BLOW-UP OF A CLASS OF SOLUTIONS WITH FREE BOUNDARIES FOR THE NAVIER-STOKES EQUATIONS

BY
V.A. Galaktionov and J.L. Vazquez

Abstract

We study the blow-up behaviour of a class of solutions of the incompressible Navier-Stokes equations in $\mathbf{R}^2$ which are described by a parabolic free-boundary problem with a nonlocal term. We prove that the solutions blow up in finite time globally in space and calculate the corresponding rates and profiles. The asymptotic profile and the free boundary are obtained as a nonviscous limit, solution of a nonlocal first-order equation of the Hamilton-Jacobi type. This means that the final highly nonstationary stage of such flows is essentially a convection-type singularity. The viscosity term, though asymptotically negligible, provides the eventual regularity of the limit profiles.

Other types of unstable blow-up and global patterns with a non-generic behaviour are discussed.

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1. Introduction

We study the existence and behaviour of a class of blow-up solutions of the Navier-Stokes equations in $\mathbf{R}^2$ describing the evolution of an incompressible fluid:

$$\begin{aligned} &u_t + uu_x + vu_y = -\frac{1}{\rho}\,p_x + \nu(u_{xx} + u_{yy}), \\ &v_t + uv_x + vv_y = -\frac{1}{\rho}\,p_y + \nu(v_{xx} + v_{yy}), \\ &u_x + v_y = 0. \end{aligned} \tag{1.1}$$

As usual, $\mathbf{u} = (u, v)$ is the velocity field, p is the pressure, $\rho > 0$ is the constant density and $\nu > 0$ is the kinematic viscosity, also a constant. We are interested in the following class of particular solutions

$$u = \int_0^x f(z,t)dz, \quad v = -yf(x,t), \quad p = h(x,t). \tag{1.2}$$

These solutions and their generalizations are described in detail in the survey paper by Berker [B]. They describe a plane jet with a stagnation point at $(0,0)$. For $f(0,0) = c \neq 0$ its local behaviour at the origin is given by $\mathbf{u} \approx c(x, -y)$. The structure of the solutions (1.2) looks like a non-stationary plane version of the famous von Kármán solution [K].

Substituting expressions (1.2) into system (1.1) one obtains the equations for the profiles f and h:

$$\int_0^x f_t(z,t)dz + \left(\int_0^x f(z,t)dz\right) f = -\frac{1}{\rho}\,h_x + \nu f_x, \tag{1.3}$$

$$f_t + \left(\int_0^x f(z,t)dz\right) f_x - f^2 = \nu f_{xx}. \tag{1.4}$$

The second equation is independent and is used to determine f, which becomes the main variable. The first equation is then used to determine the pressure h in terms of f.

In order to determine a particular solution we need to specify boundary conditions. It was shown by Pukhnachov [P] that such solutions can be treated as solutions of a *free-boundary problem* representing the expansion (or contraction) of a plane jet in a domain of the form

$$\Omega_t = \{(x,y) : \, |x| < s(t)\}$$

with rectilinear parallel *free boundaries* $\{x = \pm s(t)\}$. To determine the conditions at the free boundary we argue as follows. Assuming that at time $t \geq 0$ the liquid is contained in a domain $\Omega_t \subset \mathbf{R}^2$ with unknown boundary $\partial\Omega_t$, the conditions on $\partial\Omega_t$ have the form

$$\mathbf{u}\cdot\mathbf{n} = V_n \qquad \text{(kinematic)}, \tag{1.5}$$

$$p\mathbf{n} - 2\rho\nu\mathbf{D}\,\mathbf{n} = 2\sigma H\mathbf{n} \quad \text{(dynamic)}, \tag{1.6}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega_t$, V_n is the velocity of the displacement of $\partial\Omega_t$ in the direction of $\mathbf{n}$, $\mathbf{D}$ is the velocity deformation tensor with elements $D_{jk} = \frac{1}{2}(\partial u_j/\partial x_k + \partial u_k/\partial x_j)$, $\sigma \geq 0$ is the coefficient of surface tension, and H is the mean curvature of the free surface. For the above solutions corresponding to flat free boundaries with $\mathbf{n} = (1, 0)$ the mean curvature H is zero and the free-boundary conditions (1.5), (1.6) reduce to the following ones at $x = s(t)$:

$$f_x = 0, \qquad \frac{ds}{dt} = u \equiv \int_0^s f(z,t)dz, \tag{1.7}$$

$$h - 2\rho\nu f = 0. \tag{1.8}$$

Observe that the first condition in (1.7) follows from equation (1.6) in view of the element $D_{21} \equiv -y f_x/2 = 0$ of the velocity deformation tensor $\mathbf{D}$. The second term on the left-hand side of (1.8) is $-2\rho\nu D_{11}$ where $D_{11} = u_x = f$. It is important that the free-boundary conditions (1.7) are stated in terms of f and are independent of other variables. Moreover, a symmetry condition is imposed at $x = 0$

$$f_x(0,t) = 0. \tag{1.9}$$

Finally, the initial conditions have the form

$$s(0) = a > 0, \quad f(x,0) = f_0(x) \in C([0,a]) \quad \text{on } (0,a). \tag{1.10}$$

We thus arrive at the free-boundary problem (FBP) (1.4), (1.7), (1.9), (1.10) for the function $f(x,t)$ and its free boundary $s(t)$. Then the problem for the pressure h consists in solving the ordinary differential equation

$$h_{xx} = -2\rho f^2 \quad \text{for } 0 < x < s(t), \tag{1.11}$$

which is derived from (1.3) by differentiation. The boundary conditions

$$h = 2\rho\nu f \quad \text{at } x = s(t) \text{ and } \quad h_x(0,t) = 0 \tag{1.12}$$

follow from (1.8) and (1.3) with $x = 0$.

Outline of Results. Equation (1.4) is a nonlinear heat equation with a quadratic reaction term and a nonlocal convection term. The case of negative f represents a jet that contracts in the x direction and expands in the y direction. It follows from the Maximum Principle, see [A4, Chapter 7], that these solutions stabilize to zero in infinite time at a rate $O(1/t)$ and the free boundary goes to zero in the same way.

We study here the positive solutions, which represent flows with mass coming in from far away in the y-axis and expanding in the x direction. These solutions blow up in finite time $T = T(f_0) < \infty$ and we will provide particular patterns and rates for the blow-up. Specifically, if the initial data are monotone decreasing and satisfy certain additional assumptions we will establish a stable blow-up pattern of the form

$$f(x,t) \sim \frac{\cos^2(x\sqrt{\alpha(T-t)})}{T-t} \quad \text{for } t \approx T, \tag{1.13}$$

where $\alpha > 0$ is a constant which can take arbitrary values. This gives for u and v the asympotic values

$$u \sim \frac{x}{2(T-t)} \left(1 + \frac{\sin(2x\sqrt{\alpha(T-t)})}{2x\sqrt{\alpha(T-t)}} \right), \quad v \sim -\frac{y\,\cos^2(x\sqrt{\alpha(T-t)})}{T-t}. \tag{1.14}$$

The free boundary behaves like

$$s(t) \sim \frac{\pi}{2\sqrt{\alpha(T-t)}}\,. \tag{1.15}$$

This analysis is presented and proved in Sections 3 to 6. The main point of the analysis is to show that the viscous terms are asymptotically negligible with respect to the convective ones, so in the limit we obtain the above formulas which solve the inviscid equations (1.1) with $\nu = 0$. The technical tool in proving the *asymptotic simplification* of the nonlocal parabolic equation (1.4) into the first-order Hamilton-Jacobi equation

$$f_t + \left(\int_0^x f(z,t)dz \right) f_x - f^2 = 0 \tag{1.16}$$

is provided by the application of a general dynamical systems approach for asymptotically small perturbations of dynamical systems developed by the authors, see the survey [GV1] and also [GV].

In Section 7 we construct unstable profiles formally corresponding to $\alpha = 0$ in (1.13). Finally, we describe another way of constructing blow-up solutions by means of a periodic continuation beyond the free boundaries. First of all, one can check that the free-boundary problem is invariant under the moving frame transformation

$$y = s(t) - x, \tag{1.17}$$

which is indeed the reflection about $x = s(t)/2$, and is also invariant under the reflection transformation

$$y = 2s(t) - x. \tag{1.18}$$

In particular, this allows to construct solutions with monotone increasing initial functions and also solutions with a periodic-like f-structure with an arbitrary finite number of humps.

2. Preliminaries

The nonlocal FBP has several good properties which are typical for uniformly parabolic equations. Namely, existence and uniqueness of a local-in-time classical solution are proved in [A4, p. 245] via the Lagrangian coordinates (ξ, t) defined by means of the transformation

$$x = X(\xi, t), \tag{2.1}$$

where X solves the problem $dX/dt = u(X,t)$ for $t>0$ and $X(\xi,0)=\xi$, i.e.,

$$\frac{dX}{dt} = \int_0^X f(z,t)dz \quad \text{for } t>0, \quad X(\xi,0)=\xi. \tag{2.2}$$

The free boundary is now fixed:

$$\xi = a \quad \text{for} \ \ t \geq 0. \tag{2.3}$$

Finally, setting $F(\xi,t)=f(X,t)$ and $X_\xi = J(\xi,t)$ we arrive at the equation

$$F_t = \nu \frac{1}{J}\left(\frac{F_\xi}{J}\right)_\xi + F^2. \tag{2.4}$$

Since by (2.2) J satisfies the ordinary differential equation

$$J_t = FJ \quad \text{for } t>0, \quad J(\xi,0)=1,$$

we have that

$$J(\xi,t) = \exp\left\{\int_0^t F(\xi,\tau)d\tau\right\}. \tag{2.5}$$

Then (2.4) reads

$$F_t = \nu \exp\left\{-\int_0^t F(\xi,\tau)d\tau\right\}\left(\exp\left\{-\int_0^t F(\xi,\tau)d\tau\right\}F_\xi\right)_\xi + F^2. \tag{2.6}$$

The boundary conditions now take the form

$$F_\xi(a,t) = F_\xi(0,t) = 0 \quad \text{for } t>0.$$

The quasilinear equation (2.6) is uniformly parabolic on bounded orbits and contains nonlocality in the time variable. It can be studied by standard methods from the parabolic theory [LSU]. Observe that local interior regularity methods of Bernstein type apply to (2.6) to ensure existence of a classical solution which is unique by the Maximum Principle.

Let us recall some preliminary properties of the global negative solutions of (2.6), cf. [A4, Chapter 7]. Because of the source term $F^2 \geq 0$, it follows from the equation that any negative solution $F<0$ remains negative for all times and hence is uniformly bounded by the Maximum Principle. Moreover, by a straightforward comparison from above and below with the exact solutions $F_c(t) = -(c+t)^{-1}$, $c>0$, which are *flat solutions* (independent of the spatial variable ξ), one can conclude that all the negative solutions have the same asymptotic behaviour as $t\to\infty$

$$F(\xi,t) = -\frac{1}{t}\left[1+O\left(\frac{1}{t}\right)\right] \quad \text{uniformly on } [0,a]. \tag{2.7}$$

This means the *asymptotic stability* of the flat solution $F_*(t) = -1/t$ in the class of global negative solutions. It also yields the following *power-like* decay of the free boundary: there exists a constant $b_0 = b_0(f_0) > 0$ such that

$$s(t) = \frac{b_0}{t}\,(1 + o(1)) \to 0 \quad \text{as } t \to \infty, \tag{2.8}$$

which is stable (generic) for global *negative* solutions. In terms of velocities it means that $u \sim x/t$ with $\|u(\cdot,t)\|_\infty = O(t^{-2})$. In the vertical direction we get $v \sim y/t$. The structure of *exponentially* decaying global solutions with *changing sign* is discussed in the final Section 9 of the present paper.

3. Blow-up of positive solutions. The monotone pattern

The purpose of this paper is to study the existence and behaviour of solutions which blow up in finite time. It can be easily deduced from the equation and boundary conditions that any positive solution $f > 0$ blows up in finite time. We formulate this problem again

$$\left.\begin{aligned} & f_t = \nu f_{xx} - \left(\int_0^x f(z,t)dz \right) f_x + f^2 \quad \text{in } Q = \{0 < x < s(t),\ t > 0\}, \\ & f_x(0,t) = f_x(s(t),t) = 0, \quad t > 0, \\ & \frac{ds}{dt} = \int_0^s f(z,t)dz, \quad t > 0, \\ & s(0) = a, \quad f(x,0) = f_0(x) \in C([0,a]). \end{aligned}\right\} \tag{FBP}$$

Indeed, there is a simple explicit *flat* blow-up solution of the form

$$f_T(t) = (T-t)^{-1}, \tag{3.1}$$

for any constant $T > 0$. The interface has the form

$$s_T(t) = aT(T-t)^{-1}. \tag{3.2}$$

We will show that the flat solutions provide a correct blow-up rate but **not** a stable blow-up profile for the positive blow-up solutions of our problem. This unstability of the flat profiles is typical for highly nonstationary solutions of quasilinear heat-type equations, contrary to what is the rule for the class of global solutions.

The occurrence of blow-up for general positive solutions is proved by comparison from below with a flat blow-up solution where $T > 0$ is assumed to be large enough so that the initial data satisfy $F_0(\xi) \geq T^{-1}$. On the other hand, it is well-established for a class of semilinear and quasilinear parabolic problems that the asymptotic finite-time behaviour as $t \to T^-$ cannot be studied by a pure comparison as in the global case. The passage to the limit needs special techniques and a stability theory related to singular perturbations.

The generic blow-up behaviour is given by the following result.

Theorem 3.1. *Assume that $f_0 \geq 0$, $f_0' < 0$ on $(0,a)$ and let $T = T(f_0) > 0$ be the finite blow-up time of the solution. Let the "steepness" hypothesis* (**S**) *on f_0, specified in Section 5.4, be valid. Then: (i) There exists a constant $\alpha = \alpha(f_0) > 0$ such that as $t \to T^-$*

$$f(x,t) = (T-t)^{-1}[\cos^2(x\sqrt{\alpha(T-t)}) + o(1)] \to \infty \tag{3.3}$$

uniformly on compact subsets in $\xi = x\sqrt{T-t} \in [0, \pi/2\sqrt{\alpha})$ (GLOBAL blow-up), and

$$s(t) = \frac{\pi}{2}[\alpha(T-t)]^{-1/2}\,(1+o(1)) \to \infty. \tag{3.4}$$

(ii) On the free boundary, the function $f(s(t),t)$ is uniformly bounded, so that blow-up IS NOT UNIFORM *on $(0, s(t))$.*

Theorem 3.1 describes the monotone space-time structure of the *stable* blow-up behaviour (3.3) which we now denote by $f = f_1(x,t)$. We observe that the equation and boundary conditions of (FBP) are invariant under the scaling transformation

$$(x,t,f) \mapsto (ax, a^2t, a^{-2}f). \tag{3.5}$$

Using such invariance we have the parameter transformation $\alpha \mapsto \alpha a^{-4}$ and therefore an arbitrary parameter $\alpha > 0$ can appear in the limit in formulas (3.3), (3.4). This means that the Ω-limit set of (FBP) in the given class of initial data is *one-dimensional family*.

Sections 4 to 6 are devoted to the proof of Theorem 3.1. We show our result as a further instance of the dynamical systems approach for singular perturbed dynamical systems [GV], [GV1]. This appoach is essentially based on proving uniform stability results for the limit equation, in this case a Hamilton-Jacobi equation. Note that the asymptotic simplification happens at the highest order and we have a singular perturbation problem.

Let us mention that the transformation $y = s(t) - x$, (1.17) allows to apply the above results to monotone increasing initial functions, $f_0' > 0$, which make no difference in the analysis. The free boundary becomes the origin, and vice versa. This allows to simplify the study of the behaviour in a layer next to the free boundary since it is transformed into a fixed layer close to $x = 0$, see Section 6.

In order to show that, in general, the hypotheses on f_0 in Theorem 3.1 are essential for the generic behaviour (3.3), (3.4), we give in Section 7 a formal construction of a countable set of other (unstable) blow-up patterns $\{f_k(x,t),\ k = 2,3,...\}$ which differ from (3.3). The profiles $f_k(x,t)$ are essentially non-monotone in x and do not have locally a periodic-like structure, see also comments below. In particular, the corresponding free boundaries satisfy as $t \to T$

$$s_k(t) = \mu_k \alpha^{-1/2k}(T-t)^{-(2k-1)/2k}(1+o(1)), \quad k = 1,2,3,\dots, \tag{3.6}$$

where $\mu_k = \pi/2k\sin(\pi/2k)$. If $k = 1$ then $\mu_1 = \pi/2$ and (3.6) coincides with (3.4).

Thus, the incompressible Navier-Stokes equations (1.1) admit a countable set of the solutions (1.2) which blow up in finite time and generate blow-up free boundaries (3.6). Nevertheless, only the first mode is proved to be stable. It is worth mentioning that the formation of the final profile and of the free boundary are governed in first approximation by the convection operator. The viscosity term, which is negligible on the asymptotic stage, is responsible for a suitable regularity of the flow and for an appropriate compactness of the orbit.

4. Semiconvexity and first estimates

We proceed next with technical preliminaries for the proof of Theorem 3.1. First of all, by the Strong Maximum Principle for uniformly parabolic equations we have that $f > 0$ in $Q_T = Q \cap \{t < T\}$. The derivative $w = f_x$ satisfies a linear uniformly parabolic equation with bounded smooth coefficients

$$w_t = \nu w_{xx} - \left(\int_0^x f \right) w_x + f w \quad \text{in } Q_T, \tag{4.1}$$

and $w \leq 0$ on the parabolic boundary ∂Q_T by the assumptions. Therefore by the Strong Maximum Principle

$$f_x < 0 \quad \text{and } f(x,t) < f(0,t) \;\; \text{in } Q_T. \tag{4.2}$$

In particular this implies that $f_{xx}(0,t) \leq 0$ and $f_{xx}(s(t),t) \geq 0$ for all $t \in (0,T)$.

Comparing $f(x,t)$ with the flat solution (3.1) in Q_T, which has the same blow-up time, we conclude that $f(\cdot,t)$ must intersect $f_T(t)$ so that the following bound holds

$$f(0,t) = \|f(\cdot,t)\|_\infty > (T-t)^{-1} \quad \text{for all } t \in (0,T). \tag{4.3}$$

Observe that the lower L^∞-bound in (4.3) holds for arbitrary initial data $f_0 \in C$, not necessarily monotone. On the other hand, since $f_{xx}(0,t) \leq 0$ and $f_x(0,t) = 0$, it follows from the equation (FBP) taken at $x = 0$ that the function $f(0,t)$ satisfies the inequality

$$f_t(0,t) \leq f^2(0,t). \tag{4.4}$$

Integrating it over (t,T) we arrive at (4.3) with the inequality sign $\geq$. The strict inequality sign follows by the Strong Maximum Principle applied to a linear parabolic equation for f_{xx}, see below.

Without loss of generality we now assume that

$$C_0 = -\inf f_0'' \tag{4.5}$$

is finite. This is true for any solution $f(x,t)$ after, if necessary, a small shifting the origin in time. We prove a uniform concavity estimate which is true for all the solutions.

Proposition 4.1. *Given arbitrary smooth f_0, a semiconvexity estimate holds:*

$$f_{xx} \geq -C_0 \quad \textit{in } Q_T. \tag{4.6}$$

Proof. Differentiating equation (4.1) once again we see that the second derivative $z = f_{xx} = w_x$ solves the following parabolic differential inequality

$$z_t = \nu z_{xx} - \left(\int_0^x f\right) z_x + (f_x)^2 \geq \nu z_{xx} - \left(\int_0^x f\right) z_x \tag{4.7}$$

in Q_T. Let us estimate z at the free boundary $x = s(t)$. Differentiating the condition $f_x(s(t), t) = 0$ we get $f_{xt} + f_{xx}s' = 0$ and therefore using equation (4.1) we obtain that $z_x = f_{xxx}(s(t), t) = 0$. The invariance under the symmetry $x \mapsto -x$ means that $z_x = 0$ at the origin. Hence the Neumann boundary condition $z_x = 0$ on the lateral boundary holds for the solutions of inequality (4.7). We then arrive at (4.6) by the Maximum Principle. □

Corollary 4.2. *Under the hypotheses of Theorem 3.1, blow-up is global: as $t \to T$*

$$f(x,t) \to \infty \quad \textit{uniformly on compact subsets in } x, \tag{4.8}$$

and

$$s(t) \geq \sqrt{\frac{3}{C_0}}(T-t)^{-1/2}[1 + O((T-t))] \to \infty. \tag{4.9}$$

Proof. Integrating inequality (4.6) twice over $(0, x)$ and using (4.3) we get

$$f(x,t) \geq f(0,t) - \frac{1}{2}C_0 x^2 \geq (T-t)^{-1} - \frac{1}{2}C_0 x^2 \quad \text{in } Q_T, \tag{4.10}$$

and (4.8) follows. Substituting (4.10) into the free-boundary condition we have

$$\frac{ds}{dt} = \int_0^s f(x,t)dz \geq (T-t)^{-1}s - \frac{1}{6}C_0 s^3. \tag{4.11}$$

Integrating this ordinary differential inequality we obtain (4.9). □

We now derive an upper L^∞-bound on the solution.

Corollary 4.3. *Under the hypotheses of Theorem 3.1, as $t \to T$ there holds*

$$f(0,t) \leq (T-t)^{-1}[1 + \nu C_0 (T-t)^2]. \tag{4.12}$$

Proof. As in the derivation of (4.4), using (4.6) in the equation (FBP) at $x = 0$ we obtain the inequality

$$f_t(0,t) \geq f^2(0,t) - \nu C_0. \tag{4.13}$$

Integrating it over (t, T) with the condition $f(0, T) = \infty$ we derive (4.12). □

It can be shown that an L^∞-bound like (4.12) is valid for arbitrary solutions.

As a preliminary upper bound on the interface, one can derive from (4.12) and the free-boundary condition that

$$s'(t) \le s(t)(T-t)^{-1}[1 + O((T-t)^2)],$$

and therefore as $t \to T$

$$s(t) \le m_0 (T-t)^{-1}, \tag{4.14}$$

where $m_0 > 0$ is a constant. Indeed, this estimate differs strongly from (3.4) and corresponds to the flat solution (3.1), (3.2). Nevertheless, we show in Section 7 that such a behaviour can be approached by a sequence of nontrivial unstable blow-up patterns which are not monotone (and non-periodic) in the space variable.

5. Rescaled equation. A singular perturbation

Throughout this section we consider a solution $f(x,t)$ with f_0 satisfying the hypotheses of Theorem 3.1.

5.1. Rescaled equation. The estimates of the previous section motivate the introduction of a scaled function θ governed by the space-time structure of (3.3):

$$f(x,t) = (T-t)^{-1}\theta(\xi,\tau), \quad \xi = x(T-t)^{1/2}, \quad \tau = -\log(T-t). \tag{5.1}$$

The function $\theta(\xi,\tau)$ satisfies the equation

$$\theta_\tau = \mathcal{H}_1(\theta) + \nu e^{-2\tau}\theta_{\xi\xi}, \quad \tau > \tau_0 = -\log T. \tag{5.2}$$

Here $\mathcal{H}_1$ is the first-order nonlocal operator

$$\mathcal{H}_1(\theta) = \left(\frac{1}{2}\xi - \int_0^\xi \theta(\eta,\tau)\,d\eta \right) \theta_\xi - \theta + \theta^2. \tag{5.3}$$

The rescaled free boundary takes the form

$$l(\tau) = s(t)(T-t)^{1/2}, \tag{5.4}$$

and the free-boundary condition is transformed into

$$\frac{dl}{d\tau} = -\frac{l}{2} + \int_0^l \theta(\eta,\tau)\,d\eta, \quad \tau > \tau_0. \tag{5.5}$$

Equation (5.2) is a singular (viscosity) perturbation of the nonlocal first-order equation of the Hamilton-Jacobi type

$$h_s = \mathcal{H}_1(h), \quad s > 0. \tag{5.6}$$

Recall that the solution θ satisfies the following estimates proved in Section 4: for all $\tau > \tau_0$

$$1 < \|\theta(\cdot,\tau)\|_\infty = \theta(0,\tau) \le 1 + O(e^{-2\tau}) \le C_*, \tag{5.7}$$
$$\theta_{\xi\xi} \ge -C_0, \tag{5.8}$$
$$l(\tau) \ge C_1. \tag{5.9}$$

In addition, (5.7) and (5.8) imply that for $\tau \gg 1$

$$|\theta_\xi| \le C_2 = \sqrt{2C_0C_*}. \tag{5.10}$$

Now and later on we denote by $M_0,\ M_1, ...$ and $c,\ c_0,\ c_1, ...$ different positive constants.

In order to perform a large-time analysis we define the ω-limit set of the orbit $\{\theta(\tau),\ \tau > \tau_0\}$ with initial data θ_0 as follows

$$\omega(\theta_0) = \{g \in C :\ \exists\ \{\tau_k\} \to \infty,\ \theta(\tau_k) \to g \text{ uniformly on compact subsets}\}. \tag{5.11}$$

We now prove the following result which is equivalent to (3.3).

Theorem 5.1. *Under the hypotheses of Theorem 3.1 there exists a constant $\alpha > 0$ such that*

$$\omega(\theta_0) = \{\cos^2(\sqrt{\alpha}\xi) \quad \text{on } [0, \pi/2\sqrt{\alpha})\}. \tag{5.12}$$

5.2. Interior regularity.

Lemma 5.2. *There exists an interval $I_\delta = [0,\delta]$, $\delta > 0$, such that*

$$|\partial^k\theta/\partial\xi^k| \le M_k \quad \text{in } I_\delta \times (\tau_1, \infty) \text{ for all } k = 1, 2, ... \,. \tag{5.13}$$

Proof. The estimate for $k = 1$ is already available in (5.10). The proof for $k \ge 2$ is based on the Bernstein-type estimates and follows the lines of the proof of Proposition 5.4 in [GV]. $\square$

Corollary 5.3. *There exists a finite limit*

$$\theta_{\xi\xi}(0,\tau) \to -2\alpha \le 0 \quad \text{as } \tau \to \infty. \tag{5.14}$$

Proof. We know that $\theta_{\xi\xi}(0,\tau) \le 0$. Rescaling equation (4.7) we obtain that the second derivative $w = \theta_{\xi\xi}$ solves the equation

$$w_\tau = \left(\frac{1}{2}\xi - \int_0^\xi \theta\right) w_\xi + (\theta_\xi)^2 + \nu e^{-2\tau} w_{\xi\xi}.$$

Setting here $\xi = 0$ (then $\theta_\xi = w_\xi = 0$) and using (5.13) with $k = 4$ we have that for $\tau \gg 1$

$$\frac{d}{d\tau} w(0,\tau) = O(e^{-2\tau}), \tag{5.15}$$

hence the convergence (5.14). $\square$

From (5.7), (5.13) and (5.14) we conclude that

Corollary 5.4. *If $g \in \omega(\theta_0)$ then*

$$g(\xi) = 1 - \alpha\xi^2 + \phi(\xi) \quad \textit{with } \ |\phi(\xi)| \le c\,\xi^3 \ \textit{ for } \xi \in I_\delta. \tag{5.16}$$

In addition, we now can derive an optimal L^∞-estimate.

Corollary 5.5. *As $\tau \to \infty$*

$$\|\theta(\cdot,\tau)\|_\infty = \theta(0,\tau) = 1 + \frac{2\alpha\nu}{3}e^{-2\tau} + o(e^{-2\tau}).$$

Proof. Setting $\xi = 0$ in (5.2) and using (5.14) we obtain that

$$\theta_\tau(0,\tau) = \theta^2(0,\tau) - \theta(0,\tau) - 2\alpha\nu e^{-2\tau} + o(e^{-2\tau}).$$

Setting here $\theta(0,\tau) = 1+Y(\tau)$, where by (5.7) $Y(\tau) = O(e^{-2\tau}) \to 0$, we have $(e^{-\tau}Y)_\tau = -2\alpha\nu e^{-3\tau} + o(e^{-3\tau})$, whence the result by integration over (τ,∞). $\square$

5.3. Passage to the limit. Fix an arbitrary monotone sequence $\{\tau_j\} \to \infty$. Using Lemma 5.2 in passing to the limit in equation (5.2) we deduce that $\theta(\cdot,\tau_j + s) \to h(\cdot,s)$ as $j \to \infty$ uniformly on compact subsets in $I_\delta \times \mathbf{R}_+$, where a smooth function h solves equation (5.6) with the initial condition

$$h(\xi,0) = g(\xi) \in \omega(\theta_0) \quad \text{in } I_\delta. \tag{5.17}$$

See further details of this procedure in [GV].

In order to solve (5.6), (5.17) explicitly, let us remind that equation (5.6) is the nonviscous scaled form of equation (FBP) (the viscosity term will thus be negligible as $t \to T$):

$$H_t = H^2 - \left(\int_0^x H(z,t)dz\right)H_x, \quad x>0,\ \ t>0; \quad H(x,0) = g(x). \tag{5.18}$$

Using again the method of characteristics we introduce the Lagrangian coordinates by setting $H(X(\eta,t),t) = \omega(\eta,t)$, where

$$X_t = \int_0^X H(z,t)dz,\ \ t>0; \quad X(\eta,0) = \eta. \tag{5.19}$$

From (5.18) we have

$$\omega_t = H_t + H_X X_t = \omega^2, \quad t>0; \quad \omega(\eta,0) = g(\eta), \tag{5.20}$$

and therefore

$$\omega(\eta,t) = \frac{g(\eta)}{1 - tg(\eta)} \equiv H(X,t). \tag{5.21}$$

On the other hand, from (5.19) we have that X_η satisfies

$$(X_\eta)_t = (X_t)_\eta = \left(\int_0^X H\right)_\eta = HX_\eta, \quad t > 0; \quad X_\eta(\eta, 0) = 1,$$

and hence by (5.21)

$$X_\eta = \frac{1}{1 - tg(\eta)},$$

i.e.

$$X(\eta, t) = \int_0^\eta \frac{d\zeta}{1 - tg(\zeta)}. \tag{5.22}$$

We thus obtain a unique solution to (5.18) in the form (5.21), (5.22). We now rescale it bearing in mind that due to (5.16) the blow-up time is $T = 1$. We set (cf. (5.1)) $H = (1-t)^{-1}h$, $\xi = x(1-t)^{1/2}$, $s = -\log(1-t)$, and obtain the solution in parametric form:

$$h(\xi, s) = e^{-s}\frac{g(\eta)}{1 - (1 - e^{-s})g(\eta)}, \quad \xi = e^{-s/2}\int_0^\eta \frac{dy}{1 - (1 - e^{-s})g(y)}. \tag{5.23}$$

Substituting the function $g(\eta)$ from (5.16) into (5.23) we get the following uniform stability result for the Hamilton-Jacobi equation (5.6) in the class of smooth initial data satisfying (5.16).

Proposition 5.6. *As $s \to \infty$, the solution to (5.6), (5.17) satisfies*

$$h(\xi, s) = \cos^2(\sqrt{\alpha}\xi) + O(e^{-s}) \tag{5.24}$$

uniformly on compact subsets from $[0, \pi/2\sqrt{\alpha})$. The convergence is uniform in the class (5.16).

Observe that the functions $S_\alpha(\xi) = \cos^2(\sqrt{\alpha}\xi)$, $\alpha \geq 0$, are classical stationary solutions to (5.6): $\mathcal{H}_1(S_\alpha) = 0$. If $\alpha = 0$ then (5.24) implies that

$$h(\xi, \tau) \to 1 \quad \text{uniformly on compact subsets.}$$

Proposition 5.6 completes the proof of a part of Theorem 3.1 in the sense that we have proved that

$$\text{there exists an } \alpha \geq 0 \text{ such that (5.12) holds.} \tag{5.25}$$

Comment on ω-limit analysis. The above analysis is a simplified version of a general stability approach to perturbed dynamical systems developed in [GV] for the study of singular perturbed Hamilton-Jacobi equations. Let us note that Proposition 5.6 together with interior regularity results imply that the *reduced* ω-limit set [GV1] of the Hamilton-Jacobi equation, restricted to initial profiles satisfying (5.16), is *uniformly stable* in the singular weighted space $C_\rho(\mathbf{R}_+)$, with the weight function $\rho(\xi) = \xi^{-2}$. This conclusion plays a key role in the dynamical systems approach of [GV]. See other applications to extinction and blow-up problems in [GV1].

5.4. Proof of strict concavity at the origin. The following result is left to be proved.

Lemma 5.7. *Under the hypotheses of Theorem 3.1 we have*

$$\alpha > 0. \tag{5.26}$$

Proof. It is based on an analysis of equation (4.7) for $z = f_{xx}$ by means of the Maximum Principle. We first construct an upper uniform in x bound for f_{xx}. It follows from (5.1) and (5.10) that

$$|f_x| = (T-t)^{-1/2}|\theta_\xi| \le C_2(T-t)^{-1/2}. \tag{5.27}$$

From equation (4.7) we derive that $z = f_{xx}$ satisfies in Q_T the inequality

$$z_t \le \nu z_{xx} - \left(\int_0^x f\right) z_x + C_2^2(T-t)^{-1}, \tag{5.28}$$

and $z = 0$ on the lateral boundary, so that in Q_T

$$z = f_{xx} \le \bar{z}(t) = C_3 + C_2^2 \log\left(\frac{T}{T-t}\right), \quad C_3 = \sup f_0''. \tag{5.29}$$

Due to (4.10) we have

$$\int_0^x f(z,t)dz \ge c_*(T-t)^{-1}x \tag{5.30}$$

(with, say, $c_* = 5/6$) uniformly in the domain $Q_T^* = \{0 < x < x_0(t) = C_0^{-1/2}(T-t)^{-1/2},\quad t \in (0,T)\}$. It follows from (4.6) and (4.2) that

$$|f_x| \le C_0 x \quad \text{in } Q_T. \tag{5.31}$$

We now consider in Q_T^* a supersolution of equation (4.7) depending on x which takes a negative value at $x = 0$, $t = T$. It follows from (5.30) and (5.31) that we need to construct a suitable monotone increasing in x supersolution of the linear parabolic differential inequality in Q_T^* satisfied by $z = f_{xx}$. Actually, from

$$z_t = \nu z_{xx} - \left(\int_0^x f\right) z_x + (f_x)^2$$

we derive when $z_x \ge 0$ the inequality

$$z_t \le \nu z_{xx} - c_*(T-t)^{-1}xz_x + C_0^2x^2. \tag{5.32}$$

We thus construct a solution of the differential inequality

$$W_t \ge \nu W_{xx} - c_*(T-t)^{-1}xW_x + C_0^2x^2, \tag{5.33}$$

with the following condition on the lateral boundary

$$W(x_0(t),t) \geq \bar{z}(t), \quad t \in (0,T), \tag{5.34}$$

which follows from (5.29). We set

$$W(x,t) = -a(t) + bx^2. \tag{5.35}$$

Substituting W into (5.33) we get

$$-a' \geq 2\nu b + x^2[C_0^2 - 2bc_2(T-t)^{-1}],$$

which is true if

$$a(t) = C_0 - 2\nu bt \quad \text{and} \quad b \geq \frac{C_0^2 T}{2c_*}. \tag{5.36}$$

Then $W(0,T) < 0$ if $a(T) > 0$, i.e.

$$b < \frac{C_0}{2\nu T}. \tag{5.37}$$

Inequality (5.34) means that

$$\Phi(t) \equiv 2\nu bt + \frac{b}{C_0}(T-t)^{-1} - C_2^2 \log\left(\frac{T}{T-t}\right) - (C_0 + C_3) \geq 0, \quad t \in (0,T). \tag{5.38}$$

It is true if $\Phi(0) \geq 0$ and $\Phi'(t) \geq 0$ for $t \in (0,T)$ which give the following conditions on the parameters

$$b \geq C_0(C_0 + C_3)T, \quad b \geq C_0 C_2^2 T. \tag{5.39}$$

Finally we conclude that

$$z = f_{xx} \leq W(x,t) \quad \text{in } Q_T^*,$$

and in particular

$$f_{xx}(0,T^-) \leq W(0,T) = 2\nu bT - C_0 < 0, \tag{5.40}$$

provided that the initial function f_0 is sufficiently "steep" to satisfy the second-order inequality

$$\begin{aligned} & f_0''(x) \leq \bar{W}(x,0) = -C_0 + bx^2 \quad \text{in } (0,a), \text{ where} \\ & \max\{C_0^2 T/2c_*,\, C_0(C_0+C_3)T,\, C_0 C_2^2 T\} \leq b < C_0/2\nu T. \end{aligned} \tag{\textbf{S}}$$

It is easy to see that given positive C_0 and b such that the second derivative f_0'' satisfies the first inequality, the hypothesis (**S**) holds provided that the blow-up time T is sufficiently small, so that if $f_0 > 0$ is large enough.

In order to finish the proof of Lemma 5.7 we remind that due to rescaling (5.1) $f_{xx}(0,t) \equiv \theta_{\xi\xi}(0,\tau)$ so that (5.40) and (5.14) imply (5.26). $\square$

5.5. Convergence of the free boundary. Consider the behaviour of the rescaled free boundary $l(\tau)$. Recall that $l_\alpha = \pi/2\sqrt{\alpha}$ satisfies (cf. (5.5))

$$\frac{1}{2}l_\alpha = \int_0^{l_\alpha} \cos^2(\sqrt{\alpha}\eta)d\eta. \tag{5.41}$$

This is just saying that the solution S_α given in (5.24) has a stationary free boundary. It follows from Proposition 5.6 and the interior regularity that

$$\liminf_{\tau\to\infty} l(\tau) \geq l_\alpha. \tag{5.42}$$

On the other hand, arguing by contradiction and fixing an arbitrary monotone sequence $\{\tau_j\} \to \infty$ such that $l(\tau_j) \to \bar{l} > l_\alpha$, from the free-boundary condition (5.5) and (5.41) we conclude that for $j \gg 1$

$$l'(\tau_j) = -\frac{1}{2}l(\tau_j) + \int_0^{l(\tau_j)} \theta(\eta, \tau_j)d\eta < 0 \tag{5.43}$$

since $\theta(\eta, \tau_j) \approx \cos^2(\sqrt{\alpha}\eta)$ on $[0, l_\alpha)$ and $\theta(\eta, \tau_j) \leq \theta(l_\alpha, \tau_j) = o(1)$ for $\eta \in [l_\alpha, l(\tau_j)]$ by monotonicity. Indeed, (5.43) contradicts the assumption that $\bar{l} > l_\alpha$. Thus

$$\limsup_{\tau\to\infty} l(\tau) \leq l_\alpha,$$

and together with (5.42) this completes the proof of Theorem 3.1, (i). $\square$

6. On the free-boundary layer

Proof of Theorem 3.1, (ii). We examine next the asymptotics near the free boundary and establish the existence of a free-boundary layer as predicted in Theorem 3.1, (ii). Set

$$F(t) = f(s(t), t) = \inf_x f(x, t). \tag{6.1}$$

From equation (FBP) we have that

$$F' = \nu f_{xx}(s, t) + F^2. \tag{6.2}$$

Using (5.29) in (6.2) we deduce that

$$F' < \gamma|\log(T-t)| + F^2 \quad \text{for } t \approx T, \ \ \gamma = 2\nu C_2^2. \tag{6.3}$$

Let us compare $F(t)$ with the solutions $G(t)$ of the corresponding ordinary differential equation

$$G' = \gamma|\log(T-t)| + G^2, \quad t \approx T. \tag{6.4}$$

Since the first term on the right-hand side of (6.4) is integrable, by a standard asymptotic analysis one can derive that it admits a rapidly growing solution which blows up at $t = T$ (a separatrix) with the asymptotic behaviour

$$G_\infty(t) = (T-t)^{-1} + O(1) \quad \text{as } t \to T, \tag{6.5}$$

and a family of uniformly bounded on $(0, T)$ slowly growing solutions satisfying

$$G_C(t) = C + o(1) \quad \text{as } t \to T, \tag{6.6}$$

where $C > 0$ is an arbitrary parameter. From (3.3) we have that $F(t) = o((T-t)^{-1})$, $t \to T$, and therefore $F(t) \ll G_\infty(t)$ for $t \approx T$. Hence, given a $t_1 \approx T$ there exists a $C \gg 1$ such that $F(t_1) \le G_C(t_1)$. It then follows from (6.3) and (6.4) that

$$F(t) \le G_C(t) \quad \text{for all } t \in (t_1, T), \tag{6.7}$$

and hence $F(t)$ stays uniformly bounded on $(0, T)$. This completes the proof of Theorem 3. $\square$

In order to finish our study of the "free-boundary layer" which is formed for $x \approx s(t)$ as $t \to T$, we perform the invariant reflection transformation $y = s(t) - x$ to get for $f = f(y,t)$ the same equation

$$f_t = \nu f_{yy} - \left(\int_0^y f \right) f_y + f^2. \tag{6.8}$$

Since $f_y \ge 0$ we have that given $y_0 > 0$ the function $F(t) = f(y_0, t)$ satisfies the same inequality (6.3), so that $F(t)$ is uniformly bounded. We thus deduce that

$$f(y,t) \quad \text{is uniformly bounded in } t \in (0,T) \text{ on compact subsets in } y. \tag{6.9}$$

This means that the standard parabolic regularity theory applies to equation (6.8) to conclude that there exists the final-time profile

$$f(y,T) = \lim_{t \to T} f(y,t), \tag{6.10}$$

thus describing the asymptotic behaviour in the boundary layer.

7. Non-monotone blow-up patterns

In this section we construct a set of blow-up solutions $\{f_k(x,t),\ k = 2, 3, ...\}$ satisfying the (FBP). We expect that unlike the stable spatially monotone blow-up pattern $f = f_1(x,t)$ studied in the previous sections, all the higher-order non-monotone (and non-periodic) patterns are evolutionary unstable. Nevertheless, these patterns exhibit interesting new features of free-boundary flows described by the incompressible Navier-Stokes equations.

Our formal construction consists of two steps.

7.1. Inner parabolic region. Consider equation (5.2). First, in order to specify the behaviour of $\theta(\cdot,\tau)$ about the stationary flat solution $\theta\equiv 1$ we set

$$\theta = 1+Y, \tag{7.1}$$

where Y satisfies the equation

$$Y_\tau = -\frac{1}{2}\xi Y_\xi + Y + Y^2 - \left(\int_0^\xi Y\right) Y_\xi + \nu e^{-2\tau} Y_{\xi\xi}. \tag{7.2}$$

Second, we introduce the space variable

$$\eta = \xi e^{\tau}, \tag{7.3}$$

such that (7.2) on compact subsets in η becomes a linear *uniformly parabolic* equation with a quadratic perturbation

$$Y_\tau = \mathcal{A}Y + Y^2 - \left(\int_0^\eta Y\right) Y_\eta, \tag{7.4}$$

where $\mathcal{A}$ is the linear second-order operator of the hypergeometric type

$$\mathcal{A}Y = \nu Y_{\eta\eta} - \frac{3}{2}\eta Y_\eta + Y \equiv \frac{\nu}{\rho}(\rho Y_\eta)_\eta + Y,\ \eta>0; \quad Y'(0)=0. \tag{7.5}$$

The weight function is $\rho = e^{-3\eta^2/4\nu}$.

The operator $\mathcal{A}$ is well-known to be self-adjoint in $L^2_\rho(\mathbf{R}_+)$ with domain $D(\mathcal{A}) = H^2_\rho \cap H^1_{\rho,0}$ where the subscript 0 corresponds to the symmetry condition at the origin. It has a discrete spectrum. When restricted to symmetric functions the spectrum is

$$\lambda_k = 3k-1, \quad k=0,1,2,..., \tag{7.6}$$

and the orthonormal in L^2_ρ system of eigenfunctions takes the form

$$\psi_k(\eta) = a_k H_{2k}(\eta\sqrt{3/\nu}), \tag{7.7}$$

where H_{2k} are the $2k$-th Hermite polynomials and a_k being the normalization constants. In particular, $\lambda_0=-1$ describes the unstable mode with $\psi_0\equiv a_0$ corresponding to the unstability with respect to perturbations of the blow-up time. On the other hand, for $k=1$ we have $\lambda_1=2$ with the monotone second-order Hermite polynomial $\psi_1(\eta) = a_1(1-9\eta^2/2\nu)$. This first-mode stable behaviour for $\tau\gg 1$ has the form $Y(\eta,\tau) = Ce^{-2\tau}\psi_1(\eta)+...$ uniformly on compact subsets in η. Such a stable pattern has been studied in the previous sections, cf. (5.7), Corollary 5.5 and (5.14).

We now consider the higher-order k-th mode and assume that there exists a constant $C\neq 0$ depending on the initial data such that as $\tau\to\infty$

$$Y(\eta,\tau) = Ce^{-\lambda_k\tau}\psi_k(\eta)+... \tag{7.8}$$

uniformly on compact subsets in η.

7.2. Extension to the outer Hamilton-Jacobi region. Since by the properties of the Hermite polynomials $H_{2k}(\eta) = O(\eta^{2k})$ as $\eta \to \infty$, it follows from (7.8) that

$$Y(\eta, \tau) = -\alpha e^{-\lambda_k \tau} \eta^{2k} + \ldots = -\alpha \left(\eta e^{-(3k-1)\tau/2k} \right)^{2k} + \ldots , \tag{7.9}$$

where $\alpha > 0$ is a constant. Therefore according to this asymptotic expansion we introduce the new spatial variable

$$\zeta = \eta e^{-(3k-1)\tau/2k} \equiv \xi e^{-(k-1)\tau/2k} \equiv x e^{-(2k-1)\tau/2k}. \tag{7.10}$$

Equation (5.2) for the rescaled function $\theta = \theta(\zeta, \tau)$ takes the form

$$\theta_\tau = \mathcal{H}_k(\theta) + \nu e^{-(3k-1)\tau/k} \theta_{\zeta\zeta}, \tag{7.11}$$

$\mathcal{H}_k$ being the Hamilton-Jacobi operator (cf. (5.3) with $k = 1$)

$$\mathcal{H}_k(\theta) = \left(\frac{2k-1}{2k} \zeta - \int_0^\zeta \theta \right) \theta_\zeta - \theta + \theta^2. \tag{7.12}$$

We now apply the same asymptotic analysis as in Section 5. First of all, the interior regularity results are true for equation (7.11). Indeed, as it is shown in [GV], Bernstein-type techniques apply provided that the coefficient of the first derivative θ_ξ in the Hamilton-Jacobi operator (7.12) is of order $-A\zeta$ for small $\zeta > 0$ with a positive A. (Then the exponentially small parabolic perturbation in (7.11) makes no difficulties since it preserves the current regularity). One can see that since by (5.7) $\theta(0, \tau) = 1 + o(1)$ and (5.10) holds, we have that $A = 1/2k > 0$ for all finite k. Using the interior regularity estimates and passing to the limit in (7.11) as $\tau_j \to \infty$ we have that $\theta(\cdot, \tau_j + s) \to h(\cdot, s)$ uniformly on compact subsets, where h solves the nonlocal Hamilton-Jacobi equation

$$h_s = \mathcal{H}_k(h), \quad s > 0. \tag{7.13}$$

Due to (7.9) the initial data satisfy

$$h(0, \zeta) = g(\zeta) = 1 - \alpha \zeta^{2k} + O(\zeta^{2k+1}) \quad \text{for small } \zeta > 0. \tag{7.14}$$

As in Section 5, the Hamilton-Jacobi problem (7.13), (7.14) is solved explicitly. Namely, using the new rescaled variable (7.10) we derive from the general solution (5.21), (5.22) that the solution has the form (cf. (5.23))

$$h(\zeta, s) = e^{-s} \frac{g(\eta)}{1 - (1 - e^{-s}) g(\eta)}, \quad \zeta = e^{-(2k-1)s/2k} \int_0^\eta \frac{dy}{1 - (1 - e^{-s}) g(y)}. \tag{7.15}$$

Moreover, substituting (7.14) into (7.15) we have that as $s \to \infty$, uniformly on compact subsets (cf. (5.24))

$$h(\zeta, s) \to \left\{ 1 + [Q_k(\zeta \alpha^{1/2k})]^{2k} \right\}^{-1}, \tag{7.16}$$

where Q_k is the inverse function to

$$q_k(\rho) = \int_0^\rho \frac{dz}{1+z^{2k}} \quad (k = 2, 3, ...). \tag{7.17}$$

It follows from (7.16) that the rescaled free-boundary position ζ_{0k} is characterized by the equation $Q_k(\zeta_{0k}\alpha^{1/2k}) = \infty$, i.e.,

$$\zeta_{0k} = \mu_k \alpha^{-1/2k}, \quad \text{where} \quad \mu_k = \int_0^\infty \frac{dz}{1+z^{2k}} = \frac{\pi}{2k\sin(\pi/2k)}. \tag{7.18}$$

Finally, it follows from (7.18) that by rescaling (7.10) the free boundary $s(t)$ moves according to (3.6). Observe that as $k \to \infty$, the rate of divergence (3.6) formally approaches the rate of blow-up in (3.2), which corresponds to the flat solution. Since the higher modes with $k = 2, 3, ...$ in the inner region are generated by non-monotone, non-periodic structures, as usual in asymptotic parabolic theory, these patterns are expected to be evolutionary unstable, so that small perturbations lead to the structural convergence to the first monotone blow-up pattern.

8. Periodic patterns

The stable patterns constructed in Theorem 3.1 can be combined with the invariant reflection transformation $y = 2s(t) - x$ introduced in (1.18) in order to obtain different blow-up patterns in the form of solutions having a fixed but arbitrary finite number $M = 1, 2, ...$ of humps of the type (3.3), which are repeated periodically in x. The free boundary is located at $x = Ms(t)$ and satisfies the same boundary conditions. Note that as in (3.3), this local periodic-like structure is stable under periodic perturbations. This represents a smooth solution of the Navier-Stokes equations where the pressure problem (1.11) is solved independently with the free-boundary condition from (1.12) stated at $x = Ms(t)$.

9. Final remark about global decaying patterns

It is clear that the "most trivial solution" $f = 0, \; s = 0,$ is a border between global negative solutions with the generic behaviour (2.7), (2.8) and the stable blow-up behaviour given in Theorem 3.1. Nevertheless, there exist other global (unstable) patterns with a different space-time structure, but these solutions can be treated as belonging to the stable set of another "less trivial solution" $f \equiv 0, \; s \equiv a > 0$.

Using similar simple ideas, we now briefly show how to construct global patterns with faster exponential large-time decay. These patterns can be obtained via a standard linearization of equation (FBP) about $f \equiv 0, \; s \equiv a$. Then both nonlinear terms which are quadratic in f can be viewed as small perturbations and this yields the linearized operator $\mathcal{B} = \nu d^2/dx^2$. The corresponding "eigenvalue" problem $\mathcal{B}\psi = -\lambda\psi$ with an arbitrary $\lambda > 0$ has a solution $\psi(x) = \cos(\sqrt{\lambda/\nu}x)$. The rescaled function $\theta(x,t) = e^{\lambda t} f(x,t)$ satisfies the heat equation with an exponential perturbation of the form

$$\theta_t = \nu\theta_{xx} + \lambda\theta + e^{-\lambda t}\left[\theta^2 - \left(\int_0^x \theta\right)\theta_x\right], \tag{9.1}$$

with the free-boundary condition

$$s' = e^{-\lambda t} \int_0^s \theta(z,t)dz. \tag{9.2}$$

Therefore, a uniform L^∞-bound $|\theta| \le C$ implies that $s' = O(e^{-\lambda t})$ for $t \gg 1$, so that the free boundary converges and its eventual position $s_\infty = s(\infty) > 0$ must satisfy $\psi'(s_\infty) = 0$, i.e., $\sqrt{\lambda/\nu}\, s_\infty = \pi k$, $k = 1, 2, \ldots$. Under the above conditions one can expect the exponential decay of the solution

$$f(x,t) = C_* e^{-\lambda t}\psi(x) + \ldots \quad \text{as } t \to \infty \tag{9.3}$$

uniformly in $x \in (0, s(t))$, with a constant $C_* \neq 0$. Note that in contrast with Section 7, we see a "continuous" spectrum of possible exponential patterns. Moreover, these global solutions are entirely parabolic and no degenerate Hamilton-Jacobi limits are involved into the construction.

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Victor A. Galaktionov
School of Mathematical Sciences, University of Bath
Claverton Down, Bath BA2 7AY, UK
and Keldysh Inst. Appl. Math., Miusskaya Sq. 4, 125047 Moscow, RUSSIA
E-mail: vag@maths.bath.ac.uk

Juan L. Vazquez
Departamento de Matemáticas, Universidad Autónoma de Madrid
28049 Madrid, SPAIN
E-mail: juanluis.vazquez@uam.es