On the Symmetry Theory for Stokes Waves
of Finite and Infinite Depth

J.F. Toland [1]

Background

In 1965 Garabedian [5] was among the first to suggest a modern global variational approach to the theory of steady water waves on flows with finite depth. The independent variable was a periodic function representing the boundary of the unknown flow domain and necessary conditions for an extremum gave a stream function which satisfied the correct kinematic and dynamic boundary conditions for water waves in the absence of surface tension. Based on the variational formulation, §3 in [5] considered the symmetry of waves with one crest and one trough per wavelength on each streamline.

For reasons discussed at the end of this note we began a study of [5] with a view to understanding its contribution in the light of recent developments, [1, 8]. In particular, we were interested in the symmetry theory because of the proof in [1] that symmetry-breaking bifurcations do not occur on the primary branch of Stokes waves on infinite depth. What has emerged is that the key use made of symmetrisation and convexity in [5] can be combined with the divergence theorem and Dirichlet's principle for harmonic functions to yield a simple direct proof of the symmetry result, which has a natural extension to the case of infinite depth.

Surprisingly perhaps, although Garabedian's variational principle motivates and underlies the present treatment, it *per se* plays no rôle in the symmetry question here. However, the continuous symmetrisation introduced by Garabedian remains central to the discussion.

Symmetrisation and Dirichlet's Principle

Let $\eta > 0$ be a 2Λ-periodic function on $\mathbb{R}$ which is differentiable with Hölder continuous derivative (that is, $\eta \in C^{1,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$) and let

$$\mathcal{S}(\eta) = \{(x, \eta(x)) : x \in \mathbb{R}\} \quad \text{and} \quad \Omega(\eta) = \{(x, y) : 0 < y < \eta(x), x \in \mathbb{R}\}.$$

Now let $C(\eta)$ denote the set of all infinitely differentiable functions on $\Omega(\eta)$ which are 2Λ-periodic in x and zero in a neighbourhood of the x-axis $\{y = 0\}$ and of $\mathcal{S}(\eta)$. Let

[1] Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, U.K. jft@maths.bath.ac.uk

$H(\eta)$ be the metric completion of $C(\eta)$ with respect to the norm defined by

$$\|u\|^2 = \int_{\Omega_\eta} |\nabla u|^2 dxdy,$$

where Ω_η denotes $\{(x,y) \in \Omega(\eta) : -\Lambda < x < \Lambda\}$. Put

$$\mathcal{A}(\eta) = \{v : \Omega(\eta) \to \mathbb{R} : v(x,y) = u(x,y) + y/\eta(x),\ u \in H(\eta),\ (x,y) \in \Omega(\eta)\}.$$

The elements of $\mathcal{A}(\eta)$ satisfy the boundary conditions $v \equiv 0$ on the x-axis and $v \equiv 1$ on $\mathcal{S}(\eta)$ in the sense of trace. Let

$$D(\eta) = \inf_{v \in \mathcal{A}(\eta)} \int_{\Omega_\eta} |\nabla v|^2 dxdy. \tag{1}$$

It is a familiar observation that this infimum is attained at a unique point $v_\eta \in \mathcal{A}(\eta)$ which satisfies the periodic-Dirichlet boundary value problem:

$$v \in C^{1,\alpha}(\overline{\Omega(\eta)}) \cap C^2(\Omega(\eta)); \tag{2a}$$

$$\Delta v = 0 \text{ on } \Omega(\eta); \tag{2b}$$

$$v(x,y) = v(x+2\Lambda, y), \quad (x,y) \in \Omega(\eta); \tag{2c}$$

$$v \equiv 1 \text{ on } \mathcal{S}(\eta); \tag{2d}$$

$$v \equiv 0 \text{ on } y = 0. \tag{2e}$$

Conversely (2) has a unique solution which is to be found in $\mathcal{A}(\eta)$ and which is the minimiser of $D(\eta)$. This correspondence is known as Dirichlet's Principle [2].

Suppose now that $\eta = \eta_0$ where η_0 has exactly two critical points, a maximiser and a minimiser, in each half-open interval of the form $[b, b+2\Lambda)$, $b \in \mathbb{R}$. Without further loss of generality suppose that $\pm\Lambda$ are local minimisers of η_0 and that $\Lambda^* \in (-\Lambda, \Lambda)$, a maximiser, is the only critical point of η_0 in $(-\Lambda, \Lambda)$. Let $\underline{\eta} = \eta_0(\pm\Lambda)$ and $\overline{\eta} = \eta_0(\Lambda^*)$. Then there are two injective continuous functions $x^\pm : [\underline{\eta}, \overline{\eta}] \to \mathbb{R}$, each of which is locally Hölder continuously differentiable on $(\underline{\eta}, \overline{\eta})$, such that

$$x^+(y) \geq x^-(y), \quad \text{with equality only if } y = \overline{\eta},$$

$$x^{\pm}(\underline{\eta}) = \pm\Lambda, \quad x^{\pm}(\overline{\eta}) = \Lambda^* \text{ and } \eta_0(x^{\pm}(y)) = y, \quad y \in [\underline{\eta}, \overline{\eta}].$$

For any $\theta \in [0,1]$, let

$$x_\theta^{\pm}(y) = (1-\theta)x^{\pm}(y) - \theta x^{\mp}(y), \quad y \in [\underline{\eta}, \overline{\eta}].$$

Note that, for $\theta \in (0,1)$, $x_\theta^+(y) = x_\theta^-(y)$ only if $(1-\theta)(x^+(y) - x^-(y)) = \theta(x^-(y) - x^+(y))$, which is possible only if $y = \overline{\eta}$, irrespective of the value of θ. Note also that $|(\partial/\partial y)x_\theta^{\pm}(y)| < \infty, \quad y \in (\underline{\eta}, \overline{\eta})$, and that $|(\partial/\partial y)x_\theta^{\pm}(y)| \to \infty$ as $y \to \underline{\eta}, \overline{\eta}$.

The curve comprised of the two branches $\{(x_\theta^+(y), y), (x_\theta^-(y), y) : y \in [\underline{\eta}, \overline{\eta}]\}$ can therefore be extended as the graph of a 2Λ-periodic, continuously differentiable function $\eta_\theta : \mathbb{R} \to \mathbb{R}$. It is easily confirmed that $\eta_\theta \in C^{1,\alpha}(\mathbb{R})$ and that in this notation η_1 is the reflection in the y-axis of η_0.

Before proceeding, note from the Maximum Principle that $0 < v_{\eta_0} < 1$ on $\Omega(\eta_0)$ and hence, by Hopf's Boundary Point Lemma, $(\partial/\partial y)v_{\eta_0} > 0$ on the x-axis and on $\mathcal{S}(\eta_0)$. Therefore, again by the Maximum Principle, $(\partial/\partial y)v_{\eta_0} > 0$ on $\Omega(\eta_0)$. It follows, from the Implicit Function Theorem and the hypotheses introduced so far, that each point of $\overline{\Omega(\eta_0)}$ lies on a unique level line of v_{η_0}. Such a level line, $\{(x,y) : v_{\eta_0}(x,y) = \alpha\}$ for $\alpha \in [0,1]$ say, is the graph of a function $\{(x, Y(\alpha, x)) : x \in \mathbb{R}\}$, where Y is 2Λ-periodic in x, infinitely differentiable on $(0,1) \times \mathbb{R}$ and continuous on $[0,1] \times \mathbb{R}$. (Note that in this notation $\eta_0(x) = Y(1,x)$ is a Hölder continuously differentiable function of x.)

Suppose, in addition, that every level line of v_{η_0} (except for the bottom one) has one maximum and one minimum per wavelength. More precisely, for each $\alpha \in (0,1]$ suppose that $\chi^{\pm}$, $\chi^* : (0,1] \to \mathbb{R}$ are continuous functions such that $\chi^{\pm}(1) = \pm\Lambda$, $\chi^*(1) = \Lambda^*$,

$$\chi^+(\alpha) - \chi^-(\alpha) = 2\Lambda, \quad (\partial/\partial x)Y(\alpha, \chi^{\pm}(\alpha)) = 0,$$

and

$$(\partial/\partial x)Y(\alpha, x) = 0 \text{ for } x \in (\chi^-(\alpha), \chi^+(\alpha)) \text{ implies that } x = \chi^*(\alpha).$$

Let $\underline{Y}(\alpha) = Y(\alpha, \chi^-(\alpha)) = Y(\alpha, \chi^+(\alpha)) < Y(\alpha, \chi^*(\alpha)) = \overline{Y}(\alpha)$.

Proposition 1. *There exists a convex function $d : [0,1] \to \mathbb{R}$ such that, with D defined in* (1) *and η_θ defined above,*

$$d(0) = D(\eta_0) = D(\eta_1) = d(1) \quad \textit{and} \quad D(\eta_\theta) \le d(\theta), \quad \theta \in [0,1].$$

Proof. It will suffice to find a family of functions $v^\theta \in \mathcal{A}(\eta_\theta)$, $\theta \in [0,1]$, with

$$v^0 = v_{\eta_0} \text{ and } v^1 = v_{\eta_1}$$

for which v^1 is the reflection in the y-axis of $v^0 = v_{\eta_0}$ and

$$d(\theta) = \int_{\Omega_{\eta_\theta}} |\nabla v^\theta|^2 dxdy$$

is a convex function of $\theta \in [0, 1]$. The result of the proposition with then follow from the definition of $D(\eta_\theta)$. The following construction is in Garabedian [5]. Let $v^0 = v_{\eta_0}$, let

$$\mathcal{R} = \{(\alpha, y) : y \in (\underline{Y}(\alpha), \overline{Y}(\alpha)), \alpha \in (0, 1)\}$$

and let $X^\pm : \mathcal{R} \to \mathbb{R}$ be defined by

$$X^+(\alpha, y) = \max\{x \in [\chi^-(\alpha), \chi^+(\alpha)] : v^0(x, y) = \alpha\}$$

and

$$X^-(\alpha, y) = \min\{x \in [\chi^-(\alpha), \chi^+(\alpha)] : v^0(x, y) = \alpha\}.$$

In other words, $\{(X^\pm(\alpha, y), y) : \ y \in (\underline{Y}(\alpha), \overline{Y}(\alpha))\}$, are the two branches which describe one period of the level set $\{v^0(x, y) = \alpha\}$ as the graph of a function of $y \in [\underline{Y}(\alpha), \overline{Y}(\alpha)]$, where $X^\pm(1, y) = x^\pm(y)$ introduced earlier. Note that $X^\pm(\alpha, \underline{Y}(\alpha)) = \chi^\pm(\alpha)$ and $X^\pm(\alpha, \overline{Y}(\alpha)) = \chi^*(\alpha)$

For $\theta \in [0, 1]$ let

$$X^\pm_\theta(\alpha, y) = (1 - \theta)X^\pm(\alpha, y) - \theta X^\mp(\alpha, y), \quad (\alpha, y) \in \mathcal{R}, \quad \Omega^\pm_\theta = X^\pm_\theta(\mathcal{R}),$$

and let

$$\Omega_\theta = \Omega_{\eta_\theta} = \Omega^+_\theta \cup \Omega^-_\theta \cup \{(\chi^*(\alpha), \overline{Y}(\alpha)) : \alpha \in (0, 1)\}.$$

Then Ω_1 is the reflection of Ω_0 in the y-axis and the mappings from $\mathcal{R}$ to $\Omega^\pm_\theta$ given by $(x, \alpha) \mapsto (X^\pm_\theta(\alpha, y), y)$ are smooth bijections. Now define $v^\theta : \Omega_\theta \to \mathbb{R}$ by

$$v^\theta(x, y) = \alpha \text{ if } (x, y) \in \Omega^+_\theta \cup \Omega^-_\theta \text{ and } x = X^\pm_\theta(\alpha, y),$$

$$v^\theta(\chi^*(\alpha), \overline{Y}(\alpha)) = \alpha.$$

When Ω_θ is identified with its 2Λ-periodic extension in the x-direction, the function v^θ has a continuous 2Λ-periodic extension to the closure of Ω_θ so extended. Moreover, elementary calculus yields that

$$(\partial/\partial x)v^\theta|_{(X^\pm_\theta(\alpha,y),y)} = \frac{1}{(\partial/\partial\alpha)X^\pm_\theta(\alpha, y)}$$

and

$$(\partial/\partial y)v^\theta|_{X_\theta^\pm(\alpha,y),y)} = -\frac{(\partial/\partial y)X_\theta^\pm}{(\partial/\partial\alpha)X_\theta^\pm}(\alpha, y).$$

By the change of variables formula,

$$\int_{\Omega_\theta^\pm} |\nabla v^\theta|^2 dydx = \pm\int_{\mathcal{R}} \frac{1+((\partial/\partial y)X_\theta^\pm)^2}{(\partial/\partial\alpha)X_\theta^\pm} d\alpha dy$$

so that, by periodicity,

$$d(\theta) = \int_{\Omega_{\eta_\theta}} |\nabla v^\theta|^2 dydx = \int_{\mathcal{R}} \left\{ \frac{1+((\partial/\partial y)X_\theta^+)^2}{|(\partial/\partial\alpha)X_\theta^+|} + \frac{1+((\partial/\partial y)X_\theta^-)^2}{|(\partial/\partial\alpha)X_\theta^-|} \right\} d\alpha dy. \quad (3)$$

That

$$D(\eta_0) = d(0) = d(1) = D(\eta_1)$$

is immediate from the construction.

The convexity of d on $[0,1]$ follows because $(s,t) \mapsto (1+t^2)/|s|$ is convex on the half-planes $\{(s,t) \in \mathbb{R}^2 : s > 0\}$ and $\{(s,t) \in \mathbb{R}^2 : s < 0\}$, and because the derivatives with respect to y and α of $X_\theta^\pm$ are one-signed affine functions of θ. (This follows from the geometrical hypothesis that each level line of v_0 has only one maximum and one minimum per wavelength.)

Finally, we need to show that for $\theta \in (0,1)$, $v^\theta \in \mathcal{A}(\eta_\theta)$. Since $\eta_\theta \in C^{1,\alpha}(\mathbb{R})$ and $d(\theta) < \infty$, putting $w(x,y) = v^\theta(x,y) - y/\eta_\theta(x)$ defines a function $w \in H^1_{0,loc}(\Omega_{\eta_\theta})$. Therefore $v^\theta \in \mathcal{A}(\eta_\theta)$ and hence , from the definitions, $D(\eta_\theta) \le d(\theta)$. This completes the proof. $\square$

In connection with the convexity of $(s,t) \mapsto (1+t^2)/|s|$, note in particular that

$$\frac{1+(\frac{1}{2}(t_1+t_2))^2}{\frac{1}{2}|s_1+s_2|} \le \tfrac{1}{2}\frac{1+t_1^2}{|s_1|} + \tfrac{1}{2}\frac{1+t_2^2}{|s_2|}, \quad s_1,\, s_2 > 0, \quad (4)$$

and equality holds if and only if $s_1 = s_2$ and $t_1 = t_2$

Stokes Waves

A regular Stokes wave (on a flow with finite depth over a horizontal bottom) is given by a function $\eta \in C^{1,\alpha}(\mathbb{R})$ and a function v which, in addition to satisfying (2) on $\Omega(\eta)$ also satisfies the constant-pressure condition given by Bernoulli's theorem, namely that

$$|\nabla v| \neq 0 \text{ and } \tfrac{1}{2}|\nabla v|^2 + gy \equiv P \text{ on } \mathcal{S}(\eta), \quad (5)$$

where $P \in \mathbb{R}$ is a constant and $g > 0$ denotes the acceleration due to gravity. The word regular refers to the non-vanishing of ∇v on $\mathcal{S}(\eta)$. The function v is called the stream function and its level lines are streamlines. According to Lewy's Theorem [7], (2a), (2b), (2d) and (5) together ensure that the function η is real-analytic on $\mathbb{R}$ and that the function v has a harmonic (and therefore real-analytic) extension to an open neighbourhood of $\overline{\Omega(\eta)}$ in $\mathbb{R}^2$.

In what follows suppose, with the hypotheses and in the notation of the preceding section, that η_0 and $v_0 = v_{\eta_0}$ together comprise a solution of the Stokes wave problem (2) and (5) and that v_0 has a harmonic extension $\hat{v}$ to $\widehat{\Omega}$ (an open, periodic in x, neighbourhood of Ω_0).

Then let η_θ and Ω_θ, $\theta \in [0,1]$, be the functions and the sets defined earlier, and suppose that $\hat{\theta} \in (0,1]$ is such that $\overline{\Omega_\theta} \subset \widehat{\Omega}$ for all $\theta \in [0,\hat{\theta}]$. Let $\mathcal{S}_\theta$ be the boundary portion of $\partial\Omega_\theta$ given by $\{(x,\eta_\theta(x)) : x \in (-\Lambda,\Lambda)\}$ and let n_θ denote the outward normal to $\partial\Omega_\theta$ at points of $\mathcal{S}_\theta$.

Proposition 2. *There exists a function* $c : [0,\hat{\theta}] \to \mathbb{R}$ *such that* $c(0) = d(0) = D(\eta_0)$*,*

$$c(\theta) \le D(\eta_\theta), \quad \theta \in [0,\hat{\theta}], \quad \text{and} \quad \lim_{\theta \searrow 0} \frac{c(\theta) - c(0)}{\theta} = 0. \tag{6}$$

Proof. Note that from the definition of v_{η_θ} (we do not mean v^θ) and the divergence theorem,

$$\begin{aligned}\left|\int_{\mathcal{S}_\theta} \frac{\partial\hat{v}}{\partial n_\theta} dS\right| &= \left|\int_{\mathcal{S}_\theta} v_{\eta_\theta} \frac{\partial\hat{v}}{\partial n_\theta} dS\right| = \left|\int_{\Omega_\theta} \mathrm{div}(v_{\eta_\theta}\nabla\hat{v})\, dxdy\right| \\ &= \left|\int_{\Omega_\theta} \nabla v_{\eta_\theta}\cdot\nabla\hat{v}\, dxdy\right| \le \sqrt{D(\eta_\theta)}\left\{\int_{\Omega_\theta} |\nabla\hat{v}|^2 dxdy\right\}^{\frac{1}{2}}.\end{aligned}$$

Let

$$c(\theta) = \frac{\left|\int_{\mathcal{S}_\theta} \frac{\partial\hat{v}}{\partial n_\theta} dS\right|^2}{\int_{\Omega_\theta} |\nabla\hat{v}|^2 dxdy}.$$

Clearly $c(0) = d(0) = D(\eta_0)$ and $c(\theta) \le D(\theta)$, $\theta \in [0,\hat{\theta}]$. To complete the proof of (6) we use the the Bernoulli boundary condition (5). Notice first that, by the divergence theorem and the periodicity of $\hat{v}$,

$$\int_{\mathcal{S}_\theta} \frac{\partial\hat{v}}{\partial n_\theta} dS \text{ is independent of } \theta \in [0,\hat{\theta}].$$

Now let $C(\theta) = \int_{\Omega_\theta} |\nabla \hat{v}|^2 dxdy$. Then

$$C(\theta) = \int_0^{\underline{Y}(1)} \int_{-\Lambda}^{\Lambda} |\nabla \hat{v}|^2 dxdy + \int_{\underline{Y}(1)}^{\overline{Y}(1)} \int_{x_\theta^-(y)}^{x_\theta^+(y)} |\nabla \hat{v}|^2 dxdy.$$

Therefore, by the Bernoulli condition,

$$\begin{aligned} \frac{\partial C}{\partial \theta}(0) &= -\int_{\underline{Y}(1)}^{\overline{Y}(1)} (x^+(y) + x^-(y)) |\nabla \hat{v}|^2 \Big|_{x^-(y)}^{x^+(y)} dy \\ &= -2 \int_{\underline{Y}(1)}^{\overline{Y}(1)} (x^+(y) + x^-(y)) \{P - gy\} \Big|_{x^-(y)}^{x^+(y)} dy = 0. \end{aligned}$$

This completes the proof of (6), and the proposition is proven. □

Stokes Wave Symmetry

Now we can follow Garabedian's reasoning to infer the symmetry of Stokes waves with one maximum and one minimum per wavelength on every streamline, as follows.

The two propositions imply that the function $d : [0, 1] \to \mathbb{R}$ is convex, has $d(0) = d(1)$ and $d(\theta) \geq c(\theta)$, $\theta \in [0, \hat{\theta}]$, where $c'(0) = 0$. It follows immediately that $d(\theta)$ is independent of $\theta \in [0, 1]$. In particular, $d(0) = d(\frac{1}{2}) = d(1)$. When these values are substituted into the expression (3) for the convex function d it follows from equality in (4) that

$$(\partial/\partial y) X_0^+(\alpha, y) = (\partial/\partial y) X_1^+(\alpha, y),$$

for all α and y. Therefore

$$(\partial/\partial y)\{X^+(\alpha, y) + X^-(\alpha, y)\} = 0, \quad (\alpha, y) \in \mathcal{R}.$$

Hence $X^+(\alpha, y) + X^-(\alpha, y)$ is independent of y and the wave profile is symmetrical. □

Infinite depth

The problem of Stokes waves on flows of infinite depth can be stated as follows. Let $\eta \in C^{1,\alpha}(\mathbb{R})$ be 2Λ-periodic as before and let $\Omega(\eta) = \{(x, y) : y < \eta(x)\}$. We want a function defined on $\Omega(\eta)$ which satisfies (2)and (5), but with (2e) replaced by the condition at infinite depth that

$$\nabla v(x, y) - (0, q) \to (0, 0) \text{ as } y \to -\infty. \tag{7}$$

Here q is the flow speed of the steady wave at infinite depth.

For each $\eta \in C^{1,\alpha}(\mathbb{R})$, a Dirichlet principle for the fixed boundary value problem (2a)-(2d), (7) may be formulated as follows. Let

$$\mathcal{A}(\eta) = \{v \in W^{1,2}_{loc}(\Omega(\eta)) : v \equiv 1 \text{ on } \mathcal{S}(\eta) \text{ in trace } \int_{\Omega_\eta} |\nabla v - (0,q)|^2 dxdy < \infty\}$$

and put

$$D(\eta) = \inf_{v \in \mathcal{A}(\eta)} \int_{\Omega_\eta} |\nabla v - (0,q)|^2 dxdy. \tag{8}$$

The Dirichlet principle is that $D(\eta)$ is attained at a point $v_\eta \in \mathcal{A}(\eta)$ which is the unique solution of (2a)-(2d), (7). By the Maximum Principle, the convergence of $|\nabla v_\eta(x,y) - (0,q)|$ to 0 as $y \to -\infty$ is exponentially fast and uniform in x. Also there exists a constant Q such that, uniformly in x,

$$v_\eta(x,y) - qy \to Q \text{ as } y \to -\infty. \tag{9}$$

A Stokes wave on flows of infinite depth is therefore a solution (η, v_η) of (2a)-(2d), (7) which, in addition, satisfies the dynamic boundary condition given by (5). The previous symmetrisation construction and convexity argument can be adapted to prove the analogous result on the symmetry of certain Stokes waves on flows of infinite depth as follows.

Let η_0 and $v_0 = v_{\eta_0}$ denote a solution of the Stokes wave problem on infinite depth with one maximum and one minimum per wavelength per streamline, let $\Omega(\eta)$ be defined as in the present section, and let $\hat{v}$ denote a harmonic extension of v_0 to an open neighbourhood $\hat{\Omega}$ of $\Omega_0 = \Omega_{\eta_0}$, where $\Omega_\theta \subset \hat{\Omega}$ for $\theta \in [0, \hat{\theta}]$. For $\theta \in [0,1]$, let v^θ, η_θ, $\mathcal{S}_\theta$, n_θ and $\Omega_\theta = \Omega_{\eta_\theta}$, be defined by analogy with the notation in Proposition 1 and let

$$d(\theta) = \int_{\Omega_\theta} |\nabla v^\theta - (0,q)|^2 dydx \geq D(\eta_\theta).$$

Proposition 1 still holds and it suffices to find a function $c(\theta)$ with the properties described in Proposition 2. We are led to it by the following considerations.

Because $\hat{v}$ is harmonic and periodic, it follows from the definition of v_{η_θ} that

$$0 = \int_{\mathcal{S}_\theta} \frac{\partial}{\partial n_\theta}(\hat{v} - qy) dS = \int_{\mathcal{S}_\theta} v_{\eta_\theta} \frac{\partial}{\partial n_\theta}(\hat{v} - qy) dS,$$

by the divergence theorem,

$$\begin{aligned}
&= \int_{\Omega_\theta} \operatorname{div}(v_{\eta_\theta} \nabla(\hat{v} - qy))\, dxdy = \int_{\Omega_\theta} \nabla v_{\eta_\theta} \cdot \nabla(\hat{v} - qy)\, dxdy \\
&= \int_{\Omega_\theta} \nabla(v_{\eta_\theta} - qy) \cdot \nabla(\hat{v} - qy)\, dxdy + q \int_{\Omega_\theta} (\hat{v} - qy)_y dxdy \\
&= \int_{\Omega_\theta} \nabla(v_{\eta_\theta} - qy) \cdot \nabla(\hat{v} - qy)\, dxdy + q \int_{-\Lambda}^{\Lambda} (\hat{v}(x, \eta_\theta(x)) - q\eta_\theta(x))dx - 2\Lambda q Q,
\end{aligned} \tag{10}$$

by (9). Therefore, by the Cauchy-Schwarz inequality,

$$\left| q \int_{-\Lambda}^{\Lambda} (\hat{v}(x, \eta_\theta(x)) - q\eta_\theta(x))dx - 2\Lambda q Q \right| \leq \left\{ D(\eta_\theta) \int_{\Omega_\theta} |\nabla(\hat{v} - qy)|^2 dydx \right\}^{\frac{1}{2}}.$$

Let

$$f(\theta) = -q \int_{-\Lambda}^{\Lambda} (\hat{v}(x, \eta_\theta(x)) - q\eta_\theta(x))dx + 2\Lambda q Q \tag{11}$$

and

$$\begin{aligned}
g(\theta) &= \int_{\Omega_\theta} |\nabla(\hat{v} - qy)|^2 dxdy = \int_{\Omega_\theta} (|\nabla \hat{v}|^2 - q^2)dxdy - 2q \int_{\Omega_\theta} (\hat{v} - qy)_y dydx \\
&= \int_{\Omega_\theta} (|\nabla \hat{v}|^2 - q^2)dxdy - 2q \int_{-\Lambda}^{\Lambda} (\hat{v}(x, \eta_\theta(x)) - q\eta_\theta(x))dx + 4\Lambda q Q \\
&= \int_{\Omega_\theta} (|\nabla \hat{v}|^2 - q^2)dxdy + 2f(\theta).
\end{aligned} \tag{12}$$

Therefore $c(\theta) \leq D(\eta_\theta)$, where

$$c(\theta) = \frac{f(\theta)^2}{g(\theta)}, \quad \theta \in [0, \hat{\theta}].$$

By (10), $f(0) = g(0) \neq 0$, and it follows that $c'(0) = 2f'(0) - g'(0)$. (That $c'(0)$ exists is straightforward from the quotient rule and the calculation which follows.) To complete the proof we use (12) and observe that, because $\hat{v}$ satisfies the Bernoulli condition (5), it follows, exactly as in the proof of Proposition 2, that

$$\frac{d}{d\theta} \left\{ \int_{\Omega_{\eta_\theta}} (|\nabla \hat{v}|^2 - q^2)dxdy \right\} \Bigg|_{\theta=0} = 0.$$

Thus $2f'(0) = g'(0)$, whence $c'(0) = 0$ and the required symmetry result for Stokes waves on infinitely deep flows follows as before.

Solitary Waves

One might be tempted to try extending these arguments to cover the case of solitary waves. However there is little point since Craig & Sternberg [3] have used the method of moving planes [4, 6] to prove the much stronger result that for *all* solitary waves each streamline has a unique critical point which is a maximiser about which it is symmetrical. A similarly strong statement cannot be made for periodic Stokes waves in general, without some further constraint such as the one about each streamline having one maximum and one minimum per wavelength.

Discussion

Our primary motivation was to understand the contribution of the variational principle for the water wave problem in [5] in the light of [1] and [8]. The simplified argument for symmetry given above arose because of the following considerations.

In [5] a functional M is first defined by equation (2) on page 162 as the conformal modulus of a domain D, but then used in equation (23), page 167, as if it were a functional which depends, not on D alone, but also on a function ψ (which in general does not realise the conformal modulus of D). The variational principle for water waves depends on the former definition, whereas the theory for symmetry hangs on [5, equation (24)], which refers to the latter interpretation. By showing that the symmetry argument may be based on simpler and more direct considerations, it is hoped to remove any possibility of confusion about this point, and to bring the result to a wider audience.

Some further clarification is in order. Garabedian [5, last two sentences of §3] remarks that uniqueness of symmetric waves with given crest heights and trough heights is a corollary of the method which gave symmetry. It seems clear from the proposed proof that this must refer to waves for which *every* streamline (not just the free streamline) has crests and troughs at prescribed heights. From the viewpoint of uniqueness theory the distinction is irrelevant, since either hypothesis is difficult to establish for water waves. If it were correct, then Stokes waves would lie on a two-dimensional surface (parametrised by the crest and trough heights of the free surface), and this would be important. (That remark in [5, §3] may however have been slightly misleading; see, for example, [3, last paragraph, §1].)

While Garabedian's variational principle for water waves on finite depth is not in any question there remain aspects of [5] which lack rigorous justification; the interesting

Section 2 on existence theory using minimax principles is an example. Here are further questions.

Open Questions

1. If the free surface profile has one maximum and one minimum per wavelength, does it follow that every other streamline has similar geometry?

2. Can the methods be adapted to show *the convexity* of the Stokes Wave of Greatest Height, periodic or solitary?

Acknowledgements. I am grateful to my colleague Boris Buffoni for his interest. This work was supported by an EPSRC Senior Fellowship.

References

[1] B. Buffoni, E. N. Dancer, and J. F. Toland, *A variational theory of Stokes waves and their sub-harmonic bifurcations*, Preprint (accessible anonymously via ftp.maths.bath.ac.uk:pub/preprints), 1998.

[2] R. Courant, *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, John Wiley Interscience, New York, 1950.

[3] W. Craig and P. Sternberg, *Symmetry of solitary waves*, Comm. Partial Differential Equations **13 (5)** (1988), 603–633.

[4] L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, (In Press) Cambridge University Press, 1998.

[5] P. R. Garabedian, *Surface waves of finite depth*, J. d'Anal. Math. **14** (1965), 161–169.

[6] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related topics via the maximum principle*, Commun. Math. Phys. **68** (1979), 209–243.

[7] H. Lewy, *A note on harmonic functions and a hydrodynamic application*, Proc. Amer. Math. Soc. **3** (1952), 111–113.

[8] P. I. Plotnikov, *Nonuniqueness of solutions of the problem of solitary waves and bifurcation of critical points of smooth functionals*, Math. USSR Izvestiya **38 (2)** (1992), 333–357.