Boundary Integral Methods for Singularly Perturbed Boundary Value Problems

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Abstract

In this paper we consider boundary integral methods applied to boundary-value problems for the positive definite Helmholtz-type problem $-\triangle U + \alpha^2 U = 0$ in a bounded or unbounded domain, with the parameter α real and possibly large. Applications arise in the implementation of space-time boundary integral methods for the heat equation, where α is proportional to $1/\sqrt{\delta t}$, and δt is the time step. The corresponding layer potentials arising from this problem depend nonlinearly on the parameter α and have kernels which become highly peaked as $\alpha \to \infty$, causing standard discretization schemes to fail. We propose a new collocation method with a robust convergence rate as $\alpha \to \infty$. Numerical experiments on a model problem verify the theoretical results.

Keywords: singular perturbation, boundary integral method, Helmholtz equation, large imaginary wavenumber, heat equation, collocation, trigonometric polynomial.

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1 Introduction

In this paper we discuss the boundary integral method for the homogeneous positive definite problem

$$-\triangle U + \alpha^2 U = 0 \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$, or on $\Omega' = \mathbb{R}^2 \setminus \bar{\Omega}$, subject to Dirichlet or Neumann conditions on Γ, the boundary of Ω. This is the Helmholtz equation

$$\triangle U + \omega^2 U = 0$$

with purely imaginary wavenumber $\omega = i\alpha$. For large α the problem (1.1) is singularly perturbed, and boundary layers will arise in U if the imposed boundary condition is not compatible with the solution $U = 0$ of the reduced problem ((1.1) with $\alpha = \infty$).

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Boundary integral reformulations of (1.1) are well-known. Define the fundamental solution of (1.1) by

$$\Phi_\alpha(\boldsymbol{x},\boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(i\alpha|\boldsymbol{x}-\boldsymbol{y}|), \quad \boldsymbol{x} \neq \boldsymbol{y}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and the corresponding single and double layer potentials by

$$\mathcal{L}_\alpha v(\boldsymbol{x}) = \int_\Gamma \Phi_\alpha(\boldsymbol{x},\boldsymbol{y}) v(\boldsymbol{y}) d\boldsymbol{y}$$

and

$$\mathcal{K}_\alpha v(\boldsymbol{x}) = \int_\Gamma \frac{\partial \Phi_\alpha(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} v(\boldsymbol{y}) d\boldsymbol{y} \tag{1.2}$$

respectively, where $\partial/\partial n(\boldsymbol{y})$ is the derivative with respect to the unit outward normal at $\boldsymbol{y} \in \Gamma$. We remark that Φ_α is real valued for $\alpha \in \mathbb{R}$ (see for example Abramowitz and Stegun (1972)).

Then, for Γ sufficiently smooth, Green's Third Identity asserts that for any solution of (1.1) we have

$$\mathcal{K}_\alpha U(\boldsymbol{x}) - \mathcal{L}_\alpha \frac{\partial U}{\partial n}(\boldsymbol{x}) = \begin{cases} -U(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\ -\frac{1}{2}U(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma \\ 0, & \boldsymbol{x} \in \mathbb{R}^2 \setminus (\Omega \cup \Gamma). \end{cases} \tag{1.3}$$

In this paper we consider only second-kind boundary integral equations arising from (1.1). These are of the form

$$\lambda u + \mathcal{K}_\alpha u = f \qquad \text{on } \Gamma \tag{1.4}$$

with f a given function on Γ and $\lambda \neq 0$ given. This equation arises when (1.1) is supplemented by Neumann data $\frac{\partial U}{\partial n} = g$ on Γ, in which case, using (1.3), we have to solve (1.4) with $\lambda = 1/2$, $u = U|_\Gamma$ and $f = \mathcal{L}_\alpha g$. An analogous equation arises from the exterior Neumann problem for (1.1) but with $\lambda = -1/2$. On the other hand, interior and exterior Dirichlet problems for (1.1) can also be solved using integral equations of the form (1.4). To do this we use the "indirect" representation $U = \mathcal{K}_\alpha u$ on Ω, and then employ the standard jump relations for the double layer potential to obtain (1.4) where now f represents the Dirichlet data, and $\lambda = -1/2$ (for the interior problem) or $\lambda = 1/2$ (for the exterior problem). This background is all well-known, see for example Kress (1989).

Here, we are concerned with the formulation of methods for (1.4) which work well for α large. This is motivated by generic examples arising from the discretisation of parabolic PDEs which are described at the end of this introduction. In order to focus attention on the difficulties which arise when α is large, we study problem (1.4) in the simplest geometric context where the contour Γ is smooth. We emphasise that our ideas could also be used when Γ is non-smooth, but additional considerations for dealing with corner singularities would also be necessary, see for example Graham and Chandler (1988) or Kress (1990). We avoid these here.

When Γ is smooth it is natural to solve (1.4) by a global approximation technique. Such methods are able to exploit the smoothness of the solution, yielding superalgebraic or even exponential convergence rates.

Throughout this paper then, we assume that Γ is parametrised by a $\mathcal{C}^\infty$ 2π-periodic bijection $\boldsymbol{\gamma} : [0, 2\pi] \mapsto \Gamma$ with the property $|\boldsymbol{\gamma}'(t)| \geq C > 0$, for all $t \in [0, 2\pi]$. Equation (1.4) can then be written

$$\lambda u(t) + K_\alpha u(t) = f(t), \quad t \in [0, 2\pi], \tag{1.5}$$

where (with a slight abuse of notation) u and f now denote $u \circ \boldsymbol{\gamma}$ and $f \circ \boldsymbol{\gamma}$ respectively, and

$$K_\alpha u(t) = \int_0^{2\pi} k_\alpha(t, \tau) u(\tau) d\tau, \tag{1.6}$$

where

$$\begin{aligned} k_\alpha(t,\tau) &= \left\{ \frac{\partial}{\partial n(\boldsymbol{y})} \Phi_\alpha(\boldsymbol{x}, \boldsymbol{y}) \bigg|_{\boldsymbol{x}=\gamma(t); \boldsymbol{y}=\gamma(\tau)} \right\} |\boldsymbol{\gamma}'(\tau)| \\ &= \frac{1}{4} \alpha \left| \boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tau) \right| H_1^{(1)}(i\alpha \left| \boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tau) \right|) \left[\frac{(\boldsymbol{\gamma}(\tau) - \boldsymbol{\gamma}(t)).\hat{\boldsymbol{n}}(\boldsymbol{\gamma}(\tau))}{\left| \boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tau) \right|^2} \right] |\boldsymbol{\gamma}'(\tau)|, \end{aligned} \tag{1.7}$$

and where $H_1^{(1)}$ is the Hankel function of the first kind of order one.

We now introduce a class of collocation methods in the $2n$-dimensional space

$$T_n := \operatorname{span}\{\phi_k : k = 0, \ldots, 2n-1\},$$

where

$$\phi_k(t) = \begin{cases} \cos kt, & k = 0, \ldots, n \\ \sin(k-n)t, & k = n+1, \ldots, 2n-1. \end{cases} \tag{1.8}$$

The collocation points will be the equally spaced points in $[0, 2\pi]$:

$$t_j^{(n)} = \frac{j\pi}{n}, \quad j = 0, \ldots, 2n-1.$$

For these points, the corresponding Lagrange interpolating functions in T_n are

$$l_k^{(n)}(t) = \frac{1}{2n} \left\{ 1 + 2 \sum_{m=1}^{n-1} \cos(m(t - t_k^{(n)})) + \cos(n(t - t_k^{(n)})) \right\}, \quad k = 0, \ldots, 2n-1,$$

which satisfy

$$l_k^{(n)}(t_j^{(n)}) = \delta_{jk}, \quad j, k = 0, \ldots, 2n-1.$$

Let $\mathcal{C}$ denote the space of 2π-periodic continuous functions on $\mathbb{R}$. Then the operator $\mathcal{P}_n : \mathcal{C} \mapsto T_n$ defined by

$$\mathcal{P}_n v = \sum_{k=0}^{2n-1} v(t_k^{(n)}) l_k^{(n)} \tag{1.9}$$

is the interpolatory projection, ie. $\mathcal{P}_n v \in T_n$ interpolates v at the points $\{t_j^{(n)}\}_{j=0}^{2n-1}$.

The standard collocation method for (1.5) then seeks a solution

$$u_n = \sum_{k=0}^{2n-1} a_k^{(n)} \phi_k \in T_n,$$

where the coefficients $\{a_k^{(n)}\}_{k=0}^{2n-1}$ are defined by requiring that

$$(\lambda u_n + K_\alpha u_n)(t_j^{(n)}) = f(t_j^{(n)}), \quad j = 0, \ldots, 2n-1. \tag{1.10}$$

Equivalently the $a_k^{(n)}$ are found by solving the linear system

$$\sum_{k=0}^{2n-1} \left\{ \lambda \phi_k(t_j^{(n)}) + K_\alpha \phi_k(t_j^{(n)}) \right\} a_k^{(n)} = f(t_j^{(n)}), \quad j = 0, \ldots, 2n-1. \tag{1.11}$$

This method is only semidiscrete and is not practical until we specify how the integrals

$$K_\alpha \phi_k(t_j^{(n)}) = \int_0^{2\pi} k_\alpha(t_j^{(n)}, \tau) \phi_k(\tau) d\tau, \quad j, k = 0, \ldots, 2n-1 \tag{1.12}$$

should be computed. These are weakly singular integrals, in fact

$$k_\alpha(t, \tau) = O(|t-\tau|^2 \log |t-\tau|), \qquad \text{as } t \to \tau \text{ for fixed } \alpha,$$

and, for fixed α, there is a well-established "product integration" technique for handling this singularity (see Kress (1989)). This involves splitting k_α into the sum

$$k_\alpha(t, \tau) = k_{1,\alpha}(t, \tau) \log 4 \sin^2 \frac{t-\tau}{2} + k_{2,\alpha}(t, \tau) \tag{1.13}$$

with $k_{1,\alpha}$ and $k_{2,\alpha}$ smooth and 2π-periodic. Then the operator K_α can be replaced by the quadrature approximation:

$$(K_{\alpha,m} v)(t) = \int_0^{2\pi} \mathcal{P}_m\{k_{1,\alpha}(t, \cdot) v(\cdot)\}(\tau) \log 4 \sin^2 \frac{t-\tau}{2} d\tau + \int_0^{2\pi} \mathcal{P}_m\{k_{2,\alpha}(t, \cdot) v(\cdot)\}(\tau) d\tau,$$

for some $m \in \mathbb{N}$ which has to be chosen. The computation of the right hand side requires the integrals of each of the basis functions $\phi_k(\tau)$ and their products with $\log 4 \sin^2 \frac{t-\tau}{2}$ over the domain $[0, 2\pi]$, all of which are known analytically (Kress 1989).

Using this in (1.10) we get a discrete collocation method with solution $u_{n,m} \in T_n$ satisfying

$$\lambda u_{n,m}(t_j^{(n)}) + K_{\alpha,m} u_{n,m}(t_j^{(n)}) = f(t_j^{(n)}), \quad j = 0, \ldots, 2n-1. \tag{1.14}$$

For the special choice $m = n$, $u_{n,n}$ is closely related to the Nyström method of Kress (1989), which approximates (1.5) by

$$\lambda \tilde{u}_{n,n}(t) + K_{\alpha,n} \tilde{u}_{n,n}(t) = f(t) \qquad \text{for all } t \in [0, 2\pi] \tag{1.15}$$

and has solution satisfying $\tilde{u}_{n,n}(t_j^{(n)}) = u_{n,n}(t_j^{(n)})$, for each j.

Unfortunately, methods based on the splitting (1.13) degenerate badly as $\alpha \to \infty$. In fact

$$\|u - \tilde{u}_{n,n}\|_{L_2} \le C(\alpha)\frac{1}{n^p}\|u\|_{H^p} \tag{1.16}$$

for all $p > 1/2$, where u is the solution of (1.5), and $\|\cdot\|_{H^p}$ is the usual Sobolev norm, defined by (2.6) below. For fixed α the convergence is superalgebraic with respect to n. However, $C(\alpha)$ grows exponentially in α as $\alpha \to \infty$. This is because $k_\alpha(t,\tau)$ becomes highly peaked near $t = \tau$, and the functions $k_{1,\alpha}(t,\tau)$ and $k_{2,\alpha}(t,\tau)$ in (1.13) have opposite signs but blow up in modulus like $\alpha|\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)|I_1(\alpha|\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)|)$ away from $t = \tau$, where $I_1(z)$ is the modified Bessel function of order one which grows exponentially in z.

The following numerical example demonstrates how the method can break down when α is large. We solve (1.5), with $\lambda = -1/2$ and $f(t) = 1$, with Γ taken to be the unit circle, using the method (1.15). We evaluate $U := \mathcal{K}_\alpha u$ at the point $(0.5, 0)$ for $\alpha = 10$ and $\alpha = 20$. The exact value in each case is $I_0(\alpha/2)/I_0(\alpha)$, which has numerical values $9.7e-3$ for $\alpha = 10$ and $6.5e-5$ for $\alpha = 20$. Table 1 shows the error for several values of n. The superalgebraic convergence rate can be clearly seen for $\alpha = 10$, but for $\alpha = 20$ there is no convergence, and the error is of the same magnitude as the true solution. This may be because n is not large enough to demonstrate the asymptotic convergence rate, or else it may be that arithmetic overflow is destroying the convergence of the method. This and all examples in this paper were done using the MATLAB package on a Sun Ultra 2.

n	error($\alpha = 10$)	error($\alpha = 20$)
8	9.7e-3	6.5e-5
16	1.7e-7	6.5e-5
32	2.0e-10	7.4e-5
64		9.1e-5
128		6.1e-5
256		5.7e-5
512		7.2e-5
1024		6.7e-5

Table 1: Numerical results for method (1.15).

In this paper we describe a new discrete collocation method which avoids the splitting (1.13) and is robust when α is large. This is achieved by using a rescaling technique to transform the integrals into a more benign form. These are then approximated using a variant of the product trapezoidal rule on a suitably graded mesh. The new quadrature rules have fixed rate of convergence $1/m^2$ with respect to the number of quadrature points m, but, crucially, the resulting convergence estimate has an asymptotic constant independent of α. We use these rules to define a new quadrature approximation of $K_\alpha v$. We again denote this by $K_{\alpha,m}v$ and the new discrete collocation solution is again given by (1.14). We prove in Section 5 that, for this new method, $u_{n,m}$ satisfies

$$\|u - u_{n,m}\|_{L_2} \le C\left\{\frac{1}{n^p}\|u\|_{H^p} + \frac{1}{m^2}\|u\|_{L_2}\right\} \tag{1.17}$$

for all $p > 1/2$. Here C is independent of α, n, m and u, ie. the method is robust to large α, and algebraically convergent with respect to m but still superalgebraically convergent with respect to n. Higher order quadrature rules may also be applied to the rescaled integrals, with the result that the $1/m^2$ term in (1.17) will be replaced by $1/m^q$ with $q > 2$. To avoid a lengthy exposition we do not give this generalisation here.

As mentioned above, one motivation for considering (1.1) with α large comes from the implementation of boundary integral methods for parabolic PDEs such as the heat equation

$$\frac{\partial u}{\partial t} = c \bigtriangleup u \quad \text{in } \Omega \times (0, \infty), \tag{1.18}$$

or in $\Omega' \times (0, \infty)$. When Ω is complicated it is of practical interest to consider discretisations defined on the cylindrical boundary $\Gamma \times (0, \infty)$ and there is a considerable literature on the derivation of space-time layer potentials for such problems (eg. Arnold and Noon (1989), Costabel (1990), Lubich and Schneider (1992), Hsaio and Saranen (1993)). These boundary integral equations may be discretised in time by linear multistep methods applied to the time dependent part of the integral equation (Lubich and Schneider 1992), leading to an infinite triangular system of spatial boundary integral equations for which the execution of each time step requires the solution of a single boundary integral equation of the form (1.5) (or its first kind analogue involving the operator $\mathcal{L}_\alpha$), with α proportional to $\sqrt{1/\delta t}$, where δt is the timestep.

A transparent exposition of this technique in the special case of the backward Euler method is given in Chapko and Kress (1997). There, the time discretisation of (1.18) is considered first (this is "Rothe's method"), leading to the coupled system of Helmholtz problems:

$$- \bigtriangleup u_N + \frac{1}{c\delta t} u_N = \frac{1}{c\delta t} u_{N-1} \quad \text{in } \Omega,\ N \geq 0. \tag{1.19}$$

A fundamental solution of this system is derived in Chapko and Kress (1997), which allows its reduction to an infinite sequence of boundary integral equations of the type described above.

In Lubich and Schneider (1992) and Chapko and Kress (1997) the problem of implementation of quadrature methods for (1.5) which are robust as $\alpha \to \infty$ (ie. $\delta t \to 0$) is not addressed, although this problem is identified as a difficulty in Chapko and Kress (1997). Therefore the results of this paper have direct relevance in the implementation of integral equation methods for parabolic problems.

It is worth noting that in the engineering literature there is also considerable interest in "boundary-domain integral" methods for discretisations of parabolic problems. For example in Ramesh and Lean (1991) each equation in the system (1.19) is solved using such a method. There it is proposed to compute the boundary integrals for large α using series expansions. However, an analysis of this method is not given, and the number of terms to be summed in the expansions will grow rapidly as $\delta t \to 0$.

The layout of this paper is as follows. In Section 2 we state and prove some key properties of the double layer operator K_α which will be required in the error analysis. In Section 3 we present an error analysis for the (semi-discrete) collocation method which demonstrates that the rate of convergence of this method is independent of α. In Section 4 we explain how we evaluate the integrals resulting from the collocation method, using a non-standard quadrature scheme, and then in Section 5 we present a fully discrete error analysis for the method, obtaining the estimate (1.17). Finally in Section 6 we provide some numerical examples to demonstrate the theoretical properties of our method.

2 Properties of the double layer potential

In this section we will use without proof various well known properties of modified Bessel functions. The reference for all of these is Abramowitz and Stegun (1972).

We shall use these to derive some key properties of the double layer potential (1.2), which will be required for the error analysis of our method. Recall the parametrised form of the double layer potential (1.6), (1.7) and define

$$\begin{aligned}\Psi(x) &= xH_1^{(1)}(ix)\\ r(t,\tau) &= |\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)|\\ w(t,\tau) &= \frac{1}{4}\left[\frac{(\boldsymbol{\gamma}(\tau)-\boldsymbol{\gamma}(t)).\hat{\boldsymbol{n}}(\boldsymbol{\gamma}(\tau))}{|\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)|^2}\right]|\boldsymbol{\gamma}'(\tau)|.\end{aligned}$$

Then we can write the kernel $k_\alpha(t,\tau)$ as

$$k_\alpha(t,\tau) = \Psi(\alpha r(t,\tau))w(t,\tau). \tag{2.1}$$

Note that

$$w(t,\tau) = \frac{\pi}{2}\frac{\partial}{\partial n(\boldsymbol{y})}\left\{\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}|\right\}\Big|_{\boldsymbol{x}=\gamma(t);\boldsymbol{y}=\gamma(\tau)}.$$

Hence w is a constant multiple of the kernel of the double layer potential arising from Laplace's equation. We shall assume throughout that

$$\boldsymbol{\gamma} \in \mathcal{C}^\infty[0,2\pi], \tag{2.2}$$

from which it can be shown using a Taylor series argument that

$$w \in \mathcal{C}^\infty([0,2\pi]\times[0,2\pi]). \tag{2.3}$$

We shall now state and prove some key properties of the functions Ψ and r which we shall require later on. Throughout the rest of this paper, various constants whose values may vary from line to line, but which are always independent of α, n and m will appear. These are labelled C, C_1, C_2, etc..

Proposition 2.1 *There exists a constant C such that*

$$\left|\frac{\partial}{\partial t}r(t,\tau)\right| \leq C$$

for all $t,\tau\in[0,2\pi]$.

Proof: Using elementary calculus and the Cauchy-Schwarz inequality we have

$$\left|\frac{\partial}{\partial t}r(t,\tau)\right| = \left|\frac{(\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)).\boldsymbol{\gamma}'(t)}{|\boldsymbol{\gamma}(t)-\boldsymbol{\gamma}(\tau)|}\right| \leq |\boldsymbol{\gamma}'(t)|, \tag{2.4}$$

for all $t,\tau\in[0,2\pi]$, and (2.2) ensures the required result. □

The next two propositions can be deduced from standard references on Bessel functions, eg. Abramowitz and Stegun (1972).

Proposition 2.2 *(i)* $|\Psi(x)|$ *is decreasing on* $[0,\infty)$.
(ii) For all $\rho \in [0,1)$, *there exists a constant* $C = C(\rho)$ *such that* $|\Psi(x)| \leq Ce^{-\rho x}$, $x \in [0,\infty)$.

Proposition 2.3 *(i) There exists a constant* C *such that* $|\Psi'(x)|$ *is decreasing on* $[C,\infty)$.
(ii) For all $\rho \in (0,1)$, *there exists a constant* $C = C(\rho)$ *such that* $|\Psi'(x)| \leq Ce^{-\rho x}$, $x \in [0,\infty)$.

Proposition 2.4 *For all* $t \in [0,2\pi]$, $\tau \in [t-\pi, t+\pi]$, *there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1|t-\tau| \leq r(t,\tau) \leq C_2|t-\tau|. \tag{2.5}$$

Proof: The right hand inequality follows immediately from (2.2). To prove the left hand inequality, set $\Upsilon(t,\tau) = r(t,\tau)/|t-\tau|$, $t \neq \tau$. By (2.2) $\lim_{t\to\tau}\Upsilon(t,\tau) = |\boldsymbol{\gamma}'(t)| > 0$, and so $\Upsilon(t,\tau)$ is positive valued everywhere on $\mathbb{R}\times\mathbb{R}$, and hence the result. □

We now need to define some normed spaces. For any interval $J \subseteq \mathbb{R}$, denote the usual L_p norm on J by $\|\cdot\|_{L_p(J)}$. When $J = [0,2\pi]$ we will omit the J from the notation. For $s \geq 0$, let $H^s = H^s[0,2\pi]$ denote the space of 2π-periodic functions $v : \mathbb{R} \mapsto \mathbb{C}$ equipped with the norm

$$\|v\|^2_{H^s} := \sum_{m=-\infty}^{\infty} (1+m^2)^s |\hat{v}_m|^2 < \infty \tag{2.6}$$

where

$$\hat{v}_m = \frac{1}{2\pi}\int_0^{2\pi} v(t)e^{-imt}dt, \quad m = 0, \pm 1, \pm 2, \ldots$$

are the Fourier coefficients of v. For a detailed introduction to these Sobolev spaces, see for example Kress (1989).

Before we can establish some bounds on the operator K_α in these spaces, we need the following classical result, see Prössdorf (1978, p.86) and the references therein.

Theorem 2.5 *Let* $\theta \in L_1(\mathbb{R})$. *Then for all* p *with* $1 \leq p \leq \infty$, *the operator*

$$(\Theta\phi)(t) := \int_{-\infty}^{\infty} \theta(t-s)\phi(s)ds \tag{2.7}$$

is bounded on $L_p(\mathbb{R})$ *and*

$$\|\Theta\|_{L_p(\mathbb{R})} \leq \|\theta\|_{L_1(\mathbb{R})}.$$

Remark: Note that the case $p = 2$ can be easily obtained by the Convolution Theorem and Parseval's equality.

We shall now use these results to establish some bounds on the operator K_α defined by (1.6).

Lemma 2.6 *There exists a constant* C *such that, for all* $\alpha \geq 1$,

$$\|K_\alpha\|_{L_2} \leq \frac{C}{\alpha}.$$

Proof: Recall that $\|\cdot\|_{L_2}$ denotes the norm over $[0,2\pi]$. By periodicity, and (2.1) we can write

$$(K_\alpha v)(t) = \int_{t-\pi}^{t+\pi} \Psi(\alpha r(t,\tau)) w(t,\tau) v(\tau) d\tau.$$

Hence, using (2.3), and Propositions 2.2(ii) and 2.4 we have

$$\begin{aligned}
|(K_\alpha v(t))| &\leq C \int_{t-\pi}^{t+\pi} |\Psi(\alpha r(t,\tau))||v(\tau)| d\tau \leq C \int_{t-\pi}^{t+\pi} e^{-\rho\alpha r(t,\tau)} |v(\tau)| d\tau \\
&\leq C \int_{t-\pi}^{t+\pi} e^{-\rho\alpha C_1 |t-\tau|} |v(\tau)| d\tau.
\end{aligned}$$

Since $t \in [0,2\pi]$ we then have

$$|(K_\alpha v)(t)| \leq C \int_{-\pi}^{3\pi} e^{-\rho\alpha C_1|t-\tau|} |v(\tau)| d\tau = C \int_{-\infty}^{\infty} e^{-\rho\alpha C_1 |t-\tau|} |v_e(\tau)| d\tau := (\Theta_\alpha(|v_e|))(t)$$

where the function v_e is defined to be

$$v_e(\tau) = \begin{cases} v(\tau), & \tau \in [-\pi, 3\pi] \\ 0 & \tau \in [-\infty,\infty]/[-\pi,3\pi]. \end{cases}$$

and Θ_α is of the form (2.7) with $\theta_\alpha(x) = e^{-\rho\alpha C_1|x|}$, $x \in \mathbb{R}$. Thus,

$$\begin{aligned}
\|K_\alpha v\|_{L_2} &= \left\{ \int_0^{2\pi} |(K_\alpha v)(t)|^2 dt \right\}^{\frac{1}{2}} \leq C \left\{ \int_0^{2\pi} |(\Theta_\alpha(|v_e|))(t)|^2 dt \right\}^{\frac{1}{2}} \\
&\leq C \left\{ \int_{-\infty}^{\infty} |(\Theta_\alpha(|v_e|))(t)|^2 dt \right\}^{\frac{1}{2}} = C \|(\Theta_\alpha(|v_e|))\|_{L_2(\mathbb{R})} \\
&\leq C \|\theta_\alpha\|_{L_1(\mathbb{R})} \|v_e\|_{L_2(\mathbb{R})} \leq C \int_{-\infty}^{\infty} e^{-\rho\alpha C_1|x|} dx \|v_e\|_{L_2(\mathbb{R})},
\end{aligned}$$

where the last inequality follows from Theorem 2.5. Thus, by the periodicity of v,

$$\|K_\alpha v\|_{L_2} \leq \sqrt{2} C \int_0^{\infty} e^{-\rho\alpha C_1 x} dx \|v\|_{L_2} \leq C \frac{1}{\rho\alpha C_1} \|v\|_{L_2}$$

and hence the result. $\square$

Lemma 2.7 *There exists a constant C such that for all $\alpha \geq 1$,*

$$\|K_\alpha\|_{L_2 \to H^1} \leq C.$$

Proof: First note that $\|v\|_{H^1}^2 = \|v\|_{L_2}^2 + \|v'\|_{L_2}^2$. Now,

$$\begin{aligned}
\frac{d}{dt}(K_\alpha v)(t) &= \int_0^{2\pi} \left\{ \frac{\partial}{\partial t} k_\alpha(t,\tau) \right\} v(\tau) d\tau \\
&= \int_0^{2\pi} \alpha \left\{ \frac{\partial}{\partial t} r(t,\tau) \right\} \Psi'(\alpha r(t,\tau)) w(t,\tau) v(\tau) d\tau + \int_0^{2\pi} \Psi(\alpha r(t,\tau)) \frac{\partial}{\partial t} w(t,\tau) v(\tau) d\tau \\
&=: I_{1,\alpha} v(t) + I_{2,\alpha} v(t).
\end{aligned}$$

Then, by periodicity and (2.3) we get

$$|I_{2,\alpha}v(t)| \le C \int_{t-\pi}^{t+\pi} |\Psi(\alpha r(t,\tau))||v(\tau)|d\tau$$

and using Proposition 2.1 as well, we also get

$$|I_{1,\alpha}v(t)| \le C\alpha \int_{t-\pi}^{t+\pi} |\Psi'(\alpha r(t,\tau))||v(\tau)|d\tau.$$

Then recalling Propositions 2.2(ii) and 2.3(ii), and using exactly the same argument as for Lemma 2.6 we get:-

$$\|I_{2,\alpha}v\|_{L_2} \le \frac{C}{\alpha}\|v\|_{L_2}$$

and

$$\|I_{1,\alpha}v\|_{L_2} \le C\|v\|_{L_2}.$$

Hence, for all $\alpha \ge 1$

$$\left\|\frac{d}{dt}K_\alpha v\right\|_{L_2} \le C\|v\|_{L_2}$$

and hence the result. □

We remark that by operator interpolation theory, Lemmas 2.6 and 2.7, can be generalised to obtain

$$\|K_\alpha\|_{L_2\to H^\lambda} \le \|K_\alpha\|_{L_2\to L_2}^{1-\lambda}\|K_\alpha\|_{L_2\to H^1}^{\lambda} \le C\left(\frac{1}{\alpha}\right)^{1-\lambda},$$

for all $\lambda \in [0,1]$. However this more general result is not needed for the error analysis of the next section.

3 Error analysis for the collocation method

In this section we present an error analysis for the collocation method (1.10) without numerical integration, and we demonstrate the robustness of its convergence rate with respect to α. This result is then needed in Section 5 to obtain the analogous robustness of the fully discrete method. The main result of this section is as follows:-

Theorem 3.1 *Assuming the integrals $K_\alpha\phi_k$ in (1.11) are known exactly, then for n sufficiently large, the collocation equations (1.10) have a unique solution $u_n \in T_n$, and we have the following error estimate for all $p > 1/2$, where u is the solution of (1.5) and C is independent of α and p:*

$$\|u - u_n\|_{L_2} \le C\frac{1}{n^p}\|u\|_{H^p}.$$

Proof: Observe that by the definition (1.9) of $\mathcal{P}_n$, u_n satisfies

$$(\lambda I + \mathcal{P}_n K_\alpha) u_n(t) = \mathcal{P}_n f(t).$$

So, if $(\lambda I + \mathcal{P}_n K_\alpha)^{-1}$ exists, then standard arguments show

$$u - u_n = u - (\lambda I + \mathcal{P}_n K_\alpha)^{-1} \mathcal{P}_n f = \lambda(\lambda I + \mathcal{P}_n K_\alpha)^{-1}(u - \mathcal{P}_n u).$$

From Kress and Sloan (1993, p.202) we have the following estimate for the approximation power of $\mathcal{P}_n$:

$$\|v - \mathcal{P}_n v\|_{H^q} \leq C \frac{1}{n^{p-q}} \|v\|_{H^p}, \quad 0 \leq q \leq p, \quad \frac{1}{2} < p. \tag{3.1}$$

Hence

$$\|u - u_n\|_{L_2} \leq C \|(\lambda I + \mathcal{P}_n K_\alpha)^{-1}\|_{L_2} \frac{1}{n^p} \|u\|_{H^p},$$

for all $p > 1/2$, and the result follows provided we show that, for n sufficiently large, $(\lambda I + \mathcal{P}_n K_\alpha)^{-1}$ exists, and $\|(\lambda I + \mathcal{P}_n K_\alpha)^{-1}\|_{L_2}$ is bounded by a constant independently of α and n. This result will be proved in Theorem 3.4 below. □

First we prove two preliminary results.

Lemma 3.2 *There exists a constant C independent of n and α such that*

$$\|(I - \mathcal{P}_n) K_\alpha\|_{L_2} \leq C n^{-1}$$

Proof: Let $v \in L_2$. Then by (3.1) and Lemma 2.7, we have

$$\|(I - \mathcal{P}_n) K_\alpha v\|_{L_2} \leq C n^{-1} \|K_\alpha v\|_{H^1} \leq C n^{-1} \|v\|_{L_2}$$

with C independent of α, and the result follows. □

Lemma 3.3 *For all α sufficiently large, $(\lambda I + K_\alpha)^{-1}$ exists, and $\|(\lambda I + K_\alpha)^{-1}\|_{L_2}$ is bounded independently of α.*

Proof: Recall that, from Lemma 2.6, there exists a constant C independent of n and α such that $\|K_\alpha\|_{L_2} \leq C/\alpha$. Hence, for α sufficiently large (specifically $\alpha > C/\lambda$), $\|K_\alpha\|_{L_2} < \lambda$. Hence by Banach's lemma, $(\lambda I + K_\alpha)^{-1}$ exists as a bounded linear operator from L_2 onto L_2, and

$$\|(\lambda I + K_\alpha)^{-1}\|_{L_2} \leq \frac{1}{\lambda - \|K_\alpha\|_{L_2}} \leq \frac{1}{\lambda - C/\alpha} \leq C',$$

provided $\alpha \geq C/(\lambda - (C')^{-1})$. □

Remark: This proof requires α to be sufficiently large. In fact, the result can be shown to be true for all $\alpha \geq 1$, but the proof of this is somewhat more complicated and we avoid it here. Details will be in Langdon (To appear 1998).

We are now in a position to complete the proof of Theorem 3.1.

Theorem 3.4 *For n and α sufficiently large $(\lambda I + \mathcal{P}_n K_\alpha)^{-1}$ exists, and there exists a constant C independent of n and α such that*

$$\|(\lambda I + \mathcal{P}_n K_\alpha)^{-1}\|_{L_2 \mapsto L_2} \leq C.$$

Proof: Since by Lemma 3.2,

$$\|(\lambda I - K_\alpha) - (\lambda I - \mathcal{P}_n K_\alpha)\|_{L_2} = \|(I - \mathcal{P}_n) K_\alpha\|_{L_2} \to 0$$

uniformly in α, the result follows from Lemma 3.3 and standard operator perturbation theory, see for example Atkinson (1976). □

4 Evaluating the integrals

In this section we address the question of how to evaluate the matrix entries in (1.11). This requires computation of integrals of the form

$$(K_\alpha v)(t) = \int_0^{2\pi} \Psi(\alpha r(t,\tau)) w(t,\tau) v(\tau) d\tau, \tag{4.1}$$

where

$$v(\tau) \in T_n = \text{span}[\{\cos(j\tau), \sin(j\tau), j = 1, \dots, n-1\} \cup \{1, \cos(n\tau)\}]$$

and t is one of the node points $t_j^{(n)}$. These integrals are hard to evaluate because as α grows the integrand becomes very highly peaked near $t = \tau$. In Figure 1 we plot the integrand in (4.1) for the case $v = 1$, $t = \pi$, and $\boldsymbol{\gamma}$ the unit circle, and for $\alpha = 10$ and $\alpha = 100$. One can clearly see how the integrand becomes more peaked as α increases.

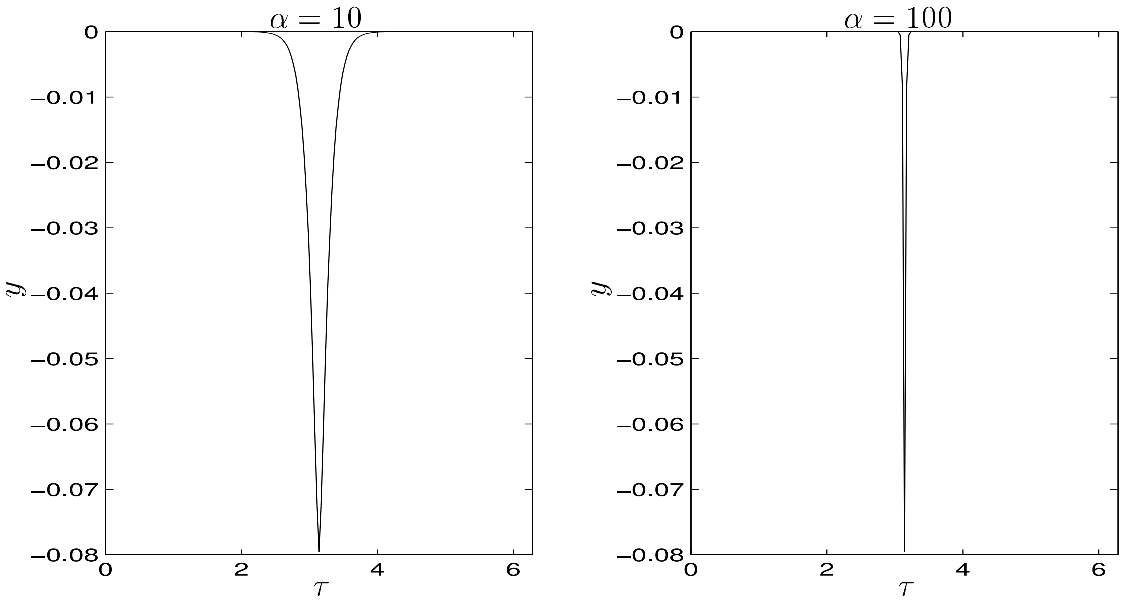

Figure 1: Graph of $y(\tau) := \Psi(\alpha r(\pi,\tau)) w(\pi,\tau)$ against $\tau \in [0, 2\pi]$.

The basic idea of our method is to use a transformation to stretch the region of integration, and to remove the α-dependence of the kernel. We can then use a standard quadrature rule on a graded mesh to evaluate the integral.

To evaluate (4.1) note that by periodicity we can write:-

$$(K_\alpha v)(t) = \int_{t-\pi}^{t+\pi} \Psi(\alpha r(t,\tau)) w(t,\tau) v(\tau) d\tau$$

and then we can make the following transformation

$$\tau = t - \frac{\tilde{\tau}}{\alpha}$$

to get

$$(K_\alpha v)(t) = \frac{1}{\alpha}\int_{-\alpha\pi}^{\alpha\pi} \Psi\left(\alpha r\left(t, t-\frac{\tilde{\tau}}{\alpha}\right)\right) w\left(t, t-\frac{\tilde{\tau}}{\alpha}\right) v\left(t-\frac{\tilde{\tau}}{\alpha}\right) d\tilde{\tau}. \tag{4.2}$$

We can then split (4.2) into the integral over $[0, \alpha\pi]$ plus the integral over $[-\alpha\pi, 0]$, and then, making the substitution $\tau = -\tilde{\tau}$ in the second integral, and removing the tildes for clarity, we get:-

$$\begin{aligned}(K_\alpha v)(t) &= \frac{1}{\alpha}\int_0^{\alpha\pi} \left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right) w\left(t, t-\frac{\tau}{\alpha}\right) v\left(t-\frac{\tau}{\alpha}\right)\right. \\ &\qquad \left. +\Psi\left(\alpha r\left(t, t+\frac{\tau}{\alpha}\right)\right) w\left(t, t+\frac{\tau}{\alpha}\right) v\left(t+\frac{\tau}{\alpha}\right)\right\} d\tau.\end{aligned} \tag{4.3}$$

Observe the effect of this transformation on the integral (4.1). As α gets large, the region of integration gets stretched, and the derivatives of the integrand with respect to τ away from $\tau = 0$ are now small as $\alpha \to \infty$. Noting that w and v are both smooth functions, and that $\Psi(x)$ decays exponentially as $x \to \infty$ and has a logarithmic singularity in its second order derivative at $x = 0$, it is clear that the integrand in (4.3) has a mild singularity near $\tau = 0$ and is exponentially decaying away from $\tau = 0$ when α is large. An effective method for evaluating this sort of integral is the graded mesh technique developed in Mendes (1988), and used in Graham and Mendes (1989).

First, we define the composite mesh, $\Pi_m = x_0, x_1, \dots, x_{m-1}$, which consists of the points

$$\left[\frac{i}{N}\right]^{q_1}, \quad i = 0, \dots, N, \tag{4.4}$$

for some $q_1 \geq 1$, $N \in \mathbb{N}$, together with the points

$$\left\{q_2 \log\left[\frac{rN}{rN-k}\right], k = 1, \dots, rN-1\right\} \cap (1, \alpha\pi] \tag{4.5}$$

for some $q_2 > 0$ and $r \in \mathbb{N}$. If the last point is less than $\alpha\pi$, we add an extra point at $\alpha\pi$ to complete the mesh. Note that the mesh parameters q_1, q_2, and r can be varied, to adjust the mesh. Increasing q_1 bunches the points on $[0, 1]$ closer to zero, increasing q_2 spreads out the points on $(1, \alpha\pi)$, and increasing r increases the ratio of the number of points in $(1, \alpha\pi)$ to the number of points in $[0, 1]$.

Now for any $f : [0, \infty) \mapsto \mathbb{R}$, let $\mathcal{Q}_m f$ denote the function which is linear on each $[x_{i-1}, x_i]$, and which interpolates f at each x_i, $i = 0, \dots, m-1$, with $\mathcal{Q}_m f \equiv 0$ on (x_{m-1}, ∞). Then define

$$\begin{aligned}(K_{\alpha,m} v)(t) &= \frac{1}{\alpha}\int_0^{\alpha\pi} \mathcal{Q}_m\left\{\Psi\left(\alpha r\left(t, t-\frac{(\cdot)}{\alpha}\right)\right) w\left(t, t-\frac{(\cdot)}{\alpha}\right)\right\}(\tau) v\left(t-\frac{\tau}{\alpha}\right) \\ &\qquad +\mathcal{Q}_m\left\{\Psi\left(\alpha r\left(t, t+\frac{(\cdot)}{\alpha}\right)\right) w\left(t, t+\frac{(\cdot)}{\alpha}\right)\right\}(\tau) v\left(t+\frac{\tau}{\alpha}\right) d\tau.\end{aligned} \tag{4.6}$$

For the computation of the matrix in (1.11) we shall replace K_α by $K_{\alpha,m}$ and thus we require in practice the evaluation of $K_{\alpha,m}\phi_k$, with ϕ_k given in (1.8). This is easily realised in practice since it requires the evaluation only of integrals of the form

$$\int_{x_{j-1}}^{x_{j+1}} \psi_j(\tau)\phi_k\left(t \pm \frac{\tau}{\alpha}\right) d\tau,$$

where ψ_j, $j = 1, \dots, m-2$ are the "hat" functions on $[x_{j-1}, x_{j+1}]$ centred at x_j. These integrals are easy to evaluate analytically.

This completes the description of the method employed to evaluate the integrals needed in the linear system (1.11), we shall now present an error analysis, leading to the main result of this section, which is given in Theorem 4.4 below.

Before getting this result we first need to make a few definitions:- Let $l \in \mathbb{N}$, $\mu > 0$ and $0 < \beta \le l$. Then for any $v : \mathbb{R}^+ \mapsto \mathbb{C}$ define

$$\|v\|_{l,\beta} = \max \left\{ \sup_{x\in[0,1]} |v(x)|, \ \sup_{x\in[0,1]} |x^{[j-\beta]} D^j v(x)|, \ j = 1, \dots, l \right\},$$

where

$$[j-\beta] = \begin{cases} j-\beta, & \text{if } j \ge \beta \\ 0, & \text{if } j \le \beta \end{cases}$$

and for any $v : \mathbb{R}^+ \mapsto \mathbb{C}$ define

$$\|v : e^{-\mu x}\|_l = \max \left\{ \sup_{x\in[1,\infty)} |e^{\mu x} D^j v(x)|, \ j = 0, \dots, l \right\}.$$

Now set

$$\|v : e^{-\mu x}\|_{l,\beta} = \max \left\{ \|v\|_{l,\beta}, \|v : e^{-\mu x}\|_l \right\}.$$

Then we have the following definition:-

Definition 4.1 *Let $\mathcal{C}^l_\beta[e^{-\mu x}]$ denote the set of all functions $v : \mathbb{R}^+ \mapsto \mathbb{C}$ which have l continuous derivatives on $(0,\infty)$ and for which $\|v : e^{-\mu x}\|_{l,\beta} < \infty$.*

We can now quote the following theorem, which appears in a more general form as Theorem 2.6 in Mendes (1988).

Theorem 4.2 *There exists a constant C independent of α and m such that for any $v \in \mathcal{C}^2_\beta[e^{-\mu x}]$, $0 < \beta < 1$, $\mu > 0$,*

$$\|(I - \mathcal{Q}_m)v\|_{L_\infty[0,\alpha\pi]} \le C \frac{1}{m^2} \|v : e^{-\mu x}\|_{2,\beta},$$

provided $q_1 \ge 2/\beta$ and $q_2 \ge 2/\mu$.

Proof: See Mendes (1988, p.30).

Now, in our quadrature approximation (4.6) to K_α, we are applying the operator $\mathcal{Q}_m$ to the functions

$$f_{\alpha,t,-}(\tau) = \Psi\left(\alpha r\left(t, t - \tau/\alpha\right)\right) w\left(t, t - \tau/\alpha\right)$$

and

$$f_{\alpha,t,+}(\tau) = \Psi\left(\alpha r\left(t, t + \tau/\alpha\right)\right) w\left(t, t + \tau/\alpha\right).$$

In order to make use of Theorem 4.2 in our error analysis, we need to establish that these functions fit the assumptions of this theorem. The analysis is exactly the same for $f_{\alpha,t,-}$ and $f_{\alpha,t,+}$, and so for simplicity we will just present it for the former.

Theorem 4.3 *There exists a constant C independent of t and $\alpha \geq 1$ such that*

$$\|f_{\alpha,t,-} : e^{-\mu\tau}\|_{2,\beta} \leq C$$

for all $0 < \beta \leq 1$, $0 < \mu < C_1$, where C_1 is the constant on the left hand side of (2.5) (note that by Proposition 2.4, $C_1 > 0$).

Proof: First, note the following:

(i) By Propositions 2.2(ii) and 2.3(ii), Ψ and Ψ' are both bounded on $[0, \infty)$. As a result, there exist constants C_3 and C_4 independent of α and $t, \tau \in \mathbb{R}$ such that

$$\left|\Psi\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| \leq C_3$$

and

$$\left|\Psi'\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| \leq C_4.$$

(ii) $\boldsymbol{\gamma}$ and all its derivatives are $\mathcal{C}^\infty$ and 2π-periodic, and thus $|\boldsymbol{\gamma}(t)|$ is continuous and 2π-periodic in t. Hence, $\sup_{t\in[0,2\pi]} |\boldsymbol{\gamma}(t)| \leq C$, and so $\sup_{t\in\mathbb{R}} |\boldsymbol{\gamma}(t)| \leq C$. Thus, for all $t, \tau \in \mathbb{R}$,

$$r(t, \tau) = |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tau)| \leq |\boldsymbol{\gamma}(t)| + |\boldsymbol{\gamma}(\tau)| \leq C.$$

(iii) $w(t, \tau)$ is also 2π-periodic in t, τ, and infinitely continuously differentiable, hence

$$\sup_{t,\tau\in\mathbb{R}} \left|\frac{\partial^j}{\partial\tau^j}\left\{w\left(t, t - \frac{\tau}{\alpha}\right)\right\}\right| \leq C$$

independently of $\alpha \geq 1$, for all $j = 0, 1, 2, \ldots$.

(iv) Noting that $r(t, \tau) = r(\tau, t)$, we have from Proposition 2.1 that there exists a constant C such that

$$\left|\frac{\partial}{\partial\tau} r(t, \tau)\right| \leq C.$$

Throughout the rest of this proof we will freely use facts (i)-(iv) above without explicitly referring to them. Now, recalling the definition,

$$\|f_{\alpha,t,-} : e^{-\mu\tau}\|_{2,\beta} = \max\{\|f_{\alpha,t,-}\|_{2,\beta}, \|f_{\alpha,t,-} : e^{-\mu\tau}\|_2\}.$$

First we will show that $\|f_{\alpha,t,-}\|_{2,\beta}$ is bounded independently of α and t for all $\beta \leq 1$, by looking at each of the terms on the right-hand side of the following expression:

$$\|f_{\alpha,t,-}\|_{2,\beta} = \max\left\{\underbrace{\sup_{\tau\in[0,1]} |f_{\alpha,t,-}(\tau)|}_{(a)}, \underbrace{\sup_{\tau\in[0,1]} |\tau^{1-\beta}\frac{\partial}{\partial\tau} f_{\alpha,t,-}(\tau)|}_{(b)}, \underbrace{\sup_{\tau\in[0,1]} |\tau^{2-\beta}\frac{\partial^2}{\partial\tau^2} f_{\alpha,t,-}(\tau)|}_{(c)}\right\}.$$

We will bound each of (a), (b), (c) in turn. First (a) and (b), which are relatively easy.

(a) For all α and $t \in [0, \alpha\pi]$

$$\begin{aligned}
\sup_{\tau\in[0,1]} |f_{\alpha,t,-}(\tau)| &\leq \sup_{\tau\in[0,1]} \left|\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right| \sup_{\tau\in[0,1]} \left|w\left(t, t-\frac{\tau}{\alpha}\right)\right| \\
&\leq C.
\end{aligned}$$

(b) Using the chain rule and proceeding analogously

$$\begin{aligned}
\sup_{\tau\in[0,1]} \left|\tau^{1-\beta}\frac{\partial}{\partial\tau} f_{\alpha,t,-}(\tau)\right| &\leq \sup_{\tau\in[0,1]} \left\{\left|\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right| \left|\frac{\partial}{\partial\tau}\left\{w\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right|\right. \\
&\qquad \left. +\alpha \left|w\left(t, t-\frac{\tau}{\alpha}\right)\right| \left|\frac{\partial}{\partial\tau}\left\{r\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right| \left|\Psi'\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right|\right\} \qquad (4.7) \\
&\leq C.
\end{aligned}$$

(c) Similarly, we have

$$\begin{aligned}
\left|\frac{\partial^2}{\partial\tau^2} f_{\alpha,t,-}(\tau)\right| &\leq \left|\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right| \left|\frac{\partial^2}{\partial\tau^2}\left\{w\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right| \\
&\quad +2\left|\frac{\partial}{\partial\tau}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\}\right| \left|\frac{\partial}{\partial\tau}\left\{w\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right| \\
&\quad +\left|\frac{\partial^2}{\partial\tau^2}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\}\right| \left|w\left(t, t-\frac{\tau}{\alpha}\right)\right| \\
&\leq C\left[1 + \left|\frac{\partial^2}{\partial\tau^2}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\}\right|\right]. \qquad (4.8)
\end{aligned}$$

Now, to bound the last term we must proceed more carefully. First note that

$$\frac{\partial}{\partial\tau}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\} = \Psi'\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right) \frac{\partial}{\partial\tau}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\}.$$

It is clear from (2.4) that $\frac{\partial}{\partial\tau} r(t,\tau)$ is discontinuous as $t \to \tau$, so we can't differentiate again at $\tau = 0$. However, for all $\tau > 0$ the derivative is well defined, and, in that case,

$$\begin{aligned}
\frac{\partial^2}{\partial\tau^2}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\} &= \Psi''\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\left[\frac{\partial}{\partial\tau}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right]^2 \\
&\quad +\Psi'\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right) \frac{\partial^2}{\partial\tau^2}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\}.
\end{aligned}$$

Since

$$\left|\frac{\partial}{\partial\tau}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right| = \left|\frac{\partial}{\partial\tilde{\tau}} r(t,\tilde{\tau})\right|,$$

(with $\tilde{\tau} = t - \tau/\alpha$) which is bounded independently of t and α, we have

$$\left|\tau^{2-\beta}\frac{\partial^2}{\partial\tau^2}\left\{\Psi\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right\}\right| \leq C\left[\left|\tau^{2-\beta}\Psi''\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right| + \left|\tau^{2-\beta}\frac{\partial^2}{\partial\tau^2}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\}\right|\right]. \qquad (4.9)$$

Consider first $\Psi''(\alpha r(t, t-\frac{\tau}{\alpha}))$. Recalling the definition of $\Psi(x)$, it can be easily shown (see for example Abramowitz and Stegun (1972)) that $\Psi''(x) = \Psi(x) + iH_0^{(1)}(ix)$. Now since Ψ is bounded on $[0,\infty)$ we have

$$|\Psi''(x)| \le C\{1 + |iH_0^{(1)}(ix)|\},$$

where (see Abramowitz and Stegun (1972)), $iH_0^{(1)}(ix)$ is real valued and decreasing on $[0,\infty)$ and blows up logarithmically as $x \to 0$. Moreover, from Proposition 2.4 we also know that there exists a constant C_1 independent of α such that

$$\alpha r\left(t, t-\frac{\tau}{\alpha}\right) \ge C_1\tau.$$

These last two expressions tell us that, for $\tau \in [0,1]$, $0 < \beta \le 1$,

$$\begin{aligned}\left|\tau^{2-\beta}\Psi''\left(\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right| &\le C\left\{1 + \left|\tau^{2-\beta} iH_0^{(1)}\left(i\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right)\right|\right\} \\ &\le C\left\{1 + \left|\tau^{2-\beta} iH_0^{(1)}(iC_1\tau)\right|\right\} \le C \qquad (4.10)\end{aligned}$$

where C is independent of α.

This bounds the first term in (4.9). Consider now the second term. For $\tau \in (0,1]$,

$$\frac{\partial^2}{\partial\tau^2}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\} = \frac{1}{\alpha}\frac{\partial^2}{\partial\tilde{\tau}^2} r(t,\tilde{\tau}), \qquad (4.11)$$

with $\tilde{\tau}$ as above. From the definition

$$\frac{\partial}{\partial\tilde{\tau}} r(t,\tilde{\tau}) = \frac{(\boldsymbol{\gamma}(\tilde{\tau}) - \boldsymbol{\gamma}(t)).\boldsymbol{\gamma}'(\tilde{\tau})}{|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tilde{\tau})|} = \frac{(\gamma_1(\tilde{\tau}) - \gamma_1(t))\gamma_1'(\tilde{\tau}) + (\gamma_2(\tilde{\tau}) - \gamma_2(t))\gamma_2'(\tilde{\tau})}{|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tilde{\tau})|}.$$

It follows that for $\tilde{\tau} < t$,

$$\frac{\partial^2}{\partial\tilde{\tau}^2} r(t,\tilde{\tau}) = -\frac{\{(\boldsymbol{\gamma}(\tilde{\tau}) - \boldsymbol{\gamma}(t)).\boldsymbol{\gamma}'(\tilde{\tau})\}^2}{|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tilde{\tau})|^3} + \frac{|\boldsymbol{\gamma}'(\tilde{\tau})|^2 + (\boldsymbol{\gamma}(\tilde{\tau}) - \boldsymbol{\gamma}(t)).\boldsymbol{\gamma}''(\tilde{\tau})}{|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tilde{\tau})|}.$$

Hence

$$\left|\frac{\partial^2}{\partial\tilde{\tau}^2} r(t,\tilde{\tau})\right| \le C|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(\tilde{\tau})|^{-1} \le C|t-\tilde{\tau}|^{-1},$$

(recalling Proposition 2.4) and so, for $\tau \in (0,1]$, (4.11) yields

$$\frac{\partial^2}{\partial\tau^2}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\} \le \frac{1}{\alpha} C|t-\tilde{\tau}|^{-1} \le C\frac{1}{\tau}.$$

This gives us, for $\tau \in [0,1]$, $0 < \beta \le 1$,

$$\tau^{2-\beta}\frac{\partial^2}{\partial\tau^2}\left\{\alpha r\left(t, t-\frac{\tau}{\alpha}\right)\right\} \le C\tau^{1-\beta} \le C. \qquad (4.12)$$

Inserting (4.10) and (4.12) into (4.9) and recalling (4.8) gives us the result that $\|f_{\alpha,t,-}\|_{2,\beta}$ is bounded independently of t and α.

Now it just remains to show that $\|f_{\alpha,t,-} : e^{-\mu\tau}\|_2$ is bounded independently of α and t for all $\mu \in (0, C_1)$. This turns out to be quite straightforward. First write

$$\|f_{\alpha,t,-} : e^{-\mu\tau}\|_2 = \max\left\{ \underbrace{\sup_{\tau\in[1,\infty)} |e^{\mu\tau} f_{\alpha,t,-}(\tau)|}_{(d)}, \underbrace{\sup_{\tau\in[1,\infty)} |e^{\mu\tau} \frac{\partial}{\partial\tau} f_{\alpha,t,-}(\tau)|}_{(e)}, \underbrace{\sup_{\tau\in[1,\infty)} |e^{\mu\tau} \frac{\partial^2}{\partial\tau^2} f_{\alpha,t,-}(\tau)|}_{(f)} \right\}.$$

We bound each of these terms in turn.

(d) Using Proposition 2.2(ii) and Proposition 2.4 we have for all $\rho \in [0, 1)$,

$$\left|\Psi\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| \le Ce^{-\rho\left(\alpha r\left(t,t-\frac{\tau}{\alpha}\right)\right)} \le Ce^{-\rho C_1 \tau}.$$

Hence if $\mu \in (0, C_1)$ then for all $\tau \in [1, \infty)$,

$$|e^{\mu\tau} f_{\alpha,t,-}(\tau)| \le Ce^{\tau(\mu-\rho C_1)}.$$

By choosing $\rho = \mu/C_1 \in (0, 1)$ we obtain the result that (d) is bounded independently of α and t.

(e) Using (4.7) we have, for all $\tau \in [1, \infty)$,

$$\left|\frac{\partial}{\partial\tau} f_{\alpha,t,-}(\tau)\right| \le C\left\{\left|\Psi\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| + \left|\Psi'\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right|\right\}.$$

Now using Propositions 2.2(ii) and 2.3(ii), the same argument as was used in (d) can be applied to show that (e) is bounded independently of α and t.

(f) Using (4.8), we get for $\tau \in [1, \infty)$

$$\left|\frac{\partial^2}{\partial\tau^2} f(\tau)\right| \le C\left\{\left|\Psi\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| + \left|\Psi'\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right| + \left|\Psi''\left(\alpha r\left(t, t - \frac{\tau}{\alpha}\right)\right)\right|\right\}.$$

Considering the asymptotic expansion of $iH_0^{(1)}(ix)$ for large x it can be deduced that

$$|\Psi''(x)| \le Be^{-\rho x},$$

for all $\rho \in [0, 1)$, and hence it can be proven that (f) is bounded independently of α and t as above.

This completes the proof of Theorem 4.3. □

We are now in a position to prove the main result of this section.

Theorem 4.4 *For all $v \in L_2$ there exists a constant C independent of α and m such that*

$$\|K_{\alpha,m} v - K_\alpha v\|_{L_\infty} \le C\frac{1}{m^2}\|v\|_{L_2}.$$

Proof: By (4.3), (4.6)

$$\begin{aligned}
\|K_{\alpha,m}v - K_\alpha v\|_{L_\infty} &= \sup_{t\in[0,2\pi]} \left| \frac{1}{\alpha}\int_0^{\alpha\pi} \left\{ (\mathcal{Q}_m - I)\{f_{\alpha,t,-}(\cdot)\}(\tau) v\left(t-\frac{\tau}{\alpha}\right) \right.\right. \\
&\qquad \left.\left. + \, (\mathcal{Q}_m - I)\{f_{\alpha,t,+}(\cdot)\}(\tau) v\left(t+\frac{\tau}{\alpha}\right)\right\} d\tau \right| \\
&\le \sup_{t\in[0,2\pi]} \frac{1}{\alpha}\left\{\int_0^{\alpha\pi} |(\mathcal{Q}_m - I)\{f_{\alpha,t,-}(\cdot)\}(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \left\{\int_0^{\alpha\pi} \left|v\left(t-\frac{\tau}{\alpha}\right)\right|^2 d\tau\right\}^{\frac{1}{2}} \\
&\quad + \sup_{t\in[0,2\pi]} \frac{1}{\alpha}\left\{\int_0^{\alpha\pi} |(\mathcal{Q}_m - I)\{f_{\alpha,t,+}(\cdot)\}(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \left\{\int_0^{\alpha\pi} \left|v\left(t+\frac{\tau}{\alpha}\right)\right|^2 d\tau\right\}^{\frac{1}{2}} \\
&\le \sup_{t\in[0,2\pi]} \frac{1}{\alpha}\left\{\alpha\pi\|[(\mathcal{Q}_m - I)\{f_{\alpha,t,-}(\cdot)\}(\tau)]^2\|_{L_\infty[0,\alpha\pi]}\right\}^{\frac{1}{2}} \left\{\alpha\int_{t-\pi}^{t} |v(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \\
&\quad + \sup_{t\in[0,2\pi]} \frac{1}{\alpha}\left\{\alpha\pi\|[(\mathcal{Q}_m - I)\{f_{\alpha,t,+}(\cdot)\}(\tau)]^2\|_{L_\infty[0,\alpha\pi]}\right\}^{\frac{1}{2}} \left\{\alpha\int_{t}^{t+\pi} |v(\tau)|^2 d\tau\right\}^{\frac{1}{2}} \\
&\le C\frac{1}{m^2}\|v\|_{L_2}
\end{aligned}$$

where the last line follows by Theorems 4.2 and 4.3. This completes the proof. □

5 Fully discrete collocation

With $K_{\alpha,m}$ defined as in Section 4, the fully discrete collocation method for solving (1.5) is then to find $u_{n,m} \in T_n$ satisfying

$$(\lambda I + K_{\alpha,m})u_{n,m}(t_j^{(n)}) = f(t_j^{(n)}), \quad j = 0, \ldots, 2n-1. \tag{5.1}$$

In this section we will present a fully discrete error analysis for this method. We start by quoting the following theorem, which appears in a much more general form as Theorem 1 in Sloan (1995).

Theorem 5.1 *There exists a constant C independent of n such that for all $f \in \mathcal{C}[0, 2\pi]$ the trigonometric interpolating polynomial $\mathcal{P}_n : \mathcal{C} \to T_n$ given by (1.9) satisfies*

$$\|\mathcal{P}_n f\|_{L_2} \le C\|f\|_{L_\infty}.$$

Proof: For a full proof of the more general theorem see Sloan (1995).

We can now use this theorem to prove the following result.

Lemma 5.2 *For n and m sufficiently large there exists a constant C independent of α, m and n such that*

$$\|\mathcal{P}_n(K_{\alpha,m} - K_\alpha)v\|_{L_2} \le C\frac{1}{m^2}\|v\|_{L_2}, \quad \textit{for all } v \in L_2.$$

Proof: By Theorems 5.1 and 4.4,

$$\|\mathcal{P}_n(K_{\alpha,m} - K_\alpha)v\|_{L_2} \le C\|(K_{\alpha,m} - K_\alpha)v\|_{L_\infty} \le C\frac{1}{m^2}\|v\|_{L_2}. \qquad \square$$

We now need one more result before we can establish stability.

Lemma 5.3 *For n and m sufficiently large there exists a constant C independent of α, m and n such that*

$$\|\mathcal{P}_n K_{\alpha,m} - K_\alpha\|_{L_2} \le C\left[\frac{1}{m^2} + \frac{1}{n}\right].$$

Proof:

$$\begin{aligned}\|\mathcal{P}_n K_{\alpha,m} - K_\alpha\|_{L_2} &= \|\mathcal{P}_n K_{\alpha,m} - \mathcal{P}_n K_\alpha + \mathcal{P}_n K_\alpha - K_\alpha\|_{L_2} \\ &\le \|\mathcal{P}_n(K_{\alpha,m} - K_\alpha)\|_{L_2} + \|\mathcal{P}_n K_\alpha - K_\alpha\|_{L_2} \\ &\le C\left\{\frac{1}{m^2} + \frac{1}{n}\right\}\end{aligned}$$

where the last line follows by Lemma 5.2 and Lemma 3.2. □

Now we have enough to prove that the method is stable.

Theorem 5.4 *For n, m and α sufficiently large, $(\lambda I + \mathcal{P}_n K_{\alpha,m})^{-1}$ exists, and is bounded in $\|\cdot\|_{L_2}$ independently of α, n and m.*

Proof: Since

$$\|(\lambda I - K_\alpha) - (\lambda I - \mathcal{P}_n K_{\alpha,m})\|_{L_2} = \|\mathcal{P}_n K_{\alpha,m} - K_\alpha\|_{L_2} \to 0 \quad \text{as } n, m \to \infty,$$

uniformly in α by Lemma 5.3, the result follows by Lemma 3.3 and standard operator perturbation theory. □

Finally, we prove the convergence of the method.

Theorem 5.5 *For n, m and α sufficiently large, the discrete collocation equation (5.1) has a unique solution $u_{n,m} \in T_n$, and we have the following error estimate, where u is the solution of (1.5), and the constant C is independent of α, n and m:-*

$$\|u - u_{n,m}\|_{L_2} \le C\left[\frac{1}{n^p}\|u\|_{H^p} + \frac{1}{m^2}\|u\|_{L_2}\right]. \tag{5.2}$$

Proof: Noting that two functions coincide at the $t_j^{(n)}$ if and only if their interpolants coincide, and, since $\mathcal{P}_n u_{n,m} = u_{n,m}$, (1.14) is equivalent to

$$(\lambda I + \mathcal{P}_n K_{\alpha,m})u_{n,m} = \mathcal{P}_n f.$$

By Theorem 5.4 we then have

$$u_{n,m} = (\lambda I + \mathcal{P}_n K_{\alpha,m})^{-1}\mathcal{P}_n f.$$

Then a little algebra shows

$$u - u_{n,m} = (\lambda I + \mathcal{P}_n K_{\alpha,m})^{-1}[\lambda(u - \mathcal{P}_n u) - \mathcal{P}_n(K_\alpha - K_{\alpha,m})u],$$

and by Theorem 5.4

$$\begin{aligned}\|u - u_{n,m}\|_{L_2} &\le C\|(\lambda I + \mathcal{P}_n \mathcal{K}_{\alpha,m})^{-1}\|_{L_2}[\|u - \mathcal{P}_n u\|_{L_2} + \|\mathcal{P}_n(K_\alpha - K_{\alpha,m})u\|_{L_2}] \\ &\le C\left[\frac{1}{n^p}\|u\|_{H^p} + \frac{1}{m^2}\|u\|_{L_2}\right]\end{aligned}$$

where the last line follows by (3.1) and Lemma 5.2. This proves the convergence rate of our method. □

Remark: By the remark following Lemma 3.3, Theorems 5.4 and 5.5 are actually true for all $\alpha \ge 1$.

6 Numerical examples

In this section, as an example, we will solve the second kind integral equation

$$-\frac{1}{2}u + K_\alpha u = 1 \tag{6.1}$$

using the method (5.1). This integral equation arises from solving the Helmholtz problem (1.1) on a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet data $U = 1$ on the boundary, and then the solution U of (1.1) can be expressed as

$$U = \mathcal{K}_\alpha u \tag{6.2}$$

where u is the solution of (6.1).

As domain boundaries we choose the simple ellipse

$$\boldsymbol{\gamma}(t) = (5\cos(t), \sin(t)), \quad t \in [0, 2\pi], \tag{6.3}$$

and then the more complicated boundary curve

$$\boldsymbol{\gamma}(t) = (\cos t(1 + \cos^2 4t), \sin t(1 + \cos^2 4t)), \quad t \in [0, 2\pi], \tag{6.4}$$

which is illustrated in Figure 2.

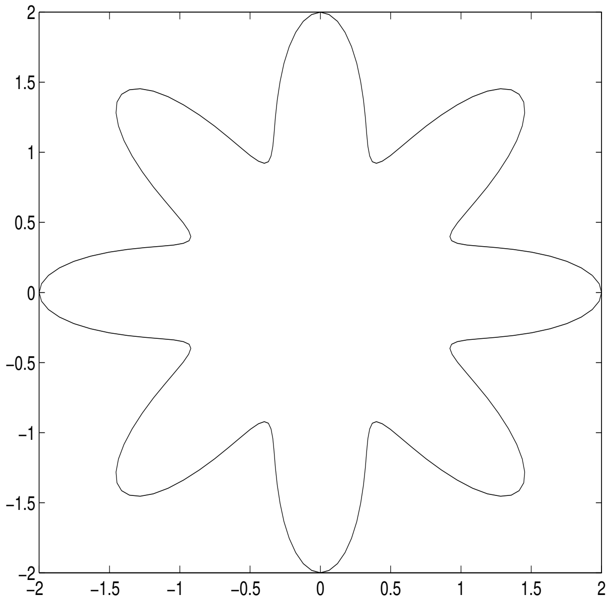

Figure 2: Boundary curve (6.4)

It is the ability of the boundary integral method to handle complicated regions such as this which makes it attractive for this problem. Note that when computing the matrix entries arising from method (5.1) for boundary curve (6.4), a Taylor series expansion must be used to evaluate the function $w(t, \tau)$ near $t = \tau$, whereas for (6.3) we can explicitly separate out the zeros in the numerator and the denominator, and thus compute $w(t, \tau)$ analytically.

For each domain we solve (6.1) for $\alpha = 10$ and $\alpha = 100$. For each experiment the quadrature points are as given in (4.4) and (4.5) with $q_1 = 2$, $q_2 = \alpha$ and $r = 10$. In each case we evaluate the approximate value of the density function $u_{n,m}$ at $t = 0$ and $t = \pi/3$. The exact solution is not known, so we compute a "near exact" solution by applying the method with a large number of collocation and quadrature points, and then for the purpose of computing errors we treat this as the exact solution. For each example we double the number of quadrature points N on the interval $[0, 1]$, leading to approximately doubling of the total number of quadrature points m on the interval $[0, \alpha\pi]$, and then we increase the number of collocation points n logarithmically in N, ie. $n = C \log N$. The estimated order of convergence (EOC) is then computed by

$$EOC = \log_2 \left(\frac{\text{error}_{2N}}{\text{error}_N} \right),$$

and the error estimate (5.2) predicts that this should be approximately equal to two, ie. the convergence is superalgebraic in n and second order in N. Results are given in Tables 2-5.

In Table 2 the "exact" solution is computed taking $n = 128$, $N = 640$, and $m = 6156$. The expected order of convergence (5.2) is clearly achieved.

In Table 3 the "exact" solution is computed taking $n = 128$, $N = 640$, and $m = 6702$. Again, the expected order of convergence (5.2) is achieved, demonstrating the robustness of the method with respect to α.

In Tables 4 and 5 the "exact" solution is computed taking $n = 224$, $N = 128$, and $m = 1233$ and $m = 1342$ respectively. Many more basis functions are needed for this example, due to the larger curvature of the boundary curve. Although the convergence is more erratic than for boundary curve (6.3), the estimated order of convergence is still broadly achieved.

On the more complicated boundary (6.4) we also evaluate (6.2) using the trapezoidal rule with $2n$ nodes, to compute the solution of the Helmholtz problem (1.1) with boundary data $U = 1$ at the points $(0, 0)$ and $(1, 1)$. Again we compute errors by applying the method with a large number of collocation and quadrature points, and take this to be the "exact" solution. Results are given in Tables 6 and 7. For $\alpha = 100$ the solution of (1.1) will be zero almost everywhere, and highly peaked in a boundary layer. The computed solution at the two points $(0, 0)$ and $(1, 1)$ can be seen to be nearly zero (Table 7), and so to confirm the convergence rate of our method we also compute the solution nearer the boundary, at the point $(1.95, 0)$ (Table 8). To compute the solution at this point we use the trapezoidal rule with $4n$ nodes. We need more nodes because the integrand in (6.2) becomes more peaked as the point of evaluation approaches the boundary.

In Table 6 the "exact" solution is computed taking $n = 224$, $N = 128$, and $m = 1233$. In Tables 7 and 8 the "exact" solution is computed taking $n = 224$, $N = 128$, and $m = 1342$. In each table the convergence rate of the method can be clearly seen, and is roughly $O(N^{-2})$ as expected.

In Langdon (To appear 1998) this method will be used in the implementation of boundary integral methods for parabolic problems.

α	n	N	m	$u_{n,m}(0)$	error(0)	EOC	$u_{n,m}(\pi/3)$	error($\pi/3$)	EOC
10	10	10	98	-1.65391455	1.4e-3	-1.3	-2.00230066	8.3e-3	-9.3
	12	20	194	-1.65316703	6.0e-4	-2.1	-1.99395046	1.3e-5	-2.3
	15	40	386	-1.65271099	1.4e-4	-2.7	-1.99396070	2.6e-6	-1.7
	18	80	771	-1.65258941	2.2e-5	-2.8	-1.99396250	8.4e-7	-2.5
	21	160	1540	-1.65257093	3.2e-6	-3.4	-1.99396319	1.5e-7	-1.9
	24	320	3079	-1.65256799	3.0e-7		-1.99396330	4.0e-8	
	128	640	6156	-1.65256769			-1.99396334		

Table 2: Density function for the boundary curve (6.3), $\alpha = 10$

α	n	N	m	$u_{n,m}(0)$	error(0)	EOC	$u_{n,m}(\pi/3)$	error($\pi/3$)	EOC
100	10	10	107	-1.95030178	1.1e-3	-2.0	-2.00106129	1.7e-3	-8.2
	12	20	212	-1.95114131	2.7e-4	-2.1	-1.99939058	5.9e-6	-2.0
	15	40	421	-1.95134900	6.6e-5	-2.2	-1.99939492	1.5e-6	-2.0
	18	80	840	-1.95139990	1.5e-5	-2.5	-1.99939606	3.8e-7	-2.1
	21	160	1677	-1.95141208	2.5e-6	-4.4	-1.99939634	9.0e-8	-2.3
	24	320	3352	-1.95141471	1.2e-7		-1.99939642	1.8e-8	
	128	640	6702	-1.95141460			-1.99939643		

Table 3: Density function for the boundary curve (6.3), $\alpha = 100$

α	n	N	m	$u_{n,m}(0)$	error(0)	EOC	$u_{n,m}(\pi/6)$	error($\pi/3$)	EOC
10	64	4	41	-1.50636903	9.7e-3	-2.0	-2.13906281	9.0e-3	-6.3
	96	8	79	-1.51358770	2.5e-3	-2.0	-2.12994074	1.2e-4	-4.4
	128	16	156	-1.51542259	6.2e-4	-2.1	-2.13006433	5.6e-6	+1.2
	160	32	310	-1.51589092	1.5e-4	-2.3	-2.13004533	1.3e-5	-3.4
	192	64	618	-1.51600854	3.0e-5		-2.13005739	1.3e-6	
	224	128	1233	-1.51603822			-2.13005868		

Table 4: Density function for the boundary curve (6.4), $\alpha = 10$

α	n	N	m	$u_{n,m}(0)$	error(0)	EOC	$u_{n,m}(\pi/6)$	error($\pi/3$)	EOC
100	64	4	44	-1.90957749	9.6e-3	-1.2	-2.02203770	2.3e-2	-9.6
	96	8	86	-1.91504627	4.1e-3	-1.9	-1.99886217	3.0e-5	+5.9
	128	16	170	-1.91806231	1.1e-3	-1.8	-1.99712031	1.8e-3	-2.3
	160	32	337	-1.91885787	3.3e-4	-2.3	-1.99924289	3.5e-4	-6.0
	192	64	672	-1.91911880	6.7e-5		-1.99889806	5.5e-6	
	224	128	1342	-1.91918539			-1.99889256		

Table 5: Density function for the boundary curve (6.4), $\alpha = 100$

α	n	N	m	$U_{n,m}(0,0)$	error$(0,0)$	EOC	$U_{n,m}(1,1)$	error$(1,1)$	EOC
10	64	4	41	1.09097301e-4	2.3e-7	-3.1	1.08467931e-1	1.3e-4	-2.4
	96	8	79	1.08896358e-4	2.7e-8	-2.2	1.08314618e-1	2.5e-5	-1.9
	128	16	156	1.08875238e-4	5.8e-9	-2.1	1.08332630e-1	6.9e-6	-2.1
	160	32	310	1.08870788e-4	1.4e-9	-2.3	1.08337877e-1	1.6e-6	-2.3
	192	64	618	1.08869687e-4	2.8e-10		1.08339195e-1	3.3e-7	
	224	128	1233	1.08869405e-4			1.08339526e-1		

Table 6: PDE solution for the boundary curve (6.4), $\alpha = 10$

α	n	N	m	$U_{n,m}(0,0)$	error$(0,0)$	EOC	$U_{n,m}(1,1)$	error$(1,1)$	EOC
100	64	4	44	7.13256905e-44	1.2e-44	-3.5	3.51222936e-13	1.1e-13	-2.6
	96	8	86	6.05540698e-44	1.1e-45	-3.3	2.62520077e-13	1.7e-14	-3.4
	128	16	170	5.95980125e-44	1.1e-46	-2.5	2.47062494e-13	1.6e-15	-4.1
	160	32	337	5.95065827e-44	1.9e-47	-2.4	2.45538423e-13	9.7e-17	-4.7
	192	64	672	5.94907935e-44	3.6e-48		2.45444904e-13	3.7e-18	
	224	128	1342	5.94872143e-44			2.45441211e-13		

Table 7: PDE solution for the boundary curve (6.4), $\alpha = 100$

α	n	N	m	$U_{n,m}(1.95,0)$	error$(1.95,0)$	EOC
100	64	4	44	8.85192110e-3	3.6e-5	-1.0
	96	8	86	8.79889776e-3	1.7e-5	-1.8
	128	16	170	8.81141689e-3	4.9e-6	-1.8
	160	32	337	8.81492525e-3	1.4e-6	-2.3
	192	64	672	8.81606404e-3	2.9e-7	
	224	128	1342	8.81635258e-3		

Table 8: PDE solution for the boundary curve (6.4), $\alpha = 100$

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