Dynamical Systems of Inequalities and Nonlinear Parabolic Equations

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To Professor Sergey P. Kurdyumov on his 70th Birthday

Abstract

A nonlinear second-order uniformly parabolic equation in $\mathbf{R} \times \mathbf{R}_+$

$$u_t = F(u, u_x, u_{xx}), \quad F_r(p, q, r) > 0,$$

with sufficiently smooth right-hand side, is shown to generate a four or two-dimensional dynamical system of inequalities (DSIs) describing evolution properties of a class of solutions $u = u(x,t)$. In general, the two-dimensional DSIs takes into account the joint evolution in time of the one-sided bounds on the solution $u(\cdot,t)$ (say, its L^∞_x-norm) and on the second spatial derivative $u_{xx}(\cdot,t)$. In two dimensions the evolutions orbits of the associated dynamical systems (DSs) form invariant regions for the DSIs. The DSIs and the invariant regions are derived for a general quasilinear heat equation and a fully nonlinear equations. The well-known semiconvexity approach by Aronson-Bénilan for the porous medium equation corresponds to a particular case of a discoupled DSIs where the inequalities can be studied separately. Radial solutions of the quasilinear heat equation $v_t = \Delta\phi(v) + F(v)$ in $\mathbf{R}^N$ are studied. Sharp bounds on the blow-up solutions are derived for classes of quasilinear and fully nonlinear equations.

Key Words: nonlinear parabolic equations, Maximum Principle, dynamical systems, dynamical systems of inequalities, invariant regions, blow-up.

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1 Introduction: DS and DSIs

1.1 On the basic approach: two-dimensional DSIs

In the present paper we answer the question how to determine a certain finite-dimensional evolution via a *dynamical system of inequalities* (DSIs in short) and the associated dynamical system (DS) for nonlinear partial differential equations of parabolic type. Consider a second-order uniformly parabolic equation of the general form

$$u_t = \mathbf{F}(u) \equiv F(u, u_x, u_{xx}), \quad x \in \mathbf{R},\ t > 0, \tag{1}$$

with a sufficiently smooth function $F(p, q, r)$, $F(0, 0, 0) = 0$, satisfying the parabolicity condition

$$F_r(p, q, r) > 0.$$

We consider classical solutions of the Cauchy problem with smooth bounded initial data. Given an equation (1), we show that there exists *two-dimensional* or, in general, *four-dimensional* DSIs and DS describing some evolution properties of the classical bounded solutions of the parabolic PDE.

The main idea in two-dimensions is as follows. Let $u = u(x, t) \geq 0$, $u(\cdot, t) \not\equiv 0$, be a sufficiently smooth uniformly bounded solution having for a fixed $t \geq 0$ uniformly bounded second spatial derivative, with the ranges $u(\cdot, t) \in [0, U(t)]$, $u_{xx}(\cdot, t) \in [W(t), \infty)$ where $U(t) \geq 0$ (resp. $W(t) \leq 0$) is the corresponding least upper (lower) bound in x of $u(x, t)$ (resp. $u_{xx}(x, t)$). The range of the first spatial derivative then can be easily estimated: $|u_x(\cdot, t)| \leq V(U(t), W(t), u) \equiv [2W(t)(u - U(t))]^{1/2}$.

In order to construct the DSIs for the two *range functions* $\{U(t), W(t)\}$ and the associated two-dimensional DS we first consider the (formal) *extended system* of the PDEs composed from equation (1) and the parabolic equation for the second derivative $w = u_{xx}$:

$$\begin{cases} u_t = F(u, u_x, w), \\ w_t = \Phi(u, u_x, w, w_x, w_{xx}) \equiv (\partial^2/\partial x^2) F(u, u_x, w)|_{u_{xx}=w}. \end{cases} \tag{2}$$

The second equation, which can be treated as a differential constraint, is a differential consequence of the parabolic equation (1) and the extended system is consistent (it is in involution). The equations in (2) are strongly coupled via both the variables u and w. We next introduce the corresponding *lower extended subsystem* where the second equation is less coupled with the first one by means of the single variable w and the range functions only:

$$\begin{cases} u_t = F(u, u_x, w), \\ w_t = \Phi_-(w, w_x, w_{xx}; U, W) \equiv \inf_{p,q}\ \Phi(p, q, w, w_x, w_{xx}), \end{cases} \tag{3}$$

where $\inf_{p,q}$ is calculated over the domain $\{p \in [0, U(t)], \, |q| \le V(U(t), W(t), p)\}$ which, by construction, contains the range of $\{u(\cdot,t), u_x(\cdot,t)\}$. We now determine the DS for the functions $\{\bar U(t), \bar W(t)\}$ which is a *homogeneous solution* of the lower subsystem:

$$
\begin{cases} \bar U' = F(\bar U, 0, \bar W), \\ \bar W' = \Phi_-(\bar W, 0, 0; \bar U, \bar W), \quad t > 0. \end{cases} \tag{4}
$$

The DSIs for the range functions of the solution, $\{U(t), W(t)\}$, is then proved to be deduced by the Maximum Principle from the lower subsystem (3) and is obtained from (4) with the appropriate inequality signs:

$$
\begin{cases} U' \ge H_1(U, W) \equiv F(U, 0, W), \\ W' \ge H_2(U, W) \equiv \Phi_-(W, 0, 0; U, W), \end{cases} \tag{5}
$$

where, due to the standard parabolic regularity, the time-derivatives of the range functions are understood in the a.e. sense for $t > 0$. The pair of the range functions $\{U(t), W(t)\}$ is thus a *homogeneous supersolution* of the lower extended subsystem.

We will show that a similar derivation is available for the four range functions $\{U_\pm(t), W_\pm\}$ which are the least upper and lower bounds on $u(\cdot,t)$ and $u_{xx}(\cdot,t)$. Then both lower and *upper* extended subsystems (which are coupled in terms of all the four range functions) should be taken into account. The analysis of the DSIs and of the associated DS in four dimensions seems to be extremely difficult in the general case. We mainly deal with the two-dimensional DSIs of the form (5) for the solutions with a suitable spatial geometric shape. We show that choosing a peculiar spatial geometric form of the solutions and slightly changing the definition of the second range function $W(t)$, the solutions of the DSIs become C^1-functions and the inequalities are understood in the usual sense. The evolution orbits of the associated DSs for the functions $\{\bar U(t), \bar W(t)\}$ consisting of two ordinary differential equations (5) with the equality signs are shown to form *invariant regions* on the $\{U, W\}$-plane for the corresponding DSIs.

1.2 Main results, comparison and plan of the paper

In Section 2 we begin with particular examples of exact reductions of quasilinear parabolic equations to one and two-dimensional DSs for special families of solutions. Such DSs are the parabolic equations restricted to linear subspaces invariant under corresponding nonlinear operators. There are many examples where nonlinear partial differential equations reduce to finite-dimensional DSs on linear invariant subspaces or sets, see references in [16] - [18] and [24]. For the parabolic equations such DSs (describing a finite-dimensional subset of the particular solutions) create DSIs on the class of much more general solutions.

In Section 3 (quasilinear equations in the one space dimension, $N = 1$), Section 4 (fully nonlinear equations for $N = 1$) and Section 5 ($N > 1$) we perform the derivation of the *two-dimensional* DSIs for the general quasilinear reaction-diffusion equation in the case when it admits no exact finite-dimensional reduction on linear invariant subspaces. As a typical example, we consider nonnegative solutions of the quasilinear heat equation with smooth coefficients

$$v_t = \mathbf{A}(v) \equiv \Delta\phi(v) + F(v). \tag{6}$$

The DSIs can be derived for rather arbitrary nonlinear parabolic equations. Using a nonlinear pointwise transformation $v = g(u)$ the equation reduces to a different parabolic one which admits a different DSIs. Given a nonlinear parabolic equation, there exist infinitely many different DSIs. We then arrive at the problem of an optimal DSIs, i.e., of that one which most adequately describes the evolution properties of the solutions.

The two-dimensional DSs look similar to those derived in the papers by Elenin, Kurdyumov, Plokhotnikov and Samarskii by the formal method of nonstationary averaging for quasilinear heat equations, see [11], [10], pp. 201-210, 301-306 in [35] and references therein. In both representations the DSs are constructed to describe evolution properties of essentially nonstationary solutions of the quasilinear heat equations. See also applications to a nonlinear problem of laser thermochemistry [8]. According to the averaging method, an approximate solutions of (6) takes the standard self-similar form

$$\bar{v}(x,t) = a(t)\theta(\xi), \quad \xi = x/b(t), \tag{7}$$

where $\theta \geq 0$ is a fixed typical bell-shaped (compactly supported) function. The averaged DS on the nonstationary functions $\{a(t), b(t)\}$ of the form $a' = H_1(a,b)$, $b' = H_2(a,b)$ is then determined by integration by parts from two *integral* identities, say, $(\bar{v}_t - \mathbf{A}(\bar{v}), 1) = 0$, $(\bar{v}_t - \mathbf{A}(\bar{v}), \bar{v}) = 0$, where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbf{R}^N)$. The calculations and the results for the power-like functions $\phi(v) = v^m$, $F(v) = v^\beta$ are presented in [35], pp. 202-210. It turned out that the averaged DS perfectly describes some delicate asymptotic properties of different classes of blow-up and global in time solutions to (6). It follows from the representation (7) that there exists a direct correspondence between the amplitude-semiwidth functions $\{a(t), b(t)\}$ and the range functions given above:

$$U(t) \sim a(t), \quad W(t) \sim \Delta_x \bar{v}(\cdot,t) = a(t)\Delta_\xi\theta/b^2(t) \sim a(t)/b^2(t).$$

We thus claim that the formal averaging method admits a rigorous justification on the basis that the corresponding averaged DS should be treated as a DSIs for the range functions.

It turns out that for the quasilinear equation (6), as in the well-known semiconvexity approach [1] for the porous medium equation (PME) with $\phi(v) = v^m$ $(m > 1)$, $F(v) \equiv 0$, a kind of optimal DSIs is often obtained by using the following equivalent "pressure" form of the parabolic equation with a separate quadratic Hamilton-Jacobi operator

$$u_t = \varphi(u)\Delta u + |\nabla u|^2 + f(u), \quad x \in \mathbf{R}^N, \ t > 0 \quad (\varphi(u) > 0).$$

This corresponds to the slow-diffusion case. In the fast-diffusion case the sign of the Hamilton-Jacobi term is negative, $-|\nabla u|^2$. The semiconvexity estimate by Aronson-Bénilan for the PME corresponds to the case where $\varphi(u) = (m-1)u$ is a linear function and hence the second differential inequality for $W(t)$ in (5) (which is a lower bound on $\Delta u(\cdot, t)$) can be solved independently. It has a typical form $W' \geq H_2(W)$, $t > 0$, where $H_2(W)$ is a quadratic function (see Section 3.2). Nevertheless, one can see that in the general case with $f \not\equiv 0$ (and even for a slightly nonlinear $\varphi(u)$), a *separate* study of the evolution of the single function $W(t) \sim \inf_x \Delta u(x,t)$ is not possible, and at least a two-dimensional DSIs should be invented.

We show that a *generalized semiconvexity/semiconcavity* approach leads to a *system of ordinary differential inequalities* (DSIs) for at least two range functions $\{U, W\}$. We prove that in the case of general nonlinearities, the evolution of the functions $\{U(t),\ W(t)\}$ is strongly coupled, and we arrive at a *joint* evolution of the range functions given by the two-dimensional DSIs of the form (5). In Section 4 we also perform calculations in order to derive the general two-dimensional DSIs by means of the corresponding lower extended subsystem for a fully nonlinear parabolic equation.

In Section 6 we compare the orbits of the DSIs (5) and the associated DS:

$$\begin{cases} \bar{U}' = H_1(\bar{U}, \bar{W}), \\ \bar{W}' = H_2(\bar{U}, \bar{W}), \quad t > 0, \end{cases} \tag{8}$$

which reduces to the first-order ordinary differential equation

$$\frac{d\bar{U}}{d\bar{W}} = \frac{H_1(\bar{U}, \bar{W})}{H_2(\bar{U}, \bar{W})}.$$

We prove the results concerning the *invariant regions* for DSIs on the $\{U, W\}$-plane given by the orbits $\{\bar{U}, \bar{W}\}$ of the associated DS (8).

The DSIs approach is designed to derive sharp bounds on the solutions $u(\cdot, t)$ and the second derivative $\Delta_x u(\cdot, t)$ near finite-time singularity of the blow-up, extinction or quenching type for highly nonlinear parabolic equations from combustion theory. This is reflected in the spatial geometric forms of the solutions presented in Section 3.1. In [19] this method applies to a fully nonlinear equation from detonation theory (see also general computations in Section 4). In Section 7 we present some other applications of the DSIs to the derivation of sharp bounds on the blow-up solutions or a wide class of semilinear, quasilinear and fully nonlinear heat equations. Actually, such optimal bounds make it possible to give a complete description of the stable generic blow-up behaviour for equations with general nonlinearities. It turns out that the estimates derived by comparison with the DS-orbits are so sharp that we can claim that in several cases the associated DS becomes the asymptotic (blow-up) form of the corresponding DSIs.

In Section 8 we consider the questions concerning multi-dimensional DSIs. For a general fully nonlinear equation using both the lower and upper extended subsystems we perform the derivation of the four-dimensional DSIs, and discuss multi-dimensional DSIs for a greater number of the range functions.

Our analysis is based on the Maximum Principle for uniformly parabolic equations [14], [33], and on the construction of suitable super and subsolutions. We always assume that such a comparison is available in the classes of smooth solutions to equations under consideration.

2 Examples of one and two-dimensional DSIs

We begin with simple particular examples explaining the main features of the derivation of the DSIs. In fact we give an interpretation of some known results which makes it possible to introduce and justify the notion of the DSIs.

2.1 Elementary single inequality

Consider a general one-dimensional semilinear heat equation of the form

$$u_t = u_{xx} + f(u, u_x), \quad x \in \mathbf{R}, \;\; t > 0, \tag{9}$$

where $f(p, q)$ is a smooth function, $f(0, 0) = 0$. Let $u = u(x, t) \geq 0$ be a classical sufficiently smooth solution in $Q_T = \mathbf{R} \times [0, T)$, $T \leq \infty$, of the Cauchy problem with smooth initial data $u(x, 0) = u_0(x) \geq 0$, $x \in \mathbf{R}$. The parabolic partial differential equation (PDE) can be treated as a dynamical system of an infinite dimension. We then state the following question: is there a hope to ascribe to a nonlinear parabolic equation like (9) a kind of nontrivial essentially finite-dimensional dynamics of some minimal dimension?

We first discuss a straightforward case of a *one-dimensional* dynamics. We assume that the function u_0 has a bell-shaped form. Then under necessary hypotheses on the nonlinearity, the solution $u(\cdot, t)$ has a similar bell-shaped form on $(0, T)$. Indeed, this follows from the Maximum Principle applied to the linear parabolic equation satisfied by the first derivative $v = u_x$:

$$v_t = \mathbf{L}v \equiv v_{xx} + f'_v(u, v)v_x + f'_u(u, v)v,$$

so that $\mathbf{L}$ is a linear elliptic operator. Therefore, if $v_0(x) = u_0'(x)$ has a unique sign change, then by the Strong Maximum Principle $v(x,t)$ has at most one sign change in x for any fixed $t > 0$ so that the bell-shaped form of the solution is preserved in time. In fact, the bell-shaped form is necessary to ensure the existence of

$$U(t) = \max_x u(x,t) > 0, \quad t \in [0,T), \tag{10}$$

where the maximum is attained at a unique finite point for $t \in [0,T)$. By the standard regularity results for uniformly parabolic equations, $U(t)$ is Lipschitz continuous and $U'(t)$ exists a.e., see e.g. [15]. Since $u_x = 0$ and $u_{xx} \le 0$ at the maximum point, it follows from (9) that $U(t)$ satisfies the *ordinary differential inequality* (ODI)

$$U' \le f(U,0) \;\text{ a.e. in }\; (0,T). \tag{11}$$

This is an example of the simplest DSIs consisting of a *single* inequality.

On the other hand, the parabolic equation (9) admits a family of exact spatially homogeneous "flat" solutions independent of x, $\{\bar{U}(t)\}$, satisfying the ODE

$$\bar{U}' = f(\bar{U},0), \quad t > 0. \tag{12}$$

Inequality (11) and equation (12) have the same right-hand sides. Nevertheless, there exists a striking difference between them. The ODE is valid only for a one-dimensional family of particular flat solutions while the ODI holds for any (bell-shaped) solution of the PDE. Observe that both the ODE and the ODI often play an important role in the derivation of the L^∞_x-estimates on general solutions of the quasilinear heat equations including singular cases of blow-up or extinction in finite time. Moreover, the intersection comparison with the family of flat solutions $\{\bar{U}\}$ proves a general effect of the *eventual bell-shaped form* of initially highly non-monotone solutions. See Section 10 in [26].

Obviously, the same one-dimensional construction is valid for a general parabolic equation (1). The solutions $u(x,t) \ge 0$ are assumed to be smooth in Q_T. Under the same hypothesis on the bell-shaped form, the ODI is given by

$$U' \le F(U,0,0) \;\text{ a.e. in }\; (0,T), \tag{13}$$

and the ODE is

$$\bar{U}' = F(\bar{U},0,0), \quad t > 0. \tag{14}$$

There exists a certain duality between the ODI for general solutions and the associated ODE for the family of particular flat solutions. Nevertheless, it is clear that the ODIs cannot give an essential information concerning the regularity of the general solutions since the only measure of the solution involved is its L^∞_x-norm.

2.2 DSIs generated by two-dimensional invariant subspace

We now consider an example of equation (1) which admits a precise reduction to a two-dimensional DS on a linear subspace invariant under a nonlinear operator, and construct the DSIs. It is important to mention that the exact flat solutions $\bar{U}(t)$ in (14) can be formally treated as the solutions on the *one-dimensional* linear subspace

$$S_1 = \mathrm{Span}\{1(x)\},$$

which is *invariant* under the nonlinear (stationary) operator $\mathbf{F}$ in (1). Indeed, setting $\bar{u} = \bar{U}(t) \cdot 1 \in S_1$ we have that $\mathbf{F}(\bar{u}) = F(\bar{U}, 0, 0) \cdot 1 \in S_1$. Therefore, the ODE (14) is the PDE (1) restricted to the invariant subspace S_1. Such an elementary one-dimensional reduction to the ODEs always exists for such PDEs, and this gives no novelties in the analysis. Of course, a two-dimensional extension of the linear invariant subspace cannot exist for general nonlinear equations.

Let us give an example of such a nontrivial reduction on a *two-dimensional* subspace. Consider the following quasilinear heat equation with the porous medium operator and a nonlinear lower-order term:

$$v_t = (v^m)_{xx} + \alpha v^{2-m} + \beta v, \tag{15}$$

where the exponent $m > 0$, $m \neq 1$, and the coefficients α, β are fixed constants. This equation is known in mechanics, plasma physics and filtration theory. For $\alpha < 0$ and $m > 1$ it describes a filtration-absorption process of seepage on a permeable bed [2]. For $\alpha > 0$ and $m \in (0, 1)$ this equation can be treated as a reaction-diffusion equation with the fast-diffusion propagation and a superlinear reaction term $\sim v^{2-m}$ with the exponent $2 - m > 1$ which admits blow-up of solutions in finite time.

On the other hand, this is a special equation admitting a class of exact solutions on an invariant subspace which are constructed as follows. Without loss of generality we consider the porous medium case $m > 1$. Introducing the pressure variable

$$u = \frac{m}{m-1}\, v^{m-1} > 0$$

we arrive at a quasilinear equation of the form

$$u_t = \mathbf{A}(u) \equiv (m-1)uu_{xx} + (u_x)^2 + \alpha m + \beta(m-1)u. \tag{16}$$

The *quadratic* operator $\mathbf{A}$ is known to admit the two-dimensional linear subspace

$$S_2 = \mathrm{Span}\,\{1,\, x^2\}, \tag{17}$$

see [16] and more general results in [17], [18]. Observe that S_1 is a subspace of S_2 and $S_2 = S_1 \oplus \mathrm{Span}\{x^2\}$. Indeed, one can check that for any

$$\bar{u} = \bar{U} + \frac{1}{2}\bar{W}x^2 \in S_2, \tag{18}$$

where $\{\bar{U}, \bar{W}\}$ are the coefficients of the expansion of the solutions $u(x,t)$ on the subspace S_2, there holds

$$\mathbf{A}(\bar{u}) = (m-1)\bar{U}\bar{W} + \alpha m + \beta(m-1)\bar{U} + \frac{1}{2}[(m+1)\bar{W}^2 + \beta(m-1)\bar{W}] \in S_2, \tag{19}$$

so that S_2 is invariant under $\mathbf{A}$:

$$\mathbf{A}(S_2) \subseteq S_2.$$

Then substituting expression (18) into equation (16) and using (19), we arrive at the conclusion: *the quasilinear parabolic equation* (16) *restricted to the invariant subspace (17) is equivalent to the following two-dimensional DS on the expansion coefficients* $\{\bar{U}(t), \bar{W}(t)\}$*:*

$$\begin{cases} \bar{U}' = (m-1)\bar{U}\bar{W} + \alpha m + \beta(m-1)\bar{U}, \\ \bar{W}' = \frac{1}{2}(m+1)\bar{W}^2 + \frac{1}{2}\beta(m-1)\bar{W}, \quad t > 0. \end{cases} \tag{20}$$

Due to (18), the coefficients of the expansion on S_2 are the first coefficients of the Taylor expansion of the solution in x:

$$\bar{U}(t) = \bar{u}(0,t), \quad \bar{W}(t) = \bar{u}_{xx}(0,t). \tag{21}$$

The second equation in (20) is independent and the DS is easily solved explicitly to give a family of exact solutions. Observe that Kersner's solution [31], with $\alpha = -1$, $\beta = 0$, belongs to this family. Such exact solutions are not invariant with respect to Lie or Lie-Bäcklund groups of transformations. The existence of such solutions is the result of a "nonlinear" invariance under the operator $\mathbf{A}$ [17], which also can be treated as a kind of "nonlinear separation of variables" [16]. These particular solutions played a fundamental role in the study of *singular asymptotic behaviour* of extinction, $\alpha < 0$ [26], [27] and blow-up, $\alpha > 0$ [22], [29].

We do not consider the question of the asymptotic behaviour where the two-dimensional subspace S_2 plays a role of an *attractor* of infinite-dimensional subsets of general orbits for $t \gg 1$. We state the following question: does the structure of the DS (20) reflect the properties of general solutions to the PDE (16) for $t > 0$? Namely, we consider for (16) with, say, $\alpha < 0$ (the absorption case), the Cauchy problem with smooth bounded initial data $u(x,0) = u_0(x) \geq 0$. Let u_0 have a bell-shaped form. We assume that the positive classical solution $u(x,t)$ exists in Q_T. Consider the following DSIs for the range functions

$$\begin{cases} U' \geq (m-1)UW + \alpha m + \beta(m-1)U, \\ W' \geq \frac{1}{2}(m+1)W^2 + \frac{1}{2}\beta(m-1)W \quad \text{a.e. } t \in (0,T), \end{cases} \tag{22}$$

where the right-hand side coincides with that in the DS (20). We will prove that the DS (20), being the two-dimensional projection of (16) onto the invariant subspace S_2, forms the corresponding DSIs (22) which describes the evolution of the general solution to the Cauchy problem in terms of the range functions (cf. (21)) $U(t) = \max_x u(x,t) > 0$ and $W(t) = \min_x u_{xx}(x,t)$. In Section 6 we prove that the orbits of the DS (20) form invariant regions for the orbits satisfying the DSIs (22). This makes it possible to derive useful bounds on the general solutions of the quadratic equation (16).

Observe that the elementary opposite inequality $U' \leq \alpha m + p(m-1)U$ (which as we have shown above corresponds to the DS for the solutions on the one-dimensional invariant subspace S_1) may be added to the system (22) to give an extra evolution estimate on $U(t)$. Sometimes such an extra inequality plays an important role in the asymptotic analysis, see examples in Section 7. A new example of the DSIs derived from the associated DS on a peculiar invariant subspace is given in Section 3.6 for an essentially non-uniformly parabolic equation with the gradient diffusivity operator.

3 Derivation of DSIs for quasilinear equations, $\mathbf{N = 1}$

Consider classical solutions $v = v(x,t) \geq 0$ in Q_T of the general quasilinear heat equation with smooth coefficients in one dimension

$$v_t = (\phi(v))_{xx} + F(v), \tag{23}$$

where $F(0) = 0$ and ϕ satisfies the parabolicity condition

$$\phi'(v) > 0. \tag{24}$$

For convenience, we first consider the "slow-diffusion case" where we apply the pressure transformation known in the filtration theory:

$$u = g(v) \equiv \int^v \frac{\phi'(z)}{z}\, dz, \tag{25}$$

with $g(\infty) = \infty$, and without loss of generality we assume that $g : \mathbf{R}_+ \to \mathbf{R}_+$. We arrive at the equation

$$u_t = \varphi(u)u_{xx} + (u_x)^2 + f(u), \tag{26}$$

with smooth coefficients

$$\varphi(u) = \phi'(g^{-1}(u)) > 0, \quad f(u) = \frac{F(g^{-1}(u))\phi'(g^{-1}(u))}{g^{-1}(u)}. \tag{27}$$

In general, $u(x,t)$ may take negative values which is not important for the analysis. In particular, for the heat operator in (23) with $\phi(v) = v$ we have the transformation $v = e^u$ where u solves (26) with $\varphi \equiv 1$. We always assume that due to the regularity of the equation coefficients, the solutions to (26) are sufficiently smooth in Q_T to perform necessary manipulations. In the case of the degenerate equations with finite solution interfaces we always assume that a suitable approximation can be performed, i.e., we first apply the computations to the family of smooth regularized solutions and the results then follow by passage to the limit.

3.1 Geometric forms and range functions

We consider for (26) the Cauchy problem with smooth bounded initial data $u(x,0) = u_0(x) \geq 0$, $x \in \mathbf{R}$. We begin our construction of suitable two-dimensional DSIs for special classes of solutions for which the "range" functions are C^1-smooth so that the derivatives in the DSIs are understood in the usual sense on $[0,T)$. The DSIs for the precise range functions of the general solution are derived later on.

We distinguish two classes of symmetric initial data and the associated range functions on the solution $u = u(x,t) \geq 0$ which are assumed to be well-defined in $[0,T)$.

Bell-shaped geometry (B-shaped): $u_0 = u_0(|x|)$ is strictly decreasing. Then, by the Maximum Principle, $u = u(|x|,t)$ is decreasing in $|x|$ for all $t \in (0,T)$, $u_x \leq 0$ for $x > 0$, $u_x(0,t) \equiv 0$, and the first range function is the L^∞_x-norm of the solution:

$$U(t) = \max_x u(x,t) \equiv u(0,t), \quad t \in [0,T). \tag{28}$$

The second range function is a *lower bound* on the second spatial derivative:

$$u_{xx} \geq W(t), \quad t \in [0,T); \quad W(0) = \inf_x u_0''. \tag{29}$$

In most cases we assume that $W(t) \leq 0$. For each $t \in [0,T)$ the spatial profile $u(x,t)$ has the absolute maximum at $x = 0$. It is expected that $W(t)$ measures the second derivative at the origin as well, so $W(t) \sim u_{xx}(0,t)$ for $t > 0$, and the behaviour of $U(t)$ and $W(t)$ is of special interest if t approaches a singularity in time T, say blow-up, extinction or quenching. See Section 7.

Inverse bell-shaped geometry (IB-shaped): $u_0 = u_0(|x|)$ is strictly increasing. Then $u_x \geq 0$ for $x > 0$ and $u_x(0,t) \equiv 0$. The first range function is

$$U(t) = \min_x u(x,t) \equiv u(0,t), \quad t \in [0,T). \tag{30}$$

The second range function is an *upper bound* on the second spatial derivative:

$$u_{xx} \leq W(t), \quad t \in [0,T); \quad W(0) = \sup_x u_0''. \tag{31}$$

Usually we assume that $W(t) \geq 0$. This corresponds to the case where $u(x,t)$ has absolute minimum at $x = 0$ and it is expected that $W(t) \sim u_{xx}(0,t) \geq 0$.

It follows from the Maximum Principle that both B-shape and IB-shape of solutions are preserved in time. In the one-dimensional case we can study both cases, unlike the multi-dimensional case $N > 1$, see Section 5. The imposed spatial symmetry of the solutions is not essential. If we omit this restriction, the differential inequalities are understood in the a.e. sense, see below. This does not affect the comparison argument and the derivation of the invariant regions in Section 6.

3.2 Derivation of the two-dimensional DSIs

We now derive the DSIs for two range functions $\{U(t), W(t)\}$. In order to give such a joint evolution of the solutions in terms of one-sided bounds on the solution, $U(t)$, and on its second spatial derivative, $W(t)$, we need a bound on the intermediate first derivative.

Proposition 1 *For any B-shaped or IB-shaped solution $u(x,t)$ in Q_T there holds*

$$(u_x)^2 \leq 2W(u-U). \tag{32}$$

Proof. Assume u is B-shaped and $u_{xx} \geq W$. Multiplying this inequality by $u_x \leq 0$ for $x > 0$ and integrating over the interval $(0,x)$ we arrive at (32). The same proof applies to the IB-shaped solutions. ■

For the B-shaped solutions $u \geq 0$, (32) with $W \leq 0$ implies a simpler estimate

$$(u_x)^2 \leq 2(-W)U, \tag{33}$$

where the right-hand side depends on the range functions $\{U, W\}$ and is independent of the solution u itself.

B-shaped solutions. The first inequality of the corresponding DSIs follows from the equation (26) and (29). Setting in the equation $x = 0$ we have

$$U'(t) = \varphi(U(t))u_{xx}(0,t) + f(U(t)) \geq \varphi(U)W + f(U) \equiv H_1(U,W), \quad t \in (0,T), \tag{34}$$

where $H(\cdot, W)$ is a linear function of W. Since $U(t) \equiv u(0,t)$, by the standard regularity results for uniformly parabolic equations, the function $U(t)$ is a classical C^1-solution of the inequality (34).

We now derive the second inequality. Differentiating twice equation (26) we obtain for $w = u_{xx}$ the following parabolic equation

$$w_t = \mathbf{L}w + (2+\varphi')w^2 + [\varphi''(u_x)^2 + f']w + f''(u_x)^2, \tag{35}$$

where $\mathbf{L}$ is a linear differential operator

$$\mathbf{L}w = \varphi w_{xx} + 2(1+\varphi')u_x w_x.$$

We now construct a *smooth flat subsolution* of (35), $W = W(t) < 0$. Then we have to determine the corresponding lower extended subsystem. By the Maximum Principle, we choose $W(t)$ satisfying for $t \in (0,T)$ the following differential inequality

$$\begin{aligned} W' \le G_-(W,t) \equiv \inf_x \{&[2+\varphi'(u(x,t))]W^2 \\ &+ [\varphi''(u(x,t))(u_x(x,t))^2 + f'(u(x,t))]W + f''(u(x,t))(u_x(x,t))^2\}. \end{aligned} \tag{36}$$

The range of the nonnegative B-shaped solutions is $u(\cdot,t) \in [0,U(t)]$. It follows from the gradient bound (32) that we can use the representation

$$(u_x)^2 = 2q(-W) \quad \text{where } q \in [0, U-u].$$

We then arrive at the inequality: for $W \le 0$

$$G_-(W,t) \ge H_2(U,W) \equiv h_2(U)W^2 + h_1(U)W, \tag{37}$$

where

$$h_1(U) = \sup_p f'(p) - 2\inf_{p,q}\{f''(p)q\}, \quad h_2(U) = 2 + \inf_p \varphi'(p) - 2\sup_{p,q}\{\varphi''(p)q\}.$$

The $\inf_p$ and $\sup_p$ are computed over the range of the B-shaped solution $p \in [0,U]$, and $\inf_{p,q}$ and $\sup_{p,q}$ are taken over the triangle $\{p \in [0,U],\ q \in [0, U-p]\}$. We see that $H_2(\cdot, W)$ is a quadratic polynomial in W. It follows from (33) that the right-hand side can be estimated in terms of a simpler function:

$$H_2(U,W) \ge \tilde{H}_2(U,W) \equiv \tilde{h}_2(U)W^2 + \tilde{h}_1(U)W, \tag{38}$$

where

$$\tilde{h}_1(U) = \sup_p f' - 2U\inf_p [f'']_-, \quad \tilde{h}_2(U) = 2 + \inf_p \varphi' - 2U\sup_p [\varphi'']_+,$$

and $[\cdot]_\pm$ denote the positive or negative parts respectively. As in (37), $\inf_p$ and $\sup_p$ are computed over $[0,U]$. One can see that (38) corresponds to $\inf_{p,q}$ and $\sup_{p,q}$ over the square $\{0 \le p, q \le U\}$.

We thus obtain the desired DSIs:

$$\begin{cases} U' \ge H_1(U,W), \quad t \in (0,T), \\ W' \le H_2(U,W). \end{cases} \tag{39}$$

The function H_2 in the second inequality can be replaces by $\tilde{H}_2$, in which case we call such a system the *reduced* DSIs.

We can fix the equality sign in the second inequality in (39). Then, under the regularity assumptions on the equation coefficients, the right-hand side is a continuous function, $U \in C^1$ and the corresponding solution $W(t)$ is a C^1-function. We call (39) the DSIs corresponding to the quasilinear parabolic equation (26).

Thus we have proved the following result.

Theorem 2 *Under the given hypotheses, for a B-shaped solution $u \geq 0$ in Q_T the range functions $\{U, W\}$ satisfy the DSIs*

$$\begin{cases} U' \geq H_1(U, W), \\ W' = H_2(U, W), \quad t \in (0, T). \end{cases} \tag{40}$$

where the functions H_1 and H_2 are given in (34) and (37).

Observe that the function $U(t)$ satisfies also the elementary inequality with the opposite sign (Section 2.1) $U' \leq H_1(U, 0) \equiv f(U),\ \ t \in (0, T)$.

The reduced DSIs takes the form

$$\begin{cases} U' \geq H_1(U, W) \equiv \varphi(U)W + f(U), \\ W' = \tilde{H}_2(U, W) \equiv \tilde{h}_2(U)W^2 + \tilde{h}_1(U)W. \end{cases} \tag{41}$$

In particular, under the assumptions

$$\varphi'' \leq 0, \quad f'' \geq 0, \tag{42}$$

we deduce the following reduced DSIs

$$\begin{cases} U' \geq \varphi(U)W + f(U), \\ W' = [2 + \varphi'(U)]W^2 + f'(U)W, \quad t \in (0, T), \end{cases} \tag{43}$$

with the associated DS reduced to the first-order ODE

$$\frac{d\bar{U}}{d\bar{W}} = \frac{\varphi(\bar{U})\bar{W} + f(\bar{U})}{(2 + \varphi'(\bar{U}))\bar{W}^2 + f'(\bar{U})\bar{W}}, \quad \bar{U} > 0,\ \ \bar{W} < 0, \tag{44}$$

which plays an important role in the study of blow-up in the quasilinear heat equations, see examples in Section 7.

DSIs for the PME with reaction-absorption. Consider a particular case of the PME operator in (23) with $\phi(v) = v^m$ where $m > 1$, so that the quasilinear heat equation has the form

$$v_t = (v^m)_{xx} + F(v). \tag{45}$$

Then the standard pressure transformation (25),

$$u = \frac{m}{m-1}\, v^{m-1}, \tag{46}$$

gives the quasilinear equation (26),

$$u_t = (m-1)uu_{xx} + (u_x)^2 + f(u), \tag{47}$$

where

$$f(u) = \frac{m-1}{\alpha} u^{(m-2)/(m-1)} F(\alpha u^{1/(m-1)}), \;\; \alpha = \left(\frac{m-1}{m}\right)^{1/(m-1)}. \tag{48}$$

Since $\varphi' \equiv m-1$ and $\varphi'' \equiv 0$, the reduced DSIs (41) corresponding to the parabolic equation (47) is given by

$$\begin{cases} U' \geq (m-1)UW + f(U), \\ W' = (m+1)W^2 + \tilde{h}_1(U)W, \end{cases} \tag{49}$$

where $\tilde{h}_1(U) = \sup_p f' - 2U \inf_p [f'']_-$. Assuming that $f'' \geq 0$ we have $[f'']_- \equiv 0$ and $\tilde{h}_1(U) \equiv \sup_p f'(p) = f'(U)$. Then the DSIs (49) takes the form (cf. (43))

$$\begin{cases} U' \geq (m-1)UW + f(U), \\ W' = (m+1)W^2 + f'(U)W. \end{cases} \tag{50}$$

In a connected component C_+^2 on the $\{U, W\}$-plane, where $H_2(U, W) > 0$ (Section 6), this DSIs gives the first-order ODI

$$\frac{dU}{dW} \geq \frac{(m-1)UW + f(U)}{(m+1)W^2 + f'(U)W}, \quad U > 0, \;\; W < 0. \tag{51}$$

We show in Section 6 that the orbits of the associated DS, (44) determine invariant regions for the DSIs (51).

For the PME with a power-type nonlinearity in the lower-order term,

$$v_t = (v^m)_{xx} + \beta v^p, \tag{52}$$

we obtain the equation

$$u_t = (m-1)uu_{xx} + (u_x)^2 + \gamma u^\delta, \tag{53}$$

where

$$\gamma = (m-1)\beta\alpha^{p-1}, \quad \delta = \frac{p+m-2}{m-1}. \tag{54}$$

The reduced DSIs then takes the form

$$\begin{cases} U' \geq (m-1)UW + \gamma U^\delta, \\ W' = (m+1)W^2 + \tilde{h}_1(U)W, \end{cases} \tag{55}$$

where $\tilde{h}_1(U) = \sup_p \{\gamma\delta p^{\delta-1}\} - 2U \inf_p \{[\gamma\delta(\delta-1)p^{\delta-2}]_-\}$. In particular, if $\gamma > 0$ and $\delta \ge 1$ ($p \ge 1$) then $\tilde{h}_1(U) = \gamma\delta U^{\delta-1}$ and the DSIs is given by

$$\begin{cases} U' \ge (m-1)UW + \gamma U^\delta, \\ W' = (m+1)W^2 + \gamma\delta U^{\delta-1}W. \end{cases}$$

Comparison with the semiconvexity approach. In the case $\gamma = 0$ in (53) the second equation for the subsolution $W(t)$ in (55),

$$W' = H_1(W) \equiv (m+1)W^2,$$

is independent of the first range function U and gives a typical solution

$$W(t) = -\frac{1}{(m+1)t} < 0.$$

By the Maximum Principle and the standard approximation of the weak solutions, it follows that nonnegative integrable weak solutions of the PME satisfy the estimate

$$u_{xx} \ge -\frac{1}{(m+1)t}, \quad t > 0.$$

This corresponds to the semiconvexity approach [1]. The first inequality in (55) then gives

$$U' \ge (m-1)UW = -\frac{m-1}{m+1}\,\frac{U}{t},$$

whence the estimate from below of the bell-shaped solutions

$$U(t) = \max_x u(x,t) \ge C\,t^{-(m-1)/(m+1)}, \quad t > 0.$$

Comparison with the famous exact Zel'dovich-Kompaneetz-Barenblatt solution of the PME shows that both estimates on u and u_{xx} are optimal in this class of solutions. A similar analysis was performed in [26] for the PME with strong absorption (45) where $F(v) = -v^{2-m}$. This gave optimal estimates of u and u_{xx} near the strong extinction singularity occurring in finite time.

IB-shaped solutions. The derivation of the DSIs is similar. For such solutions we have that $u \ge U(t) > 0$ and $w = u_{xx} \le W(t)$ where $W(t)$ is a positive supersolution of the parabolic equation (35).

Theorem 3 *For IB-shaped solutions the DSIs takes the form*

$$\begin{cases} U' \le H_1(U,W) \equiv \varphi(U)W + f(U), \\ W' = H_2(U,W), \quad t \in (0,T), \end{cases} \tag{56}$$

where

$$H_2(U,W) = h_2(U)W^2 + h_1(U)W, \tag{57}$$

$$h_1(U) = \sup_p f'(p) + 2\sup_{p,q} \{f''(p)q\}, \quad h_2(U) = 2 + \sup_p \varphi'(p) + 2\sup_{p,q}\{\varphi''(p)q\}.$$

The $\inf_p$ and $\sup_p$ are computed over $p \ge U$, and $\sup_{p,q}$ is taken over the triangle $\{p \ge U,\ q \in [0, p-U]\}$.

Example: other range functions, nonuniqueness of the DSIs. We consider the semilinear heat equation with strong absorption

$$u_t = u_{xx} - u^p, \quad p < 1. \tag{58}$$

Assume that $u(\cdot, t)$ has an IB-shaped form. We show that the type of the range functions depend on the equation. Namely for equation (58) we slightly change the definition of the range functions and will find a *lower* bound on u_{xx}: $u_{xx} \ge W(t) > 0$ in Q_T. Then the first inequality of the DSIs is given by

$$U' \ge W - U^p. \tag{59}$$

The parabolic equation for $w = u_{xx}$ has the form

$$w_t = w_{xx} - pu^{p-1}w + p(1-p)u^{p-1}(u_x)^2.$$

Therefore there exists a *positive* subsolution, $w \ge W(t)$, where $W > 0$ solves

$$W' = pW \inf_x \{-u^{p-1}(x,t)\} = -pU^{p-1}W < 0. \tag{60}$$

From the DSIs (59), (60) we derive the first-order ODE

$$\frac{dU}{dW} \le \frac{U^p - W}{p\,U^{p-1}W}, \quad U > 0,\ W > 0.$$

Let us derive a new DSIs for equation (58). We perform the natural change $u = \tilde{u}^{2/(1-p)}$, so that $\tilde{u} > 0$ in Q_T satisfies the equation with quadratic nonlinearities

$$\tilde{u}_t = \tilde{u}_{xx} + \gamma\,\frac{(\tilde{u}_x)^2}{\tilde{u}} - \frac{1-p}{2}\,\frac{1}{\tilde{u}}, \quad \gamma = \frac{1+p}{1-p}. \tag{61}$$

It is easy to derive the equation for the second derivative $w = \tilde{u}_{xx}$:

$$w_t = \mathbf{L}w - 5\gamma\frac{(\tilde{u}_x)^2}{\tilde{u}^2}w + 2\gamma\frac{w^2}{\tilde{u}} + 2\gamma\frac{(\tilde{u}_x)^4}{\tilde{u}^3} + \frac{1-p}{2}\frac{w}{\tilde{u}^2} - (1-p)\frac{(\tilde{u}_x)^2}{\tilde{u}^3}, \tag{62}$$

where

$$\mathbf{L}w = w_{xx} + 2\gamma\,\frac{\tilde{u}_x}{\tilde{u}}\,w_x.$$

We now use the standard definition of the range functions for the IB-shaped solutions as an upper bound on the second spatial derivative: $w \le \tilde{W}$, where $\tilde{W} > 0$ is a supersolution of (62). Then by the gradient estimates $(\tilde{u}_x)^2 \le 2\tilde{W}\tilde{u}$ and $(\tilde{u}_x)^4 \le 4\tilde{W}^2\tilde{u}^2$ applied to the nonnegative terms on the right-hand side of (62) we obtain the following reduced DSIs

$$\begin{cases} \tilde{U}' \le \tilde{W} - \frac{1-p}{2}\,(1/\tilde{U}), \\ \tilde{W}' = 10\gamma\,(\tilde{W}^2/\tilde{U}) + \frac{1-p}{2}\,(\tilde{W}/\tilde{U}^2). \end{cases} \tag{63}$$

Therefore the DSIs is not unique, see also comments below.

3.3 DSIs in the fast-diffusion case

In the *fast-diffusion case*, e.g. for

$$\phi(v) = v^m, \quad 0 < m < 1, \tag{64}$$

we perform the transformation

$$u = g(v) \equiv \int_v \frac{\phi'(z)}{z}\, dz, \tag{65}$$

with $g(0) = \infty$, and assuming that $g : \mathbf{R}_+ \to \mathbf{R}_+$, we obtain the equation

$$u_t = \varphi(u) u_{xx} - (u_x)^2 + f(u). \tag{66}$$

The Hamilton-Jacobi term in the right-hand side has the negative sign. This means that in the DSIs presented in Theorems 2 and 3 the coefficient 2 in the multiplier $h_2(U)$ of W^2 in the function $H_2(U, W)$ is replaced by -2.

3.4 DSIs in a.e. sense for solutions without definite shape

Let us show that DSIs are derived in the general case of the solutions without a definite shape.

(I) First, consider a solution $u(x, t) \geq 0$ in Q_T which does not have a definite B-shape but, say,

$$U(t) = \max_x u(x, t) > 0$$

exists for all $t \in [0, T)$. By the regularity for uniformly parabolic equations, $U(t)$ is Lipshchitz continuous and differentiable a.e. The first inequality in the DSIs is then understood in the a.e. sense. Moreover, if at a point $t = t_0 > 0$ there exist both the left derivative $D^- U$ and the right one $D^+ U$, then

$$D^- U(t_0) \leq D^+ U(t_0). \tag{67}$$

Similarly, for a non-IB-shaped solution with $U(t) = \min_x u(x, t) > 0$ for all $t \in [0, T)$, there holds

$$D^- U(t_0) \geq D^+ U(t_0) \tag{68}$$

a.e., and the first inequality of the DSIs reads in the a.e. sense.

(II) In the most general case the two-dimensional DSIs are determined in terms of the precise range functions

$$U(t) = \max_x u(x, t), \quad W(t) = \min_x u_{xx}(x, t), \tag{69}$$

which are uniquely given by the solution having a kind of B-shape so that these functions are assumed to exist on $[0, T)$. It then follows from (35) that at the minimum point of the function $w = u_{xx}$ we have $\mathbf{L}w \geq 0$ and we then arrive at the inequality

$$W' \geq H_2(U, W). \tag{70}$$

The inequality for $U(t)$ then stays the same:

$$U' \geq H_1(U, W). \tag{71}$$

The range functions $\{U(t), W(t)\}$ form a supersolution of the lower extended subsystem. In the general case both inequalities in the DSIs (71), (70) have to be understood in the a.e. sense. This does not change the main conclusions of the comparison in the invariant regions given in Section 6.

3.5 Different forms of the DSIs

The DSIs can be constructed for any quasilinear equation (23). As the form of the equation is not fixed, the DSIs is not uniquely determined. Namely, performing an arbitrary smooth transformation of equation (23)

$$v = E(u) \tag{72}$$

we arrive at the equation

$$u_t = \varphi(u)u_{xx} + \psi(u)(u_x)^2 + f(u), \tag{73}$$

with the coefficients $\varphi = \phi'(E)$, $\psi = (\phi'(E)E')'/E'$, $f = F(E)/E'$. The only difference between (26) and (73) is that in general $\psi \not\equiv 1$. Let us show how this affects the resulting form of the corresponding DSIs. Consider the B-shaped solutions. The first inequality of the DSIs (40) does not depend on ψ and stays the same. In order to derive the second one, we perform similar computations to get that $w = u_{xx}$ solves the parabolic equation

$$w_t = \tilde{\mathbf{L}}w + (2\psi + \varphi')w^2 + [(\varphi'' + 5\psi')(u_x)^2 + f']w + f''(u_x)^2 + \psi''(u_x)^4, \tag{74}$$

where $\tilde{\mathbf{L}}w = \mathbf{L}w + 2\psi u_x w_x$. Therefore, the function H_2 in the second inequality takes the form

$$H_2(U, W) = h_2(U)W^2 + h_1(U)W, \tag{75}$$

$$h_1(U) = \sup_p f'(p) - 2\inf_{p,q}\{f''(p)q\},$$

$$h_2(U) = \inf_p\{2\psi + \varphi'\} - 2\sup_{p,q}\{(\varphi'' + 5\psi')(p)q\} + 4\inf_{p,q}\{\psi''(p)q^2\},$$

with the same definitions of inf, sup as above. In the reduced DSIs we obtain

$$\tilde{H}_2(U, W) = \tilde{h}_2(U)W^2 + \tilde{h}_1(U)W, \tag{76}$$

$$\tilde{h}_1(U) = \sup\, f' - 2U \inf\{[f'']_-\},$$

$$\tilde{h}_2(U) = \inf_p \{2\psi + \varphi'\} - 2U \sup_p \{[\varphi'' + 5\psi']_+\} + 4U^2 \inf\{[\psi'']_-\}.$$

We derive a family of different DSIs depending on the free functional parameter $E(u)$ in (72). We arrive at an optimisation problem of the DSIs with respect to E, a question which can be considered in terms of a certain "minimal" evolution for $t > 0$ described by the range functions $\{U(t), W(t)\}$. On the other hand, the function E can be chosen so that the DSIs describes the correct asymptotic behaviour of the B-shaped solutions as $t \to \infty$ or as $t \to T^- < \infty$. The latter case means finite time blow-up or extinction. See examples in Section 7. It turns out that in several cases optimal estimates from the DSIs correspond to the pressure-like transformation which gives $\psi \equiv \pm 1$. We have considered this case in detail.

3.6 Range functions and DSIs for the p-Laplacian operator

As we have shown, the choice of suitable range functions essentially depends on the equation and, in fact, on the differential properties of weak solutions (which are assumed to be constructed by approximation). We present an example of a non-uniformly parabolic equation where the range functions and the DSIs are determined by the structure of the linear invariant subspace.

Consider the quasilinear parabolic equation with the gradient diffusion term (the p-Laplace operator)

$$v_t = (|v_x|^\sigma v_x)_x + \alpha v^{1/(\sigma+1)} + \beta v, \quad \sigma > -1. \tag{77}$$

Setting $v = u^{(\sigma+1)/\sigma}$ we obtain the equation

$$u_t = \mathbf{B}(u) \equiv \mu(\sigma+1)|u_x|^\sigma \left[u u_{xx} + \frac{1}{\sigma}(u_x)^2 \right] + A + Bu, \tag{78}$$

where $\mu = |(\sigma+1)/\sigma|^\sigma$, $A = \alpha\sigma/(\sigma+1)$, $B = \beta\sigma/(\sigma+1)$. The operator $\mathbf{B}$ admits the invariant subspace [23]

$$S_2 = \mathrm{Span}\{1,\, |x|^\gamma\}, \quad \gamma = (\sigma+2)/(\sigma+1), \tag{79}$$

i.e., $\mathbf{B}(S_2) \subseteq S_2$. Therefore equation (78) has a family of the exact solutions

$$\bar{u}(x,t) = \bar{U}(t) + \bar{W}(t)\,|x|^\gamma, \tag{80}$$

where the expansion coefficients $\{\bar{U}, \bar{W}\}$ satisfy the DS

$$
(81) \qquad \begin{cases} \bar{U}' = \mu\gamma^{\sigma+1}\bar{U}|\bar{W}|^{\sigma}\bar{W} + A + B\bar{U}, \\ \bar{W}' = \mu\gamma^{\sigma+1}(1+\gamma+\gamma/\sigma)|\bar{W}|^{\sigma+2} + B\bar{W}, \quad t>0. \end{cases}
$$

It follows from (80) that, as usual, $\bar{U}(t) = u(0,t)$, unlike the second expansion coefficient which now has the following meaning for B-shaped solutions:

$$
\bar{W}(t) = -\frac{1}{\gamma}[-(|\bar{u}_x|^{\sigma}\bar{u}_x)_x]^{1/(\sigma+1)} < 0.
$$

This gives the appropriate definition of the range functions for equations like (78). Namely, given a general B-shaped solutions $u(x,t) \geq 0$, the range functions

$$
U(t) = \max_x \, u(x,t), \quad W(t) = \min_x \, \{-\frac{1}{\gamma}[-(|u_x|^{\sigma}u_x)_x]^{1/(\sigma+1)}\},
$$

satisfy the following DSIs

$$
(82) \qquad \begin{cases} U' \geq \mu\gamma^{\sigma+1}U|W|^{\sigma}W + A + BU, \\ \bar{W}' \geq \mu\gamma^{\sigma+1}(1+\gamma+\gamma/\sigma)|W|^{\sigma+2} + BW \quad \text{a.e. } t>0. \end{cases}
$$

The DSIs can be constructed for equations of the type (77) with general nonlinearities. The semiconvexity approach based on the Maximum Principle is well-established [12] for the more general doubly nonlinear operator of the equation $v_t = (|(v^m)_x|^{\sigma}(v^m)_x)_x$, where $m, \sigma > 0$. Then the second range function takes the form $|W|^{\sigma+1}(t) \sim (|u_x|^{\sigma}u_x)_x$, where $u = v^{m-1/(\sigma+1)}$ if $m(\sigma+1) \neq 1$ and $u = \ln v$ if $m(\sigma+1) = 1$.

4 Fully nonlinear equations

4.1 Derivation of DSIs, examples

Consider a fully nonlinear parabolic equation

$$
(83) \qquad u_t = F(u, u_x, u_{xx}),
$$

with a smooth function $F(p,q,r)$, $F(0,0,0) = 0$, satisfying the parabolicity condition $F_r(p,q,r) > 0$. Existence, uniqueness and regularity results for such equations are well-known, see [9], [32], [34] and references therein. Without loss of generality we derive the DSIs on $(0,T)$ for smooth symmetric solutions of a definite shape. As in Section 3.4, for general solutions the inequalities of the DSIs are understood in the a.e. sense on $(0,T)$. Fix a B-shaped smooth solution $u(x,t) \ge 0$ in Q_T with the range functions $\{U(t), W(t)\}$. The first inequality of the DSIs takes the standard form

$$U' \ge H_1(U,W) \equiv F(U,0,W). \tag{84}$$

It is worth mentioning that $U(t)$ also satisfies the inequality with the opposite sign

$$U' \le H_1(U,0) \equiv F(U,0,0). \tag{85}$$

In order to derive the second inequality of the DSIs, differentiating twice (83), we deduce that $w = u_{xx}$ satisfies the equation

$$w_t = \mathbf{L}w + F_{qq}w^2 + (2F_{pq}u_x + F_p)w + F_{pp}(u_x)^2, \tag{86}$$

where $\mathbf{L}$ is a linear differential operator of the form

$$\mathbf{L}w = F_r w_{xx} + (F_{rx} + F_q + F_{pr}u_x + F_{qr}w)w_x.$$

From Proposition 1 we have that

$$|u_x| \le [2W(u-U)]^{1/2}, \tag{87}$$

and hence a subsolution $W(t) \le 0$ of (86) satisfies the ODE

$$W' = H_2(U,W) \equiv h_2(U,W)W^2 + h_1(U,W)W + h_0(U,W), \tag{88}$$

where

$$h_0 = \inf_{p,q}\{F_{pp}(p,q,W)q^2\}, \quad h_2 = \inf_{p,q}\{F_{qq}(p,q,W)\},$$
$$h_1 = \sup_{p,q}\{2F_{pq}(p,q,W)q + F_p(p,q,W)\}.$$

Here $\inf_{p,q}$ and $\sup_{p,q}$ are computed over the domain $\{0 \le p \le U,\ |q| \le [2W(p-U)]^{1/2}\}$.

This completes the derivation of the DSIs. Consider two examples.

Dual PME. DSIs take the most simple form in the case of a fully nonlinear equation of the form

$$u_t = g(u_{xx}) + f(u) \quad (g'(r) > 0). \tag{89}$$

The monotone power-like function $g(r) = |r|^{m-1}r$, $m > 1$, corresponds to the so-called *dual porous medium operator* [5]. Since $w = u_{xx}$ solves the parabolic equation

$$w_t = \mathbf{L}w + f'w + f''(u_x)^2, \quad \mathbf{L}w = (g(w))_{xx},$$

the DSIs for the B-shaped solutions takes the form

$$\begin{cases} U' \ge g(W) + f(U), \\ W' = [\sup_p f'(p)]W + \inf_{p,q}\{f''(p)q^2\}. \end{cases} \tag{90}$$

Equation from detonation theory. As the second example, we consider the nonlinear parabolic equation with a different fully nonlinear term

$$u_t = g(uu_{xx}) - \alpha(u_x)^2 + f(u) \quad (g'(s) > 0). \tag{91}$$

In the detonation theory [7] this equation with the functions

$$g(s) = \ln\left(\frac{e^s - 1}{s}\right), \quad f(p) = \ln p, \quad \alpha > 0, \tag{92}$$

describes the instability of the famous Zel'dovich-von Neumann-Doering (ZND) square detonation wave. The smooth solutions $u > 0$ in Q_T are assumed to be of the IB-shape, and a singularity occurs in a finite time T when $u(x,T)$ first touches the singular level $\{u = 0\}$. This is a quenching singularity. Observe that for $u = 0$ both the lower-order term $f(u)$ in (92) and the parabolic differential operator are not well-defined. Several aspects of the asymptotic behaviour as $t \to T^-$ as well as $t \to T^+$ (a possible continuation of the solutions beyond the singularity) have been studied in [19], where the idea of the DSIs was essentially used in the case of the coefficients (92).

Let us derive the DSIs in the case of the general nonlinearities for the IB-solutions. The first inequality follows from the parabolicity condition:

$$U' \le g(UW) + f(U). \tag{93}$$

The second one is derived from the lower extended subsystem: $w = u_{xx}$ solves

$$w_t = \mathbf{L}w + [g'(uw) + g''(uw)(u_x)^2 - 2\alpha]w^2 + f'(u)w + f''(u)(u_x)^2,$$

where

$$\mathbf{L}w = g'uw_{xx} + (g''u^2w_x + 2g''uu_xw + 2g'u_x - 2\alpha u_x)w_x.$$

Using the gradient bound (87) we obtain that the flat supersolution $W(t) > 0$ solves the ODE

$$W' = [\sup_{p,q}\{g'(pW)) + g''(pW)q^2\} - 2\alpha]W^2 + [\sup_p f'(p)]W + \sup_{p,q}\{f''(p)q^2\}. \tag{94}$$

It was shown in [19] that the DSIs (93), (94) for the functions (92) gives sharp estimates on the solutions u and its second derivative u_{xx} near the singular quenching point of the unstability of the ZND detonation wave.

4.2 DSIs is generated by the lower subsystem of PDEs

We now briefly present a geometric interpretation of the two-dimensional DS, which is already discussed in the introduction, for the B-shaped solutions as the tangency conditions to the family of the homogeneous solutions of the lower extended subsystem. Let $u \ge 0$ in Q_T be a smooth B-shaped solution of the equation (83). Consider the system of the PDEs (83), (86) for the functions u, $w = u_{xx}$ of the form (2), where $\inf_{p,q}$ is calculated over the range of $p = u(x,t)$ and $q = u_x(x,t)$ which is contained in the set $\{p \in [0,U],\ |q| \le [2W(p-U)]^{1/2}\}$. The range functions $U(t) = \max_x\, u(x,t),\ \ W(t) = \min_x\, u_{xx}(x,t)$, are known to satisfy a.e. the inequalities

$$U' = F(U,0,w) \ge F(U,0,W) \tag{95}$$

and

$$W' = \Phi(u,u_x,W,0,w_{xx}) \ge \Phi(u,u_x,W,0,0) \ge \Phi_-(W,0,0;U,W). \tag{96}$$

These inequalities form the DSIs for a.e. $t \in (0,T)$. The associated DS,

$$\begin{cases} \bar U' = F(\bar U,0,\bar W), \\ \bar W' = \Phi_-(\bar W,0,0;\bar U,\bar W), \quad t > 0, \end{cases} \tag{97}$$

then gives spatially homogeneous solutions of the lower extended subsystem and the inequalities in (95), (96) are the tangency conditions of the general solution $\{u(x,t), w(x,t)\}$ of the PDEs system (2) with the solution $\{\bar U(t), \bar W(t)\}$ of the DS (97).

5 DSIs in the multi-dimensional case

In this section we briefly describe the main features of the derivation of the two-dimensional DSIs for $N > 1$. Consider the quasilinear heat equation

$$v_t = \Delta\phi(v) + F(v), \quad x \in \mathbf{R}^N,\ \ t > 0. \tag{98}$$

Without loss of generality, we consider the slow-diffusion case where we perform transformation (25) to obtain the equation

$$u_t = \varphi(u)\Delta u + |\nabla u|^2 + f(u). \tag{99}$$

We introduce a similar definition of the B-shaped, sufficiently smooth, symmetric solutions $u = u(|x|,t)$ nonincreasing in $|x|$, with the range functions $\{U, W\}$ given by

$$U(t) = \max_x\, u(x,t) > 0; \quad \Delta u(\cdot,t) \ge W(t) \le 0, \quad W(0) = \inf_x\, \Delta u_0. \tag{100}$$

For the IB-shaped smooth symmetric solutions we have

$$U(t) = \min_x\, u(x,t) > 0; \quad \Delta u(\cdot,t) \le W(t) \ge 0, \quad W(0) = \sup_x\, \Delta u_0. \tag{101}$$

Proposition 4 *For B-shaped or IB-shaped symmetric solutions $u(x,t)$ in Q_T there holds*

$$|\nabla u|^2 \le 2W(u-U). \tag{102}$$

Proof. It is similar to the proof of Proposition 1. In the symmetric case of B-shaped solutions we have

$$\Delta u = u_{rr} + \frac{N-1}{r}\, u_r \ge W, \quad r=|x|>0.$$

Multiplying this inequality by $u_r \le 0$ and integrating over $(0,r)$ we obtain (102).

■

Unlike the one-dimensional case, for $N>1$ in the slow-diffusion case we can consider the B-shaped solutions only.

B-shaped solutions. Assume (100) holds. The first inequality which follows from (99) stays the same:

$$U' \ge \varphi(U)W + f(U). \tag{103}$$

The equation for $w = \Delta u$ takes the form

$$w_t = \mathbf{L}w + \varphi' w^2 + (\varphi''|\nabla u|^2 + f'(u))w + f''(u)|\nabla u|^2 + 2\sum_{(k)} |\nabla u_{x_k}|^2, \tag{104}$$

with the linear operator

$$\mathbf{L}w = \varphi \Delta w + 2(1+\varphi')\nabla u \cdot \nabla w.$$

As in [1], using the Cauchy-Buniakowskii-Schwarz inequality

$$\sum_{(k)} |\nabla u_{x_k}|^2 \equiv \sum_{(i,j)} (u_{x_i x_j})^2 \ge \frac{1}{N}\, w^2, \tag{105}$$

we obtain the parabolic differential inequality

$$w_t \ge \mathbf{L}w + (2/N + \varphi')w^2 + (\varphi''|\nabla u|^2 + f'(u))w + f''(u)|\nabla u|^2, \tag{106}$$

and the Maximum Principle applies. Therefore, in constructing the subsolution $W(t) \le 0$ we replace the coefficient 2 in the coefficient $h_2(U)$ of the first term of H_2 (and in $\tilde{H}_2$) in (37) by $2/N$, and this is the only change in the derivation of the DSIs. Observe that, since the inequality (105) is essentially one-sided, in general we cannot derive a DSIs for the IB-shaped solutions.

A quite similar procedure applies to the fast-diffusion case when the equation is

$$u_t = \varphi(u)\Delta u - |\nabla u|^2 + f(u). \tag{107}$$

In view of the negative sign of the first-order operator on the right-hand side, the DSIs is constructed for the IB-shaped solutions where the inequality (105) can be used. One can extend the above approach to a fully nonlinear parabolic equation of the form $u_t = F(u, |\nabla u|, \Delta u)$, with a smooth function F satisfying the parabolicity condition.

6 Invariant regions: comparison of the DSIs and DSs

In this section we compare the orbits $\gamma = \{U(t), W(t)\}$ of the two-dimensional DSIs and the corresponding orbits $\bar\gamma = \{\bar U(t), \bar W(t)\}$ of the associated DSs. We consider the DSIs in the maximal generality so that this analysis covers all types of the nonlinear parabolic equations discussed above.

6.1 B-shaped solutions

Consider the DSIs

$$\begin{cases} U' \ge H_1(U,W), \\ W' = H_2(U,W), \quad t \in (0,T), \end{cases} \tag{108}$$

generated by a B-shaped solution $u \ge 0$ in Q_T. We may assume that the range functions $U(t)$ and $W(t)$ are smooth on $[0,T)$. We consider the orbit $\gamma_0 = \{U(t), W(t),\ t \in [0,T)\}$ on the $\{U,W\}$-plane with the initial point $P_0 = \{U_0, W_0\}$ where

$$U_0 = U(0) = \max_x u_0 > 0, \quad W_0 = W(0) = \inf u_0'' < 0. \tag{109}$$

The corresponding orbit $\bar\gamma_0 = \{\bar U(t), \bar W(t),\ t \ge 0\}$ of the associated DS:

$$\begin{cases} \bar U' = H_1(\bar U, \bar W), \\ \bar W' = H_2(\bar U, \bar W), \quad t > 0, \end{cases} \tag{110}$$

starts at the same point P_0. We assume that the first-order ordinary differential equation

$$\frac{d\bar U}{d\bar W} = H(\bar U, \bar W) \equiv \frac{H_1(\bar U, \bar W)}{H_2(\bar U, \bar W)}, \quad \bar P \in \Omega_2 = \{\bar U > 0,\ \bar W < 0\}, \tag{111}$$

admits a unique orbit $\bar\gamma_0$ passing through the initial point P_0 which is a regular point of the ODE (111), so that the uniqueness is true for any point P_0' in a neighbourhood of P_0.

Let us introduce some notations. Denote by C_+^1 and C_+^2 (resp. C_-^1 and C_-^2) the connected components of the positivity (resp. negativity) set of the functions H_1 and H_2 in Ω_2, which contain the initial point P_0. Similarly, D_+^1 and D_+^2 (resp. D_-^1 and D_-^2) denote the connected components of the positivity (resp. negativity) set of the partial U-derivatives of these functions H_{1U}' and H_{2U}' in Ω_2, which contain P_0. Finally, E_+^1 and E_+^2 (resp. E_-^1 and E_-^2) denote the connected components of the positivity (resp. negativity) set of the partial W-derivatives H_{1W}' and H_{2W}' in Ω_2 containing P_0.

We now state the main result on the comparison of the orbits given by the DSIs and the associated DS. We have that in C_+^2 or C_-^2 the region above the DS-curve $\bar\gamma_0$ on the $\{U,W\}$-plane is *invariant* for the corresponding DSIs.

Theorem 5 *Consider the orbits γ_0 and $\bar{\gamma}_0$ on the $\{U, W\}$-plane.*

(i) If $P_0 \in C^2_+$ then the DSIs-orbit γ_0 stays above the DS-orbit $\bar{\gamma}_0$ for $W > W_0$ in C^2_+, i.e., if $P = \{U, W\} \in \gamma_0$, $\bar{P} = \{\bar{U}, W\} \in \bar{\gamma}_0$ for $W > W_0$ and $P, \bar{P} \in C^2_+$, then $U \geq \bar{U}$.

(ii) If $P_0 \in C^2_-$ then the DSIs-orbit γ_0 stays above the DS-orbit $\bar{\gamma}_0$ for $W < W_0$ in C^2_+.

Proof. Consider the case (i). While $P \in C^2_+$ we have that $dW = H_2 dt > 0$ so that (108) implies that

$$\frac{dU}{dW} \geq H(U, W), \quad W > W_0, \;\; P \in C^2_+. \tag{112}$$

Therefore, the result follows by the standard comparison for the first-order ODEs with smooth right-hand sides. ∎

Comparison for inequalities in the a.e. sense. Assume that the first inequality in (108) is understood in the a.e. sense which corresponds to solutions without definite B or IB-shapes, see Section 3.4. Then Theorem 5 is still true. Indeed, if in the proof we assume that the ODI (112) holds for $W > W_0$ a.e., then the orbit γ_0 cannot intersect $\bar{\gamma}_0$ from above in the corresponding domain on the $\{U, W\}$-plane. Such an intersection would mean that γ_0 would intersect infinitely many other integral curves $\{\bar{\gamma}\}$ of the ODE (111) which stay in a small neighbourhood of $\bar{\gamma}_0$. Since in any such neighbourhood of $\bar{\gamma}_0$ there exist a dense subset of integral curves of a non-zero measure on $\mathbf{R}$, this would contradict the fact that dU/dW exists a.e. The same result applies to the DSIs (5) where both inequalities are true a.e.

Let us describe further possibilities to extend the comparison in Theorem 5 to the estimates on the range functions $U(t)$ and $W(t)$. Without loss of generality we consider the case $P_0 \in C^2_+$. The following conclusions hold.

I. Comparison of $U(W)$ and $\bar{U}(W)$. Under the conditions of Theorem 5 (i), denoting by

$$\bar{U} = \bar{U}(W), \quad W > W_0, \;\; \bar{\gamma}_0 \subset C^2_+, \tag{113}$$

the representation of monotone (in W) segment of $\bar{\gamma}_0$, we have

$$U(W) \geq \bar{U}(W), \quad W > W_0, \;\; \gamma_0, \bar{\gamma}_0 \subset C^2_+. \tag{114}$$

II. Estimate on $W(t)$. It then follows from the second inequality of the DSIs that the following autonomous inequality holds

$$W' = H_2(U, W) \geq H_2(\bar{U}(W), W), \;\; W > W_0, \;\; \{\bar{U}(W) \leq U \leq U(W), W\} \subset D^2_+, \tag{115}$$

or

$$W' \leq H_2(\bar{U}(W), W), \;\; W > W_0, \;\; \{\bar{U}(W) \leq U \leq U(W), W\} \subset D^2_-. \tag{116}$$

Integrating these inequalities gives bounds on the function $W(t)$ in terms of the solution $\bar{W}(t)$ of the DS.

III. ESTIMATE ON $U(t)$. Assuming that γ_0, $\bar{\gamma}_0 \subset C_+^2 \cap C_+^1$ we obtain that $dU > 0$ and (108) implies that

$$\frac{dW}{dU} \le \frac{H_2}{H_1}, \quad U > U_0.$$

Therefore,

$$W(U) \le \bar{W}(U), \quad U > U_0. \tag{117}$$

Then the first inequality of the DSIs yields the autonomous inequality

$$U' \ge H_1(U, W) \ge H_1(U, \bar{W}(U)), \quad U > U_0, \tag{118}$$

provided that $\{U, W(U) \le W \le \bar{W}(U)\} \subset E_+^1$. Integrating it we obtain an estimate on $U(t)$ in terms of the DS-solution $\bar{U}(t)$.

The analysis of the case $P_0 \in C_-^2$ and related ones is similar.

6.2 IB-shaped solution

The main results are quite similar. The DSIs takes the form

$$\begin{cases} U' \le H_1(U, W), \\ W' = H_2(U, W), \quad t \in (0, T), \end{cases} \tag{119}$$

where in the first inequality the sign is replaced by the opposite one. This obviously changes the geometric interpretation of the invariant regions in Theorem 5.

7 Applications to singular blow-up behaviour

7.1 Semilinear and quasilinear heat equations

We now apply the two-dimensional DSIs to derive sharp bounds on the singular blow-up solutions of the quasilinear heat equation (98), which in the pressure representation takes the form

$$u_t = \varphi(u)\Delta u + |\nabla u|^2 + f(u), \quad x \in \mathbf{R}^N, \ t > 0. \tag{120}$$

As usual in the applications to combustion problems, we consider a symmetric B-shaped solution $u \ge 0$ in Q_T which blows up in time T:

$$\sup_x u(x, t) \equiv u(0, t) \to \infty, \quad t \to T^-.$$

There exist several important blow-up problems concerning the asymptotic behaviour as $t \to T^-$. A survey of recent results on blow-up in quasilinear parabolic equations can be found in [35]. Some of the problems in the maximal generality are still open. In particular, the problem on the classification of asymptotic stable combustion patterns in terms of the nonlinear coefficients $\varphi(u)$ and $f(u)$ of the equation. Below we study such a problem on the basis of DSIs for a wide class of equations. We impose the natural conditions

$$\varphi'' \le 0, \quad f \ge 0, \quad f' \ge 0, \;\; f'' \ge 0.$$

Then the DSIs for B-shaped solutions takes the form (see Sections 3.2 and 5)

$$\begin{cases} U' \ge \varphi(U)W + f(U) \equiv H_1(U,W), \\ W' = [2/N + \varphi'(U)]W^2 + f'(U)W \equiv H_2(U,W), \quad t > 0. \end{cases} \tag{121}$$

We are interested in the behaviour of the solutions to (121) near the blow-up time where

$$U = U(t) \to +\infty. \tag{122}$$

Consider the blow-up orbits of the corresponding DS:

$$\begin{cases} \bar{U}' = \varphi(\bar{U})\bar{W} + f(\bar{U}), \\ \bar{W}' = [2/N + \varphi'(\bar{U})]\bar{W}^2 + f'(\bar{U})\bar{W}, \quad t > 0, \end{cases} \tag{123}$$

which reduces to the first-order ODE

$$\frac{d\bar{U}}{d\bar{W}} = \frac{\varphi(\bar{U})\bar{W} + f(\bar{U})}{[2/N + \varphi'(\bar{U})]\bar{W}^2 + f'(\bar{U})\bar{W}}. \tag{124}$$

(DSs of a similar structure occur in the application of the formal averaging method, see p. 203 in [35].) In the second quarter $\Omega_2 = \{\bar{U} > 0, \bar{W} < 0\}$ there exist two curves: the zero-isocline $Z_0 : \;\; \bar{W} = -f(\bar{U})/\varphi(\bar{U}) < 0$, and the infinity-isocline $Z_\infty : \;\; \bar{W} = -f'(\bar{U})/[2/N + \varphi'(\bar{U})] < 0$. We assume that both isoclines are monotone in $\bar{U}$ and $\bar{W}$ in the domain where $\bar{U} \gg 1$ and end at the point $\bar{U} = +\infty$, $\bar{W} = -\infty$. It then follows from the phase-plane portrait of equation (124) that the blow-up orbits satisfying (122) exist provided that

$$Z_\infty \;\text{ stays above }\; Z_0 \;\text{ for }\; \bar{U} \gg 1. \tag{125}$$

We then need to impose the necessary condition of blow-up on the combustion term

$$\int^\infty dz/f(z) < \infty. \tag{126}$$

We first show that the DSIs gives sharp estimates on the solutions of the semilinear equation, and finally extend them to a wide class of quasilinear equations.

Semilinear equation. If $\varphi \equiv 1$ then the first-order ODE

$$\frac{d\bar{U}}{d\bar{W}} = \frac{\bar{W} + f(\bar{U})}{(2/N)\bar{W}^2 + f'(\bar{U})\bar{W}} \tag{127}$$

is easily integrated, and in Ω_2 the general solution is given by

$$\bar{W}(\bar{U}) = \rho(\bar{U}) f(\bar{U}), \tag{128}$$

where the function $\rho < 0$ satisfies the ODE

$$\rho' = \frac{\rho^2}{\rho + 1}\left(\frac{2}{N} - \frac{f'}{f}\right)$$

and hence the algebraic equation

$$|\rho| e^{1/|\rho|} = \frac{C}{f(\bar{U})}\, e^{2\bar{U}/N}, \tag{129}$$

$C > 0$ being a free constant of integration.

By Theorem 5 (ii), for any solution of (120) with the initial data satisfying $P_0 = (U_0, W_0) \in C_-^2$ (i.e., P_0 is above the infinity-isocline Z_∞ on the $\{U, W\}$-plane) the DSIs orbit $\gamma_0 = \{U(t), W(t)\}$ stays above the corresponding DS-orbit $\bar{\gamma}_0$ starting at the same initial point P_0. Therefore,

$$W(U) \ge \bar{W}(U), \quad U \gg 1, \tag{130}$$

where $\bar{W}$ is a solution to (127) satisfying the initial condition $\bar{W}(U_0) = W_0$. Then from the first inequality in (121), $\varphi \equiv 1$, we get

$$U' \ge W(U) + f(U) \ge \bar{W}(U) + f(U) > 0, \quad t \approx T^-. \tag{131}$$

Integrating it over (t, T) we obtain an *upper* estimate on the L^∞-norm of the solution, $U(t) = \max_x u(x,t)$, as $t \to T$:

$$\int_{U(t)}^{\infty} \frac{dz}{\bar{W}(z) + f(z)} \ge T - t. \tag{132}$$

On the other hand, a lower estimate on $U(t)$ follows from the elementary inequality $U' \le f(U)$ so that

$$\int_{U(t)}^{\infty} \frac{dz}{f(z)} \le T - t. \tag{133}$$

Finally, (130), (132) or (133) imply a *lower* bound $W(t) \ge \bar{W}(U(t))$ on the second derivative $W(t) < 0$ near the blow-up time. We will show that these estimates are *optimal*. Consider first two typical examples.

Example 1. If in (98) $F(v) = v^p, \ p > 1$, then $v = e^u$ and the equation takes the form

$$u_t = \Delta u + |\nabla u|^2 + e^{(p-1)u}.$$

The algebraic equation (129), $|\rho| e^{1/|\rho|} = C e^{(2/N+1-p)\bar{U}}$, has a blow-up solution $\rho < 0$ defined for all $\bar{U} \gg 1$ provided that $p < p_F = 1 + 2/N$, where p_F is the well-known critical Fujita exponent for the semilinear heat equation and for $p \in (1, p_F)$ any solution $u \not\equiv 0$ blows up. This is reflected by the phase portrait of (127) in this case. We then conclude that for $\bar{U} \gg 1$

$$\rho(\bar{U}) = -\frac{c_0}{\bar{U}} \left[1 + O\left(\frac{\ln \bar{U}}{\bar{U}}\right)\right], \quad c_0 > 0.$$

The blow-up orbits (128) of the DS satisfy

$$\bar{W}(\bar{U}) = -\frac{c_0}{\bar{U}}\, e^{(p-1)\bar{U}} \left[1 + O\left(\frac{\ln \bar{U}}{\bar{U}}\right)\right], \quad \bar{U} \gg 1. \tag{134}$$

Therefore, the first term in the right-hand side of (131) gives a small correction:

$$U' \geq e^{(p-1)U} \left[1 - \frac{c_0}{U}(1 + o(1))\right], \quad t \approx T^-.$$

Together with (133) we then obtain the estimate

$$U(t) = -\frac{1}{p-1} \ln[(p-1)(T-t)] + O\left(\frac{1}{|\ln(T-t)|}\right), \quad t \to T^-. \tag{135}$$

On the other hand, at the same time we obtain a lower bound on $\Delta u(\cdot, t)$. Namely, (134) and (135) imply that

$$\Delta u(x,t) \geq W(t) \geq \bar{W}(t) = \bar{W}(U(t)) = -\frac{c_1}{(T-t)|\ln(T-t)|}\,(1 + o(1)) \tag{136}$$

as $t \to T^-$. Both estimates (135) and (136) are known to be sharp and correspond to the actual generic blow-up asymptotic behaviour for the semilinear heat equation with the source term $F(v) = v^p$. See the results in [3], [6], [13], [30], [20], [36] and a survey in Chapter 4, [35].

Example 2. Setting $v = e^u$ in the semilinear heat equation with "weak" nonlinearity

$$v_t = \Delta v + v \ln^\beta v \quad (v \geq 1), \quad \beta > 1, \tag{137}$$

we deduce the equation

$$u_t = \Delta u + |\nabla u|^2 + u^\beta. \tag{138}$$

Observe that $\beta = 2$ corresponds to the *regional* blow-up where the blow-up set of B-shaped solutions has the measure 2π, see p. 295 in [35] and [25]. We then obtain from (129) with $f(u) = u^\beta$ that for $\bar{U} \gg 1$

$$\rho(\bar{U}) = -\frac{N}{2\bar{U}} \left(1 + \frac{(\beta-1)N}{2}\,\frac{\ln \bar{U}}{\bar{U}} + ...\right).$$

Finally, using similar calculations we arrive at the estimates of the blow-up orbits of the DSIs $U(t) = [(\beta - 1)(T - t)]^{-1/(\beta-1)}(1 + o(1))$ and $W(t) \geq -c_1(T - t)^{-1}(1 + o(1))$. Both estimates are sharp and correspond to the blow-up behaviour as $t \to T^-$ described by the Hamilton-Jacobi equation $u_t = |\nabla u|^2 + u^\beta$, see p. 292 in [35] and [25], [28].

Quasilinear equations. Due to the standard results of the continuous dependence for the DSs, one can conclude that the above analysis is extended to the quasilinear equations with sufficiently arbitrary "slow varying" functions $\varphi(u)$ in (120). Let us perform some simple computations in order to derive the asymptotics of the blow-up orbits. It follows from (124) under the hypothesis (125) that if

$$\varphi'(u) \to 0, \quad u \to \infty \tag{139}$$

monotonically, then the blow-up orbits of the DS satisfy

$$\frac{d\bar{W}}{d\bar{U}} \approx \bar{W}\frac{(2/N)\bar{W} + f'}{\varphi\bar{W} + f}.$$

Setting as in (128) $W = \rho f$ we obtain

$$\rho' \approx \frac{\rho^2}{\varphi\rho + 1}\left(\frac{2}{N} - \varphi\frac{f'}{f}\right) \approx \rho^2\left(\frac{2}{N} - \varphi\frac{f'}{f}\right)$$

provided that

$$\varphi(u)\rho(u) \to 0, \quad u \to \infty. \tag{140}$$

We then deduce that

$$\rho(\bar{U}) \approx -\left[\frac{2\bar{U}}{N} - \int^{\bar{U}} \frac{\varphi(z)f'(z)}{f(z)}\,dz\right]^{-1}.$$

In particular, if

$$\frac{\varphi(u)f'(u)}{f(u)} \to 0, \quad u \to \infty, \tag{141}$$

we conclude that $\rho(\bar{U}) = -(N/2\bar{U})(1 + o(1))$ as $\bar{U} \to \infty$. Observe that in this case, (140) follows from (139) by the L'Hospital rule. Finally, under the assumptions (139), (141) we conclude that the blow-up orbits of the DS satisfy

$$\bar{W}(\bar{U}) = -\frac{N}{2}\frac{f(\bar{U})}{\bar{U}}(1 + o(1)), \quad \bar{U} \gg 1.$$

By Theorem 5 (ii), as in the semilinear case, we get for $t \approx T^-$

$$U' \geq \varphi(U)W(U) + f(U) \geq \varphi(U)\bar{W}(U) + f(U) = f(U)\left[1 - \frac{N}{2}\frac{\varphi(U)}{U}(1 + o(1))\right],$$

where $\varphi(U)/U \to 0$ as $U \to \infty$ by (139). Together with the opposite inequality $U' \leq f(U)$ this gives the sharp result on the blow-up rate

$$G(U(t)) \equiv \int_{U(t)}^{\infty} \frac{dz}{f(z)} = (T-t)\left[1 + O\left(\frac{\varphi(G^{-1}(T-t))}{G^{-1}(T-t)}\right)\right]. \tag{142}$$

Then the inequality

$$W(t) \geq \bar{W}(t) = -\frac{N}{2}\frac{f(U(t))}{U(t)}(1+o(1)) \tag{143}$$

gives a sharp lower bound on the second derivative.

The analysis of blow-up patterns for the general *quasilinear* heat equation (120) is technically rather complicated unlike the *semilinear* case with $\varphi \equiv 1$, where the linear semigroup approach based on the convolution for the linear heat operator $\partial/\partial t - \Delta$ applies, see the lists of references in Chapter 4 of [35] and in the papers [3], [13], [30]. For the case $\varphi(u) \not\equiv 1$ the blow-up analysis is available for asymptotically small quasilinear perturbations with $\varphi(u) = 1 + o(1)$ as $u \to \infty$ [3]. On the other hand, different blow-up patterns were constructed in [4] for rather arbitrary slow varying functions $\varphi(u)$ (the question of the stability of the patterns was not considered). We claim that the above estimates on the range functions $U(t)$ and $W(t)$ of the two-dimensional DSIs which, as we have shown, are available for a wide class of the quasilinear heat equations, make it possible to complete the stability analysis of the first blow-up pattern. The estimates provide us with the correct rates of rescaling and give the compactness of the rescaled orbits, defines according to the asymptotic behaviour, see e.g. [35], [3] and [28]. The convergence to a *nontrivial* (non-uniform) limit state then follows from the gradient bound first established in [15] for the semilinear equations and extended to arbitrary quasilinear equations in [21], see also Section 4 in [28]. We present such general blow-up results for quasilinear heat equations in a forthcoming paper.

7.2 Blow-up and degeneracy for a fully nonlinear equation

We now study the problem on the blow-up behaviour for a fully nonlinear parabolic equation from combustion theory. Consider the one-dimensional equation similar to that studied in Section 4.1 with the quadratic Hamilton-Jacobi operator

$$u_t = g(u_{xx}) + (u_x)^2 + f(u), \quad x \in \mathbf{R}, \ t > 0, \tag{144}$$

where g, f are smooth functions, $g(0) = f(0) = 0$, $g' > 0$, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$ and the necessary blow-up condition (126) holds. We fix a B-shaped solution $u \geq 0$ in Q_T which blows up at $t = T$. We pose the following question: for which nonlinear coefficients g and f the blow-up behaviour as $t \to T^-$ does not essentially depends of the second order operator

$g(u_{xx})$, i.e., the asymptotic blow-up behaviour is described by the first-order Hamilton-Jacobi equation

(145) $$u_t = (u_x)^2 + f(u).$$

Let us remind that such an asymptotic degeneracy of the equation near blow-up point is known for some particular quasilinear heat equations, see Chapter 4 in [35] and [25], [28].

The DSIs corresponding to the nonlinear equation (144) takes the form (cf. (90))

(146) $$\begin{cases} U' \geq g(W) + f(U), \\ W' = 2W^2 + f'(U)W. \end{cases}$$

The phase portrait of the associated DS,

(147) $$\frac{d\bar{U}}{d\bar{W}} = \frac{g(\bar{W}) + f(\bar{U})}{\bar{W}[2\bar{W} + f'(\bar{U})]},$$

contains the zero-isocline $Z_0 : \bar{W} = g^{-1}(-f(\bar{U}))$ and the infinity-isocline $Z_\infty : \bar{W} = -\frac{1}{2}f'(\bar{U})$ (we are assuming that $f'(u) \to \infty$ as $u \to \infty$), and we impose the condition of blow-up (125), i.e.,

(148) $$g^{-1}(-f(u)) < -\frac{1}{2}f'(u) \quad \text{for } u \gg 1.$$

In order to find the asymptotics of the blow-up orbits of (147) we perform the same change $\bar{W}(\bar{U}) = \rho(\bar{U})f$, and obtain the ODE

$$\rho' = 2\rho^2 \left(1 - \frac{g(\rho f)f'}{2\rho f^2}\right)\left(1 + \frac{g(\rho f)}{f}\right)^{-1} = 2\rho^2(1 + o(1))$$

provided that

$$\frac{g(\rho f)f'}{2\rho f^2}(u), \quad \frac{g(\rho f)}{f}(u) \to 0, \quad u \to \infty.$$

It then follows that for $\bar{U} \gg 1$

(149) $$\rho(\bar{U}) = -\frac{1}{2\bar{U}}(1 + o(1))$$

does not essentially depend on the function g if

(150) $$\frac{ug(-f(u)/2u)f'}{f^2(u)}, \quad \frac{g(-f(u)/2u)}{f(u)} \to 0, \quad u \to \infty.$$

In particular, if $g(u) = |u|^{m-1}u$ and $f(u) = u^\beta$, $m > 0$, $\beta > 1$, then all the three conditions (148), (150) are valid either if $m \leq 1$ or for $m > 1$ with $\beta < m/(m-1)$.

Using (149) we derive the estimates on the blow-up orbits of the DS following from the expansion $\bar{W} = -[f(\bar{U})/2\bar{U}](1 + o(1))$ independent of g, which by Theorem 5 are extended to the blow-up solutions to (144) exactly in the same way as above for the quasilinear heat equations. We finally conclude that the rescaled factors of the blow-up solutions to (144) do not depend on the operator $g(u_{xx})$. This means that in the limit $t \to T^-$ the B-shaped blow-up rescaled solutions converge to the rescaled orbits of the first-order equation (145). Therefore, in the asymptotic analysis we fall in the scope of the theory of singularly perturbed dynamical systems similar to that proposed in [25], [28].

In a similar way, the effect of the degeneracy of the blow-up singularity is studied for the nonlinear equation from the detonation theory $u_t = g(uu_{xx}) + (u_x)^2 + f(u)$, where the fully nonlinear term depends also on the solution u. A detailed analysis of the degenerate quenching phenomenon based on the singular perturbation theory for an equation like that is presented in [19].

8 Multi-dimensional DSIs

8.1 Derivation of the four-dimensional DSIs

We have shown that, in general, two is the minimal dimension of the DSIs which can be derived for a wide class of parabolic equations and it describes a one-sided evolution of the solution and its second-order spatial derivative. The dimension of the DSIs increases if we include bounds from above and below. As an example, consider the fully nonlinear uniformly parabolic equation (1) and let us introduce four range functions. We assume that for $t \geq 0$ there exist bounded ranges

$$\operatorname{Im} u(\cdot,t) = [U_-(t), U_+(t)], \quad \operatorname{Im} u_{xx}(\cdot,t) = [W_-(t), W_+(t)], \tag{151}$$

where $W_- \leq 0$, $W_+ \geq 0$. It then follows that, as in the proof of Proposition 1, the range of the first spatial derivative is estimated as

$$|u_x| \leq V(U_\pm, W_\pm, u) \equiv \min\{[2W_+(u-U_-)]^{1/2},\ [2W_-(u-U_+)]^{1/2}\}.$$

We now consider the extended system (2) with the operator Φ given in the equation (86) and introduce two extended subsystems, the lower and upper ones:

$$\begin{cases} u_t = F(u, u_x, w), \\ w_t = \Phi_-(w, w_x, w_{xx}; U_\pm, W_\pm) \equiv \inf_{p,q}\ \Phi(p, q, w, w_x, w_{xx}), \end{cases} \tag{152}$$

and

$$\begin{cases} u_t = F(u, u_x, w), \\ w_t = \Phi_+(w, w_x, w_{xx}; U_\pm, W_\pm) \equiv \sup_{p,q}\ \Phi(p, q, w, w_x, w_{xx}), \end{cases} \tag{153}$$

where $\inf_{p,q}$, $\sup_{p,q}$ are calculated over the domain $\{p \in [U_-(t), U_+(t)],\ |q| \le V(U_\pm, W_\pm, p)\}$. These two subsystems are not independent and are coupled by means of all the four range functions on the right-hand sides.

Let us determine the DS for the functions $\{\bar U_\pm(t), \bar W_\pm(t)\}$. The first pair $\{\bar U_+(t), \bar W_-(t)\}$ is a homogeneous solution of the lower subsystem, while the second pair $\{\bar U_-(t), \bar W_+(t)\}$ solves the upper subsystem:

$$
\text{(154)} \qquad \begin{cases} \bar U_+' = F(\bar U_+, 0, \bar W_-), \\ \bar W_-' = \Phi_-(\bar W_-, 0, 0; \bar U_\pm, \bar W_\pm), \\ \bar U_-' = F(\bar U_-, 0, \bar W_+), \\ \bar W_+' = \Phi_+(\bar W_+, 0, 0; \bar U_\pm, \bar W_\pm), \quad t > 0. \end{cases}
$$

The DSIs for the range functions of the solution, $\{U_\pm(t), W_\pm(t)\}$, then follows by the Maximum Principle and, as in the two-dimensional case, is deduced from the subsystems (152), (153) and is obtained from (154) with the appropriate inequality signs:

$$
\text{(155)} \begin{cases} U_+' \ge F(U_+, 0, W_-), \\ W_-' \ge \Phi_-(W_-, 0, 0; U_\pm, W_\pm) \equiv \inf_{p,q} \{(2F_{pq}q + F_p)W_- + F_{pp}q^2\}, \\ U_-' \le F(U_-, 0, W_+), \\ W_+' \le \Phi_+(W_+, 0, 0; U_\pm, W_\pm) \equiv \sup_{p,q} \{(2F_{pq}q + F_p)W_+ + F_{pp}q^2\} \quad \text{a.e. } t > 0. \end{cases}
$$

Thus, the range functions $\{\bar U_+(t), \bar W_-(t)\}$ (resp. $\{\bar U_-(t), \bar W_+(t)\}$) are a homogeneous supersolution (resp. subsolution) of the lower (resp. upper) subsystem. Observe that the following elementary inequalities can be added to the DSIs:

$$
\text{(156)} \qquad U_+' \le F(U_+, 0, W_+), \quad U_-' \ge F(U_-, 0, W_-),
$$

which play a role of the additional inequalities similar to (85), with $W_+ \equiv 0$, in the two-dimensional case.

8.2 Extended expansions

Multi-dimensional DSIs may occur if one looks for an extended expansion consisting in more than two coefficients (range functions), say, $U(t) \sim u$, $W(t) \sim D_x^2 u$, $Z(t) \sim D_x^4 u$, etc. Moreover, there exists examples of the multi-dimensional DSs representing quasilinear heat equations on the invariant subspaces. For instance, consider the quasilinear equation with the fast diffusion and a superlinear combustion source term

$$
v_t = (v^{-4/3} v_x)_x + v^{7/3} \quad (v > 0).
$$

Setting $v = u^{-3/4}$ we arrive at the equation

$$u_t = \mathbf{A}(u) \equiv uu_{xx} - \frac{3}{4}(u_x)^2 - \frac{4}{3}. \tag{157}$$

The operator $\mathbf{A}$ admits the *five-dimensional* invariant subspace $S_5 = \text{Span}\{1, x, x^2, x^3, x^4\}$, so that there exists the exact solution

$$\bar{u}(x,t) = C_0(t) + C_1(t)x + C_2(t)x^2 + C_3(t)x^3 + C_4(t)x^4 \in S_5, \tag{158}$$

see p. 234 in [17]. Substituting it into (157) we obtain the five-dimensional DS

$$\begin{cases} C_0' = 2C_0C_2 - \frac{3}{4}C_1^2 - \frac{4}{3}, \\ C_1' = -C_1C_2 + 6C_0C_3, \\ C_2' = -C_2^2 + 12C_0C_4 + \frac{3}{2}C_1C_3, \\ C_3' = -C_2C_3 + 6C_1C_4, \\ C_4' = 2C_2C_4 - \frac{3}{4}C_3^2, \quad t > 0. \end{cases} \tag{159}$$

Indeed, (158) is the Taylor expansion of the solutions $\bar{u}(x,t)$ on S_5. Equation (157) has remarkable invariant properties. The structure of the linear invariant subspace S_5 shows that the linear operator $Hu = D_x^5 u$ is *invariant* under the flow generated by the parabolic equation (157): if u is a suitable solution and $Hu(x,0) = 0$ then $Hu(x,t) = 0$ for $t > 0$. Moreover, this operator is a *sign-invariant* [18] of (157) in the sense that under suitable hypotheses on the application of the Maximum Principle, $Hu(x,0) \geq 0$ (≤ 0) implies that $Hu(x,t) \geq 0$ (≤ 0) for $t > 0$. Nevertheless, it is not clear if any DSIs with the right-hand side from (159) (this corresponds to the case where all the derivatives u, $D_x u$, $D_x^2 u$, $D_x^3 u$ and $D_x^4 u$ are involved) may describe evolution properties of more general solutions to (157). This problem of a possible *five-dimensional* DSIs remains open.

Consider now the case where the multi-dimensional DSIs corresponding to the extended expansion are derived for the even spatial derivatives only by differentiation of the equation to get the parabolic equations for $w = D_x^2 u$, $z = D_x^4 u$, etc. The necessary range of the corresponding odd spatial derivatives $D_x u$, $D_x^3 u$, etc., then can be obtained from the bounds on the neighbour even derivatives similar to the estimate in Proposition 1. We then arrive at a multi-dimensional DSIs, where the first two inequalities for $\{U, W\}$ can be considered independently and the estimates of the higher-order derivatives follow from the lower order ones on u and $D_x^2 u$. It is a natural property of the second-order parabolic equations. The two-dimensional DSIs considered in the present paper are the main (determining) ones for the nonlinear second-order parabolic equations.

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