint problem?

There are two cases to consider:

(i) $c(x)$ non-constant. In subsection 3.1, we describe a rigorous numerical procedure for answering (a). This could also be adapted to find estimates for (b) and (c).

(ii) $c(x) = \lambda$, a constant. In subsection 3.2, we show that the answer to (a) turns on whether $\lambda > -\lambda_1$, where λ_1 is as in Corollary 3 when $c \equiv 0$. We find estimates for $\lambda_1(\nu,\, M,\, R)$ and an explicit formula for $H(r)$, the solution of (2.1), in terms of Kummer functions. We also obtain exact answers to question (c) for some special values of p and q.

3.1 Checking condition (P)

For second-order elliptic boundary value problems in general domains, the equivalence between the existence of a single positive u such that $\mathcal{L}[u] \geq 0$, a maximum principle and the positivity of the principal eigenvalue has been considered by Berestycki, Nirenberg and Varadhan [7]. Condition (P) is therefore equivalent to the existence of a strictly positive function $g \in C^1[0,\, R]$ such that

$$L^*[g] \geq 0, \qquad r \in (0,\, R), \qquad g'(0) = 0. \tag{3.1}$$

Below, we describe how one might use a numerical procedure to *rigorously* verify the existence of such a g.

Let Ω be covered by a Cartesian grid of N-dimensional cubes of width $h > 0$. Let Ω_h be the union of all cubes with non-empty intersection with Ω. Let c_h be the piecewise constant function equal to the essential minimum of c in each cube and let $\tilde{c}_h = (c_h^+)_\sharp - (c_h^-)^\sharp$ be the (simply constructed) rearrangement of c_h over B_{R_h}, where $|B_{R_h}| = |\Omega_h|$. Define

$$L_h^*[u] \equiv -\nu \frac{d^2 u}{dr^2} - \left(\frac{\nu(N-1)}{r} - M \right) \frac{du}{dr} + \overline{c}_h(r) u, \qquad r \in [0, R_h], \tag{3.2}$$

where $\overline{c}_h(r) = c_h(x)$ for $|x| = r$.

One might now try using a numerical time-marching initial value solver for

$$L_h^*[u] = 1, \qquad u(0) = 1,\ u'(0) = 0,$$

followed by polynomial interpolation. Alternatively, one might use a boundary value problem (BVP) solver, e.g. finite element method with piecewise quadratic elements, or cubic spline collocation, to try to solve

$$L_h^*[u] = 1, \qquad u'(0) = 0, \quad u(1) = 1.$$

(Note that one eventually needs a BVP solver to find numerical approximations to $\overline{v}$.) In all cases, one needs to have nodes at the points where $\overline{c}_h(r)$ is discontinuous. Except in the finite element case, where a weak formulation is used, one needs to replace the term $u'(r)/r$ by $u''(r)$ at $r = 0$. A suitable reference for the above schemes is [5].

Proposition 3.1 *Suppose that there is a strictly positive $g \in C^1[0,\, R]$ such that*

$$L_h^*[g] \geq 0, \qquad r \in (0,\, R), \qquad g'(0) = 0, \tag{3.3}$$

then (P) is satisfied.

Proof: Let $C_h(x)$ be the function $c(x)$, extended by $\operatorname{ess.max}_\Omega c$ in $\Omega_h \setminus \Omega$, and let $\tilde{C}_h(x)$ denote the rearrangement corresponding to (1.8) of $C_h(x)$ over B_{R_h}. Clearly, $\tilde{C}_h(x) = \tilde{c}(x)$, $x \in B_R$. On the other hand, since $C_h(x) \geq c_h(x)$, $x \in \Omega_h$, and Schwarz symmetrization is monotonic, see [16], $\tilde{C}_h(x) \geq \tilde{c}_h(x)$. We conclude that $\tilde{c}(x) \geq \tilde{c}_h(x)$, $x \in B_R$. Hence,

$$L^*[g] = L_h^*[g] + (\overline{c} - \overline{c}_h)g \geq 0, \qquad r \in [0,\, R].$$

Thus, g satisfies (3.1) and (P) is satisfied. $\square$

Since any of the numerical procedures suggested above yields a C^1 piecewise polynomial of low order, it is easy to check if it is positive and satisfies (3.3). Should (3.3) not be satisfied, one may try again with smaller h. Rigorous upper bounds for $\overline{v}$ might also be found by a similar procedure, using a BVP solver. This would involve an approximate rearrangement of f similar to that described above for c. Note also that the left-hand boundary condition changes to $\overline{v}'(0) = -M\overline{v}(0)$ in this case.

3.2 Calculations when $c(x)$ is constant

Here, we suppose that $c(x) = \lambda \in \mathbb{R}$ for all $x \in \Omega$. We make the scalings $\hat{r} = r/R$, $\hat{M} = MR/\nu$, $\hat{\lambda} = R^2\lambda/\nu$, $\hat{\overline{f}}(\hat{r}) = R^2\overline{f}(r)$ and $\hat{\overline{v}}(\hat{r}) = \overline{v}(r)$, where a hat '$\hat{\ }$' denotes a new variable. We may rewrite (1.9) in the new variables, (omitting hats), as

$$(L_0 + \lambda)[\overline{v}] \equiv -\frac{d^2\overline{v}}{dr^2} - \left(\frac{(N-1)}{r} + M\right)\frac{d\overline{v}}{dr} - \left(\frac{M(N-1)}{r} - \lambda\right)\overline{v} = \overline{f}, \tag{3.4}$$

for $r \in (0,\, 1)$. Let the operator A for L_0 be denoted A_0, and in future let λ_1 specifically denote $\lambda_1(A_0)$, where $\lambda_1(A)$ is described in Corollary 3.

Proposition 3.2 *Condition (P) is satisfied by $L^* = L_0^* + \lambda$ if and only if $\lambda > -\lambda_1$.*

Proof: When $L^* = L_0^*$, $H(r) \equiv 1$ and so (P) is satisfied. We conclude from Corollary 3 that $\lambda > -\lambda_1$ implies (P) for $L^* = L_0^* + \lambda$. For $\lambda = -\lambda_1$, $H(r) = \phi_1(r)$, the positive eigenfunction of A_0^*, implying $H(R) = 0$, so (P) is not satisfied.

If $\lambda < -\lambda_1$ and (P) were satisfied, then, by the proof of Corollary 3, $(A_0^* + \lambda I)^{-1}$ would be a positive operator. But $(A_0^* + \lambda I)^{-1}\phi_1 = (\lambda_1 + \lambda)^{-1}\phi_1 < 0$, so this could not be true. $\square$

From the classical theory of ODEs Erdélyi *et al.* [10, p. 250], we obtain the following representation for $H(r)$.

Proposition 3.3 *For $L^* = L_0 + \lambda$, $H(r)$ the solution of (2.1), (not normalized to satisfy $H(0) = 1$), is given by the following formulae. When $N = 1$,*

$$H(r) = \begin{cases} \sigma(\lambda)\left(\frac{e^{(M/2-\gamma)r}}{M/2-\gamma} - \frac{e^{(M/2+\gamma)r}}{M/2+\gamma}\right), & \lambda \neq -M^2/4,\ \lambda \neq 0, \\ e^{Mr/2}(1 - Mr/2), & \lambda = -M^2/4 \neq 0, \\ 1, & \lambda = 0, \end{cases} \tag{3.5}$$

where $\gamma = \sqrt{M^2/4 + \lambda}$ and $\sigma(\lambda) \in \{\pm 1, \pm i\}$ is chosen so that $H(1) > 0$. When $N \geq 2$,

$$H(r) = \begin{cases} e^{(M/2-\gamma)r}\Phi((N-1)\delta,\ (N-1),\ 2\gamma r), & \lambda \neq -M^2/4,\ \lambda \neq 0, \\ r^{-(N-2)/2}J_{N-2}(\sqrt{2(N-1)Mr}), & \lambda = -M^2/4,\ M \neq 0, \\ 1, & \lambda = 0, \end{cases} \tag{3.6}$$

where $\delta = 1/2 - M/(4\gamma)$.

Here, $J_n(x)$ is the Bessel function of the first kind and

$$\Phi(a,\ b,\ x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b)x^n}{\Gamma(b+n)\Gamma(a)n!} \tag{3.7}$$

is the Kummer confluent hypergeometric function of the first kind, see [1, 10]. We note that the Kummer transformation, $\Phi(a,\ b,\ x) = e^x\Phi(b-a,\ b,\ -x)$, implies that the right-hand side of (3.6) is an even function of γ for $N \geq 2$, and thus real even when γ is imaginary.

Proposition 3.2 begs the question as to the value of λ_1. In the next result, we consider $\lambda_1 = \lambda_1(M)$ and partially answer this question.

Proposition 3.4 *When $N = 1$, $\lambda_1(0) = \pi^2/4$, $\lambda_1(2) = 1$ and for large M,*

$$\lambda_1(M) = M^2e^{-M} + 2M^2(M-1)e^{-2M} + o(e^{-(3-\epsilon)M}), \qquad \text{all } \epsilon > 0. \tag{3.8}$$

When $N \geq 2$, $\lambda_1(0) = j_{\mu,1}^2$, where $\mu = (N-2)/2$ and $j_{\mu,1}$ is the first zero of J_μ, see [1, pp. 370, 467],

$$\lambda_1\left(\frac{j_{N-2,1}^2}{N-1}\right) = \frac{j_{N-2,1}^4}{16(N-1)^2}, \tag{3.9}$$

and for large M,

$$\lambda_1(M) \sim M^2 \left(\sum_{n=1}^{\infty} \frac{(N-1)!M^n}{(N+n-2)!n} \right)^{-1}. \tag{3.10}$$

Proof: Since the results for $N = 1$ were established in [14], we consider $N \geq 2$ here. When $\lambda = -\lambda_1$, $r = 1$ is the first zero of $H(r) = \phi_1(r)$. For $M = 0$, considering the identity [1, p. 504]

$$e^{-ir}\Phi((N-1)/2,\, (N-1),\, 2ir) = \Gamma(1+\mu)(r/2)^{-\mu} J_\mu(r),$$

we deduce from (3.6) that $\sqrt{\lambda_1} = j_{\mu,1}$. Similarly, we obtain (3.9) by considering the case $\lambda = -M^2/4$.

Proposition 3.2 implies $H(1) > 0$ for $\lambda > -\lambda_1$. Hence, by (3.6), $\lambda = -\lambda_1(M)$ is the least negative solution of

$$\Phi((N-1)\delta,\, (N-1),\, M(1-2\delta)^{-1}) = 0, \qquad \delta(\lambda) = \frac{1}{2} - \frac{M}{4\sqrt{M^2/4+\lambda}}. \tag{3.11}$$

By (3.7), the left-hand side of (3.11) is equal to

$$1 + \sum_{n=1}^{\infty} \frac{(N-1)!(n-1)!}{(N-2+n)!n!} \left(\frac{M}{1-2\delta} \right)^n \left\{ \prod_{k=1}^{n-1} (1 + (N-1)\delta/k) \right\}.$$

For $\delta \in (-1/(N-1),\, 0) \cap (-1/2,\, 0)$, we find that

$$f_-(\delta) < \Phi((N-1)\delta,\, (N-1),\, M(1-2\delta)^{-1}) < f_+(\delta),$$

where $f_-(\delta) = 1 + b\delta$, $f_+(\delta) = 1 + b\delta + a\delta^2$,

$$b = \sum_{n=1}^{\infty} \frac{(N-1)!M^n}{(N-2+n)!n}, \qquad a = \sum_{n=1}^{\infty} \frac{(N-1)!M^n}{(N-2+n)!n} \left\{ 2n + (N-1) \sum_{k=1}^{n-1} \frac{1}{k} \right\}.$$

Clearly, $\delta_+ < \delta(-\lambda_1) < \delta_-$, where $\delta_\pm$ are the respective least negative zeros of $f_\pm$. Some straightforward estimates yield

$$C_1(N) M^{-(N-1)} e^M < a,\ b < C_2(N) M^{-(N-1)} e^M,$$

and thus $b^2 \gg a$ for large M. We deduce that

$$\frac{-2}{b + \sqrt{b^2 - 4a}} = \delta_+ < \delta(-\lambda_1) < \delta_- = -\frac{1}{b},$$

and so $\delta(-\lambda_1) \sim -1/b$. Since $\lambda_1 = -M^2\delta(-\lambda_1) + O(\delta^2)$, we obtain (3.10). $\square$

Corollary 2 gives upper bounds for the norms of the solution operator for (1.1) and its adjoint in terms of those for a symmetrized problem. Here, still under the assumption that $c(x) = \lambda$ for all $x \in \Omega$, we calculate values for the latter in some special cases. Below, L_p denotes $L_p(B_1)$ and $q' = q/(q-1)$.

Example 1: $\|(A_0+\lambda I)^{-1}\|_{L_1\to L_q} = \|(A_0+\lambda I)^{-1}\|_{L_{q'}\to L_\infty}$, $\lambda > -\lambda_1$, $1/q > 1-2/N$. Consider $\overline{v}$, the solution of

$$r^{N-1}(L_0+\lambda)[\overline{v}] = \delta(r)/C_N, \qquad \overline{v}(1) = 0.$$

Theorem 1 and the arguments leading to (1.11) imply that if $\overline{u} = (A_0+\lambda I)^{-1}\overline{f}$, where $\|f\|_{L_1} = 1$, then $\|u\|_{L_q} \le \|v\|_{L_q}$ for all q such that $1/q > 1-2/N$. Thus, $\|(A_0+\lambda I)^{-1}\|_{L_q} = \|v\|_{L_q}$. As may be verified by substitution,

$$\overline{v}(r) = e^{-Mr}\left(G(r) - \frac{G(1)}{H(1)}H(r)\right), \tag{3.12}$$

where $H(r)$ is as in (3.5, 3.6) and

$$G(r) = \begin{cases} \dfrac{e^{-(\gamma-M/2)r}}{2\gamma - M}, & N = 1, \\ c_M e^{-(\gamma-M/2)r}\Psi((N-1)\delta,\, (N-1),\, 2\gamma r), & N \ge 2. \end{cases} \tag{3.13}$$

Here, $c_M = \dfrac{(2\gamma)^{N-2}\Gamma((N-1)\delta)}{C_N(N-2)!}$, and $\Psi(a,\, b,\, x)$, see [10], is the Kummer confluent hypergeometric function of the second kind, singular at $x = 0$. (In [1] $\Psi(a,\, b,\, x)$ is denoted $U(a,\, b,\, x)$.)

Example 2: $\|(A_0+\lambda I)^{-1}\|_{L_\infty\to L_q} = \|(A_0+\lambda I)^{-1}\|_{L_{q'}\to L_1}$, $\lambda > -\lambda_1$, $q \ge 1$.
It is evident from (2.3) that if $\|f\|_{L_\infty} = 1$, $\overline{v}(r)$ will be largest when $\overline{f}(r) \equiv 1$. Hence, $\|(A_0+\lambda I)^{-1}\|_{L_\infty\to L_q} = \|v\|_{L_q}$, where $\overline{v}$ satisfies

$$(L_0+\lambda)[\overline{v}] = 1, \qquad \overline{v}(1) = 0.$$

The solution $\overline{v}$ maybe represented in terms of the double integral (2.3). Alternatively, one may construct the Green's function from the linearly independent pair of solutions of $(L_0+\lambda)[\overline{u}] = 0$, $e^{-Mr}H(r)$ and the right-hand side of (3.12), which respectively satisfy $\overline{u}'(0) = 0$ and $\overline{u}(1) = 0$. This yields a single integral representation of $\overline{v}$.

Example 3: $\|(A_0^*+\lambda I)^{-1}\|_{L_\infty\to L_q} = \|(A_0+\lambda I)^{-1}\|_{L_{q'}\to L_1}$, $\lambda > -\lambda_1$, $q \ge 1$.
Corollary 3 implies that $L_0^*+\lambda$ satisfies a maximum principle for $\lambda > -\lambda_1$. Hence,

$\|(A_0^* + \lambda I)^{-1}\|_{L_\infty \to L_q} = \|v\|_{L_q}$, where

$$(L_0^* + \lambda)[\overline{v}] = 1, \qquad \overline{v}(1) = 0.$$

This problem has solution

$$\overline{v}(r) = \begin{cases} \dfrac{H(1) - H(r)}{\lambda H(1)}, & \lambda \neq 0, \\ \displaystyle\int_r^1 \int_0^s (t/s)^{N-1} e^{M(s-t)}\, dt\, ds, & \lambda = 0. \end{cases} \tag{3.14}$$

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