Regularity of Stokes Waves in Hardy Spaces and in Spaces of Distributions

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Abstract

This paper deals with some functional-analytic questions which arise when the Stokes-wave problem, for the free boundary of a steady irrotational water wave, is formulated as a quadratic equation for a 2π-periodic, real-valued function w on $\mathbb{R}$ which need not be differentiable, even in a weak sense.

It is shown that any solution w of bounded variation which lies in the fractional order Sobolev space $H^{\frac{1}{2}}$ must be real-analytic and describes the profile of a steady water wave.

The investigation involves elements of complex Hardy space theory and an examination of the behaviour of the Hilbert transform on spaces of 2π-periodic integrable functions in the context of the Stokes wave equation.

1 Introduction

By a Stokes wave is meant a steady periodic wave propagating horizontally without without change of form with constant speed c on the surface of an infinitely deep irrotational flow. It is assumed that gravity g acts vertically and the free surface is determined by a combination of Laplace's equation, kinematic and periodic boundary conditions and by a dynamic boundary condition given by Bernoulli's theorem. The present study, which requires no previous knowledge of hydrodynamics, focuses on the functional analysis of a nonlinear equation, the solutions of which give solutions of the Stokes wave problem. Here is the necessary mathematical background.

For a locally integrable 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ let $\mathcal{C}f$ denote its periodic Hilbert transform (also known as its conjugate function; see [16] where $\mathcal{C}f$ is written $\tilde{f}$. Notation is defined in the next section.) Suppose that $w : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic locally square-integrable function with locally square-integrable weak derivative ($w \in W^{1,2}_{2\pi}$ in the notation of §2). It was shown in [6] that if $(\lambda, w) \in \mathbb{R} \times W^{1,2}_{2\pi}$ satisfies

$$\mathcal{C}w' = \lambda\{w + w\mathcal{C}w' + \mathcal{C}(ww')\}, \quad \lambda > 0, \tag{1.1a}$$

$$1 - 2\lambda w(t) \geq \sigma \text{ for some } \sigma > 0, \tag{1.1b}$$

$$\text{and } t \mapsto (t + \mathcal{C}w(t), w(t)) \text{ is injective on } \mathbb{R}, \tag{1.1c}$$

then w is real-analytic on $\mathbb{R}$ and corresponds to a classical Stokes wave with wavelength 2Λ, where the Froude number $\pi c^2/g\Lambda$ is given by $1/\lambda$. In dimensionless coordinates the profile of the corresponding Stokes wave is given by

$$\mathcal{S} = \{(x + \mathcal{C}w(x), w(x)) : x \in \mathbb{R}\} \tag{1.2}$$

and the excess area of one period of the wave, relative to that of the corresponding uniform horizontal flow, is proportional to

$$\mathcal{A} = \int_{-\pi}^{\pi} w(1 + \mathcal{C}w')dx. \tag{1.3}$$

In these variables, the dynamic boundary condition given by Bernoulli's theorem is the requirement that

$$(1 - 2\lambda w)\{(w')^2 + (1 + \mathcal{C}w')^2\} = 1 \text{ almost everywhere on } \mathbb{R}. \tag{1.4}$$

In [6, §2.4] (see also Theorem 4.2 below) it is shown that (1.4) is a consequence of (1.1a) and (1.1b) provided that $w \in W^{1,2}_{2\pi}$. It then follows from a bootstrap argument and (1.1b) that w lies in the Hölder space $C^{1,\alpha}(\mathbb{R})$, $\alpha \in (0,1)$, from which, by a famous theorem of Lewy [8], w is real-analytic on $\mathbb{R}$. Thus, when (1.1a) and (1.1b) hold, regularity follows from knowing that $w \in W^{1,2}_{2\pi}$.

However, when w is absolutely continuous, the individual terms in (1.1a) (although not necessarily integrable because the Hilbert transform of an integrable function may not be integrable) are functions which are well defined pointwise almost everywhere. In that case (1.2) defines a (generalised) curve in the plane which is rectifiable (meaning, as usual, that w and $\mathcal{C}w$, although not necessarily continuous, have bounded variation) if and only if both w and $\mathcal{C}w$ are absolutely continuous [16, Vol. I, (8.2), page 285]. Also, $\mathcal{A}$ is finite if and only if w lies in the fractional order Sobolev space $H^{\frac{1}{2}}$.

Our first regularity result, proved in §5, is that every absolutely continuous solution w of (1.1a) and (1.1b) belongs to $W^{1,2}_{2\pi}$ and hence is real-analytic.

The next section, §6, is concerned with a more general formulation of the Stokes wave problem in which the function w is not required to be differentiable, even in a weak sense. This formulation arises from the observation that (1.1a) is the Euler-Lagrange of a continuously differentiable functional $\mathcal{J}$ defined by (6.1) when $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. This formula has a natural extension as a continuously differentiable functional $\mathfrak{J}$ defined by (6.2) on $Y = L^\infty_{2\pi} \cap H^{\frac{1}{2}}$. We show that any critical point w of $\mathfrak{J}$ in Y satisfies (6.6) and, if w has bounded variation and satisfies (1.1b), that w is real-analytic. (A variational theory of the equation $\mathfrak{J}'(w) = 0$ is under investigation and will appear elsewhere.)

The regularity theory presented here and in [6] makes no use of the boundary condition (1.1c), the rôle of which is to define the region occupied by the wave as the conformal image of a fixed region in the complex plane. On the other hand the constraint (1.1b), which may at first appear artificial, is essential. In §7, the Stokes wave of extreme form [12] is used as an example to show the existence of a solution $w \in W^{1,2}_{2\pi}$ of (1.1a), which satisfies $1 - 2\lambda w < 0$ except at a discrete set of points, but which is not Lipschitz continuous (although it is in $W^{1,3-\epsilon}_{2\pi}$ for all $\epsilon \in (0,1)$).

It should be remarked from §7 that this extreme-wave solution has the property that $(1-2\lambda w)^{-1}$ is integrable on $[-\pi,\pi]$. Note also that if w satisfies (1.4), then $w \in W^{1,2}_{2\pi}$ if and only if $(1-2\lambda w)^{-1}$ is positive almost everywhere and integrable on $[-\pi,\pi]$. This motivates the study in §8, where regularity theory in the case when (1.1b) is replaced with

$$1 - 2\lambda w(t) > 0 \text{ for almost all } t \in \mathbb{R} \text{ and } \int_{-\pi}^{\pi} \frac{dt}{1 - 2\lambda w(t)} < \infty \tag{1.5}$$

is developed. The result is that a generalised solution w of the Stokes wave problem which satisfies (1.5) is in $W^{1,2}_{2\pi}$ and is real-analytic on the (open) set where it is non-zero. This, Theorem 8.1, is the most general result in the paper.

The discussion involves the Hilbert transform in real function spaces, in particular the Lebesgue space $L^1_{2\pi}$, the Hardy space $\mathcal{H}^1_{\mathbb{R}}$ and the fractional order Sobolev space $H^{\frac{1}{2}}$. This theory is recorded in §2.

The mapping $\mathcal{Q}$, from the definition [15] of the fractional order Slobodecki space $W^{\frac{1}{2}}$, arises naturally in the Stokes wave problem and has crucial properties which are discussed in §3.

At the centre of our treatment of regularity questions is a simple application of the Riesz Representation theorem combined with an outer-function construction given in §4. (These are the outer functions which arise in the factorisation theory of functions which, holomorphic in the unit disc, are in one of the complex Hardy spaces $\mathcal{H}^p_{\mathbb{C}}$.)

The remaining sections discuss the regularity of solutions of the Stokes wave problem at various levels of generality.

2 Real Functions and the Hilbert Transform

Here we recall the classical real-function spaces and how the Hilbert transform acts on them. Let $L^p_{2\pi}$, $0 < p \leq \infty$, denote the linear space of 2π-periodic, real-valued, measurable functions on $\mathbb{R}$ with

$$\|u\|_{L^p_{2\pi}} = \left\{ \int_{-\pi}^{\pi} |u|^p dx \right\}^{1/p} < \infty \text{ if } p < \infty,$$

and essentially bounded if $p = \infty$. Of course, $L^p_{2\pi}$ is a Banach space if $p \geq 1$.

The periodic Hilbert transform of an $L^1_{2\pi}$-function u is defined for almost all $x \in \mathbb{R}$ by the Cauchy principle value integral

$$\mathcal{C}u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(y) \cot\left(\tfrac{1}{2}(x-y)\right) dy. \tag{2.1}$$

It is well known [16, Vol. I, (2.4), page 253] that $\mathcal{C} : L^p_{2\pi} \to L^p_{2\pi}$ is a bounded linear operator if $1 < p < \infty$ (this a theorem of M. Riesz) and that $\mathcal{C}$ does not map $L^p_{2\pi}$ into itself when $p = 1$ and ∞. However [16, Vol. I, (2.6), page 254],

$$\mathcal{C}u \in L^p_{2\pi}, \text{ for all } p \in (0,1) \text{ when } u \in L^1_{2\pi}. \tag{2.2}$$

The Hilbert transform of an infinitely differentiable function is infinitely differentiable.

Let the Fourier coefficients of $u \in L^1_{2\pi}$ be denoted by

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{\phi_k(x)}dx, \text{ where } \phi_k(x) = \exp(ikx), \quad k \in \mathbb{Z}. \tag{2.3}$$

Then $\mathcal{C}$ is a Fourier multiplication operator on $L^p_{2\pi}$, $p > 1$, in the sense that

$$\widehat{\mathcal{C}u}(0) = 0 \text{ and } \widehat{\mathcal{C}u}(k) = -i\text{sgn}(k)\hat{u}(k), \quad k \in \mathbb{Z}\setminus\{0\} \text{ when } u \in L^p_{2\pi},\ p > 1. \tag{2.4}$$

The real Hardy space $\mathcal{H}^1_{\mathbb{R}}$ is the space of functions $u \in L^1_{2\pi}$ for which $\mathcal{C}u$ is also in $L^1_{2\pi}$. It is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{H}^1_{\mathbb{R}}} = \int_{-\pi}^{\pi} |u(x)| + |\mathcal{C}u(x)|dx.$$

Note that (2.4) holds for functions $u \in \mathcal{H}^1_{\mathbb{R}}$. It will be important for us to note that $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u \in L^1_{2\pi}$ and the negative part of $\mathcal{C}u$ is integrable. (See Theorem 5.4 and the Remark which follows its proof.)

Let $L\log^+ L$ denote the set of functions u with $\int_{-\pi}^{\pi} |u|\log^+|u|dx < \infty$, where $\log^+(t) = \max\{0, \log t\}$ for $t > 0$. Results of F. Riesz and Zygmund [16, Vol. I, (2.8) & (2.10), page 254] imply that that if $u \in L^1_{2\pi}$ and $u \geq 0$, then

$$u \in \mathcal{H}^1_{\mathbb{R}} \text{ if and only if } u \in L\log^+ L. \tag{2.5}$$

The fractional order Sobolev space $H^{\frac{1}{2}}$ is the Hilbert space of functions $u \in L^1_{2\pi}$ with norm given by

$$\|u\|^2_{H^{\frac{1}{2}}} = \sum_{k\in\mathbb{Z}} (1 + |k|)|\hat{u}(k)|^2 < \infty. \tag{2.6}$$

From this definition and (2.4) it is clear that $\mathcal{C}$ maps $H^{\frac{1}{2}}$ boundedly into itself. The periodic Slobodecki space $W^{\frac{1}{2}}$ [15] is the set of measurable functions u with Hilbert-space norm given by

$$\|u\|^2_{W^{\frac{1}{2}}} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dxdy + \int_{-\pi}^{\pi}|u(x)|^2dx < \infty. \tag{2.7}$$

It is well known (and we will see a proof in §3) that the spaces $W^{\frac{1}{2}}$ and $H^{\frac{1}{2}}$ are the same and that the infinitely differentiable 2π-periodic functions form a dense subspace of this space.

Lemma 2.1. *Suppose $u \in W^{\frac{1}{2}}$ and $a \in \mathbb{R}$. Then*

$$\liminf_{x\searrow a} u(x) \le \limsup_{y\nearrow a} u(y) \text{ and } \liminf_{y\nearrow a} u(y) \le \limsup_{x\searrow a} u(x).$$

Proof. Suppose that

$$\liminf_{x\searrow a} u(x) > \limsup_{y\nearrow a} u(y)$$

where $u \in W^{\frac{1}{2}}$ and $a \in \mathbb{R}$. Then there exist ϵ, $\delta > 0$ such that

$$u(x) - u(y) \ge \delta \text{ if } a-\epsilon \le y < a < x \le a+\epsilon.$$

Therefore

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dxdy &\ge \int_{a-\epsilon}^{a}\int_{a}^{a+\epsilon}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dxdy\\ &\ge \delta^2\int_{a-\epsilon}^{a}\int_{a}^{a+\epsilon}\left\{\frac{1}{\sin\frac{1}{2}(x-y)}\right\}^2 dxdy = \infty.\end{aligned}$$

This contradicts (2.7) and proves the first inequality. The proof of the second inequality is the same. □

For $1 \le p \le \infty$ the Sobolev space $W^{1,p}_{2\pi}$ is the usual Banach space of functions which along with their weak derivative are in $L^p_{2\pi}$. Here the norm is given by

$$\|u\|^p_{W^{1,p}_{2\pi}} = \|u\|^p_{L^p_{2\pi}} + \|u'\|^p_{L^p_{2\pi}}.$$

The Hardy space $\mathcal{H}^{1,1}_{\mathbb{R}}$ is the subspace of $\mathcal{H}^1_{\mathbb{R}}$ comprised of functions whose weak derivative is in $\mathcal{H}^1_{\mathbb{R}}$. It also is a Banach space with the norm

$$\|u\|_{\mathcal{H}^{1,1}_{\mathbb{R}}} = \|u\|_{\mathcal{H}^1_{\mathbb{R}}} + \|u'\|_{\mathcal{H}^1_{\mathbb{R}}}.$$

Thus

$$W^{1,p}_{2\pi} \subset \mathcal{H}^{1,1}_{\mathbb{R}} \subset L^\infty_{2\pi}, \quad 1 < p \leq \infty, \tag{2.8}$$

but

$$W^{1,1}_{2\pi} \not\subset H^{\frac{1}{2}} \text{ and } \mathcal{H}^{1,1}_{\mathbb{R}} \subset H^{\frac{1}{2}} \not\subset W^{1,1}_{2\pi}. \tag{2.9}$$

The first inclusion in (2.8) is immediate since $\mathcal{C}$ commutes with weak differentiation on $W^{1,p}_{2\pi}$, $p > 1$, and because $\mathcal{C}(L^p_{2\pi}) \subset L^p_{2\pi} \subset L^1_{2\pi}$, $1 < p < \infty$. The second is immediate because every element of $\mathcal{H}^{1,1}_{\mathbb{R}}$ is absolutely continuous. To see (2.9), note first that

$$f(x) = \sum_{k=3}^\infty \frac{\cos kx}{(\log k)^{\frac{1}{2}}}$$

defines an element of $L^1_{2\pi}$ [16, Vol. I, (1.5), page 183]. Since the mean of f is zero its primitive F is 2π-periodic with Fourier series

$$\sum_{k\geq 3} \frac{\sin kx}{k(\log k)^{\frac{1}{2}}}.$$

Therefore $F \in W^{1,1}_{2\pi} \setminus H^{\frac{1}{2}}$, since $\sum_{k\geq 3}[k \log k]^{-1} = \infty$. This gives the first non-inclusion in (2.9). Along the same elementary lines,

$$\sum_{k=3}^\infty \frac{\cos kx}{k \log k}$$

is the Fourier series of an unbounded discontinuous function $g \in L^1_{2\pi}$ [16, Vol. I, (2.3), page 253]. Hence $g \notin W^{1,1}_{2\pi}$. But $\sum_{k\geq 3}[k(\log k)^2]^{-1} < \infty$ means that $g \in H^{\frac{1}{2}} \setminus W^{1,1}_{2\pi}$. (See (2.10) for a stronger statement.) That $\mathcal{H}^{1,1}_{\mathbb{R}} \subset H^{\frac{1}{2}}$ is immediate from (2.6) and a theorem of G. H. Hardy [16, Vol. 1, (8.7), page 286] which says that

$$2\sum_{k=0}^\infty \frac{|\hat{u}(k)|}{k+1} \leq \|u\|_{\mathcal{H}^1_{\mathbb{R}}} \text{ for all } u \in \mathcal{H}^1_{\mathbb{R}}.$$

The following result , which is implicit in general theory, connects membership of w in $\mathcal{H}^{1,1}_{\mathbb{R}}$ with the absolute continuity of $w + i\mathcal{C}w$. Thus it says, in the water-wave context, that the Stokes wave free surface is rectifiable if and only if $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$.

Lemma 2.2. *A function* $w \in \mathcal{H}^1_{\mathbb{R}}$ *belongs to* $\mathcal{H}^{1,1}_{\mathbb{R}}$ *if and only if the weak derivative* $(w + i\mathcal{C}w)'$ *exists in* $L^1_{2\pi} + iL^1_{2\pi}$*. In particular,* $|w + i\mathcal{C}w| \in L^\infty_{2\pi}$ *if* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$*.*

Proof. First suppose that the weak derivative $(w+i\mathcal{C}w)'$ exists in $L^1_{2\pi}+iL^1_{2\pi}$. Then its Fourier series is

$$\sum_{k\in\mathbb{Z}} a_k e^{ikx} \text{ where } a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}(w+i\mathcal{C}w)'\overline{\phi_k}dx = \frac{ik}{2\pi}\int_{-\pi}^{\pi}(w+i\mathcal{C}w)\overline{\phi_k}dx.$$

Therefore $a_k = 0$ if $k \le 0$ since $w \in \mathcal{H}^1_{\mathbb{R}}$. From this observation and an elementary calculation equating real and imaginary parts, there results that the Fourier series S for $(\mathcal{C}w)'$ is the trigonometric series which is conjugate to the Fourier series for w'. Now the fact that $w' \in L^1_{2\pi}$ implies [16, Vol. I, (1.5), page 253] that S is Abel convergent almost everywhere to $\mathcal{C}w'$. On the other hand S, being the Fourier series of $(\mathcal{C}w)' \in L^1_{2\pi}$, is Abel convergent almost everywhere to $(\mathcal{C}w)'$ [16, Vol. I, (7.9)(i), page 101]. Therefore $\mathcal{C}w' = (\mathcal{C}w)' \in L^1_{2\pi}$ when $(w+i\mathcal{C}w)' \in L^1_{2\pi}$

Now suppose that $w' \in \mathcal{H}^1_{\mathbb{R}}$. Then there is a sequence $\{h_n\}$ of smooth functions in $\mathcal{H}^{1,1}_{\mathbb{R}}$ such that $h_n + \mathcal{C}h_n \to w' + i\mathcal{C}w'$ in $L^1_{2\pi}$ as $n \to \infty$. (For example, let $h_n = W_{1/n}$ where $W^* = w' + i\mathcal{C}w'$, as defined in (iii) of the discussion of complex Hardy spaces in §4.) Therefore, for any smooth $\phi \in L^1_{2\pi}$,

$$\begin{aligned}\int_{-\pi}^{\pi}\phi'\mathcal{C}w dx &= -\int_{-\pi}^{\pi} w\mathcal{C}\phi' dx = -\int_{-\pi}^{\pi} w(\mathcal{C}\phi)' w dx\\ &= \int_{-\pi}^{\pi} w'\mathcal{C}\phi = \lim_{n\to\infty}\int_{-\pi}^{\pi} h_n\mathcal{C}\phi dx,\end{aligned}$$

since $\mathcal{C}$ commutes with differentiation on smooth functions,

$$= \lim_{n\to\infty} -\int_{-\pi}^{\pi}\phi\mathcal{C}h_n dx = -\int_{-\pi}^{\pi}\phi\mathcal{C}w' dx.$$

Therefore $(\mathcal{C}w)' = \mathcal{C}w'$ if $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. This completes the proof. □

It has been noted already that a complex-valued function *of the form* $w + i\mathcal{C}w$ is absolutely continuous if and only if it has bounded variation. However $\mathcal{C}u$ need not have bounded variation even when u is absolutely continuous. (An example, due to Lusin, of an absolutely continuous function u with $\mathcal{C}u$ essentially unbounded on every interval of $\mathbb{R}$ is given in [5, Vol. II, §VIII.13, page 94, 95]. The following observation is relevant to the Stokes wave problem because of the relation between $\mathcal{A}$ and $W^{\frac{1}{2}} = H^{\frac{1}{2}}$.

Let $C_{2\pi}$ and BV denote the spaces of real-valued 2π-periodic functions on $\mathbb{R}$ which are, respectively, continuous on $\mathbb{R}$ and of bounded variation on $[-\pi, \pi]$. Since each element u of BV has right and left limits at every point $a \in \mathbb{R}$ it is immediate from Lemma 2.1 that u is continuous on $\mathbb{R}$ if $u \in BV \cap W^{\frac{1}{2}}$. However it follows from [16, Vol. II, (10.12), page 146] (by concatenating functions F of the type described there to make a continuous function on $\mathbb{R}$) that there is a continuous, but not absolutely continuous, 2π-periodic, real-valued function u on $\mathbb{R}$ which has bounded variation on $[-\pi, \pi]$, with $|\hat{u}(k)| = O(|k|^{\epsilon - \frac{3}{2}})$ as $|k| \to \infty$ for any $\epsilon \in (0, 1)$. Such a function is in $H^{\frac{1}{2}}$. Hence, since $W^{\frac{1}{2}} = H^{\frac{1}{2}}$,

$$H^{\frac{1}{2}} \cap BV \subset C_{2\pi} \text{ but } H^{\frac{1}{2}} \cap BV \not\subset W^{1,1}_{2\pi}. \tag{2.10}$$

As usual, $C^{k,\alpha}(I)$ denotes the Banach space of functions the k^{th} derivatives of which are uniformly Hölder continuous with index α on an interval I, and χ_A denotes the characteristic function of a set A.

3 The Quadratic Mapping $\mathcal{Q}$

Before dealing with the regularity question for solutions of the Stokes wave problem we study a quadratic mapping $\mathcal{Q}$, defined in (3.3), which plays a central rôle in the analysis. This mapping $\mathcal{Q}$ has (surprisingly) already appeared in the definition (2.7) of $W^{\frac{1}{2}}$. Here we derive $\mathcal{Q}(u)$ as a formula for the commutator of two linear operators acting on u'. For given $u \in W^{1,1}_{2\pi}$, let $A\phi = u\phi$ for $\phi \in L^1_{2\pi}$. We will see that $\mathcal{Q}(u) = (A\mathcal{C} - \mathcal{C}A)(u')$ and observe that the outcome is pointwise non-negative.

Lemma 3.1. *Suppose that $u \in W^{1,1}_{2\pi}$. Then for almost all $x \in \mathbb{R}$*

$$u(x)\mathcal{C}u'(x) - \mathcal{C}(uu')(x) = \frac{1}{8\pi} \int_{-\pi}^{\pi} \left\{ \frac{u(x) - u(y)}{\sin \frac{1}{2}(x - y)} \right\}^2 dy. \tag{3.1}$$

Hence if $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$ then $uu' \in \mathcal{H}^1_{\mathbb{R}}$.

Proof. Let $x \in \mathbb{R}$ be such that $y \mapsto \int_x^y u'(t)dt$ is differentiable with respect to y at $y = x$. According to Lebesgue's theorem this is a set of full measure in $\mathbb{R}$. For almost all such x

$$\begin{aligned} u(x)\mathcal{C}u'(x) - \mathcal{C}(uu')(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(u(x)-u(y))u'(y)\cot(\tfrac{1}{2}(x-y))dy \\ &= \frac{-1}{4\pi}\int_{\pi}^{\pi}\cot(\tfrac{1}{2}(x-y))\frac{d}{dy}\left(\int_x^y u'(t)dt\right)^2 dy \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dy. \end{aligned}$$

If $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$ it follows that $u\mathcal{C}u' \in L^1_{2\pi}$, and $u \in H^{\frac{1}{2}} = W^{\frac{1}{2}}$ by (2.9). Therefore, by (2.7), $\mathcal{C}(uu') \in L^1_{2\pi}$. Since $uu' \in L^1_{2\pi}$, this proves that $uu' \in \mathcal{H}^1_{\mathbb{R}}$. □

Remark. The identity (3.1) holds everywhere on $\mathbb{R}$ when u is 2π-periodic and infinitely differentiable. Therefore for such functions u integrating both sides of (3.1) and Parseval's identity gives

$$\sum_{k\in\mathbb{Z}}|k||\hat{u}(k)|^2 = \int_{-\pi}^{\pi}\frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dydx.$$

This observation with (2.6) and (2.7) gives $H^{\frac{1}{2}} = W^{\frac{1}{2}}$ (which we already know) and it follows, in addition, that

$$\|u\|^2_{H^{\frac{1}{2}}} = \|\mathcal{Q}(u)\|_{L^1_{2\pi}} + \|u\|_{L^2_{2\pi}} \tag{3.2}$$

where

$$\mathcal{Q}(u)(x) = \frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dy. \quad \square \tag{3.3}$$

Remark. A surprising *pointwise* lower bound for $\mathcal{Q}u(x)$ is evident from the following simple calculation. For almost all $x \in \mathbb{R}$ and $u \in H^{\frac{1}{2}}$,

$$\begin{aligned} \mathcal{Q}(u)(x) &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dy \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\{u(x)-u(y)\}^2 dy + \frac{1}{8\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\tan\frac{1}{2}(x-y)}\right\}^2 dy \\ &\geq \frac{1}{8\pi}\int_{-\pi}^{\pi}\{u(x)-u(y)\}^2 dy + \frac{1}{4}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{u(x)-u(y)}{\tan\frac{1}{2}(x-y)}\right\}dy\right\}^2, \end{aligned}$$

by Jensen's inequality,

$$
\begin{aligned}
&= \frac{1}{8\pi}\int_{-\pi}^{\pi}\{u(x)-u(y)\}^2 dy + \frac{1}{4}(\mathcal{C}u(x))^2 \\
&= \frac{1}{8\pi}\|u\|^2_{L^2_{2\pi}} - \frac{1}{2}[u]u(x) + \frac{1}{4}(u(x)^2 + \mathcal{C}u(x)^2), \qquad (3.4)\\
&\geq \frac{1}{8\pi}(\|u\|^2_{L^2_{2\pi}} - 2\pi[u]^2) + \frac{1}{4}(\mathcal{C}u(x))^2,
\end{aligned}
$$

where $[u]$ denotes the mean value of u on $[-\pi,\pi]$. Therefore, if u is not constant, $\mathcal{Q}(u)$ is bounded below by a positive constant $(1/8\pi)(\|u\|^2_{L^2_{2\pi}} - 2\pi[u]^2)$. From (3.4), for all $u \in H^{\frac{1}{2}}$,

$$
\begin{aligned}
\int_{-\pi}^{\pi}\mathcal{Q}(u)dx &\geq \frac{1}{2}\|u\|^2_{L^2_{2\pi}} - \pi[u]^2 + \frac{1}{4}\int_{-\pi}^{\pi}(\mathcal{C}u(x))^2 dx \\
&= \frac{1}{2}\|u\|^2_{L^2_{2\pi}} - \pi[u]^2 + \frac{1}{4}\{\|u\|^2_{L^2_{2\pi}} - 2\pi[u]^2\} \\
&= \frac{3}{4}\{\|u\|^2_{L^2_{2\pi}} - 2\pi[u]^2\}.
\end{aligned}
$$

Therefore it is clear from the characterisation of $H^{\frac{1}{2}}$-functions in terms of $\mathcal{Q}$ that the product of two bounded $H^{\frac{1}{2}}$-functions is a bounded $H^{\frac{1}{2}}$-function, and the composition of a $H^{\frac{1}{2}}$-function with a Lipschitz mapping on the left is in $H^{\frac{1}{2}}$. □

Note, from (2.5) and the observation that $\mathcal{Q}(u)(x) \in [0,\infty]$ for all 2π-periodic measurable functions u and all $x \in \mathbb{R}$, that

$$\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}} \text{ if and only if } \mathcal{Q}(w) \in L\log^+ L.$$

The relations (2.8) and (2.9) show the sense in which the next observation is sharp.

Lemma 3.2. *Let $u \in W^{1,p}_{2\pi}$, $1 < p < 2$. Then $\mathcal{Q}(u) \in L^{p/(2-p)}_{2\pi}$. Since $1 < p < p/(2-p)$ it follows that $\mathcal{Q}(u) \in L^1_{2\pi}$ whence $u \in H^{\frac{1}{2}}$.*

Proof. Let $u \in W^{1,p}_{2\pi}$. Then an integration by parts, as in the proof of the preceding lemma, gives that for almost all $x \in \mathbb{R}$

$$\begin{aligned}
0 \le \mathcal{Q}(u)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot(\tfrac{1}{2}(x-y))u'(y)\int_y^x u'(t)dt\,dy \\
&\le \frac{\|u'\|_{L^p_{2\pi}}}{2\pi}\int_{-\pi}^{\pi} \frac{|u'(y)||x-y|^{1/q}}{|\tan(\frac{1}{2}(x-y))|}dy, \quad \frac{1}{p}+\frac{1}{q}=1, \\
&\le \frac{\|u'\|_{L^p_{2\pi}}}{\pi}\int_{-\pi}^{\pi} \frac{|u'(y)|}{|x-y|^{1/p}}dy \\
&= \frac{\|u'\|_{L^p_{2\pi}}}{\pi}\int_{\mathbb{R}} \frac{\chi_{[-\pi,\pi]}(y)|u'(y)|}{|x-y|^{1/p}}dy. \qquad (3.5)
\end{aligned}$$

Since $\chi_{[-\pi,\pi]}u' \in L^p_{2\pi}$ it follows from the Hardy-Littlewood-Sobolev inequality [9, page 98] that, as a function of $x \in \mathbb{R}$, the right-hand side of (3.5) is r^{th}-power integrable where $r = p/(2-p)$. Hence $\mathcal{Q}(u) \in L^{p/(2-p)}_{2\pi}$. The result now follows from the preceding discussion. □

4 Complex Function Spaces

Theorem 4.4, which is a consequence of the most elementary part of complex Hardy-space theory [7] or [11, Chapter 17], will be helpful with the regularity theory. Let D denote the unit disc centred at 0 in the complex plane and let T denote its boundary, the unit circle in $\mathbb{C}$. When $f : D \to \mathbb{R}$ is holomorphic on D, let $f_r(t) = f(re^{it})$ for $t \in \mathbb{R}$ and $r \in (0,1)$. Then, for any $p \in (0,\infty]$,

$$||f||_p = \lim_{r\to 1} \|\,|f_r|\,\|_{L^p_{2\pi}} = \sup_{r\in(0,1)} \|\,|f_r|\,\|_{L^p_{2\pi}}$$

is well defined. The Hardy class $\mathcal{H}^p_{\mathbb{C}}$ is the set of all such functions f with $||f||_p < \infty$. Moreover

(i) for any $f \in \mathcal{H}^p_{\mathbb{C}}$, $p \in (0,\infty]$, $f^*(t) = \lim_{r\to 1} f_r(t)$ is well defined for almost all $t \in \mathbb{R}$, $|f^*| \in L^p_{2\pi}$, $\|\,|f^*|\,\|_{L^p_{2\pi}} = ||f||_p$ and $\log|f^*| \in L^1_{2\pi}$;

(ii) every $f \in \mathcal{H}^1_{\mathbb{C}}$ is both the Cauchy and Poisson integrals of f^* on the unit circle;

(iii) $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u + i\mathcal{C}u = U^*$ for some $U \in \mathcal{H}^1_{\mathbb{C}}$;

(iv) if $g : \mathbb{R} \to \mathbb{C}$ has $|g| \in L^1_{2\pi}$ and $\int_{-\pi}^{\pi} g(t)\phi_k(t)dt = 0$ for all $k \in \mathbb{N}$, then $g = G^*$ for some $G \in \mathcal{H}^1_{\mathbb{C}}$.

We will write $g \in \mathrm{Log}$ when $g : \mathbb{R} \to \mathbb{C}$ and $\log|g| \in L^1_{2\pi}$. (When $g \in \mathrm{Log}$ the set where g is zero has measure zero, and $f^* \in \mathrm{Log}$ when $f \in \mathcal{H}^p_{\mathbb{C}}$, $p > 0$, by (i) above.) For $g \in \mathrm{Log}$, an *outer function* $\mathcal{O}(g)$ is defined by

$$\mathcal{O}(g)(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|g|(t)dt\right\}, \quad z \in D.$$

Outer functions have the following properties [11, Theorems 17.16 & 17.7]:

(a) $\mathcal{O}(g)$ is holomorphic, has no zeros in D and $\mathcal{O}(g)(0) \in \mathbb{R}$;

(b) for g_1, $g_2 \in \mathrm{Log}$, $\mathcal{O}(g_1 g_2) = \mathcal{O}(g_1)\mathcal{O}(g_2)$;

(c) for $g \in \mathrm{Log}$ and $p \in (0, \infty]$, $|g| \in L^p_{2\pi}$ if and only if $\mathcal{O}(g) \in \mathcal{H}^p_{\mathbb{C}}$, and then $|(\mathcal{O}(g))^*| = |g|$;

(d) for $f \in \mathcal{H}^p_{\mathbb{C}}$, $p \in (0, \infty]$ and $z \in D$, $|\mathcal{O}(f^*)(z)| \geq |f(z)|$;

Since $f^* \in L^1_{2\pi}$ does not imply that $f \in \mathcal{H}^1_{\mathbb{C}}$, the following observation is useful.

Proposition 4.1. *Suppose that f_1 and f_2 are in $\mathcal{H}^p_{\mathbb{C}}$, $p \in (0, \infty]$, and that $f_3 \in L^1_{2\pi}$, where $f_3(t) = |f_1^*(t) f_2^*(t)|$, $t \in \mathbb{R}$. Then $f_1 f_2 \in \mathcal{H}^1_{\mathbb{C}}$ and $(f_1 f_2)^* = f_1^* f_2^*$.*

Proof. Note first that f_1^*, $f_2^* \in \mathrm{Log}$, since f_1, $f_2 \in \mathcal{H}^p_{\mathbb{C}}$. Hence $f_3 \in \mathrm{Log}$. Let

$$F_k(z) = \mathcal{O}(f_k^*)(z) \quad k = 1,\, 2 \quad \text{and} \quad F_3(z) = \mathcal{O}(f_3)(z), \quad z \in D.$$

Since $|f_1^*|$, $|f_2^*| \in L^p_{2\pi}$ and $f_3 \in L^1_{2\pi}$, it follows by property (c) that $F_k \in \mathcal{H}^p_{\mathbb{C}}$ for $k = 1$, 2 and $F_3 \in \mathcal{H}^1_{\mathbb{C}}$. By properties (b) and (d), therefore

$$|f_1(z) f_2(z)| \leq |F_1(z) F_2(z)| = |F_3(z)|, \quad z \in D.$$

This proves that $f_1 f_2 \in \mathcal{H}^1_{\mathbb{C}}$ since $F_3 \in \mathcal{H}^1_{\mathbb{C}}$. That $(f_1 f_2)^* = f_1^* f_2^*$ follows from the definition of f_k^*. $\square$

Now we can derive (1.4) from (1.1a) using the language of complex Hardy spaces. This gives a proof of a generalised version of [6, Theorem 2.3].

Theorem 4.2. *Suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of* (1.1a) *and*

$$\int_{-\pi}^{\pi} |1 - 2\lambda w|\{(w')^2 + (\mathcal{C}w')^2\}dx < \infty.$$

Then w satisfies (1.4). *In particular, $1 - 2\lambda w > 0$ almost everywhere.*

Consequently, if $w \in W^{1,2}_{2\pi}$ is a solution of (1.1a), *then w satisfies* (1.4) *and $1 - 2\lambda w > 0$ almost everywhere.*

Proof. Let $u = \frac{1}{2}(1 - 2\lambda w)(1 + \mathcal{C}w')$. Then, from Lemma 3.1and (1.1a), $u \in \mathcal{H}^1_{\mathbb{R}}$ and $\mathcal{C}u = \frac{1}{2}(1 - 2\lambda w)w'$, since $u = \frac{1}{2}(1 - \mathcal{C}(w' - 2\lambda ww'))$ by (1.1a). Let U, $W \in \mathcal{H}^1_{\mathbb{C}}$ be such that

$$U^* = u + i\mathcal{C}u \quad \text{and} \quad W^* = w' + i(1 + \mathcal{C}w') \text{ on } T,$$

and let $V(z) = U(z)W(z),\ z \in D$. Then for almost all t,

$$\begin{aligned} U^*W^* &= \tfrac{1}{2}i(1 - 2\lambda w)\{1 + \mathcal{C}w' + iw'\}\{1 + \mathcal{C}w' - iw'\} \\ &= \tfrac{1}{2}i(1 - 2\lambda w)\{(w')^2 + (1 + \mathcal{C}w')^2\}. \end{aligned}$$

It follows from Proposition 4.1 that $V \in \mathcal{H}^1_{\mathbb{C}}$ and $V^* = U^*W^*$ almost everywhere, since $|U^*V^*| \in L^1_{2\pi}$ by hypothesis. Because Real $V^*(t) = 0$ almost everywhere on T, V, being the Poisson integral of V^*, is imaginary, and is therefore an imaginary constant on D, by the Cauchy-Riemann equations. Now

$$V(0) = U(0)W(0) = iU(0) = \frac{i}{2\pi}\int_{-\pi}^{\pi} u(t)dt = \frac{i}{4\pi}\int_{-\pi}^{\pi}(1-2\lambda w)(1+\mathcal{C}w')dt = \frac{i}{2},$$

since $\int_{-\pi}^{\pi} \mathcal{C}w'dt = 0$ and $\int_{-\pi}^{\pi} w(1 + \mathcal{C}w')dt = 0$ by (1.1a). Therefore $V \equiv i/2$ on D and so (1.4) holds on T. This completes the proof. □

Proposition 4.3. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a bounded, non-negative, real-valued function with $g^{-p} \in L^1_{2\pi}$ for some $p > 0$. Then there exists a function $H \in \mathcal{H}^\infty_{\mathbb{C}}$ with $H(z) \neq 0$ for $z \in D$, $H(0) \in \mathbb{R}$, $H^*(t)\overline{H^*(t)} = g(t)$ for almost all $t \in \mathbb{R}$ and $1/H \in \mathcal{H}^{2p}_{2\pi}$.*

Proof. The hypotheses imply that $\sqrt{g} \in$ Log. It suffices therefore to let $H = \mathcal{O}(\sqrt{g})$ on D and to invoke properties (a)-(d) of outer functions to observe that $H \in \mathcal{H}^\infty_{2\pi}$ has no zeros in D, $|H^*|^2 = g$ and $1/H = \mathcal{O}(1/\sqrt{g}) \in \mathcal{H}^{2p}_{2\pi}$. □

The next result would be trivial if u were square-integrable.

Theorem 4.4. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a bounded, non-negative, real-valued function with $g^{-p} \in L^1_{2\pi}$ for some $p > 0$ and that $u \in \mathcal{H}^1_{\mathbb{R}}$ has the property that for every infinitely differentiable, 2π-periodic function ϕ on $\mathbb{R}$*

$$\int_{-\pi}^{\pi} g(t)\{u(t)\phi(t) + \mathcal{C}u(t)\mathcal{C}\phi(t)\}dt = 0.$$

Then $u = 0$.

Proof. Suppose that $\phi = \phi_k$, $k \in \mathbb{Z}$, as defined in (2.3). Then $k = 0$ gives

$$\int_{-\pi}^{\pi} g(t)u(t)dt = 0 \tag{4.1}$$

and from (2.4) it follows that

$$\int_{-\pi}^{\pi} g(t)\{u(t) + i\mathcal{C}u(t)\}\overline{\phi_k(t)}dt = 0 \text{ if } k > 0. \tag{4.2}$$

Let $v : \mathbb{R} \to \mathbb{R}$ be defined by

$$v(t) = g(-t)\{u(-t) + i\mathcal{C}u(-t)), \quad t \in \mathbb{R}.$$

Then $|v| \in L^1_{2\pi}$ since $g \in L^\infty_{2\pi}$ and $u \in \mathcal{H}^1_{\mathbb{R}}$. From (4.1) and (4.2), the Fourier series for v is

$$\sum_{k=0}^{\infty} a_k e^{ikt} \text{ where } a_0 \in i\mathbb{R}.$$

Let

$$V(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in D.$$

Then $V \in \mathcal{H}^1_{\mathbb{C}}$ by property (iv) of Hardy spaces, $V(0) = a_0 \in i\mathbb{R}$ and $V^*(t) = v(t)$. Let $U \in \mathcal{H}^1_{\mathbb{C}}$ be such that $U^* = u + i\mathcal{C}u$. Then $U(0) \in \mathbb{R}$ (since $\mathcal{C}u$ has zero mean) and

$$g(t)U^*(t) = V^*(-t), \quad t \in \mathbb{R}. \tag{4.3}$$

Now let H be the outer function defined in Proposition 4.3 and let K be defined by

$$K(z) = \frac{1}{\overline{H(\overline{z})}}, \quad z \in D.$$

From Proposition 4.3, $K \in H^{2p}_{\mathbb{C}}$,

$$K^*(t) \neq 0 \quad \text{and} \quad g(t) = \frac{H^*(t)}{K^*(-t)} \quad \text{for almost all } t \in \mathbb{R}.$$

Hence (4.3) may be re-written

$$H^*(t)U^*(t) = V^*(-t)K^*(-t) \quad \text{for almost all } t \in \mathbb{R}, \tag{4.4}$$

where both HU and KV are in $\mathcal{H}^1_{\mathbb{C}}$, the former because $U \in \mathcal{H}^1_{\mathbb{C}}$ and $H \in \mathcal{H}^\infty_{\mathbb{C}}$, and the latter by Proposition 4.1, since $V \in \mathcal{H}^1_{\mathbb{C}}$, $K \in \mathcal{H}^{2p}_{2\pi}$, $p > 0$, and $|V^*K^*| = |U^*H^*| \in L^1_{2\pi}$. Let

$$H(z)U(z) = \sum_{k=0}^\infty A_k z^k \text{ and } V(z)K(z) = \sum_{k=0}^\infty B_k z^k, \quad z \in D.$$

By (4.4), $A_k = B_k = 0$ for $k \in \mathbb{N}$ and $A_0 = B_0$. However $A_0 = H(0)U(0) \in \mathbb{R}$ and $B_0 = V(0)K(0) \in i\mathbb{R}$. Therefore $A_k = B_k = 0$ for all $k \in \mathbb{N}$, and so $H^*U^* \equiv 0$. Since $|H^*| > 0$ the result follows. □

5 Regularity Theory for Stokes Waves

Here as elsewhere, $\mathcal{C}w'$ means the conjugate of w'. When any other meaning is intended brackets will be used to show the distinction as, for example, in Lemma 2.2 where the equivalence of $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ and rectifiability of the free surfaces was noted. This equivalence is important for the geometric interpretation of the results of this section, as is the observation [6] that $w \in H^{\frac{1}{2}}$ means that a Stokes wave has finite mass (additional to uniform horizontal flow). Thus the additional mass is also related to $\mathcal{Q}(w)$ since it is given by

$$\int_{-\pi}^{\pi} w\mathcal{C}w'dx = \int_{-\pi}^{\pi} \mathcal{Q}(w)dx.$$

Now we will show that any solution $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of (1.1) is in $W^{1,2}_{2\pi}$ and hence corresponds to a classical Stokes wave. Before doing so, we give an easy independent proof of this result in the special case when $w \in W^{1,p}_{2\pi}$, $p > 1$. Clearly it is sufficient to consider $p \in (1, 2)$.

Lemma 5.1. *Suppose that* $p \in (1, 2)$ *and that* $w \in W^{1,p}_{2\pi}$ *satisfies* (1.1a) *and* (1.1b). *Then* $w \in W^{1,2}_{2\pi}$.

Proof. Because of Lemmas 3.1 and 3.2, equation (1.1a) can be re-written as

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda\{w - w\mathcal{C}w' + \mathcal{C}(ww')\} = \lambda\{w - \mathcal{Q}(w)\} \tag{5.1}$$

where $\mathcal{Q}(w) \in L^{p/(2-p)}_{2\pi}$ when $p \in (1, 2)$. Since w is absolutely continuous and satisfies (1.1b), it follows that $w \in W^{1,p/(2-p)}_{2\pi}$, where $p/(2-p) > p > 1$. A bootstrap argument leads to the required conclusion in a finite number of steps. □

Theorem 5.2. *Suppose that* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *satisfies* (1.1a) *and* (1.1b). *Then* $w \in W^{1,2}_{2\pi}$.

Proof. Suppose that w is as in the hypothesis of the Theorem and let ϕ be any 2π-periodic function which is infinitely differentiable on $\mathbb{R}$. Since, by Lemma 3.1, each term of (1.1a) is integrable on $[-\pi, \pi]$, multiplication by $\mathcal{C}\phi$ and integration gives

$$\begin{aligned}\int_{-\pi}^{\pi} \mathcal{C}\omega'\mathcal{C}\phi dx &= \lambda \int_{-\pi}^{\pi} \{w\mathcal{C}\phi + w\mathcal{C}w'\mathcal{C}\phi + \mathcal{C}(ww')\mathcal{C}\phi\}dx \\ &= \lambda \int_{-\pi}^{\pi} \{w\mathcal{C}\phi + w(\mathcal{C}w'\mathcal{C}\phi + w'\phi)\}dx.\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} \mathcal{C}w'\mathcal{C}\phi dx = \int_{-\pi}^{\pi} w'\phi dx,$$

this may be re-arranged to give

$$\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}w'\mathcal{C}\phi + w'\phi\}dx = \lambda \int_{-\pi}^{\pi} w\mathcal{C}\phi dx \tag{5.2}$$

for all smooth, 2π-periodic functions ϕ. Since w is absolutely continuous and satisfies (1.1b), there exist a and b with $0 < a \leq \frac{1}{2} - \lambda w \leq b < \infty$. Therefore, since $\mathcal{C}$ is bounded on $L^2_{2\pi}$, a real-Hilbert-space inner product is defined on $L^2_{2\pi}$ by

$$(u, v) = \int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}u\mathcal{C}v + uv\}dx, \quad u,\, v \in L^2_{2\pi},$$

and the corresponding norm is obviously equivalent to the original one. Now let a bounded linear functional f on $L^2_{2\pi}$ be defined by

$$f(v) = \lambda \int_{-\pi}^{\pi} w\mathcal{C}v dx, \quad v \in L^2_{2\pi}.$$

Then by the Riesz Representation Theorem there exists $u \in L^2_{2\pi}$ such that for all $v \in L^2_{2\pi}$

$$f(v) = (u, v) = \int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}u\mathcal{C}v + uv\}dx.$$

Hence $w' - u \in \mathcal{H}^1_{\mathbb{R}}$ has the property that for all smooth 2π-periodic functions ϕ

$$\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}(u - w')\mathcal{C}\phi + (u - w')\phi\}dx = 0.$$

It now follows from Theorem 4.4 that $u = w'$. Hence $w \in W^{1,2}_{2\pi}$ and the proof is complete. □

For equation (1.1a) to be meaningful for almost all $x \in \mathbb{R}$ it is sufficient that w be absolutely continuous (equivalently $w' \in L^1_{2\pi}$) since then $\mathcal{C}(ww')$ and $\mathcal{C}w'$ are well-defined pointwise almost everywhere. We have the following result.

Theorem 5.3. *Suppose that $w \in W^{1,1}_{2\pi}$, that* (1.1a) *holds almost everywhere and that* (1.1b) *hold everywhere. Then the following are equivalent:*

(i) $w \in H^{\frac{1}{2}}$ (finite area);

(ii) $w \in W^{1,2}_{2\pi}$ (real-analytic boundary);

(iii) $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ (rectifiable boundary).

Proof. By Lemma 3.1

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda\{w - \mathcal{Q}(w)\}.$$

That $\mathcal{C}w' \in L^1_{2\pi}$ (equivalently $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$) if and only if $\mathcal{Q}(w) \in L^1_{2\pi}$ is immediate and the result follows from (3.2) and Theorem 5.2. □

In the light of the equivalence just noted, the next result is the most general in this section.

Theorem 5.4. *Suppose that $w \in W^{1,1}_{2\pi}$ satisfies* (1.1a) *pointwise almost everywhere. Then $w \in H^{\frac{1}{2}}$.*

Proof. If $w \in W^{1,1}_{2\pi}$ satisfies (1.1a) almost everywhere then

$$\mathcal{C}(w' - 2\lambda w w') = \lambda\{w + w\mathcal{C}w' - \mathcal{C}(ww')\} = \lambda\{w + \mathcal{Q}(w)\} \tag{5.3}$$

holds. Since w is bounded, it follows from (5.3) that $\mathcal{Q}(w) \geq 0$ is the periodic Hilbert transform of an $L^1_{2\pi}$-function and therefore, by (2.2), $q_k \in L^1_{2\pi}$ for each $k \in \mathbb{N}$, where

$$q_k(x) = \begin{cases} \mathcal{Q}(w)(x) & \text{if } 0 \leq \mathcal{Q}(w)(x) \leq 1, \\ \mathcal{Q}(w)(x)^{1-1/k} & \text{otherwise.} \end{cases}$$

We will now prove that

$$\int_{-\pi}^{\pi} \{q_k(x) + w(x)\} dx \leq 0 \text{ for all } k \in \mathbb{N}. \tag{5.4}$$

Since $0 \leq q_k \nearrow \mathcal{Q}(w)$ as $k \to \infty$, it will follow from the Monotone Convergence Theorem that $\mathcal{Q}(w) \in L^1_{2\pi}$ and hence that $w \in H^{\frac{1}{2}}$, as required.

To prove (5.4), recall the properties of integrability-B [16, Vol. I, Ch. VII, §4, page 262]:

(i) if $u \in L^1_{2\pi}$ then u is integrable-B and the two integrals coincide (we write this as $\int_{-\pi}^{\pi} u dx = (B) \int_{-\pi}^{\pi} u dx$);

(ii) if $u \in L^1_{2\pi}$ then $\mathcal{C}u$ is integrable-B and $(B) \int_{-\pi}^{\pi} \mathcal{C}u dx = 0$;

(iii) if u, v are integrable-B and $u \geq v$, then $(B) \int_{-\pi}^{\pi} u dx \geq (B) \int_{-\pi}^{\pi} v dx$.

Now, for each $k \in \mathbb{N}$,

$$\lambda\{w + q_k\} \leq \lambda\{w + \mathcal{Q}(w)\} = \mathcal{C}(w' - 2\lambda w w').$$

Since $w + q_k$ and $w' - 2\lambda w w'$ belong to $L^1_{2\pi}$, it follows from properties (i)-(iii) of integrability-B that

$$\begin{aligned} \int_{-\pi}^{\pi} \lambda\{w + q_k\} dx = (B) \int_{-\pi}^{\pi} \lambda\{w + q_k\} dx &\leq (B) \int_{-\pi}^{\pi} \lambda\{w + \mathcal{Q}(w)\} dx \\ = (B) \int_{-\pi}^{\pi} \mathcal{C}(w' - 2\lambda w w') dx &= 0. \end{aligned}$$

This completes the proof. $\square$

Remark. More generally, for $u \in L^1_{2\pi}$ this proof gives that $\mathcal{C}u \in L^1_{2\pi}$ if and only if the negative part $(\mathcal{C}u)^- \in L^1_{2\pi}$. It then follows from (2.5) that if $u \in L^1_{2\pi}$ and $(\mathcal{C}u)^-$ is bounded below by a constant, then $\mathcal{C}u \in L \log^+ L$. $\square$

6 Stokes Waves in the Sense of Distributions

As was noted in [6], (1.1a) is the Euler-Lagrange of a functional $\mathcal{J} : \mathcal{H}^{1,1}_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} w\mathcal{C}w' - \lambda w^2\{1 + \mathcal{C}w'\}dx. \tag{6.1}$$

In other words critical points $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of $\mathcal{J}$ are solutions of (1.1a) and *vice versa*. In the last section we discussed the smoothness of the boundary of Stokes wave which arise from solutions of (1.1a) in $\mathcal{H}^{1,1}_{\mathbb{R}}$. Now we examine a more delicate but possibly more interesting version of the Stokes-wave problem which arises from considering it in an even more general variational setting. Note first that when $w \in W^{1,p}_{2\pi}$ is smooth

$$\begin{aligned}\int_{-\pi}^{\pi} w^2\mathcal{C}w'dx &= \int_{-\pi}^{\pi} w\,\mathcal{Q}(w)dx + \int_{-\pi}^{\pi} w\mathcal{C}(ww')dx \\ &= \int_{-\pi}^{\pi} w\,\mathcal{Q}(w)dx + \frac{1}{2}\int_{-\pi}^{\pi} w^2\mathcal{C}w'dx.\end{aligned}$$

Therefore, since smooth functions are dense in $\mathcal{H}^{1,1}_{\mathbb{R}}$,

$$\int_{-\pi}^{\pi} \mathcal{C}(ww')dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} w^2\mathcal{C}w'dx = 2\int_{-\pi}^{\pi} w\,\mathcal{Q}(w)dx, \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}},$$

from which it follows that

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \{(1 - 2\lambda w)\mathcal{Q}(w) - \lambda w^2\}dx, \quad w \in \mathcal{H}^{1,1}_{\mathbb{R}}.$$

Note that, when re-written like this, $\mathcal{J}$ is defined on the larger space $Y = H^{\frac{1}{2}} \cap L^{\infty}_{2\pi}$. To be precise let $\mathfrak{J} : Y \to \mathbb{R}$ be defined by

$$\mathfrak{J}(w) = \int_{-\pi}^{\pi} \{(1 - 2\lambda w)\mathcal{Q}(w) - \lambda w^2\}dx, \quad w \in Y. \tag{6.2}$$

Then $\mathfrak{J}$ is a continuous polynomial and so infinitely differentiable on Y.

Remark. Note in the formula for $\mathfrak{J}(w)$ the juxtaposition of $\mathcal{Q}(w) \in L^1_{2\pi}$ and $(1 - 2\lambda w) \in L^{\infty}_{2\pi}$ which is significant for weak continuity in the calculus of variations, since $L^{\infty}_{2\pi}$ is the dual space of $L^1_{2\pi}$. Moreover, if $\{w_n\}$ is a bounded sequence in Y then it has a subsequence $\{w_{n_k}\}$ which converges almost everywhere to a bounded limit w, say. From the definition of $\mathcal{Q}(w_n)$ and Fatou's Lemma it follows, if w_n satisfies (1.1b) for all n, that

$$\liminf \mathfrak{J}(w_n) \geq \mathfrak{J}(w). \quad \square$$

To characterise the critical points of $\mathfrak{J}$ we need the following definition. Let

$$Q(u,v) = \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{(u(x)-u(y))(v(x)-v(y))}{\sin^2(\frac{1}{2}(x-y))} dy\, dx, \quad u,\, v \in H^{\frac{1}{2}}.$$

Note that by first integrating (3.1) and then taking the Fréchet derivative with respect to $u \in \mathcal{H}^{1,1}_{\mathbb{R}}$ we obtain that

$$Q(u,v) = -\int_{-\pi}^{\pi} v' \mathcal{C} u dx$$

for all functions $u,\, v \in \mathcal{H}^{1,1}_{\mathbb{R}}$. By continuity, therefore, if v is smooth,

$$Q(u,v) = -\int_{-\pi}^{\pi} v' \mathcal{C} u \, dx \text{ for all } u \in H^{\frac{1}{2}}. \tag{6.3}$$

Theorem 6.1. *Suppose that*

$$w \in Y \text{ and } \mathfrak{J}'(w) = 0 \in Y^*. \tag{6.4}$$

Then

$$\int_{-\pi}^{\pi} Q(w - \lambda w^2, v) - \lambda v \mathcal{Q}(w) - \lambda v w\} dx = 0 \text{ for all } v \in Y. \tag{6.5}$$

In particular, $\mathcal{C}(w-\lambda w^2)$ is absolutely continuous (that is, the weak derivative of $\mathcal{C}(w-\lambda w^2)$ exists in $L^1_{2\pi}$) and

$$(\mathcal{C}(w - \lambda w^2))' = \lambda(\mathcal{Q}(w) + w). \tag{6.6}$$

Proof. Suppose $\mathfrak{J}'(w) = 0$ in the dual space of Y. Then it follows easily that for all $v \in Y$

$$\int_{-\pi}^{\pi} \{(1-2\lambda w) Q(w,v) - \lambda v \mathcal{Q}(w) - \lambda v w\} dx = 0.$$

Now

$$\int_{-\pi}^{\pi} -2\lambda w Q(w,v)dx$$

$$\begin{aligned}
&= \frac{-\lambda}{4\pi}\int_{-\pi}^{\pi} w(x)\int_{-\pi}^{\pi}\frac{(w(x)-w(y))(v(x)-v(y))}{\sin^2(\frac{1}{2}(x-y))}dy\,dx \\
&= \frac{-\lambda}{8\pi}\int_{-\pi}^{\pi}(w(x)+w(y))\int_{-\pi}^{\pi}\frac{(w(x)-w(y))(v(x)-v(y))}{\sin^2(\frac{1}{2}(x-y))}dy\,dx \\
&= \frac{-\lambda}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{(w^2(x)-w^2(y))(v(x)-v(y))}{\sin^2(\frac{1}{2}(x-y))}dy\,dx \\
&= -\lambda\int_{-\pi}^{\pi} Q(w^2,v)dx.
\end{aligned}$$

This proves (6.5). Now suppose that $v \in Y$ in (6.5) is smooth. Then (6.3) gives that

$$\int_{-\pi}^{\pi}\{v'\mathcal{C}(w-\lambda w^2)+\lambda v\mathcal{Q}(w)+\lambda vw\}dx = 0 \tag{6.7}$$

for all such smooth functions v. But $\mathcal{Q}(w)+w \in L^1_{2\pi}$ since $w \in H^{\frac{1}{2}}$. Therefore $\mathcal{C}(w-\lambda w^2)$ has a weak derivative in $L^1_{2\pi}$, and (6.6) has been established. $\square$

Remark. We do not know if $w \in W^{1,1}_{2\pi}$, or even that $w-\lambda w^2 \in W^{1,1}_{2\pi}$ (recall (2.9) and (2.10)). However we have the following observation. $\square$

Theorem 6.2. *Suppose that $w \in Y$ satisfies* (6.6). *Then $w-\lambda w^2 \in W^{1,1}_{2\pi}$ if and only if $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$, in which case*

$$\mathcal{C}((w-\lambda w^2)') = \lambda(\mathcal{Q}(w)+w). \tag{6.8}$$

Proof. Suppose that $w \in Y$ satisfies (6.6) and $w-\lambda w^2 \in W^{1,1}_{2\pi}$. Then $(w-\lambda w^2)+i(\mathcal{C}(w-\lambda w^2)$ is absolutely continuous. Therefore, by Lemma 2.2,

$$\mathcal{C}((w-\lambda w^2)') = \lambda(\mathcal{Q}(w)+w) \text{ whence } \lambda\mathcal{C}(\mathcal{Q}(w)) = (\lambda w^2-w)'-\lambda\mathcal{C}w \in L^1_{2\pi}.$$

Thus $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$ if $w-\lambda w^2 \in W^{1,1}_{2\pi}$.

Conversely, suppose that $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$, or equivalently $\mathcal{C}(\mathcal{Q}(w)) \in L^1_{2\pi}$. Then there exists a sequence of smooth function $\{h_n\} \in \mathcal{H}^1_{\mathbb{R}}$ such that

$$h_n \to \lambda(\mathcal{Q}(w)+w) \text{ and } \mathcal{C}h_n \to \mathcal{C}(\lambda(\mathcal{Q}(w)+w))$$

in $L^1_{2\pi}$ as $n \to \infty$. From (6.7) it follows that when v is smooth

$$
\begin{aligned}
&-\int_{-\pi}^{\pi} v'(w-\lambda w^2)dx = \int_{-\pi}^{\pi} \lambda(\mathcal{Q}(w)+w)\mathcal{C}v dx\\
&= \lim_{n\to\infty}\int_{-\pi}^{\pi} h_n \mathcal{C}v dx = -\lim_{n\to\infty}\int_{-\pi}^{\pi} v\mathcal{C}h_n dx\\
&= -\int_{-\pi}^{\pi} \lambda\mathcal{C}(\mathcal{Q}(w)+w)v dx.
\end{aligned}
$$

Hence $w - \lambda w^2 \in W^{1,1}_{2\pi}$ and $(w-\lambda w^2)' = -\mathcal{C}(\mathcal{Q}(w)+w)$. Therefore $\mathcal{C}(w' - 2\lambda w w') = \lambda(\mathcal{Q}(w)+w)$. The proof is complete. $\square$

Corollary 6.3. *Suppose that $w \in Y$ satisfies* (6.6) *and* (1.1b). *Then $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ if and only if $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$ which occurs if and only if $\mathcal{Q}(w) \in L\log^+ L$. In these circumstances, w satisfies* (1.1a) *and is real-analytic.*

Proof. Suppose that $w \in Y$ and $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$. That $w \in W^{1,1}_{2\pi}$ follows immediately from Theorem 6.2 and (1.1b). Therefore w satisfies (1.1a) and $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ follows from Theorem 5.3.

Now suppose that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$. Then (6.8) and $\mathcal{Q}(w) \in \mathcal{H}^1_{\mathbb{R}}$ follow by Theorem 6.2, and $w \in W^{1,2}_{2\pi}$ is a consequence of Theorem 5.2. The real-analyticity of w is proved in [6]. Finally recall that a non-negative function $q \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $q \in L\log^+ L$. $\square$

Theorem 6.4. *If $w \in BV \cap H^{\frac{1}{2}}$ is a solution of* (6.6), *then $w - \lambda w^2 \in \mathcal{H}^{1,1}_{\mathbb{R}}$. If, in addition, w satisfies* (1.1b), *w is a real-analytic solution of* (1.1a).

Proof. From (6.6) we know that $\mathcal{C}(w-\lambda w^2)$ is absolutely continuous and hence of bounded variation. Since $w - \lambda w^2$ has bounded variation then it follows ([16, Vol. I, (8.2), page 285] and Lemma 2.2) that $w - \lambda w^2 \in \mathcal{H}^{1,1}_{\mathbb{R}}$.

If, in addition, (1.1b) is satisfied, $w \in W^{1,1}_{2\pi} \cap H^{\frac{1}{2}}$ is a solution of (1.1a). The conclusion of this theorem follows from Theorem 5.3. $\square$

7 The Extreme Wave

Here we give an example of an even solution $w \in W^{1,2}_{2\pi}$ of (1.1a) with

$$
1 - 2\lambda w(0) = 0 \text{ and } 1 - 2\lambda w(t) < 0, \quad t \neq 0 \mod 2\pi; \tag{7.1a}
$$

w real-analytic in a neighbourhood of all points $t \neq 0 \mod 2\pi$; (7.1b)

$$\frac{1-2\lambda w(t)}{(\frac{3}{2}\lambda|t|)^{\frac{2}{3}}} \to 1 \text{ and } \frac{w'(t)}{(\frac{3}{2}\lambda t)^{-\frac{1}{3}}} \to -\frac{1}{2} \text{ as } |t| \to 0. \tag{7.1c}$$

Thus w is not Lipschitz continuous at 0 and hence (1.1b) is essential for the regularity theory of previous sections. This in turn motivates the local regularity discussion of the next section.

The example in mind is easily derived from the Nekrasov-equation theory for the Stokes wave of extreme form [12]. It is well known that there exists an odd function θ such that

$$\theta(s) = \frac{1}{3\pi}\int_0^\pi \log\left|\frac{\sin\frac{1}{2}(s-t)}{\sin\frac{1}{2}(s+t)}\right| \frac{\sin\theta(t)}{\int_0^t \sin\theta(r)dr}dt, \quad s \in [0,\pi], \quad \text{see [13]}; \tag{7.2a}$$

$$\lim_{t\searrow 0}\theta(t) = \pi/6, \quad \text{see [3, 10]}, \quad \text{and } 0 < \theta(t) < \pi/5, \quad \text{see [1]}; \tag{7.2b}$$

θ has a 2π-periodic extension to $\mathbb{R}$ which is

real-analytic except when $t = 0 \mod 2\pi$, see [8]. (7.2c)

Equation (7.2a) is Nekrasov's equation for the Stokes wave of extreme form. In [4, (A2), Appendix, with $1/\mu = 0$] it is shown that if θ satisfies (7.2) and we put

$$\lambda^{\frac{1}{3}} = \frac{1}{\pi}\int_0^\pi \frac{\cos\theta(t)}{(3\int_0^t \sin\theta(s)ds)^{\frac{1}{3}}}dt \text{ and } \Lambda = \int_0^\pi e^{-\mathcal{C}\theta(t)}\cos\theta(t)dt, \tag{7.3}$$

then there exists a Stokes wave of extreme form with wavelength Λ, Froude number λ^{-1} and slope θ (when suitably parametrised). In this formulation the Bernoulli free surface condition, which is guaranteed by Nekrasov's equation has the form (see [4, 14])

$$\frac{\Lambda^2 c^2}{2\pi^2 g}e^{2\mathcal{C}\theta(x)} - \int_0^x e^{-\mathcal{C}\theta(s)}\sin\theta(s)ds = 0, \quad x \in (0,\pi]. \tag{7.4}$$

From differentiating this identity, and integrating again, it follows that

$$\frac{\Lambda^2 c^2}{3\pi^2 g}e^{3\mathcal{C}\theta(t)} = \int_0^t \sin\theta(s)ds. \tag{7.5}$$

Note, from the definition of Λ in (7.3) that, since θ is odd,

$$\exp(-\mathcal{C}\theta + i\theta) - \Lambda/\pi = (e^{-\mathcal{C}\theta}\cos\theta - \Lambda/\pi) + ie^{-\mathcal{C}\theta}\sin\theta \qquad (7.6)$$

has zero mean on $[-\pi,\pi]$ and that $|\exp(-\mathcal{C}\theta + i\theta)| \in L^2_{2\pi}$, by (7.2b) and (7.5). For $x \in [-\pi,\pi]$, let

$$w(x) = \frac{1}{2\lambda} - \frac{\pi}{\Lambda}\int_0^x e^{-\mathcal{C}\theta(t)}\sin\theta(t)dt, \qquad (7.7)$$

so that w is even and by (7.6)

$$\mathcal{C}w(x) = -x + \frac{\pi}{\Lambda}\int_0^x e^{-\mathcal{C}\theta(t)}\cos\theta(t)dt. \qquad (7.8)$$

It now follows immediately that, for $x \in (0,\pi)$,

$$\tan\theta(x) = -\frac{w'(x}{1+\mathcal{C}w'(x)}, \qquad (7.9)$$

$$\begin{aligned}
1 - 2\lambda w(x) &= \frac{2\pi\lambda}{\Lambda}\int_0^x e^{-\mathcal{C}\theta(t)}\sin\theta(t)dt \\
&= \frac{2\pi\lambda}{\Lambda}\left(\frac{\Lambda^2c^2}{3\pi^2 g}\right)^{\frac{1}{3}}\int_0^x \frac{\sin\theta(t)}{(\int_0^t\sin\theta(s)ds)^{\frac{1}{3}}}dt, \text{ by (7.5)}, \\
&= \frac{3\pi\lambda}{\Lambda}\left(\frac{\Lambda^2c^2}{3\pi^2 g}\right)^{\frac{1}{3}}\left(\int_0^x \sin\theta(t)dt\right)^{\frac{2}{3}} \\
&= \left(3\lambda\int_0^x \sin\theta(t)dt\right)^{\frac{2}{3}} \sim \{\tfrac{3}{2}\lambda|x|\}^{\frac{2}{3}} \text{ as } x \to 0, \qquad (7.10)
\end{aligned}$$

and (7.1c) holds.

Since (7.2b), (7.2c), (7.9) and (7.10) hold, and since w is even, it has been shown that w satisfies (7.1), as required. (Further terms in the asymptotic expansion of w may be obtained from (7.10) and the result of [2].) From (7.7), (7.8) and (7.4), we find that

$$(1-2\lambda w)((w')^2+(1+\mathcal{C}w')^2) = \frac{2\pi\lambda}{\Lambda}\left(\int_0^x e^{-\mathcal{C}\theta(t)}\sin\theta(t)dt\right)\left(\frac{\pi}{\Lambda}\right)^2 e^{-2\mathcal{C}\theta(x)} \equiv 1.$$

That w satisfies (1.1a) now follows from the fact that $w \in W^{1,2}_{2\pi}$ and the calculation in [6, §2.4]. Notice that $w \in W^{1,3-\epsilon}_{2\pi}$ for all $\epsilon \in (0,1)$. This completes the example.

8 Local Regularity Theory

This section is concerned with regularity theory for equations (1.1a) and (6.4) when (1.1b) is replaced with the weaker condition

$$1 - 2\lambda w(t) > 0 \text{ for almost all } t \in \mathbb{R} \text{ and } \int_{-\pi}^{\pi} \frac{dt}{1-2\lambda w(t)} < \infty. \tag{8.1}$$

The goal is to explain how to adapt the proofs to give the following result.

Theorem 8.1. *Suppose that $w \in W^{1,1}_{2\pi}$ satisfies that* (1.1a)*, or that $w \in BV \cap H^{\frac{1}{2}}$ satisfies* (6.4)*, and that* (8.1) *holds. Then the Bernoulli condition*

$$(1-2\lambda w(t))\{(w'(t))^2 + (1+\mathcal{C}w'(t))^2\} = 1 \quad \textit{holds almost everywhere.} \tag{8.2}$$

(Note that $w \in W^{1,2}_{2\pi}$ follows from (8.1) *and* (8.2)*.) Suppose also that, for an open interval I and $\sigma > 0$,*

$$1 - 2\lambda w(t) \geq \sigma > 0 \textit{ for almost all } t \in I. \tag{8.3}$$

Then w is equal almost everywhere to a real-analytic function on I.

Thus w gives a weak solution of the Stokes wave problem [6], one which satisfies the Bernoulli condition in the classical sense on the open set where $w \neq 0$. This set has full measure in $[-\pi, \pi]$.

The proof, along the lines of the preceding discussion, relies on local versions of standard theorems which, although they follow by the same arguments as for their more standard counterparts, are not all recorded in the literature. It is however inappropriate repeat all the details here and we confine ourselves to giving a guide to the relevant literature.

Suppose that $w \in L^\infty_{2\pi}$ satisfies (8.1), and let Z denote the set of functions $u \in L^1_{2\pi}$ such that

$$\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{u^2 + (\mathcal{C}u)^2\} dx < \infty.$$

Note, for $u \in Z$, that the Cauchy-Schwarz inequality gives

$$\begin{aligned}&\left(\int_{-\pi}^{\pi} (|u| + |\mathcal{C}u|)dx\right)^2 \\ &\qquad \leq \left(\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)^{-1} dx\right)\left(\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)(|u|^2 + |\mathcal{C}u|^2)dx\right) < \infty,\end{aligned}$$

and hence that $\mathcal{H}^1_{\mathbb{R}} \subset Z$. It follows easily that Z is a real Hilbert space when endowed with the inner product

$$(u,v)_Z = \int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{uv + \mathcal{C}u\mathcal{C}v\}dx.$$

Now suppose that, in addition, $w \in W^{1,1}_{2\pi}$ is a solution of (1.1a). Then

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda\{w - \mathcal{Q}(w)\} \leq \lambda w,$$

since $\lambda \mathcal{Q}(w) \geq 0$. Therefore,

$$\mathcal{C}w' \leq \frac{\lambda w}{1 - 2\lambda w} \in L^1_{2\pi},$$

and so $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$, because of the remark following the proof of Theorem 5.4.

Theorem 8.2. *Suppose that* $w \in W^{1,1}_{2\pi}$ *satisfies* (1.1a) *and* (8.1). *Then* $w \in W^{1,2}_{2\pi}$ *and* w *satisfies* (8.2).

Proof. We have already observed that $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ when $w \in W^{1,1}_{2\pi}$ satisfies (1.1a) and (8.1). Therefore, just as in the proof of Theorem 5.2,

$$\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}w'\mathcal{C}\phi + w'\phi\}dx = \lambda \int_{-\pi}^{\pi} w\mathcal{C}\phi dx$$

for all smooth 2π-periodic functions ϕ. For $v \in Z$ let

$$F(v) = -\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\mathcal{C}v dx.$$

Clearly, since $w \in L^\infty_{2\pi}$, F is a bounded linear functional on Z and, by the Riesz Representation Theorem, there exists $u \in Z$ such that

$$(u,v)_Z = \int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}u\mathcal{C}v + uv\}dx = F(v), \quad v \in Z.$$

However, $F(v) = \lambda \int_{-\pi}^{\pi} w\mathcal{C}v dx$ for $v \in Z$, and therefore

$$\int_{-\pi}^{\pi} (\tfrac{1}{2} - \lambda w)\{\mathcal{C}(w' - u)\mathcal{C}\phi + (w' - u)\phi\}dx = 0$$

for all smooth 2π-periodic functions ϕ.

Since $w' - u \in \mathcal{H}^1_{\mathbb{R}}$ and $0 < (\frac{1}{2} - \lambda w)^{-1} \in L^1_{2\pi}$, Theorem 4.4 yields that $w' = u$. We have shown that every solution $w \in W^{1,1}_{2\pi}$ of (1.1a) and (8.1) belongs to Z. By Theorem 4.2, this is enough to ensure that w satisfies (8.2) and it follows from (8.1) that $w \in W^{1,2}_{2\pi}$. This completes the proof of the theorem. □

Remark. Only solutions of (1.1a) which satisfy (1.1b) are treated in [6], where $w \in W^{1,2}_{2\pi}$ is used to obtain (8.2). However in that case $1 - 2\lambda w \geq \sigma > 0$, and $w \in Z$ is equivalent to $w \in W^{1,2}_{2\pi}$. □

Before proceeding, we record local versions of the theorems of M. Riesz and Privalov.

Lemma 8.3. *Suppose that $u \in L^1_{2\pi}$ and that $[\tilde{a}, \tilde{b}] \subset (a, b) \subset [-\pi, \pi]$.*

(i) If $\alpha \in (0,1)$ and $u \in C^\alpha[a, b]$, then $\mathcal{C}u \in C^\alpha[\tilde{a}, \tilde{b}]$.

(ii) If $p \in (1, \infty)$ and $u \in L^p[a, b]$, then $\mathcal{C}u \in L^p[\tilde{a}, \tilde{b}]$.

Proof. Privalov's theorem [5] says that if $[a, b] = [-\pi, \pi]$, then part (i) holds with $[\tilde{a}, \tilde{b}] = [-\pi, \pi]$. An examination of the proof, [5, Vol. II, pages 99-101], shows that the argument is purely local and that part (i) here follows by exactly the same argument.

To prove (ii), let $f = \chi_{[a,b]}u$ and $g = u - f$. Then $g \equiv 0$ on $[a, b]$, and therefore, by part (i), $\mathcal{C}g \in C^\alpha[\tilde{a}, \tilde{b}] \subset L^p[\tilde{a}, \tilde{b}]$, $\alpha \in (0, 1)$, $p \in (1, \infty)$. Since $f \in L^p_{2\pi}$, it follows from the M. Riesz theorem (see the paragraph following (2.1)) that $\mathcal{C}f \in L^p_{2\pi}$. Thus $\mathcal{C}u = \mathcal{C}f + \mathcal{C}g \in L^p[\tilde{a}, \tilde{b}]$, as required. □

Now suppose that, for $w \in W^{1,1}_{2\pi}$ which satisfies (1.1a) and (8.1), there is an open interval I on which (8.3) holds. Let $\tilde{I}$ be any compact interval contained in I and let $\overline{I} \supset I_1 \supset \cdots \supset I_5 = \tilde{I}$ be compact intervals each of which is contained in the interior of its neighbour on the left in the list.

Then $w' \in Z$ implies that $w \in W^{1,2}_{2\pi}(I)$, and the proof of Lemma 3.3 in [6] (which involves a purely local argument) gives that $\mathcal{Q}(w) \in L^\infty_{2\pi}(I_1)$. Since

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w - \mathcal{Q}(w)), \tag{8.4}$$

it follows that $\mathcal{C}w' \in L^\infty_{2\pi}(I_1)$.

From Lemma 8.3 it follows that $w' \in L^p_{2\pi}(I_2)$ for all $p \in (1, \infty)$. An application of the local argument for Lemma 3.5 in [6] now yields that $\mathcal{Q}(w) \in$

$C^\alpha(I_3)$ for all $\alpha \in (0,1)$. Hence, from (8.4), $\mathcal{C}w' \in C^\alpha(I_3)$ and, by (8.3) and Lemma 8.3, $w' \in C^\alpha(I_4)$. An application of the argument for (a local version of) [6, Theorem 3.6] yields that $w \in C^{1,\alpha}(I_5) = C^{1,\alpha}(\tilde{I})$.

The fact that $w \in C^{1,\alpha}(\tilde{I})$ and (8.2) ensure that the hypotheses of Lewy's Theorem [8] (which is itself a local result) are satisfied on $\tilde{I}$ and gives the real-analyticity of w on the interior of $\tilde{I}$. Since$\tilde{I}$ was an arbitrary compact interval in I, this proves Theorem 8.1 under the hypothesis that $w \in W^{1,1}_{2\pi}$ satisfies (1.1a) and (8.1).

To deal with the case when $w \in BV \cap H^{\frac{1}{2}}$ satisfies (6.4) and (8.1), we need the following technical observation.

Lemma 8.4. *If u is a continuous function with bounded variation on $[-\pi,\pi]$ and u^2 is absolutely continuous, then u is absolutely continuous on $[-\pi,\pi]$.*

Proof. Since the continuous function u has bounded variation on $[-\pi,\pi]$, there exist two continuous, non-decreasing functions f and g on $[-\pi,\pi]$ such that $u = f - g$. Both f and g are differentiable almost everywhere, and the derivatives are non-negative integrable functions defined almost everywhere on $[-\pi,\pi]$.

Let $\{(a_k, b_k) : k \in \mathbb{N}\}$ denote the collection of maximal open intervals the union of which is the set N of points in $(-\pi,\pi)$ at which $u \neq 0$. If $a_k \neq -\pi$ or $b_k \neq \pi$, $u(a_k) = 0$ or $u(b_k) = 0$. For each k, u^2, and hence u, is absolutely continuous on every compact sub-interval of (a_k, b_k). Hence on (a_k, b_k), $u' = f' - g'$ almost everywhere where $f' - g' \in L^1(a_k, b_k)$. Therefore, if ϕ is a smooth function with compact support in $(-\pi,\pi)$ and $k \in \mathbb{N}$,

$$\int_{a_k}^{b_k} u(t)\phi'(t)dt = -\int_{a_k}^{b_k} u'(t)\phi(t)dt = -\int_{a_k}^{b_k} (f'(t) - g'(t))\phi(t)dt.$$

Therefore

$$\begin{aligned}\int_{-\pi}^{\pi} u(t)\phi'(t)dt &= \sum_{k\in\mathbb{N}} \int_{a_k}^{b_k} u(t)\phi'(t)dt \\ &= -\sum_{k\in\mathbb{N}} \int_{a_k}^{b_k} (f'(t) - g'(t))\phi(t)dt = -\int_{-\pi}^{\pi} \chi_N(f'(t) - g'(t))\phi(t)dt.\end{aligned}$$

Since $\chi_N(f' - g')$ is integrable on $[-\pi,\pi]$, u is absolutely continuous on $[-\pi,\pi]$, as required. $\square$

Suppose that $w \in BV \cap H^{\frac{1}{2}}$ satisfies (6.4). Then (6.5) is satisfied and, by Theorem 6.4,

$$(1 - 2\lambda w)^2 = 1 - 4\lambda(w - \lambda w^2) \in \mathcal{H}_{\mathbb{R}}^{1,1}.$$

Since $1 - 2\lambda w$ is continuous, by (2.10), and has bounded variation, w is absolutely continuous by Lemma 8.4. In other words, $w \in W_{2\pi}^{1,1}$ satisfies (1.1a) and (8.1). Thus Theorem 8.1 under the hypothesis $w \in BV \cap H^{\frac{1}{2}}$ is reduced to the first case, and the proof is complete. $\square$

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