QUATERNIONIC KÄHLER MANIFOLDS OF COHOMOGENEITY ONE

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ABSTRACT. Classification results are given for (i) compact quaternionic Kähler manifolds with a cohomogeneity-one action of a semi-simple group, (ii) complete hyperKähler manifolds with a cohomogeneity-two action of a semi-simple group preserving each complex structure, (iii) compact 3-Sasakian manifolds which are cohomogeneity one with respect to a group of 3-Sasakian symmetries. Information is also obtained about non-compact quaternionic Kähler manifolds of cohomogeneity one and the cohomogeneity of adjoint orbits in complex semi-simple Lie algebras.

CONTENTS

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1. Introduction

A Riemannian manifold (M, g) is said to be of cohomogeneity one with respect to a group G if G acts isometrically with generic orbit of real codimension one. There has been considerable interest in studying Einstein manifolds of this type, because the Einstein condition then reduces to a set of ordinary differential equations.

Our aim is to investigate quaternionic Kähler manifolds which are of cohomogeneity one. Recall that a manifold of dimension $4n$ is quaternionic Kähler if the holonomy reduces to a subgroup of $Sp(n)\,Sp(1)$ (see [6], for example). Such manifolds are always Einstein. Each quaternionic Kähler manifold M has associated to it a twistor space Z: a complex contact manifold of complex dimension $2n+1$ mapping onto M with fibers rational curves of normal bundle $2n\mathcal{O}(1)$.

Quaternionic isometries of M give rise to holomorphic symmetries of Z preserving the complex contact structure. If M has cohomogeneity one, then, in many cases, moment map techniques show that Z is related to a coadjoint orbit in a complex Lie algebra. We are thus lead to discuss the action of a real group G on $G^{\mathbb{C}}$-orbits in the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. When G is compact, we prove a series of monotonicity results that say that the cohomogeneity of such orbits varies in a natural way with respect to various partial orders on the orbits. This enables us to narrow down the possibilities for M and Z, and in the compact case, we are able to determine all examples (M, g, G) when G is semi-simple.

In the compact case, partial results were obtained by Alekseevsky & Podestà [3]. Most of our techniques are different, though we do use one of their arguments. In our paper, the geometry of nilpotent orbits plays a notable rôle and results of Brylinski & Kostant [14] on shared orbits and of Beauville [5] on Fano contact manifolds are applied.

It turns out that the compact quaternionic Kähler manifolds that we obtain are all symmetric spaces. Kollross [23] has recently classified cohomogeneity-one actions of reductive groups on compact symmetric spaces and our results are consistent with his. However, one difference to note is that he considers two such actions to be equivalent if they have the same orbits. Thus in some cases he gives a space with an action of a reductive group, where we obtain the same space with a semi-simple symmetry group. One of the main open questions in quaternionic Kähler geometry is whether there exist any compact non-symmetric examples with positive scalar curvature. Our work shows that there are no such examples of cohomogeneity one with respect to a semi-simple group.

Our techniques, using moment maps and cohomogeneities of adjoint orbits, also enable us to tackle classification problems for two other types of geometric structure: hyperKähler and 3-Sasakian. In [7], Bielawski showed how a certain class of complete hyperKähler manifolds always arises as coadjoint orbits. We analyse this situation in more detail to classify those of cohomogeneity two.

There has been much recent interest in 3-Sasakian manifolds because they provide new examples of compact Einstein manifolds with positive scalar curvature. As shown in [11], there is a close relationship between 3-Sasakian manifolds, hyperKähler structures and quaternionic Kähler orbifolds. Using knowledge of cohomogeneities of adjoint orbits and extending the results of Beauville [5], we are able to classify 3-Sasakian manifolds whose group of 3-Sasakian symmetries acts with cohomogeneity one.

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2. Hypotheses

Let M be a quaternionic Kähler manifold of dimension $4n$, with twistor space $\pi\colon Z \to M$, see [6]. Suppose that G is a compact semi-simple Lie group that acts quaternionically on M with cohomogeneity one. Without loss of generality we may also assume that G is connected. Write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ for the complexification of $\mathfrak{g}$. Let M_0 be the union of principal orbits in M. General information about G-manifolds may be found in [12].

3. The Action on the Twistor Space

The G-action lifts to a group of holomorphic contact transformations on Z. We then have a map of sheaves

$$\alpha\colon \mathcal{O} \otimes \mathfrak{g}^{\mathbb{C}} \longrightarrow TZ$$

given by mapping elements of $\mathfrak{g}^{\mathbb{C}}$ to the corresponding vector fields on Z (cf. [21]).

Lemma 3.1. *At each point z of Z, the image of α has real dimension either $4n$ or $4n+2$.*

Proof. As the action of G is of cohomogeneity one on M, the lifted action has cohomogeneity at most three. Thus the real dimension of the image of α is at least $4n-1$. However, the image of α is complex, so the real dimension is even, and thus either $4n$ or $4n+2$. $\square$

Lemma 3.2. *Let U_0 be the set of points $z \in Z$ where the dimension of the image of α is $4n+2$. Then U_0 is open and either empty or dense.*

Proof. Taking exterior powers, we get a bundle map

$$\Lambda^{2n+1}\alpha\colon \mathcal{O} \otimes \Lambda^{2n+1}\mathfrak{g}^{\mathbb{C}} \to K_Z^{-1},$$

where $K_Z = \Lambda^{2n+1}T^*Z$ is the canonical bundle. The set U_0 is just the complement of the zero set of the analytic map $\Lambda^{2n+1}\alpha$. Thus either U_0 is empty or U_0 is open and dense. $\square$

We thus have two possibilities: either U_0 is empty or it is not. Consider the complexified group $G^{\mathbb{C}}$. This need not act on Z for reasons of completeness, but it does act in the sense of groupoids. The case $U_0 \neq \varnothing$ corresponds exactly to the existence of an open orbit for this action of $G^{\mathbb{C}}$ on Z. The treatment of this case will start in §5.

4. Twistor Spaces with No Open Orbits

Let us consider the case when the twistor space has no open $G^{\mathbb{C}}$-orbits. This is, in fact, the simplest case. Let $\widehat{M}$ be the universal cover of the union of principal orbits in M. We first show that $\widehat{M}$ admits a hypercomplex structure. This structure can taken to be G-invariant and the aim is then to show that $\widehat{M}$ fibres over a homogeneous quaternionic manifold. The assumption that G is compact implies that the quotient is a Wolf space and the structure on M can then be analysed metrically. We are then able to show that the only compact example is given by the action of $Sp(n)$ on $\mathbb{HP}(n)$.

Proposition 4.1. *If U_0 is empty, then M is locally hypercomplex and each point of $\widehat{M}$ admits a $\widetilde{G}$-invariant neighbourhood with a $\widetilde{G}$-invariant hypercomplex structure, where $\widetilde{G}$ is the universal cover of G.*

Proof. The complex rank of α is $2n$. The definition of α implies that its image is integrable in the sense of Frobenius, so we have a foliation of Z_0 by leaves of complex dimension $2n$ tangent to the image of α. If we compose α with projection to the horizontal distribution on Z_0, then we get a map $\alpha^{\mathcal{H}}$, whose real rank is at least $4n-1$ as the action on M has cohomogeneity one. However, the image of $\alpha^{\mathcal{H}}$ is complex, so $\alpha^{\mathcal{H}}$ is a surjection, and the leaves of the foliation are transverse to twistor lines. This means that locally we have a holomorphic projection $Z_0 \to \mathbb{C}\mathrm{P}(1)$ and M is locally hypercomplex.

The map α is G-equivariant, so the foliation determined by the image of α is invariant under the infinitesimal action of G. However, we have assumed that G is connected, so G preserves the foliation and the local hypercomplex structure.

Now consider $\widehat{M}$. Topologically, $\widehat{M}$ is a product $\mathbf{I} \times \widetilde{G}/H$ for some interval $\mathbf{I}$. As $\widetilde{G}$ is connected and $\widehat{M}$ is simply-connected, the exact homotopy sequence for a fibration implies that H is also connected. Fix t_0 in $\mathbf{I}$. There is a neighbourhood V of $x = t_0 \times H$ in $\widehat{M}$ on which $\widehat{M}$ has a hypercomplex structure I, J, K. Consider the $\widetilde{G}$-orbit of I. This meets the twistor line $\widehat{Z}_x$ in a set which has empty interior and is an H-orbit. As H is connected, this orbit is a single point. Thus we have a $\widetilde{G}$-invariant hypercomplex structure on $\mathbf{J} \times \widetilde{G}/H$, where $\mathbf{J} \subset \mathbf{I}$ is an open interval containing t_0. □

We say a quaternionic manifold has a compatible hypercomplex structure, if there exist global integrable sections I, J and K of $\mathcal{G}$ satisfying the quaternion identities.

Proposition 4.2. *Suppose M is a quaternionic Kähler manifold of positive scalar curvature. If M has a compatible hypercomplex structure I, J, K, then M admits an infinitesimal action of $Sp(1) \times \mathbb{R}$ with the following properties:*

(i) *$Sp(1)$ acts isometrically;*
(ii) *If V is the vector field generating the action of $\mathbb{R}$, then the action of $Sp(1)$ is generated by IV, JV and KV;*
(iii) *V preserves I, J and K;*
(iv) *$L_{IV}I = 0$ and $L_{IV}J = K$, and these formulæ remain valid for any cyclic permutation of I, J, K.*

Proof. The construction follows [36, §5]. Let ∇ be the Levi-Civita connection and let ∇' be the Obata connection. The twistor operator D is the composition

$$H \xrightarrow{\nabla''} H \otimes EH \cong ES^2H \oplus E \xrightarrow{p} ES^2H,$$

where ∇'' is any quaternionic connection and p is projection. The operator D is independent of ∇''. The hypercomplex structure defines a section h of H such that $\nabla' h = 0$. In particular, $Dh = 0$. Thus ∇h lies in the module E. Let $e = \nabla h$. By [36, Lemma 5.7], $\nabla e = \lambda h$, for some constant λ (a positive constant times the scalar curvature of M).

Let $V = e\tilde{h} - \tilde{e}h$, where $\tilde{e} = je$ and $\tilde{h} = jh$. Then $IV = eh + \tilde{e}\tilde{h}$, $JV = i(e\tilde{h}+\tilde{e}h)$ and $KV = i(eh-\tilde{e}\tilde{h})$. We claim that V is quaternionic and IV, JV and KV are Killing. Note that

$$\begin{aligned} T^*M \otimes T^*M &\cong S^2T^*M + \Lambda^2T^*M \\ &\cong (\mathbb{R} + \Lambda^2_0E + S^2ES^2H) \\ &\qquad + (S^2E + S^2H + \Lambda^2_0ES^2H). \end{aligned}$$

Killing vector fields are characterised by having covariant derivative in Λ^2T^*M and quaternionic vector fields are those with covariant derivative in

$$\mathfrak{gl}(n,\mathbb{H}) + \mathfrak{sp}(1) \cong \mathbb{R} + \Lambda^2_0E + S^2E + S^2H.$$

Now

$$\nabla V = \lambda h \wedge \tilde{h} + e \wedge \tilde{e} \tag{4.1}$$

which lies in $\mathbb{R} + \Lambda^2_0E \subset \mathfrak{gl}(n,\mathbb{H})$ so V is not only quaternionic but it also preserves the hypercomplex structure. Similarly computation of the covariant derivatives of IV, JV and KV shows that these three vector fields are Killing,

$$[IV, JV] = 2(\|e\|^2 + \lambda\|h\|^2)KV$$

and

$$L_{IV}J = 2(\|e\|^2 + \lambda\|h\|^2)K.$$

Here $\|e\|^2 = e \wedge \tilde{e}$ and $\|h\|^2 = h \wedge \tilde{h}$, which are identified with functions via the symplectic forms on E and H. Now $\nabla(h\wedge\tilde{h}) = 0$ and $\nabla(e\wedge\tilde{e}) = 0$, so $\|e\|^2 + \lambda\|h\|^2$ is a constant. Thus dividing V by the constant $2(\|e\|^2 + \lambda\|h\|^2)$ yields a vector field with the required properties. □

Remark 4.3. If the scalar curvature of M is negative then the above proof goes through unchanged provided $\lambda\|h\|^2 \neq -\|e\|^2$. If $\lambda\|h\|^2 = -\|e\|^2$ then we still get V but now IV, JV and KV commute. We suspect this case does not arise.

If M is of cohomogeneity one with an invariant hypercomplex structure, then we may choose the section h to be G-invariant. This leads to:

Proposition 4.4. *If U_0 is empty and M has positive scalar curvature, then $\widehat{M}$ is the associated bundle of a compact Wolf space.*

Proof. From the previous proof we have that ∇V lies in $S^2T^*\widehat{M}$. As $\widehat{M}$ is simply connected, this implies that the vector field V is a gradient, say $V = \operatorname{grad}\rho$. Now h is $\widetilde{G}$-invariant, so its covariant derivative e is also and hence the $\widetilde{G}$-action commutes with the (infinitesimal) action of $Sp(1) \times \mathbb{R}$ on $\widehat{M}$.

We claim that the function ρ is $Sp(1)$-invariant. As IV commutes with V, we have $d((IV)\rho) = L_{IV}V^\flat = 0$ and hence $(IV)\rho$ is constant. Similarly, $(JV)\rho$ and $(KV)\rho$ are constant. This implies $[JV, KV]\rho = (JV)((KV)\rho) - (KV)((JV)\rho) = 0$. However, $[JV, KV]$ is a non-zero constant times IV, hence ρ is $Sp(1)$-invariant. Similarly, ρ is $\widetilde{G}$-invariant, as $\widetilde{G}$ is semi-simple and so each element of its Lie algebra is a sum of commutators.

As the action of $\widetilde{G}$ on $\widehat{M}$ is cohomogeneity one, each component of a generic level set of ρ will be an orbit of $\widetilde{G}$ and hence compact. The infinitesimal action of $Sp(1)$ therefore integrates to a genuine group action, and the quotient of a component of a generic level set by $Sp(1)$ will be $\widetilde{G}$-homogeneous and hence smooth.

The proof of [36, Theorem 5.1] may now be modified to show that such a quotient is a quaternionic Kähler manifold N. (The two changes required are to replace μ by ρ and to note that the covariant derivative of ω_I is now a linear combination of ω_J and ω_K.)

By [1, 2], N is a Wolf space. By [32] the topology of $\widehat{M}$ is that of $\mathcal{U}(N)$. □

Theorem 4.5. *If U_0 is empty (i.e., Z has no open $G^{\mathbb{C}}$-orbit) and M has positive scalar curvature, then the metric on $\widehat{M}$ is a member of the one-parameter family of quaternionic Kähler metrics given in* [36, Theorem 3.5] *on the associated bundle of a compact Wolf space.*

Proof. The principal orbit of $\mathcal{U}(N)$ is G/H, where $N = G/H\,Sp(1)$ is the Wolf space. The Lie algebra $\mathfrak{g}$ of G splits under the action of $\mathrm{Ad}(H)$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1,$$

where $\mathfrak{p}_0$ is a trivial module and $\mathfrak{p}_1$ is a sum of non-trivial $\mathrm{Ad}(H)$-modules. As N is a Wolf space, $\mathfrak{p}_0$ is three-dimensional and isomorphic to $\mathfrak{sp}(1)$, and $\mathfrak{p}_1$ is isomorphic to the tangent space of N.

We may identify $\widehat{M} = \mathcal{U}(N)$ with $(t_0, t_1) \times G/H$ for some subinterval (t_0, t_1) of $\mathbb{R}$, in such a way that the metric is written as $g = dt^2 + g_t$, where g_t is a homogeneous metric on G/H for each t. By Schur's Lemma, $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are orthogonal for g_t and the hyperHermitian structure preserves the splitting

$$T\,\mathcal{U}(N) = \left(\left\langle \frac{\partial}{\partial t} \right\rangle \oplus \mathfrak{p}_0\right) \oplus \mathfrak{p}_1,$$

Thus, $\mathfrak{p}_1$ is the orthogonal complement to the quaternionic span of V. Equation (4.1) shows that on $\mathfrak{p}_1$, the vector field V acts conformally. Moreover, $Sp(1)$ acts isometrically and $\mathfrak{p}_1$ is irreducible as a representation of $H \times Sp(1)$. Thus the restriction of g_t to $\mathfrak{p}_1$ is a multiple of the metric on TN, which is itself a constant times $\langle \cdot, \cdot \rangle_{\mathfrak{p}_1}$, the restriction to $\mathfrak{p}_1$ of minus the Killing form on $\mathfrak{g}$. Let $f(t)$ be the function such that $g_t|_{\mathfrak{p}_1} = f(t)^2 \langle \cdot, \cdot \rangle_{\mathfrak{p}_1}$.

The restriction of g to $\langle \frac{\partial}{\partial t} \rangle + \mathfrak{p}_0$ defines a Bianchi IX hyperHermitian metric, so as in [21] we can diagonalise this part of the metric for all t. More concretely, we can find a basis $\{X_1, X_2, X_3\}$ for $\mathfrak{p}_0 \cong \mathfrak{sp}(1)$ satisfying $[X_1, X_2] = 2X_3$ etc., such that $g(X_a, X_b) = \delta_{ab} h_a^2$ for $a, b = 1, 2, 3$, where h_a are functions of t. Note that by Proposition 4.4, the hyperHermitian structure on $\widehat{M}$ induces the quaternionic structure on the Wolf space, and in particular, the complex structures act via $\operatorname{ad} X_a$ on $\mathfrak{p}_1$.

Writing J_1, J_2, J_3 for I, J, K, we may argue as in [18] and derive the formula

$$J_a|_{\mathfrak{p}_1} = \frac{h_a}{2ff'} \operatorname{ad} X_a \qquad \text{for } a = 1, 2, 3.$$

The only change needed in the proof, is the contribution to $\nabla_{Y_i}(J_a \partial/\partial t)$ from terms $(\nabla J_a)\partial/\partial t$. But as $\widehat{M}$ is quaternionic Kähler, each of these terms lies in $\mathfrak{p}_0$ and so does not contribute to J_a on $\mathfrak{p}_1$.

As $J_a^2 = -1$, we see that the functions h_a^2 are all equal. This means that the metric on $\widehat{M} = \mathcal{U}(N)$ is of the form considered in [36, §3]. Lemma 3.4 in [36], implies that g is of the form in [36, Theorem 3.5]. □

Theorem 4.6. *Suppose M is a compact quaternionic Kähler manifold of positive scalar curvature with a cohomogeneity-one action of a compact semi-simple group G. If $G^{\mathbb{C}}$ has no open orbit on the twistor space of M, then $M = \mathbb{HP}(n)$ with its symmetric metric and G is the subgroup $Sp(n)$ of the full isometry group $Sp(n+1)$.*

Proof. Without loss of generality we may assume that G is connected. As M is simply-connected [34], the exact homotopy sequence implies that the compact one-dimensional manifold M/G is a closed interval. Thus we have two special orbits G/H_1 and G/H_2. If the principal orbit is G/H, then the fact that M is a smooth manifold implies that H_i/H is a sphere for $i = 1, 2$.

The form of the metric in [36] implies that one of the special orbits is a point and the other is the underlying Wolf space W. Thus G/H is a sphere S^{4n-1} which implies $G = Sp(n)$ and $W = \mathbb{HP}(n-1)$, giving $M = \mathbb{HP}(n)$. However, $\mathbb{HP}(n)$ only admits one hyperKähler metric [34]. □

5. Moment Maps

We assume from now on that the groupoid action of $G^{\mathbb{C}}$ has an open orbit U_1 on the twistor space Z, or equivalently that the set U_0 of Lemma 3.2 is non-empty. The idea is to use a moment map construction to identify an open set of Z with an open set in a projectivised adjoint orbit or in a bundle over such an orbit. We then have to classify which adjoint orbits in $\mathfrak{g}^{\mathbb{C}}$ admit G actions of low cohomogeneity, before proceeding further.

The following constructions are essentially due to Lichnerowicz [29], but our treatment follows the notation of [37, 38].

The twistor space Z of a quaternionic Kähler manifold is a complex contact manifold [34]. This means that Z has a complex line bundle L and a holomorphic one-form $\theta \in \Omega^1(Z, L)$ such that $\theta \wedge (d\theta)^n$ is nowhere zero. The action of $G^{\mathbb{C}}$ preserves the complex contact structure. We may define a *moment map* $f\colon Z \to L \otimes (g^{\mathbb{C}})^*$ for the action of $G^{\mathbb{C}}$ by

$$f(z)(Y) = \theta_z(\alpha(Y)),$$

for $z \in Z$ and $Y \in \mathfrak{g}$. Choosing an Ad-invariant inner-product on $\mathfrak{g}^{\mathbb{C}}$, we identify $\mathfrak{g}^{\mathbb{C}}$ with its dual equivariantly and regard f as a map $Z \to L \otimes \mathfrak{g}^{\mathbb{C}}$.

For z in U_0 the vectors $\alpha_z(Y)$ span $T_z Z$, so non-degeneracy of θ implies that the map f is non-vanishing at z. Projectivising, we get a well-defined map $\mathbb{P}f\colon U_0 \to \mathbb{P}(\mathfrak{g}^{\mathbb{C}})$. This map is $G^{\mathbb{C}}$-equivariant and so the image $\mathbb{P}f(U_1)$ of the open orbit U_1 lies in some $G^{\mathbb{C}}$-orbit $\mathbb{P}\mathcal{O}$.

The group $\mathbb{C}^*$ acts on $\mathfrak{g}^{\mathbb{C}}$ by scaling: $X \mapsto \lambda X$, for $X \in \mathfrak{g}^{\mathbb{C}}$ and $\lambda \in \mathbb{C}^*$. We let $\mathbb{P}\colon \mathfrak{g}^{\mathbb{C}} \setminus \{0\} \to \mathbb{P}(\mathfrak{g}^{\mathbb{C}})$ denote the associated quotient map. The adjoint action of $G^{\mathbb{C}}$ commutes with the scaling action of $\mathbb{C}^*$, so the orbit $\mathbb{P}\mathcal{O}$ is $\mathbb{P}(\mathcal{O})$ for some adjoint orbit $\mathcal{O}$.

Lemma 5.1. *Let $\mathcal{O}$ be an adjoint orbit in $\mathfrak{g}^{\mathbb{C}} \setminus \{0\}$. If $\mathcal{O}$ is nilpotent, then $\mathcal{O}$ is invariant under the scaling action of $\mathbb{C}^*$. If $\mathcal{O}$ is not nilpotent, then the map $\mathcal{O} \to \mathbb{P}(\mathcal{O})$ is an unbranched finite cover.*

Proof. Let X be an element of $\mathcal{O}$. For $\lambda \in \mathbb{C}^*$, the eigenvalues of $\operatorname{ad}(\lambda X)$ are λ times the eigenvalues of X. But if λX lies in $\mathcal{O}$, the orbit of X, then $\operatorname{ad}(\lambda X)$ has the same eigenvalues as $\operatorname{ad}(X)$. Thus either all the eigenvalues of $\operatorname{ad}(X)$ are zero and X is nilpotent, or $|\lambda| = 1$. In the first case, it is well-known that nilpotent orbits are invariant under scaling: choose an $\mathfrak{sl}(2,\mathbb{C})$-triple $\langle X, Y, H\rangle$ containing X, then the action of $\exp(tH)$ on X gives the ray through X (see Carter [16] for more details; recall that an $\mathfrak{sl}(2,\mathbb{C})$-triple satisfies $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$). If X is not nilpotent, then as $\operatorname{ad}(X)$ only has finitely many eigenvalues, there are only a finite number of possible λ

such that $\lambda X \in \mathcal{O}$. Thus the map $\mathcal{O} \to \mathbb{P}(\mathcal{O})$ is finite-to-one. It is unbranched as it is $G^{\mathbb{C}}$-equivariant and the image is homogeneous. □

As G is reductive, each adjoint orbit $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ carries a complex symplectic structure $\omega_{\mathcal{O}}$. This form was defined by Kirillov, Kostant and Souriau (see for example [22]) and is given by

$$\omega_{\mathcal{O}}([X, A], [X, B]) = \langle X, [A, B] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is an invariant inner product. In particular, the complex dimension of each adjoint orbit $\mathcal{O}$ is even. The previous lemma shows that the complex dimension of $\mathbb{P}(\mathcal{O})$ is odd if $\mathcal{O}$ is nilpotent, and even otherwise.

Lichnerowicz shows that $\mathbb{P}f$ can behave in two possible ways. In the proper case, $\mathcal{O}$ is nilpotent, so $\mathbb{P}(\mathcal{O})$ is contact. Moreover, $\mathbb{P}f$ maps U_1 to $\mathbb{P}(\mathcal{O})$ with discrete fibre.

In the non-proper case, $\mathcal{O}$ is non-nilpotent, and $\mathbb{P}f(\mathcal{O})$ is of even complex dimension. Now $\mathbb{P}f$ has fibres of complex dimension one with a complex Lie group structure. Moreover, the fibres are transverse to the complex contact distribution, so we have a vector field X on U such that $\theta(X)$ never vanishes. The bundle $\mathcal{L}$ is therefore holomorphically trivial on U_1.

We shall now obtain estimates on the cohomogeneity of the adjoint orbit $\mathcal{O}$.

Lemma 5.2. *Suppose the scalar curvature of (M, g) is positive. Then, in the non-proper case, U_1 contains no compact complex submanifolds.*

Proof. If we are in the non-proper situation, $\mathcal{L}$ is trivial on U_1. However, if (M, g) has positive scalar curvature, then $\mathcal{L}$ has a positive curvature form. Hence U_1 contains no compact complex manifolds. □

In particular, the proper case must occur if U_1 contains a twistor line.

Proposition 5.3. *Suppose (M, g) has positive scalar curvature and let $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ be an orbit such that $\mathbb{P}f(U_1) \subset \mathbb{P}(\mathcal{O})$.*

If we are in the proper case, then $\mathcal{O}$ is a nilpotent orbit and has cohomogeneity at most 5 with respect to G. On the other hand, in the non-proper case, $\mathcal{O}$ is not nilpotent and has cohomogeneity at most 2 with respect to G.

Proof. Let us first consider the proper case. As explained above, the map $\mathbb{P}f\colon U_1 \mapsto \mathbb{P}(\mathcal{O})$ is equivariant with discrete fibre. Moreover the map $\mathbb{P}\colon \mathcal{O} \mapsto \mathbb{P}(\mathcal{O})$ is equivariant with fibre $\mathbb{C}^*$. Therefore

$$\operatorname{cohom}_G(Z) = \operatorname{cohom}_G U_1 = \operatorname{cohom}_G \mathbb{P}(\mathcal{O}) \geqslant \operatorname{cohom}_G(\mathcal{O}) - 2$$

Now $\operatorname{cohom}_G(M) = 1$, so $\operatorname{cohom}_G(Z) \leqslant 3$ and the above inequalities show that $\operatorname{cohom}_G(\mathcal{O}) \leqslant 5$, as required.

For the non-proper case, using Lemma 5.2 we see that the fibres of $\mathbb{P}f : U_1 \to \mathbb{P}(\mathcal{O})$ are non-compact. We therefore have

$$\begin{aligned}\operatorname{cohom}_G(\mathcal{O}) &= \operatorname{cohom}_G(\mathbb{P}(\mathcal{O})) \\ &\leqslant \operatorname{cohom}_G(U_1) - 1 = \operatorname{cohom}_G(Z) - 1 \leqslant 2.\end{aligned}$$

□

Remark 5.4. The classification of these orbits is also relevant for the classification of 3-Sasakian manifolds of cohomogeneity one and hyper-Kähler manifolds of cohomogeneity two, see §§8 and 9.

6. Cohomogeneity of Adjoint Orbits

In this section we shall first assume that G is a compact simple Lie group and give some procedures for calculating the cohomogeneity of adjoint orbits in $\mathfrak{g}^{\mathbb{C}}$ and related objects. We will use this to classify the orbits that arise from the discussion in the previous section. Results for semi-simple G will be given later in §6.4.

The adjoint orbits fall into three classes: semi-simple, nilpotent and mixed. We will discuss the cohomogeneity questions in each of these cases in turn. One feature all cases have in common is the existence of monotonicity theorems for the cohomogeneity of topologically related orbits. In the nilpotent case, there is one quaternionic Kähler metric known related to each orbit; we prove similar results for these structures.

In [18], the adjoint orbits of cohomogeneity one were classified. These were found to be the minimal nilpotent orbits and the semi-simple orbit of $\operatorname{diag}(\lambda, \dots, \lambda, -n\lambda)$ in $\mathfrak{sl}(n+1, \mathbb{C})$; this last orbit is $SU(n+1)$-equivariantly diffeomorphic to $T^*\,\mathbb{C}\mathrm{P}(n)$.

6.1. Semi-Simple Orbits.

Let $\mathcal{O}$ be a semi-simple orbit. In this case, $\mathcal{O}$ is a flag manifold and we have G-equivariant diffeomorphisms

$$\mathcal{O} \cong T(G/K) \cong G^{\mathbb{C}}/K^{\mathbb{C}},$$

where K is the centraliser of some torus in G. We will write $\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m}$ a K-module.

First we note

Lemma 6.1. *If G/K is any compact homogeneous space, then*

$$\operatorname{cohom}_G T(G/K) = \operatorname{cohom}_K \mathfrak{m}$$

and the latter is bounded below by the number of irreducible summands in $\mathfrak{m}$. □

We will obtain bounds on the cohomogeneity of flag manifolds by considering various fibrations. The following will be a useful observation.

Lemma 6.2. *Suppose $G/K \to G/H$ is a fibration of homogeneous G-spaces, with G, K and H compact and* $\dim K < \dim H$. *Then*

$$\operatorname{cohom} T(G/K) > \operatorname{cohom} T(G/H).$$

Proof. We have $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{k} \subsetneqq \mathfrak{h}$. So $\mathfrak{m} \cong \mathfrak{a} \oplus \mathfrak{n}$ for some non-zero representation $\mathfrak{a}$ of K. As K is compact, this gives

$$\operatorname{cohom}_K \mathfrak{m} > \operatorname{cohom}_K \mathfrak{n} \geqslant \operatorname{cohom}_H \mathfrak{n},$$

since each K-orbit in $\mathfrak{n}$ lies in an H-orbit. The result now follows from Lemma 6.1. □

Recall that a flag manifold $T(G/K)$ is specified by a subset $\mathfrak{K}$ of the simple roots for $\mathfrak{g}$: the elements of $\mathfrak{K}$ are the simple roots not in $\mathfrak{k}^{\mathbb{C}}$. A Dynkin diagram is defined for G/K by taking the Dynkin diagram for $\mathfrak{g}$ and putting a cross through each simple root α in $\mathfrak{K}$ (see [4], for example). We call $|\mathfrak{K}|$ the *length* of G/K and note that there is a fibration $T(G/K_1) \to T(G/K_2)$ if and only if $\mathfrak{K}_1 \supset \mathfrak{K}_2$. The following result is a simple consequence of the previous Lemma.

Lemma 6.3 (Monotonicity for Semi-Simple Orbits)**.** *If $\mathfrak{K}_1 \supset \mathfrak{K}_2$, then*

$$\operatorname{cohom} T(G/K_1) \geqslant \operatorname{cohom} T(G/K_2) + (|\mathfrak{K}_1| - |\mathfrak{K}_2|).$$

In particular, $\operatorname{cohom} T(G/K_1) \geqslant |\mathfrak{K}_1|$. □

This last estimate is extremely crude. The next result, which is particularly relevant because of our interest in flag manifolds of cohomogeneity two, serves to emphasise this.

Lemma 6.4. *If $T(G/K)$ is a flag manifold with $|\mathfrak{K}| \geqslant 2$, then*

$$\operatorname{cohom} T(G/K) \geqslant 3.$$

Proof. Suppose α_1 and α_2 are simple roots in $\mathfrak{K}$. Let α denote the top root form of $\mathfrak{g}$. Then α_1, α_2 and α do not pairwise differ by linear combinations of roots of $\mathfrak{k}^{\mathbb{C}}$. Therefore, by Kostant's criterion [9, p. 40], α_1, α_2 and α lie in distinct K-submodules of $\mathfrak{m}$ (using the notation of Lemma 6.1). In particular, $\mathfrak{m}$ has at least three summands and hence $\operatorname{cohom}_K \mathfrak{m} \geqslant 3$, giving the result. □

The above results now enable us to classify semi-simple orbits of cohomogeneity two.

Theorem 6.5. *For a compact simple group G, the semi-simple orbits of cohomogeneity two are the tangent bundles of the following homogeneous spaces:*

$$\frac{Sp(n+1)}{U(1)\,Sp(n)}, \qquad \frac{SU(n+2)}{S(U(n)\times U(2))}$$
$$\frac{SO(n+2)}{SO(n)\times SO(2)}, \qquad \frac{SO(10)}{U(5)} \qquad \textit{and} \qquad \frac{E_6}{Spin(10)\,SO(2)}.$$

Proof. If $T(G/K)$ is a flag manifold, Burstall & Rawnsley [15] show that there is a fibration $G/K \to G/H$ with G/H an inner symmetric space. We can take H to be connected. By Lemma 6.4, $T(G/K)$ is a flag manifold of length one in our case. So Lemma 6.2 implies we have two cases to consider, either the fibration $G/K \to G/H$ is non-trivial or G/K is Hermitian symmetric.

If the fibration $G/K \to G/H$ is non-trivial, then $T(G/H)$ must have cohomogeneity one. This implies that G/H is rank one (see [6] or the remarks in the last paragraph of the proof). Moreover, G/H cannot be Hermitian symmetric, otherwise the length of $T(G/K)$ would be at least two. It follows that G/H is one of

$$\text{(i)} \quad \frac{Sp(n+1)}{Sp(1)\,Sp(n)}, \qquad \text{(ii)} \quad \frac{SO(2n+1)}{SO(2n)}, \qquad \text{(iii)} \quad \frac{F_4}{Spin(9)}.$$

The rank one symmetric space $SO(2n)/\,SO(2n-1)$ does not occur, as it is not inner.

In case (i), there are two flag manifolds of length one fibering over G/H, namely $\times\circ\cdots\circ\Leftarrow\circ$ and $\circ\times\circ\cdots\circ\Leftarrow\circ$. For the first of these $K = U(1)\,Sp(n)$ and $\mathfrak{m}^{\mathbb{C}} \cong (L+\overline{L})E + L^2 + \overline{L}^2$, where $E \cong \mathbb{C}^{2n}$ and $L \cong \mathbb{C}$ are the standard representations of $Sp(n)$ and $U(1)$ respectively. In particular, $\mathfrak{m} \cong \mathbb{H}^n \oplus \mathbb{R}^2$, with $Sp(n)$ acting transitively on the unit sphere in $\mathbb{H}^n$ and $U(1)$ acting non-trivially on $\mathbb{R}^2$. So this is of cohomogeneity two and we have the first case on our list.

For $\circ\times\circ\cdots\circ\Leftarrow\circ$, we have $K = SU(2)\,U(1)\,Sp(n-1)$ and $\mathfrak{m}^{\mathbb{C}} \cong (L^2+\overline{L}^2)S^2H + (L+\overline{L})HE$, with $H \cong \mathbb{C}^2$ the standard representation of $SU(2)$ and E and L essentially as before. The real module underlying $(L^2+\overline{L}^2)S^2H$ is $\mathbb{R}^6$ with a cohomogeneity two action of $SO(2)\times SO(3)$. Thus, $\mathfrak{m}$ has cohomogeneity at least three under K and so is not on our list.

For case (ii), we take $n > 2$ to avoid overlapping with case (i). Then the only flag manifold of length one fibering over this is $\circ\cdots\circ\Rightarrow\times$, which has $K = U(n)$ and $\mathfrak{m} \cong [\Lambda^{1,0}\mathbb{C}^n] + [\Lambda^{2,0}\mathbb{C}^n]$, and which is not of cohomogeneity two. (The brackets $[\cdot]$ indicate the underlying real representation.)

Finally, for case (iii), first note that $\mathfrak{spin}(9)$ is embedded in $\mathfrak{f}_4$ with positive simple roots $01 \Rightarrow 22$, $10 \Rightarrow 00$, $01 \Rightarrow 00$, $00 \Rightarrow 10$. The only flag manifold of length one fibering over $F_4 / Spin(9)$ is $\circ\circ\Rightarrow\circ\times$. Here $\mathfrak{k} \cong \mathfrak{u}(1) \oplus \mathfrak{so}(7)$, which is 22-dimensional, whereas $\dim \mathfrak{m} = 30$, so the action is not cohomogeneity two.

We now need to consider the case when G/K is a Hermitian symmetric space. The cohomogeneity we are interested in is $\operatorname{cohom}_K \mathfrak{m}$, where $\mathfrak{m}$ is the isotropy representation for this space. The Cartan theory for symmetric spaces (see [19, Chapter VIII], for example) shows that a transversal for the action of K on $\mathfrak{m}$ is given by a chamber in a maximal Abelian subspace of $\mathfrak{m}$. It follows that $\operatorname{cohom}_K \mathfrak{m}$ is just the rank of the symmetric space G/K. The table in Besse [6, pp. 312–313] shows that the Hermitian symmetric spaces of rank two are (modulo low-dimensional coincidences, and up to covers) the following: the Grassmannians of two-planes in $\mathbb{C}^n$, the hyperquadrics $SO(n+2)/ SO(n) \times SO(2)$, the symmetric space $SO(10)/ U(5)$ and the exceptional space $E_6/ Spin(10)\, SO(2)$. This completes the proof. $\square$

For future reference, it is worth noting that the principal orbits in $\mathcal{O}$ for the five cases of Theorem 6.5 are

$$\frac{Sp(n+1)}{Sp(n-1)}, \qquad \frac{SU(n+2)}{S(U(n-2) \times U(1) \times U(1))},$$
$$\frac{SO(n+2)}{SO(n-2)}, \qquad \frac{SO(10)}{S(U(2) \times U(2))} \qquad \text{and} \qquad \frac{E_6}{U(4)}. \tag{6.1}$$

Also, from the cohomogeneity one case $T^* \mathbb{C}\mathrm{P}(n)$ we have the principal orbit

$$\frac{SU(n+1)}{U(n-1)} \tag{6.2}$$

However, not all these cases will arise from the twistor spaces of cohomogeneity one quaternionic Kähler manifolds, as we will see in §7.

6.2. **Nilpotent Orbits.** Recall that there is a partial order on the nilpotent orbits in $\mathfrak{g}^{\mathbb{C}}$ defined by $\mathcal{O}_1 \succeq \mathcal{O}_2$ if and only if $\overline{\mathcal{O}_1} \supset \mathcal{O}_2$.

Proposition 6.6 (Monotonicity for Nilpotent Orbits). *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are nilpotent orbits with $\mathcal{O}_1 \succneqq \mathcal{O}_2$, then*

$$\operatorname{cohom} \mathcal{O}_1 > \operatorname{cohom} \mathcal{O}_2.$$

Proof. Fix an element X in $\mathcal{O}_2$. For any $\mathfrak{sl}(2,\mathbb{C})$-triple $\langle X, Y, H\rangle$, set

$$S_{X,Y} = X + \mathfrak{z}(Y),$$

where $\mathfrak{z}(Y)$ denotes the centraliser of Y. Slodowy [35] showed that $S_{X,Y}$ is a transverse slice to $\mathcal{O}_2$ at X. We need to study the intersection of G-orbits with such a slice. For this we make a good choice of Y.

Let $B = \operatorname{stab}_G X$ be the real stabiliser of X and let $\mathfrak{c}$ be the centraliser in $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{b}$, the Lie algebra of B. Then $\mathfrak{c}$ is σ-invariant and hence reductive. Moreover, $\mathfrak{c}$ contains X, so there is a $Y \in \mathfrak{c}$ such that $\langle X, Y, [X, Y]\rangle$ is an $\mathfrak{sl}(2,\mathbb{C})$-triple. As Y lies in $\mathfrak{c}$, we have that Y, $\mathfrak{z}(Y)$ and $S_{X,Y}$ are all B-invariant.

Fix this choice of Y and let $H = [X, Y]$. In $\mathfrak{c}$, $\langle X, Y, H\rangle$ is conjugate to a real $\mathfrak{sl}(2,\mathbb{C})$-triple, so there exists $g \in \exp \mathfrak{c}$ such that $(\operatorname{Ad} g)Y = -\sigma((\operatorname{Ad} g)X)$. Thus for $\sigma^g = (\operatorname{Ad} g)^{-1}\sigma(\operatorname{Ad} g)$, we have $Y = -\sigma^g X$ and $\sigma^g\mathfrak{b} = \mathfrak{b}$. Define G^g to be the connected subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^g = \mathfrak{g}^{\mathbb{C}} \cap \sigma^g\mathfrak{g}^{\mathbb{C}}$. Then G^g is isomorphic to G and contains B.

Choose a σ^g-invariant $\mathfrak{sl}(2,\mathbb{C})$-triple $\langle X_1, Y_1, H_1\rangle$ with $X_1 \in \mathcal{O}_1$ and $Y_1 = -\sigma^g X_1$. Let $\mathcal{M}$ be the space of maps $(X(t), H(t))\colon \mathbb{R} \to \mathfrak{g}^{\mathbb{C}} \times i\mathfrak{g}^g$ such that

(a) $\dot X = -2X + [X, H]$, $\dot H = -2H + 2[X, \sigma^g X]$,
(b) $X(t) \to X$ and $H(t) \to H$ as $t \to +\infty$, and
(c) $X(t) \to X_{-\infty}$ and $H(t) \to H_{-\infty}$ as $t \to -\infty$, with $(X_{-\infty}, H_{-\infty})$ G^g-conjugate to (X_1, H_1).

Kronheimer [27] showed that $\mathcal{M}$ is a manifold naturally isomorphic to $S_{X,Y} \cap \mathcal{O}_1$. This isomorphism is $\operatorname{stab}_{G^g} X$ equivariant; but $B \subset \operatorname{stab}_{G^g} X$, so we have $\mathcal{M} \cong S_{X,Y} \cap \mathcal{O}_1$ B-equivariantly.

The manifold $\mathcal{M}$ admits an action of $\mathbb{R}$ given by $(X(t), H(t)) \mapsto (X(t-c), H(t-c))$. This $\mathbb{R}$-action is non-trivial and commutes with the action of G^g on the values. In particular, $\mathcal{M}/B$ has positive dimension.

Let $\mathfrak{v}$ be a B-invariant complement to $T((\operatorname{Ad} G)X)$ in $T_X\mathcal{O}_2$. Then the orbit $(\operatorname{Ad} G)X$ has a G-invariant neighbourhood

$$U \cong G \times_B (\mathfrak{v} + \mathfrak{z}(Y))$$

in $\mathfrak{g}^{\mathbb{C}}$ such that the map $U \cap \overline{\mathcal{O}_1} \to G \times_B \mathfrak{v}$ is an equivariant surjection onto a G-invariant neighbourhood of $(\operatorname{Ad} G)X$ in $\mathcal{O}_2$ and contains $S_{X,Y} \cap \mathcal{O}_1$ in the fibre over $0 \in \mathfrak{v}$. Thus for generic X in $\mathcal{O}_2$, we have

$$\begin{aligned}\operatorname{cohom}_G \mathcal{O}_1 &= (\operatorname{cohom}_B S_{X,Y} \cap \mathcal{O}_1) + \operatorname{cohom}_G \mathcal{O}_2\\ &> \operatorname{cohom}_G \mathcal{O}_2,\end{aligned}$$

as required. □

Each simple Lie group G has a unique non-trivial nilpotent orbit of smallest dimension. We call this the *minimal* nilpotent orbit $\mathcal{O}_{\min}$. The orbit $\mathcal{O}_{\min}$ has the property that $\mathcal{O}_{\min} \preceq \mathcal{O}$ for all non-trivial nilpotent orbits $\mathcal{O}$. In [18], it was shown that $\mathcal{O}_{\min}$ is the unique nilpotent orbit of cohomogeneity one.

As explained in §5, each nilpotent orbit $\mathcal{O}$ fibres over a quaternionic Kähler manifold $\mathfrak{M}(\mathcal{O}) = \mathcal{O}/\mathbb{H}^*$. In the case of $\mathcal{O}_{\min}$, the space $\mathfrak{M}(\mathcal{O}_{\min})$ is G-homogeneous and is the Wolf space with isometry group G. (The details of the fibration in this case may be found in [36, §6].) Proposition 6.6 carries over to the manifolds $\mathfrak{M}(\mathcal{O})$.

Proposition 6.7 (Monotonicity for $\mathfrak{M}(\mathcal{O})$). *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are nilpotent orbits with $\mathcal{O}_1 \gneqq \mathcal{O}_2$, then*

$$\operatorname{cohom} \mathfrak{M}(\mathcal{O}_1) > \operatorname{cohom} \mathfrak{M}(\mathcal{O}_2).$$

Proof. Fix $X \in \mathcal{O}_2$ and choose Y as in the proof of Proposition 6.6. The subgroup $\mathbb{R}^* \leqslant \mathbb{H}^*$ acts on X by $X \mapsto \lambda^2 X$ and an $\mathfrak{sl}(2,\mathbb{C})$-triple containing $\lambda^2 X$ is given by $\langle \lambda^2 X, \lambda^{-2} Y, [X,Y] \rangle$. Now $\mathfrak{z}(\lambda^{-2}Y) = \mathfrak{z}(Y)$, so the slice at $\lambda^2 X$ is given by $S_{\lambda^2 X,Y}$. Note that the set $S_{X,Y} \cap S_{\lambda^2 X,Y}$ is empty for $\lambda^2 \neq 1$. We set

$$W_{X,Y} = \mathbb{H}^*(S_{X,Y} \cap \overline{\mathcal{O}_1})/\mathbb{H}^* = Sp(1)(S_{X,Y} \cap \overline{\mathcal{O}_1})/\,Sp(1)$$

to get a slice to $\mathfrak{M}(\mathcal{O}_2) \subset \overline{\mathfrak{M}(\mathcal{O}_1)}$ at $\mathbb{H}^* X$.

The proof of Proposition 6.6 shows that $S_{X,Y} \cap \mathcal{O}_1$ is non-compact. As $Sp(1)$ and the stabiliser D of $\mathbb{H}^* X$ are both compact, we have for generic $X \in \mathcal{O}_2$,

$$\begin{aligned}\operatorname{cohom}_G \mathfrak{M}(\mathcal{O}_1) &= \operatorname{cohom}_D W_{X,Y} + \operatorname{cohom}_G \mathfrak{M}(\mathcal{O}_2)\\ &> \operatorname{cohom}_G \mathfrak{M}(\mathcal{O}_2),\end{aligned}$$

as required. □

Let us now consider the cohomogeneity-one quaternionic Kähler manifolds that arise from nilpotent orbits. We say that $\mathcal{O}$ is next to minimal in the partial order $\preceq$ if $\mathcal{O} \gneqq \mathcal{O}_{\min}$ and there are no nilpotent orbits $\mathcal{O}'$ with $\mathcal{O} \gneqq \mathcal{O}' \gneqq \mathcal{O}_{\min}$. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ may have more than one nilpotent orbit satisfying this condition, as we will see in the proof of the following result.

Theorem 6.8. *Suppose $\mathcal{O}$ is a nilpotent orbit and $\mathfrak{M}(\mathcal{O})$ is the associated quaternionic Kähler manifold. Then $\mathfrak{M}(\mathcal{O})$ is of cohomogeneity one if and only if $\mathcal{O}$ is next to minimal in the order $\preceq$ on nilpotent orbits.*

Proof. Proposition 6.7 implies that the only candidates are the next to minimal nilpotent orbits. We will verify that each of these does indeed lead to a cohomogeneity-one structure on $\mathfrak{M}(\mathcal{O})$ by considering each case in turn.

First, we recall from [38] the structure of $\mathfrak{M}(\mathcal{O})$ as a G-manifold. The nilpotent orbit $\mathcal{O}$ defines a conjugacy class of subalgebras $\mathfrak{sl}(2,\mathbb{C}) \leqslant \mathfrak{g}^{\mathbb{C}}$. Choose one such subalgebra $\mathfrak{sl}(2,\mathbb{C})$ and without loss of generality, assume that this is the complexification of a real subalgebra $\mathfrak{su}(2)$. Let K be the centraliser of $L = \exp\{\mathfrak{su}(2)\}$ in G. We may decompose $\mathfrak{g}$ under the adjoint action of $L \times K$ to get

$$\mathfrak{g} \cong \mathfrak{su}(2) \oplus \mathfrak{k} \oplus \bigoplus_{k\geqslant 1}[A_k S^k],$$

where S^k is the irreducible representation of $SU(2)$ of complex dimension $k+1$ and A_k is a K-module. Then $\mathfrak{M}(\mathcal{O})$ is G-equivariantly isomorphic to the bundle

$$W = \bigoplus_{k\geqslant 2}[A_k S^{k-2}] \longrightarrow G/LK.$$

In particular, when $\mathfrak{M}(\mathcal{O})$ is of cohomogeneity one, there is at most one non-zero module A_k for $k \geqslant 2$.

We now consider the various possible types of G in turn. Relevant facts about the description of nilpotent elements and their stabilisers have been conveniently collected in [16, Chapter 13].

Suppose G is of type A_n. Then the nilpotent orbits are described by partitions of $n+1$ giving the sizes of the Jordan blocks. The minimal nilpotent orbit is (21^{n-1}) and there is a unique next-to-minimal nilpotent orbit. When $n > 2$, the next-to-minimal orbit is $(2^2 1^{n-3})$ and for this case the Lie algebra of K is $\mathfrak{su}(2)_- + \mathfrak{su}(n-3) + \mathfrak{u}(1)$. Write the Lie algebra of L as $\mathfrak{su}(2)_+$ to distinguish it from the $\mathfrak{su}(2)$-factor in $\mathfrak{k}$. Now as a KL-module, $\mathbb{C}^{n+1} \cong S_+S_-\ell^r + V\ell^s$, where $S_\pm \cong \mathbb{C}^2$, $V \cong \mathbb{C}^{n-3}$ and $\ell \cong \mathbb{C}$. So

$$\begin{aligned}\mathfrak{su}(n+1) &= \operatorname{End}_0 \mathbb{C}^{n+1}\\ &\cong \mathfrak{su}(2)_+ + \mathfrak{k} + [S_+S_-\ell^{r-s}V^*] + [S_+^2 S_-^2].\end{aligned}$$

Thus, $W = [S_-^2]$ which is $\mathbb{R}^3$ with the standard representation of $SO(3)$. Thus $\mathfrak{M}(2^2 1^{n-3})$ is cohomogeneity one.

For $n = 2$, there are only two non-trivial nilpotent orbits. The next-to-minimal orbit is the regular orbit. We have $\mathfrak{su}(3) = \mathfrak{su}(2) + [S^4]$ and $W = [S^2]$, which is cohomogeneity one.

For type B_n, the nilpotent orbits are again described by partitions, this time of $2n+1$, but not all partitions arise. The minimal orbit is $(2^2 1^{2n-3})$. There are two next-to-minimal orbits: (31^{2n-2}) and $(2^4 1^{2n-7})$. For the first of these, $\mathfrak{k} \cong \mathfrak{so}(2n-2)$. The $SO(2n+1)$-module $\mathbb{R}^{2n+1}$ splits as $[S^2] + V$, with $V = \mathbb{R}^{2n-2}$ the standard representation of K. Using $\mathfrak{so}(2n+1,\mathbb{C}) \cong \Lambda^2\mathbb{C}^{2n-2}$, we find $W \cong V$, which is cohomogeneity one. For $(2^4 1^{2n-7})$, $\mathfrak{k} \cong \mathfrak{sp}(2) + \mathfrak{so}(2n-7)$ and $\mathbb{R}^{2n+1} \cong [\mathbb{C}^4 S^1] + \mathbb{R}^{2n-7}$, with $Sp(2)$ and $SO(2n-7)$ acting irreducibly on $\mathbb{C}^4$ and $\mathbb{R}^{2n-7}$, respectively. This gives $W \cong \mathbb{R}^5$ with $Sp(2)$-acting via the standard representation of $SO(5)$, which is again cohomogeneity one.

The nilpotent orbits for G of type C_n are classified by certain partitions of $2n$. The minimal orbit is (21^{2n-2}). There is only one next to minimal orbit: $(2^2 1^{2n-4})$. The centraliser K has Lie algebra $\mathfrak{sp}(n-2)+\mathfrak{u}(1)$ and $\mathbb{C}^{2n} \cong (\ell+\ell^{-1})S^1+\mathbb{C}^{2n-4}$, where $\mathfrak{sp}(n-2)$ acts irreducibly on the last summand and ℓ is a one-dimensional representation of $U(1)$. We have $\mathfrak{sp}(n,\mathbb{C}) \cong S^2\mathbb{C}^{2n}$ and hence $W \cong \mathbb{R}^2$ as an irreducible $SO(2)$-module. This is cohomogeneity one.

For type D_n, we have $G = SO(2n)$, the minimal orbit is $(2^2 1^{2n-4})$ and the next to minimal orbits have partitions (31^{2n-3}) and $(2^4 1^{2n-8})$. For n odd this gives two next-to-minimal orbits, however for n even, the last partition describes two orbits, which we need not distinguish, and there are three next-to-minimal orbits. The calculations for all of these partitions are the same as the B_n case and give cohomogeneity-one manifolds each time.

For the exceptional groups there is a unique next-to-minimal orbit. The centralisers, Lie algebra decompositions and representation W are given in Table 1. The calculations for the groups of type E where done using the program "Lie" [40]. The script for the case of E_8 is given in Figure 1. This script first calculates the semi-simple element `h` associated to the nilpotent orbit. It then determines the stabiliser `cs` $= \mathfrak{k}_1$ of `h` and finds its Cartan type. The decomposition of the adjoint representation of E_8 under $\mathfrak{k}_1$ is calculated, using the restriction matrix `rm`. This information, combined with direct calculation of the weight spaces for the action of $\mathfrak{su}(2)$, are enough to determine the where the nilpotent element X lies. The Lie algebra of $\mathfrak{k}$ is now the stabiliser in $\mathfrak{k}_1$ of X and decomposing each $\mathfrak{k}_1$-module under the action of $\mathfrak{k}$ leads to W. □

Note that if we calculate the cohomogeneities of the next-to-minimal nilpotent orbits themselves we do not quite get this uniform answer. Having seen the above theorem one might guess that moving up the partial order decreases the cohomogeneity of $\mathfrak{M}(\mathcal{O})$ by one at each stage. However, a look at the diagrams in [25] for $\mathfrak{sp}(4)$, $\mathfrak{so}(10)$ or $\mathfrak{su}(7)$ shows that this can not be the case, as there can be paths of different lengths in the partial order.

6.3. **Mixed Orbits.** If X is a general element of $\mathfrak{g}^{\mathbb{C}}$, we can write X uniquely as $X = X_s + X_n$ with X_s semi-simple, X_n nilpotent and $[X_s, X_n] = 0$. Similarly, if $\mathcal{O}$ is the $G^{\mathbb{C}}$-orbit of X, then we write $\mathcal{O}_s$

Type	Orbit	$\mathfrak{k}$	W	cohom($\mathcal{O}$)
A_n				
$n=2$	(3)	$\{0\}$	$[S^2]\cong\mathbb{R}^3$	4
$n\geqslant 3$	$(2^2 1^{n-3})$	$\mathfrak{su}(2)_- + \mathfrak{su}(n-3)$ $+\mathfrak{u}(1)$	$[S^2_-]\cong\mathbb{R}^3$	2
$B_{(n-1)/2}$, $D_{n/2}$				
	(31^{n-3})	$\mathfrak{so}(n-3)$	$\mathbb{R}^{n-3}$	2
	$(2^4 1^{n-8})$	$\mathfrak{so}(5)+\mathfrak{so}(n-8)$	$\mathbb{R}^5$	2
C_n	$(2^2 1^{2n-4})$	$\mathfrak{so}(2)+\mathfrak{sp}(n-2)$	$\mathbb{R}^2$	2
G_2	$0\Rrightarrow 1$	$\mathfrak{su}(2)_-$	$[S^1_+ S^1_-]\cong\mathbb{R}^4$	2
F_4	$00\Rightarrow 01$	$\mathfrak{so}(6)$	$\mathbb{R}^6$	2
E_6	$\overset{0}{10001}$	$\mathfrak{so}(2)+\mathfrak{so}(7)$	$\mathbb{R}^7$	2
E_7	$\overset{0}{010000}$	$\mathfrak{su}(2)+\mathfrak{so}(9)$	$\mathbb{R}^9$	2
E_8	$\overset{0}{0000001}$	$\mathfrak{so}(13)$	$\mathbb{R}^{13}$	2

TABLE 1. Next-to-minimal orbits, specified by partitions for classical groups and Dynkin diagrams for exceptional Lie groups. In each case $W=\mathbb{R}^n$ is the standard irreducible representation of $SO(n)$.

```
setdefault E8
print(diagram)
h=i_Cartan*[1,0,0,0,0,0,0,0]/det_Cartan;print(h)
hp=h+0;print(hp)
s=cent_roots(hp)
ct=Cartan_type(s);print(ct)
cs=closure(s)
rm=res_mat(cs)
ad=expon(adjoint,1);print(ad);print(dim(ad))
branch(ad,ct,rm)
```

FIGURE 1. "Lie" script for calculations for E_8 in the nilpotent case.

and $\mathcal{O}_n$ for the orbits of X_s and X_n, respectively. We partially order the orbits by saying $\mathcal{O}_1 \succeq \mathcal{O}_2$ if and only if $\overline{\mathcal{O}_1} \supset \mathcal{O}_2$.

Lemma 6.9. (a) *If two mixed orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ satisfy $\mathcal{O}_1 \succeq \mathcal{O}_2$ then $(\mathcal{O}_1)_s = (\mathcal{O}_2)_s$ and $(\mathcal{O}_1)_n \succeq (\mathcal{O}_2)_n$.*

(b) *If $\mathcal{O}$ is a mixed orbit, then $\mathcal{O} \succeq \mathcal{O}_s$.*

Proof. (a) Suppose $\mathcal{O}_2$ is in the closure of $\mathcal{O}_1$. Fix $X \in \mathcal{O}_2$ and choose a sequence $X^{(i)} \in \mathcal{O}_1$ such that $X^{(i)} \to X$. Now the orbit of the semi-simple part of $X^{(i)}$ is determined by the characteristic polynomial of $\operatorname{ad}(X^{(i)})$. But that polynomial is a continuous function of $X^{(i)}$ and is constant on $\mathcal{O}_1$. So the characteristic polynomial of $\operatorname{ad}(X)$ equals that of $\operatorname{ad}(X^{(i)})$ and hence $(\mathcal{O}_1)_s = (\mathcal{O}_2)_s$. The map $\pi_n \colon \overline{\mathcal{O}_1} \to \overline{(\mathcal{O}_1)_n}$ given by $\pi_n(X) = X_n = X - X_s$ is continuous and $G^{\mathbb{C}}$-equivariant. So $\pi_n(\overline{O}_1)$ is contained in $\overline{\pi_n(\mathcal{O}_1)} = \overline{(\mathcal{O}_1)_n}$. In particular, $(\mathcal{O}_2)_n$ is contained in $\overline{(\mathcal{O}_1)_n}$, as required.

(b) Write $X = X_s + X_n$. Then $X_s + \lambda X_n$ is in $\mathcal{O}$ for all non-zero $\lambda \in \mathbb{C}$. Thus X_s is in the closure of $\mathcal{O}$. □

Proposition 6.10 (Monotonicity for Mixed Orbits). *Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are mixed orbits with $\mathcal{O}_1 \supsetneqq \mathcal{O}_2$. Then*

$$\operatorname{cohom} \mathcal{O}_1 > \operatorname{cohom} \mathcal{O}_2.$$

Proof. This is a slight modification of the proof of Proposition 6.6. Fix $X \in \mathcal{O}_2$. The semi-simple orbit $(\mathcal{O}_2)_s$ is isomorphic to $T(G/K)$ for some $K \leqslant G$. Suppose X_s lies in the fibre of $T(G/K) \to G/K$ over gK. Then $\operatorname{stab}_G X_s$ is a subgroup of gKg^{-1} which is a compact real form of $gK^{\mathbb{C}}g^{-1}$. Now $gK^{\mathbb{C}}g^{-1}$ is abstractly isomorphic to $K_s^{\mathbb{C}} := \operatorname{stab}_{G^{\mathbb{C}}} X_s$, so we conclude that $\mathfrak{k}_s^{\mathbb{C}} = \mathfrak{z}(X_s)$ admits a real structure σ_s giving a compact real form K_s containing $\operatorname{stab}_G X_s$.

Let $B = \operatorname{stab}_G X = \operatorname{stab}_G X_s \cap \operatorname{stab}_G X_n$ and let $\mathfrak{c}$ be the centraliser of $\mathfrak{b}$ in $\mathfrak{k}_s^{\mathbb{C}}$. Then X, X_s and X_n all lie in $\mathfrak{c}$ and $\mathfrak{c}$ is σ-invariant, so we can find $Y \in \mathfrak{k}_s^{\mathbb{C}}$ such that $\langle X_n, Y, [X_n, Y]\rangle$ is an $\mathfrak{sl}(2,\mathbb{C})$-triple commuting with $\mathfrak{b}$. As in the proof of Proposition 6.6 we may find a real structure σ_s^g on $\mathfrak{k}_s^{\mathbb{C}}$ giving a compact form K_s^g and such that $Y = -\sigma_s^g X_n$ and $\sigma_s^g \mathfrak{b} = \mathfrak{b}$. A transverse slice to $\mathcal{O}_2$ at X is given by

$$\begin{aligned} S_{X,Y} &= X + \mathfrak{z}(Y) \cap \mathfrak{z}(X_s) \\ &= X_s + (X_n + \mathfrak{z}_{\mathfrak{k}_s^{\mathbb{C}}}(Y)). \end{aligned}$$

Orbits of G meet $S_{X,Y}$ in orbits of B and these lie in K_s^g-orbits. The inequalities on cohomogeneities now follow as in Proposition 6.6. □

Proposition 6.11. *If G is a compact simple Lie group and $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ is an orbit which is neither semi-simple nor nilpotent, then*

$$\operatorname{cohom} \mathcal{O} > 2.$$

Proof. Suppose $\operatorname{cohom} \mathcal{O} \leqslant 2$. As $\mathcal{O}$ is not semi-simple, we have $\mathcal{O} \supsetneqq \mathcal{O}_s$ and Proposition 6.10 implies $\operatorname{cohom} \mathcal{O} > \operatorname{cohom} \mathcal{O}_s$. Hence $\operatorname{cohom} \mathcal{O}_s$ is one. By [18], we have $G = SU(n+1)$ and $\mathcal{O}_s$ is the orbit of $\operatorname{diag}(\lambda, \dots, \lambda, -n\lambda)$. The mixed orbits with this semi-simple part are parameterised by nilpotent orbits of $\mathfrak{sl}(n,\mathbb{C})$, and have the same partial ordering. This follows directly from the Jordan normal form of such elements. Proposition 6.10 implies that it is sufficient to calculate the cohomogeneity of the $SL(n+1,\mathbb{C})$-orbit, or equivalently $GL(n+1,\mathbb{C})$-orbit, of

$$X = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & & 0 & \\ & & \ddots & & \\ & 0 & & \lambda & \\ & & & & -n\lambda \end{pmatrix}.$$

The stabiliser of X in $U(n+1)$ has Lie algebra $\mathfrak{k} = \mathfrak{u}(1)_+ + \mathfrak{u}(n-2) + \mathfrak{u}(1)_-$. The fibre of the normal bundle to the real orbit of X at X is

$$\begin{aligned} &(\operatorname{ad} X)(\mathfrak{sl}(n+1,\mathbb{C}))/(\operatorname{ad} X)(\mathfrak{su}(n+1)) \\ &\qquad \cong \left\{ \begin{pmatrix} 0 & a & & & \\ & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & & \\ b_1 & b_2 & \dots & b_n & 0 \end{pmatrix} : a \in \mathbb{R}, b_i \in \mathbb{C} \right\}. \end{aligned}$$

As a representation of $\mathfrak{k}$, this fibre is

$$\mathbb{R} + 2[L_+\overline{L_-}] + [\Lambda^{1,0}\overline{L_-}],$$

where $\Lambda^{1,0}$ is the standard representation of $U(n-2)$ on $\mathbb{C}^{n-2}$. This representation of $\mathfrak{k}$ has cohomogeneity 5. □

We therefore do not need to consider mixed orbits any further.

6.4. **Orbits of Semi-Simple Groups.** If G is a compact semi-simple Lie group, then the Lie algebra of G splits as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$$

with each $\mathfrak{g}_i$ simple. Let G_i be the simply-connected group with Lie algebra $\mathfrak{g}_i$. If $\mathcal{O}$ is an adjoint orbit for $G^{\mathbb{C}}$, then we can write

$$\mathcal{O} = \mathcal{O}_1 \times \dots \times \mathcal{O}_r,$$

where $\mathcal{O}_i$ is an adjoint orbit for $G_i^{\mathbb{C}}$. Now

$$\operatorname{cohom}_G \mathcal{O} = \operatorname{cohom}_{G_1} \mathcal{O}_1 + \dots + \operatorname{cohom}_{G_r} \mathcal{O}_r,$$

so we have the following result:

Proposition 6.12. *If G is a compact semi-simple group acting almost effectively on an adjoint orbit $\mathcal{O}$ with cohomogeneity at most two, then either G is simple and the orbit $\mathcal{O}$ is listed in §§6.1 and 6.2 or G has two simple factors G_1 and G_2, $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{O}_i$ is of cohomogeneity one with respect to G_i, for $i = 1, 2$.* □

So in the semi-simple, non-simple case, the new orbits we must study are $\mathcal{O}_1 \times \mathcal{O}_2$ where each $\mathcal{O}_i$ is either the semi-simple orbit $T^*\mathbb{CP}(n)$ of $SL(n+1, \mathbb{C})$ or a minimal nilpotent orbit.

7. Quaternionic Kähler Structures

The results of §§5 and 6 give us the following classification result. At this stage we are not assuming completeness of the metric.

Theorem 7.1. *Let (M, g) be a quaternionic Kähler manifold of positive scalar curvature, of cohomogeneity one with respect to a compact simple group G acting effectively.*

Then up to covers, one of the following three alternatives holds:

(i) *(M, g) is hyperHermitian, and g is one of the metrics of* [36] *as described in Proposition* 4.4;

(ii) *an open set of the twistor space of (M, g) is an open set in a projectivised nilpotent orbit $\mathbb{P}(\mathcal{O})$ as a complex contact manifold with real structure and $\mathcal{O}$ is of cohomogeneity at most* 5;

if the twistor space covers the whole of $\mathbb{P}(\mathcal{O})$, then (M, g) is locally isometric to $\mathfrak{M}(\mathcal{O})$, and these cases are listed in Theorem 6.8;

(iii) *we are in the non-proper case, and the principal orbit of G in M is either*

$$\text{(a) } \frac{SU(n+1)}{U(n-1)}, \quad \text{(b) } \frac{SU(n+2)}{U(n-2)},$$
$$\text{(c) } \frac{SO(10)}{SU(2) \times SU(2)}, \quad \textit{or} \quad \text{(d) } \frac{E_6}{SU(4)}.$$

Proof. Part (i), is the case of no open orbits on Z given in Proposition 4.4. Part (ii) is the case of Z having an open $G^{\mathbb{C}}$-orbit which is a proper complex contact manifold. The statement about the real structure is due to Nitta & Takeuchi [31]. When Z covers all of $\mathbb{P}(\mathcal{O})$, we have that Z is homogeneous under $G^{\mathbb{C}}$ and the fact that M is isometric to $\mathfrak{M}(\mathcal{O})$ was proved in [38].

It only remains to deal with case (iii). Recall from §5 that in the non-proper case, we have a map $\mathbb{P}f\colon U_1 \to \mathbb{P}(\mathcal{O})$ with fibre a non-compact complex Lie group of complex dimension one. We derived the inequalities

$$3 \geqslant \operatorname{cohom}_G(U_1) \geqslant \operatorname{cohom}_G(\mathbb{P}(\mathcal{O})) + 1 = \operatorname{cohom}_G(\mathcal{O}) + 1. \qquad (7.1)$$

By Proposition 6.11, the orbit $\mathcal{O}$ is semi-simple, and hence the principal orbit types on $\mathcal{O}$ are given by (6.1) and (6.2).

Consider the principal orbit types $Sp(n+1)/\,Sp(n-1)$ and $SO(n+2)/\,SO(n-2)$, with $n \neq 4$ in the second case. In these cases, the stabiliser acts trivially on the fibre of $\mathbb{P}f$, which has complex dimension one. So the second inequality in (7.1) is strict, and we get a contradiction. Thus we are left with the cases listed in the Theorem (up to covers, the orthogonal case with $n = 4$ gives the same principal orbit type in M as case (b) with $n = 2$). ☐

Remark 7.2. We do not know whether all the cases in the above Theorem can be realised. For case (iii), we will see below that there is a compact example with special orbit as in (iii)(a). We do not know whether (iii)(b–d) occur in non-compact examples. In case (ii), it is conceivable that open sets of the projectivised nilpotent orbits admit families of twistor lines not corresponding to the structures $\mathfrak{M}(\mathcal{O})$. This is known to happen in $\mathfrak{so}(4,\mathbb{C})$ (see [38]), but currently we have no examples with $\mathfrak{g}^{\mathbb{C}}$ simple.

Let us now consider the case when M is compact. For the nilpotent case we will need an interpretation of a result of Beauville which is of independent interest. Note this result does not assume that M is of cohomogeneity one.

Theorem 7.3. *Suppose M is a compact quaternionic Kähler manifold with an action of a compact connected Lie group G preserving this structure. If $G^{\mathbb{C}}$ has an open orbit on Z containing a twistor line then M is isometric to a Wolf space and either G is the isometry group of M or G is given by one of the shared-orbit pairs of Brylinski & Kostant* [13, 14].

Proof. The hypotheses on the action mean that M is quaternionic Kähler with positive scalar curvature, so the twistor space Z is projective with $c_1(Z) > 0$, and $G^{\mathbb{C}}$ acts on Z algebraically. The assumption that there is an open $G^{\mathbb{C}}$-orbit containing a twistor line implies that the image of $\mathbb{P}f$ is the projectivisation of a nilpotent orbit, i.e., we are in the proper case of §5. In particular, $\mathbb{P}f$ has injective differential on an open set of Z. By [30, III.5 Corollary 2, p. 252] the algebraic map $\mathbb{P}f$ is étale and hence finite.

Exactly the same remarks apply to the component of the identity G_1 of the full isometry group of M and the associated map $\mathbb{P}f_1\colon Z \dashrightarrow$

$\mathbb{P}(\mathfrak{g}_1^{\mathbb{C}})$. Beauville's results [5] apply to this $G_1^{\mathbb{C}}$-action on Z, giving that Z is the projectivised minimal nilpotent orbit $\mathbb{P}(\mathcal{O}_1)$ for $G_1^{\mathbb{C}}$ and that M is the Wolf space with isometry group G_1.

If $G = G_1$ we are finished. Otherwise $\mathbb{P}f$ induces a finite morphism $\phi\colon \mathcal{O}_1 \dashrightarrow \mathcal{O}$ which is $G^{\mathbb{C}}$-equivariant. This is the condition that $\mathcal{O}$ be a "shared orbit" and Beauville shows that only the pairs listed in [13, 14] arise. ☐

Theorem 7.4. *Let (M, g) be a compact quaternionic Kähler manifold, of cohomogeneity one with respect to a compact simple group G.*

Then (M, g) is a quaternionic Kähler symmetric space and the pair (M, G) is given in Table 2.

M	G	M	G
$\mathbb{HP}(n)$	$Sp(n)$	$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$	G_2
$\mathbb{HP}(n)$	$SU(n+1)$	$G_2 / SO(4)$	$SU(3)$
$\mathrm{Gr}_2(\mathbb{C}^n)$	$SU(n-1)$	$F_4 / Sp(3)\, Sp(1)$	$Spin(9)$
$\mathrm{Gr}_2(\mathbb{C}^{2n})$	$Sp(n)$	$E_6 / SU(6)\, Sp(1)$	F_4
$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n)$	$SO(n-1)$		

TABLE 2. Compact quaternionic Kähler manifolds M which are cohomogeneity one under an action of a simple group G.

Proof. We consider the three cases of Theorem 7.1. Case (i), when there is no open $G^{\mathbb{C}}$-orbit in Z, was dealt with in Theorem 4.6 and gives the first entry in the table.

If there is an open $G^{\mathbb{C}}$-orbit U_1, then we have $U_1 = Z \setminus D$ for some subvariety D of Z. We are either in the proper case (ii) or the non-proper case (iii).

In (ii), we can apply the proof of Theorem 7.3. This together with Theorem 7.1 implies that in order to get quaternionic Kähler manifolds of cohomogeneity one, we need to consider shared orbit pairs $(\mathcal{O}, \mathcal{O}_1) \subset (\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}_1^{\mathbb{C}})$ with $\mathcal{O}_1$ a minimal orbit and $\mathcal{O}$ next-to-minimal. Looking at the list of shared orbits in [13] we see that all but two cases give next-to-minimal orbits; the exceptions being the four-to-one covering of an orbit in $\mathfrak{so}(8, \mathbb{C})$ by the minimal orbit in $\mathfrak{f}_4^{\mathbb{C}}$ and the six-to-one covering of the subregular orbit in $\mathfrak{g}_2^{\mathbb{C}}$ by the minimal orbit of $\mathfrak{so}(8, \mathbb{C})$. This gives the last six cases in the table.

It remains to discuss case (iii) and account for the second and third entries of the table, which we do using an argument of Poon & Salamon [33]. As mentioned above, the twistor space Z is projective algebraic, with a linear action of $G^{\mathbb{C}}$. Applying the Borel fixed point theorem to the action of a maximal torus T of G, we see that T has a fixed point on Z, and hence on M. Therefore the rank of the stabiliser of some point of M equals the rank of G.

As in the proof of Theorem 4.6, we see that M/G is a compact interval, that there are two special orbits G/H_1 and G/H_2, and, writing G/K for the principal orbit, that H_i/K is a sphere for $i = 1, 2$. Examining the ranks of isotropy groups we see that cases (iii)(b), (c) and (d) cannot arise.

The remaining case is when $G = SU(n+1)$ and the principal stabiliser has Lie algebra $\mathfrak{u}(n-1)$. As the classification of compact quaternionic Kähler manifolds is known in dimensions four [20] and eight [33], we take $n > 2$.

Let us first assume that the principal stabiliser K is connected. The long exact homotopy sequence for the fibration $K \to G \to G/K$, and the Hurewicz theorem, show that the second Betti number of G/K is 1.

If G/H is a special orbit then H/K is a sphere S^r with $r > 0$, since M is simply-connected. Hence H is connected, and the homotopy sequence for $H \to G \to G/H$ shows that $b_2(G/H)$ is the rank of $\pi_2(G/H) \cong \pi_1(H)$. The latter may be computed by considering the fibration $K \to H \to S^r$ and we get that

$$b_2(G/H) = \begin{cases} 2, & \text{if } r = 1, \\ 0, & \text{if } r = 2, \\ 1, & \text{if } r > 2. \end{cases}$$

Following [3], we consider the Mayer-Vietoris sequence for an open cover of M consisting of the complements of each special orbit. We obtain

$$H^2(M) \to H^2\left(\frac{G}{H_1}\right) \oplus H^2\left(\frac{G}{H_2}\right) \to H^2\left(\frac{G}{K}\right) \to H^3(M) = 0$$

in real cohomology. If M is not the complex Grassmannian then, by [28], $b_2(M) = 0$ and so one of the special orbits has $r = 2$ and the other has $r > 2$.

Now, the complex coadjoint orbit $T^* \mathbb{C}P(n)$ has a special $SU(n+1)$-orbit of real dimension $2n$, so Z contains a special $SU(n+1)$-orbit of real dimension less than or equal to $2n + 1$. It follows that one of the special orbits in M has real codimension at least $2n - 1$.

Therefore if $n > 2$ and M is not the complex Grassmannian then the special orbits in M must be $SU(n+1)/U(n)$ and $SU(n+1)/SU(2) \times SU(n-1)$ and M is $\mathbb{HP}(n)$ [3].

If K is disconnected we can argue as follows. The homotopy sequence for $K \to H \to S^r$ shows that either $r = 1$ or H is disconnected. But if one special orbit G/H_1 has $r > 1$ then the codimension of that orbit is strictly greater than 2 and so the other special orbit G/H_2, like M, must be simply-connected. In particular, G/H_2 would have connected stabiliser and $r = 1$. The arguments of the preceding paragraphs now show that $b_2(G/H_2) = 2$ and $b_2(G/K) \leqslant 1$, so the Mayer-Vietoris argument again forces M to be the complex Grassmannian.

The remaining possibility, that $r = 1$ for both special orbits, is ruled out for $n > 1$ because we know one of the special orbits in M has codimension at least $2n - 1$. □

Remark 7.5. If $n = 1$ then disconnected stabilisers can arise. We can view S^4 as an $SO(3)$-manifold with both special orbits being $\mathbb{RP}(2)$. Also $\mathbb{CP}(2)$ admits an $SO(3)$-action with one orbit an S^2 and the other an $\mathbb{RP}(2)$.

Let us now discuss the case when G is semi-simple and M is compact.

Theorem 7.6. *Suppose M is a compact quaternionic Kähler manifold of cohomogeneity one with respect to a compact group G. If G is semi-simple but not simple, then $M = \mathbb{HP}(N)$ and $\mathfrak{g} = \mathfrak{sp}(N-m) \oplus \mathfrak{sp}(m+1)$ for some $0 \leqslant m < N$.*

Proof. The case when there is no open $G^{\mathbb{C}}$-orbit on Z is covered by the results of §4 and yields no examples with G not simple.

In the proper case, we may apply Theorem 7.3. The only shared orbit pair for a non-simple group occurs when $G = Sp(n_1) \times \cdots \times Sp(n_r)$ and $\mathcal{O}$ is the product of the minimal orbits in $\mathfrak{sp}(n_1, \mathbb{C}), \dots, \mathfrak{sp}(n_r, \mathbb{C})$. This is covered by the minimal orbit in $\mathfrak{sp}(n_1 + \cdots + n_r, \mathbb{C})$ and M is the quaternionic projective space $\mathbb{HP}(n_1 + \cdots + n_r - 1)$. This is of cohomogeneity one only when G has exactly two factors, i.e., $r = 2$.

In the non-proper case, the new orbits we must consider are, from §5, of the form $\mathcal{O}_1 \times \mathcal{O}_2$ where each of $\mathcal{O}_i$ is either the $SL(n+1, \mathbb{C})$-orbit $T^* \mathbb{CP}(n)$ or a minimal nilpotent orbit. The principal orbit of $SU(n+1)$ in the former orbit is $SU(n+1)/U(n-1)$, while for the minimal nilpotent orbits for $H^{\mathbb{C}}$ the principal H-orbit is H/K, where $H/K\,Sp(1)$ is the associated Wolf space.

In all these cases it follows that the rank of G is at least two greater than the rank of the principal stabiliser for the G-action on the orbit. Also, the orbits are cohomogeneity two with respect to G. Hence the rank of G is at least three greater than that of the principal stabiliser for the G-action on Z or M. So the Borel fixed point argument shows that we do not obtain any compact examples with G not simple. $\square$

Remark 7.7. Attempting to prove the above theorem for general reductive G requires stronger results than we were able to obtain in §4.

8. HyperKähler Structures

The results obtained in §6 can be used to make results of Bielawski [7] a little more precise. In [18] we were able to classify hyperKähler metrics of cohomogeneity one. We now wish to extend this to metrics of cohomogeneity two.

Suppose M is a hyperKähler manifold with Riemannian metric g and complex structures I, J and K satisfying $IJ = K = -KI$. Then $aI + bJ + cK$ is also a complex structure on M whenever $(a, b, c) \in S^2 \subset \mathbb{R}^3$. Let $\mathbf{S}$ denote the set of these complex structures. Suppose that a compact semi-simple Lie group G acts on M preserving both the metric and the individual complex structures and that the action is of cohomogeneity two. Fix a point x of M lying on some principal orbit. Then at x the tangent vectors generated by the G-action span a real subspace $\mathcal{D}$ of T_xM of codimension 2.

We claim that there are at most two points $\mathcal{J}$, $-\mathcal{J}$ in $\mathbf{S}$ which preserve $\mathcal{D}$. Suppose $A, B \in \mathbf{S}$ are linearly independent and preserve $\mathcal{D}$. Then $\lambda A + \mu B$ also preserves $\mathcal{D}$ for any $\lambda, \mu \in \mathbb{R}$, so we may assume A and B are orthogonal. But now $AB = -BA$ and this preserves $\mathcal{D}$, so $\mathcal{D}$ is an $\mathbb{H}$-module, contradicting the fact that $\mathcal{D}$ has real codimension 2 in the $\mathbb{H}$-module T_xM.

Thus, without loss of generality, we may assume that I does not preserve $\mathcal{D}$. Infinitesimally, we may complexify the action of G with respect to I. Then the tangent vectors to this action span T_xM and they also span the tangent space of M in a G-invariant neighbourhood of x. The assumption that G be semi-simple implies that we have a hyperKähler moment map

$$\mu = \mu_I i + \mu_J j + \mu_K k \colon M \to \mathfrak{g} \otimes \operatorname{Im} \mathbb{H},$$

where we have identified $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. The map $\mu_c := \mu_J + i\mu_K$ is a moment map for the complex-symplectic action of $G^{\mathbb{C}}$. In a neighbourhood of x, the map μ_c is a local diffeomorphism to an open subset of an adjoint orbit $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$. Moreover, μ_c is G-equivariant, so the orbit $\mathcal{O}$ is of cohomogeneity two and is given by Proposition 6.12. Note that each of these orbits actually arises, since Kronheimer, Biquard and Kovalev have shown that every adjoint orbit carries a G-invariant hyperKähler metric compatible with the Kirillov-Kostant-Souriau complex-symplectic form [27, 26, 8, 24].

If we assume that M is complete, then Bielawski [7] shows that $\mathcal{O}$ is a semi-simple orbit and that for one of the hyperKähler metrics on $\mathcal{O}$ constructed by Kronheimer [26], $\mu_c\colon M \to \mathcal{O}$ is an isometry.

Noting that the above discussion also applies to G with central factors providing we assume the existence of a hyperKähler moment map, we have the following result.

Theorem 8.1. *Suppose M is a complete hyperKähler manifold which is of cohomogeneity two with respect to an isometric action of a compact group G preserving each complex structure. If G is semi-simple or, more generally, if there is a hyperKähler moment map for the action of G on M, then M is isometric to one of:*

(1) $T^*\,\mathbb{C}\mathrm{P}(2n-1)$, *with* $G = Sp(n)$;
(2) $T^*\,\mathrm{Gr}_2(\mathbb{C}^n)$, *with* $G = SU(n)$;
(3) $T^*\,\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$, *with* $G = SO(n)$;
(4) $T^*\big(SO(10)/U(5)\big)$;
(5) $T^*\big(E_6\,/\,Spin(10)\,SO(2)\big)$;
(6) $T^*\,\mathbb{C}\mathrm{P}(n)\times T^*\,\mathbb{C}\mathrm{P}(m)$, *with* $G = SU(n+1)\times SU(m+1)$.

9. Three-Sasakian Structures

Let S be a compact 3-Sasakian manifold with Riemannian metric g. We refer the reader to [11] for a precise definition, but for our purposes the following characteristic property will be sufficient: the manifold $N_S := S \times \mathbb{R}_{>0}$ with the metric $dr^2 + r^2 g$ is hyperKähler with an action of $Sp(1)$, trivial on the $\mathbb{R}_{>0}$-factor and satisfying (i)–(iv) of Proposition 4.2. This gives examples of 3-Sasakian manifolds by taking $N_S = \mathcal{U}(M)$ with M a quaternionic Kähler manifold of positive scalar curvature. In general, $S/\,Sp(1)$ is only a quaternionic Kähler orbifold even though S is smooth.

The group of 3-Sasakian symmetries of S is exactly the group of triholomorphic isometries of N_S and is necessarily compact. Suppose G is a compact group of symmetries of S. Then there is a unique hyperKähler moment map $\mu\colon N_S \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$ for this action with the property that

$$\mu(s,r) = r^2\mu(s,1), \tag{9.1}$$

for $s \in S$, $r \in \mathbb{R}_{>0}$ [11]. Recall that μ is G-equivariant.

Suppose S is of cohomogeneity one under the action of G. Then the hyperKähler manifold N_S is of cohomogeneity two and we may apply some of the results of the previous section.

Let P_S and P_N denote the union of principal orbits of S and N_S respectively. Note that as S is compact and Einstein with positive scalar curvature, the fundamental group $\pi_1(S)$ is finite and so P_S can not be all of S.

Lemma 9.1. *The hyperKähler manifold N_S has a compatible complex structure I such that $T_xN_S = T(G\cdot x) + IT(G\cdot x)$ for all $x \in P_N$.*

Proof. In §8 we showed that at each $x \in P_N$ the set

$$\mathcal{A}_x = \{\, \mathcal{J} \in \mathbf{S} : T_xN_S \neq T(G\cdot x) + \mathcal{J}T(G\cdot x)\,\}$$

is either empty or a point of $\mathbb{RP}(2) = \mathbf{S}/\{\pm 1\}$. However, $\mathcal{A}_{(s,r)} = \mathcal{A}_{(s,1)}$ for $r \in \mathbb{R}_{>0}$ and $\mathcal{A}_{(g\cdot s,r)} = \mathcal{A}_{(s,r)}$ for $g \in G$, so we have a map from a subset of $P_S/G = P_N/(G\times\mathbb{R}_{>0}) \to \mathbb{RP}(2)$. This map is a smooth map of a one-dimensional manifold, so its image is one-dimensional and the set $\mathbf{S}\setminus\cup_{x\in P_N}\mathcal{A}_x$ of I satisfying our requirements is non-empty. □

Fix such a choice of I and let $J, K \in \mathbf{S}$ be such that $IJ = K = -JI$. Let μ_I be the component of the hyperKähler moment map μ corresponding to I, etc. The group $\mathbb{C}^*$ acts on N_S as the complexification with respect to I of the $\mathbb{R}_{>0}$-action. The quotient $Z = N_S/\mathbb{C}^*$ is the *twistor space* of S and is a compact, normal, $\mathbb{Q}$-factorial Fano variety [10] and, in particular, is projective. The hyperKähler manifold N_S is the complement of the zero section of an algebraic line bundle on Z. We now have that $G^{\mathbb{C}}$ acts on both Z and N_S.

As P_N is connected, Lemma 9.1 implies that there is an open $G^{\mathbb{C}}$-orbit U containing P_N. The scaling property (9.1) implies that $\mu_c = \mu_J + i\mu_K$ maps U to a nilpotent orbit $\mathcal{O}$. Note that μ_c is $\mathbb{C}^*$-equivariant. The usual rank argument for the moment maps shows that the fibres of μ_c on U are of dimension zero. In particular, U and $\mathcal{O}$ have the same dimension. Any other $G^{\mathbb{C}}$-orbit on N_S lies in the complement of P_N, so has smaller dimension than $\mathcal{O}$, and hence can not lie in $\mu_c^{-1}(\mathcal{O})$. Thus $U = \mu_c^{-1}(\mathcal{O})$.

Lemma 9.2. *The map μ_c is an algebraic morphism, finite over $\mathcal{O}$. The symmetry group G is semi-simple with at most two simple factors. If G is simple then $\mathcal{O}$ is next-to-minimal and $\mathfrak{g} \neq \mathfrak{su}(3)$. If G is not simple then $\mathcal{O}$ is the product of a minimal orbit in each simple factor of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. The definition of μ_c and the algebraic nature of N_S imply that μ_c is an algebraic map and so, by the above discussion, is finite-to-one on $\mu_c^{-1}(\mathcal{O})$. Thus only the semi-simple part of the component of the identity of G acts effectively. As $\mathcal{O}$ has cohomogeneity two with respect to G, the list of possible nilpotent orbits $\mathcal{O}$ follows from the results of §6. □

Let G_1 be the component of the identity of the full 3-Sasakian symmetry group of S. Then G_1 satisfies the same hypotheses as G, but might act homogeneously on S. Let $\nu\colon N_S \to \mathfrak{g}_1^* \otimes \operatorname{Im} H$ be the moment map for the action of G_1.

Lemma 9.3. *The map ν_c is a finite map without zeroes. The image of ν_c is $\overline{\mathcal{O}_1} \setminus \{0\}$, where $\mathcal{O}_1$ is a nilpotent orbit in $\mathfrak{g}_1^{\mathbb{C}}$ which is either minimal or listed in Lemma* 9.2.

Proof. If G_1 acts transitively on S then there is nothing to prove as $G_1^{\mathbb{C}}$ has only one orbit on N_S. So we suppose that G_1 acts with cohomogeneity one on S.

Let $Q = N_S \setminus \nu_c^{-1}(\mathcal{O}_1)$. Then Q is complex, as it is the pre-image of the subvariety $\overline{\mathcal{O}_1} \setminus \mathcal{O}_1$ of $\overline{\mathcal{O}_1}$. As $\nu_c^{-1}(\mathcal{O}_1)$ contains the set of principal G_1-orbits and ν_c has the scaling property (9.1), the set Q is a union of components of $(G_1/H_1 \coprod G_1/H_2) \times \mathbb{R}_{>0}$, where G_1/H_i are the two special orbits in S. As G_1 commutes with the action of $Sp(1)$, we conclude that the decomposition

$$N_S = \nu_c^{-1}(\mathcal{O}_1) \cup Q \tag{9.2}$$

is invariant under the action of $Sp(1) \times \mathbb{R}_{>0}$. In particular, the decomposition (9.2) is determined by the action of the real group G_1 and not by the particular choice of complex structure I. So Q is complex for I, J and K and the real dimension of Q is divisible by four. Note that G_1 acts on Q with cohomogeneity one, so each component of Q is a single $G_1^{\mathbb{C}}$-orbit.

The scaling property (9.1) implies that ν_c induces a map

$$\mathbb{P}\nu\colon Z \dashrightarrow \mathbb{P}(\mathfrak{g}_1^{\mathbb{C}}).$$

The set where $\mathbb{P}\nu$ fails to be defined lies in $\mathbb{P}Q$ and so has complex codimension 2 in Z. The hypotheses of [5, Lemma 3.3] now apply to $\mathbb{P}\nu$. We conclude that $\mathbb{P}\nu$ is everywhere defined and that both $\mathbb{P}\nu$ and ν_c are finite. This implies, using the scaling property, that ν_c has no zeroes. The image of $\mathbb{P}\nu$ is the closure of $\mathbb{P}\mathcal{O}_1$ and so the image of ν_c is $\overline{\mathcal{O}_1} \setminus \{0\}$. The orbit $\mathcal{O}_1$ has cohomogeneity at most two with respect to G_1 and so is either minimal or next-to-minimal. □

Theorem 9.4. *Suppose S is a compact* 3*-Sasakian manifold which is of cohomogeneity one under a compact connected Lie group G preserving the* 3*-Sasakian structure. Then S is homogeneous under its full* 3*-Sasakian symmetry group and the possible pairs (S, G) are given in Table* 3.

S	G
S^{4n+3}	$Sp(r) \times Sp(n+1-r)$
$\mathbb{R}\mathrm{P}(4n+3)$	$Sp(r) \times Sp(n+1-r)$
$SO(n+1)/\,SO(n-3)\,Sp(1)$	$SO(n)$
$SU(2n)/\,S(U(2n-2)\,U(1))$	$Sp(n)$
$SO(7)/\,SO(4)\,Sp(1)$	G_2
$F_4\,/\,Sp(3)$	$Spin(9)$
$E_6\,/\,SU(6)$	F_4

TABLE 3. Compact three-Sasakian manifolds S of cohomogeneity one under a connected compact Lie group G of 3-Sasakian symmetries.

Proof. Let $\mathcal{O}_1$ be as above. Following Beauville, we will show that $\mathcal{O}_1$ is a minimal nilpotent orbit and that $(\mathcal{O}, \mathcal{O}_1)$ is a shared-orbit pair.

The space N_S is a smooth finite branched $G_1^{\mathbb{C}}$-equivariant cover of $\overline{\mathcal{O}_1}^{\times} := \overline{\mathcal{O}_1} \setminus \{0\}$, such that $S = N_S/\mathbb{R}_{>0}$ is smooth, compact and cohomogeneity at most one under G_1. We determine all such covers for the orbits $\mathcal{O}_1$ of Lemma 9.3. A priori this is more general than $\overline{\mathbb{P}\mathcal{O}_1}$ having a smooth equivariant cover; however, much of Beauville's paper applies at the level of orbits.

Firstly, if $\nu_c\colon N_S \to \overline{\mathcal{O}_1}^{\times}$ is birational then the proof of Corollary 5.3 in [5] shows that $\mathcal{O}_1$ is either the next-to-minimal orbit in $\mathfrak{g}_2^{\mathbb{C}}$ or it is a minimal nilpotent orbit.

Now suppose that $\nu_c\colon N_S \to \overline{\mathcal{O}_1}^{\times}$ has non-trivial degree δ. Note that δ divides $|\pi_1(\mathcal{O}_1)|$.

Consider the case when G_1 is simple. Lemma 9.3 implies that $\mathcal{O}_1$ is either the minimal orbit in $\mathfrak{sp}(n, \mathbb{C})$ or a next-to-minimal orbit in a simple Lie algebra other than $\mathfrak{sl}(3, \mathbb{C})$ (since $\mathfrak{sp}(n, \mathbb{C})$ is the only simple Lie algebra whose minimal orbit is not simply connected). Using the tables in [17] to find $|\pi_1(\mathcal{O}_1)|$, we see that $\mathcal{O}_1$ is either (i) $\mathcal{O}_{(31^{n-3})}$ in $\mathfrak{so}(n, \mathbb{C})$, or (ii) $\mathcal{O}_{(2^41)}$ in $\mathfrak{so}(9, \mathbb{C})$, or the unique next-to-minimal orbit in (iii) $\mathfrak{sp}(n, \mathbb{C})$ or (iv) $\mathfrak{f}_4^{\mathbb{C}}$. In each of these cases $\pi_1(\mathcal{O}_1) = \mathbb{Z}/2$ and thus $\delta = 2$. The manifold N_S is now uniquely determined by $\mathcal{O}_1$ as the complement of the vertex in the normalisation of $\overline{\mathcal{O}_1}$ in the function field of the universal cover of $\mathcal{O}_1$ (cf. [14]). In the four cases to hand, this space is the minimal nilpotent orbit $\widehat{\mathcal{O}} \subset \widehat{\mathfrak{g}}^{\mathbb{C}}$ where $\widehat{\mathfrak{g}}^{\mathbb{C}}$ is either (i) $\mathfrak{so}(n+1, \mathbb{C})$, (ii) $\mathfrak{f}_4^{\mathbb{C}}$, (iii) $\mathfrak{su}(2n, \mathbb{C})$ or (iv) $\mathfrak{e}_6^{\mathbb{C}}$.

If G_1 is not simple, then $\mathcal{O}_1 = \mathcal{O}_a \times \mathcal{O}_b$ with $\mathcal{O}_j$ the minimal orbit in the simple Lie algebra $\mathfrak{g}_j^{\mathbb{C}}$, for $j = a, b$. Dividing by the action of $\mathbb{R}_{>0}$, we get a finite surjective G_1-equivariant map

$$\mathbb{R}\nu \colon S \to S_1 := \left(\overline{(\mathcal{O}_a \times \mathcal{O}_b)} \setminus \{(0,0)\}\right)/\mathbb{R}_{>0}$$

induced by ν_c. The special orbits of S_1 have strictly smaller dimension than the principal orbits. As $\mathbb{R}\nu$ is a finite map, we deduce that $\mathbb{R}\nu$ maps principal orbits to principal orbits and that $\mathbb{R}\nu$ is a bijection of the G_1-orbit spaces. Now S is smooth with principal orbits G_1/H and special orbits G_1/H_i, so H_i/H are spheres, $i = 1, 2$. In S_1, we have principal orbits $G_a/K_a \times G_b/K_b$ and special orbits G_a/K_a and G_b/K_b. Thus one of the spheres H_i/H finitely covers $(G_a \times K_b)/(K_a \times K_b) = G_a/K_a = \mathcal{O}_a/\mathbb{R}_{>0}$ in S_1. The other sphere covers $\mathcal{O}_b/\mathbb{R}_{>0}$. So $\mathcal{O}_j/\mathbb{R}_{>0}$, $j = a, b$, is a finite quotient of a sphere, which is only the case for $\mathfrak{g}_j^{\mathbb{C}} = \mathfrak{sp}(n_j, \mathbb{C})$. As in [5, Proposition 6.8] we now get that N_S is the double cover of the minimal orbit of $\mathfrak{sp}(n_a + n_b, \mathbb{C})$.

In summary, the above three cases show that N_S is the cover of a minimal nilpotent orbit $\widehat{\mathcal{O}} \subset \widehat{\mathfrak{g}}^{\mathbb{C}}$ with $\widehat{\mathfrak{g}}^{\mathbb{C}}$ simple: more precisely, either $N_S = \widehat{\mathcal{O}}$ or N_S is the double cover of the minimal nilpotent orbit in $\mathfrak{sp}(n, \mathbb{C})$. But now Z is $\mathbb{P}\widehat{\mathcal{O}}$ as a complex contact manifold with real structure and so the 3-Sasakian symmetry group of S is a compact real form $\widehat{G}$ of $\widehat{\mathfrak{g}}^{\mathbb{C}}$. But $\widehat{G}$ acts transitively on $\widehat{\mathcal{O}}/\mathbb{R}_{>0}$. So S is homogeneous, $G_1 = \widehat{G}$ and $\mathcal{O}_1 = \widehat{\mathcal{O}}$. The pair $(\mathcal{O}, \mathcal{O}_1)$ is now a shared orbit pair, with $\mathcal{O}_1$ a minimal nilpotent orbit and $\mathcal{O}$ as in Lemma 9.2. Table 3 now follows from the list in [13]. □

Remark 9.5. On a compact 3-Sasakian manifold S it is also natural to consider the full group of isometries $\operatorname{Isom}(S)$. By results of Tanno [39] (see [11]), if S is not of constant curvature, the Lie algebra $\mathfrak{i}$ of the isometry group is the direct sum of the Lie algebra $\mathfrak{s}$ of the group $\operatorname{Sym}(S)$ of 3-Sasakian symmetries and $\mathfrak{sp}(1)$. The $Sp(1)$-orbits are the fibres of the canonical fibration of S over a quaternionic Kähler orbifold M. In particular, the quotients of S by $\operatorname{Isom}(S)$ and of M by $\operatorname{Sym}(S)$ agree and we have

$$\operatorname{cohom}_{\operatorname{Isom}(S)} S = \operatorname{cohom}_{\operatorname{Sym}(S)} M.$$

Working on unions of principal orbits we may apply some of our results when S has a semi-simple group of isometries acting with cohomogeneity one. In this case, there is a semi-simple subgroup G of $\operatorname{Sym}(S)$ such that $G \times Sp(1)$ acts with cohomogeneity at most one on S and hence $\operatorname{cohom}_G M \leqslant 1$. When $G^{\mathbb{C}}$ has no open orbit on the twistor space, the techniques of §4 imply that there are no examples with non-constant curvature. If $G^{\mathbb{C}}$ has an open orbit on Z then in the non-proper case we may apply the Borel fixed point arguments as in §7 and deduce that $SU(n) \times SU(2)$ acts with cohomogeneity one on S. Non-homogeneous examples occur in this case as 3-Sasakian quotients of a sphere via a $U(1)$ action embedded in $Sp(n+1)$ with weights $(1, \ldots, 1, k)$ as described in [11]. We do not know whether these are the only such examples. The proper case is the hardest to analyse. We are unable to apply Beauville's techniques because there is no bound on the second Betti number of a 3-Sasakian manifold and we have not been able to find other arguments to show that the complement of the open $G^{\mathbb{C}}$-orbit has codimension at least 2.

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