MAXIMAL SIGN-INVARIANTS OF QUASILINEAR PARABOLIC EQUATIONS WITH GRADIENT DIFFUSIVITY

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To Professor Sergey P. Kurdyumov on his 70th birthday

We consider quasilinear parabolic equations with gradient diffusivity $u_t = \mathrm{div}(|\nabla u|^\sigma \nabla u) + f(u)$, $x \in \mathbf{R}^N$, $t > 0$, where $\sigma \neq -1$ is a fixed constant and $f(u)$ is a given smooth function. We also study quasilinear parabolic equations with a gradient-dependent coefficient $u_t = h(|\nabla u|)\Delta u + f(u)$, with a smooth function $h(p)$. For both classes of equations we derive first-order sign-invariants, i.e. first-order operators preserving their signs on the evolution orbits $\{u(\cdot, t),\, t > 0\}$. We give a complete description of (maximal) sign-invariants of prescribed structures. As a consequence, we construct new exact solutions of some quasilinear equations.

I. Introduction

We consider quasilinear parabolic equations with gradient-like diffusivity

$$u_t = \mathrm{div}(|\nabla u|^\sigma \nabla u) + f(u), \quad x \in \mathbf{R}^N, \; t > 0, \tag{1.1}$$

where $\sigma \neq -1$ is a fixed constant and $f(u)$ is a sufficiently smooth function to be prescribed later. It is well-known that such types of diffusion-like operators occur, for instance, in the theory of non-Newtonian liquids and in some turbulence problems, see [1] and references therein. We also study quasilinear parabolic equations with a general nonlinearity in the diffusion term of the gradient-dependent type

$$u_t = h(|\nabla u|)\Delta u + f(u), \tag{1.2}$$

with a sufficiently smooth function $h(p) > 0$. Such nonlinearities are widely used in mean curvature flow equations related to differential geometry and to problems of motion of a closed hypersurface in $\mathbf{R}^N$ by its mean curvature, see [2] and [3]. Such equations are important in phase transition phenomena [4]. The one-dimensional equation (1.1) ($N = 1$) corresponds to the function $h(p) = (\sigma + 1)p^\sigma$ in (1.2). As a typical example we consider the equation

$$u_t = \frac{u_{xx}}{1 + (u_x)^2} - \frac{1}{u}, \quad x \in \mathbf{R}, \; t > 0, \tag{1.3}$$

which describes (after a suitable surface parametrization) the evolution of cylindrically symmetric hypersurfaces moving by mean curvature in $\mathbf{R}^3$, see [5,6]. On the other hand, a one-dimensional quasilinear parabolic equation with both gradient-dependent coefficients

$$u_t = \frac{u_{xx}}{1 + (u_x)^2} + f(u)(1 + (u_x)^2)^{1/2}$$

is interpreted [7] as an equation of motion for the graph of u:

$$\Gamma_t = \{(x, y) : \;\; y = u(x, t), \;\; x \in \mathbf{R}\},$$

with the equation

$$V = k + f(y),$$

where $V = u_t/(1 + (u_x)^2)^{1/2}$ denotes the upward velocity of Γ_t and $k = u_{xx}/(1 + (u_x)^2)^{3/2}$ is the upward curvature of Γ_t. This is an example of a curve shortening equation with driving force term $f(y)$. Blow-up of the derivative u_x for a more general equation of such type

$$u_t = a(u_x)u_{xx} + f(u)g(u_x),$$

was studied in [7].

Let us describe the main techniques of the present paper. We introduce a general first-order Hamilton-Jacobi operator of the form

$$H(u) = H(u_t, \nabla u, u), \tag{HJ}$$

where the function $H(\cdot)$ is assumed to be smooth enough.

We say that the operator (HJ) is a *sign-invariant* of the given nonlinear parabolic equation if it preserves both signs, ≥ 0 and ≤ 0, on its solutions $u(x,t)$. This means that

$$H(u) \geq 0 \ (\leq 0) \text{ in } \mathbf{R}^N \text{ for } t=0 \Rightarrow H(u) \geq 0 \ (\leq 0) \text{ in } \mathbf{R}^N \text{ for } t>0. \tag{1.4}$$

These properties are understood in the sense of a formal application of the Maximum Principle for linear parabolic equations, see details in [8].

Some ideas of sign invariants were originated from the qualitative theory of nonlinear parabolic equations where similar barrier techniques (one-sided as usual) play a fundamental role in the study of existence, uniqueness, differential and asymptotic properties of wide classes of solutions. We refer to [8,9] where these questions are discussed in detail for other classes of quasilinear parabolic equations, and necessary references are available.

In this paper we present the results on the structure of the first-order sign-invariants admitted by the equations with gradient diffusivity. We study **maximal** sign-invariants describing all possible sign-invariants of a prescribed structure. We also construct the corresponding sets of exact solutions (i.e. those which can be expressed in terms of finite-dimensional dynamical systems) using the *zero-invariance property* of sign-invariants. Namely, under assumption (1.4) the operator (HJ) is also a *zero-invariant* of the evolution equation, i.e., there holds

$$H(u)=0 \quad \text{for } t=0 \ \Rightarrow \ H(u)=0 \quad \text{for } t>0. \tag{1.5}$$

This is a differential constraint, a well-known approach for constructing exact solutions of nonlinear evolution equations, see references in [8]. We claim that under certain restrictions a consistent differential constraint for a quasilinear *second-order parabolic* equation can be extended to a sign-invariant which describes properties of a much wider (than purely exact ones) class of solutions $u(x,t)$.

In Section II we consider sign-invariants of the form $u_t - \psi(u)$ and the corresponding exact solutions. Sections III and IV are devoted to the sign-invariants of the form $u_r - g(r)F(u)$, $r=|x|$ in the class of spatially symmetric solutions $u=u(r,t)$.

Sign-invariants for quasilinear parabolic equations can give new exact finite-dimensional solutions [8,9]. On the other hand, the corresponding differential inequalities $H(u) \geq 0 \ (\leq 0)$ can define a generalized property of B-convexity (concavity) of solutions preserved in time [10], which is treated as convexity (concavity) relative to the corresponding family B of exact solutions. Under certain hypotheses on initial-boundary data and the coefficients of the equation, the differential inequalities can allow us to derive some a priori bounds on more general solutions of the parabolic equation under consideration.

II. Non-stationary autonomous maximal sign-invariant

We are looking for a sign-invariant for the general equation (1.2) of the form

$$H(u) \equiv u_t - \psi(u), \tag{2.1}$$

with a smooth function $\psi(u)$ depending on the coefficients of the equation. The structure of operator (2.1) comes from the conditions of ψ-criticality of solutions to quasilinear parabolic equations[11]. Given the particular structure (2.1) of the sign-invariants depending on a single unknown function $\psi(u)$ we perform our further research in terms of the following

Definition 2.1. *The set*

$$\mathcal{H}_{\max} = \{H : \ \exists \ \psi \ \ \text{such that (2.1) is a sign-invariant of (1.2)}\}$$

is said to be a maximal sign-invariant of equation (1.2) of the type (2.1).

We begin with a preliminary description of the maximal sign-invariant.

Proposition 2.2. *(2.1) is a sign-invariant to (1.2) if and only if the functions $h(p)$ and $\psi(u)$ satisfy the identity for all $(u, p) \in \mathbf{R} \times \mathbf{R}_+$*

$$\mathcal{F}(u,p) = f'\psi - f\psi' + \frac{h'(p)}{h(p)} p(\psi - f)\psi' + h(p)p^2\psi'' \equiv 0. \tag{2.2}$$

Proof. It is not difficult to calculate that the function $J = u_t - \psi(u)$, assumed to be sufficiently smooth, solves a linear parabolic equation of the form

$$J_t = \mathcal{L}J + \mathcal{F}, \tag{2.3}$$

where $\mathcal{L}$ is a second-order elliptic operator and $\mathcal{F}$ is as given in (2.2). Therefore, by the Maximum Principle, the function $J(\cdot, t)$ preserves its sign (both the positive and negative ones) given on the parabolic boundary of the domain if $\mathcal{F} = 0$. See details in [8].

Let us now prove that

$$\mathcal{H}_{\max} = \{H(u) = u_t - \psi(u) : \quad \psi \text{ solves (2.2)}\}$$

is the maximal sign-invariant for equation (1.2). Indeed, assume for contradiction that there exist constants $\mu \in \mathbf{R}$, $\nu \in \mathbf{R}_+$ such that $\mathcal{F}(\mu, \nu) \neq 0$ and, say, > 0. Then one can construct a smooth initial function $u_0 \equiv u(x, 0)$ such that $J(x, 0) \equiv h(|\nabla u_0|)\Delta u_0 + f(u_0) - \psi(u_0) = 0$ in a neighbourhood of the origin $x = 0$, and $u_0(0) = \mu$, $|\nabla u_0|(0) = \nu$. For instance, one can take a symmetric function $u_0(r)$, $r = |x - x_0|$, with an arbitrary fixed $x_0 \in \mathbf{R}^N$, $x_0 \neq 0$. Then $u_0(|x_0|) = \mu$, $u_0'(|x_0|) = \nu$ and the existence of such a function $u_0(r)$ follows from the standard theory of ordinary differential equations. As usual, for a formal application of the Maximum Principle we assume that such a u_0 can be continued beyond a neighbourhood of the origin to provide us with a local-in-time solution $J(x, t)$ of equation (2.3). Then a standard formal application of the Maximum Principle, based on the interior regularity results for uniformly parabolic equations implies that $J(x, t) > 0$ for all small $t > 0$ and $x \approx 0$. Therefore the negative sign, $J \leq 0$, is not preserved on all the evolution orbits. $\square$

We now give a detailed complete description of the maximal sign-invariant and the corresponding exact solutions. It follows from (2.2) that the right-hand side, as a function of p only, belongs to the linear span

$$W = \mathcal{L}\left\{1, \ \frac{h'(p)}{h(p)} p, \ h(p)p^2\right\}. \tag{2.4}$$

Therefore we consider three cases.

Case 1: dim $W = 3$. Then since all the three terms in (2.2) are linearly independent we arrive at the system

$$f'\psi - f\psi' = 0, \quad (\psi - f)\psi' = 0, \quad \psi'' = 0, \tag{2.5}$$

which is easily solved with respect to the function $\psi(u)$. We have

$$\psi(u) = au + b, \quad f(u) = \lambda(au + b), \tag{2.6}$$

where either $\lambda = 1$ or $a = 0$. In particular, if $\lambda = 1$, i.e., $\psi(u) = f(u) = au + b$ we have that the equation

$$u_t = h(|\nabla u|)\Delta u + au + b \quad (a \neq 0)$$

admits the exact solution

$$u(x,t) = \frac{1}{a}[g(x)e^{at} - b],$$

where $g(x)$ is an arbitrary harmonic function. If $a = 0$ then the equation

$$u_t = h(|\nabla u|)\Delta u + b$$

has the exact solution

$$u(x,t) = g(x) + dt$$

where g solved the elliptic equation

$$h(|\nabla g|)\Delta g + b - d = 0 \quad \text{in } \mathbf{R}^N.$$

Case 2: dim $W = 2$. Without loss of generality, we may assume that there exist constants $\alpha \neq 0$, $\beta \neq 0$ such that

$$\frac{h'}{h}p + \alpha h p^2 + \beta = 0. \tag{2.7}$$

Substituting $h'p/h$ from (2.7) into (2.2) we conclude that the functions $f(u)$ and $\psi(u)$ satisfy the following system of the ODE's

$$\psi'' = \alpha(\psi - f)\psi', \quad \beta\psi\psi' = f'\psi + (\beta - 1)f\psi'. \tag{2.8}$$

The second one is easily solved with respect to the function f:

$$f = A_0\psi^{1-\beta} + \psi. \tag{2.9}$$

Here and later on A_0, $B_0, \ldots$ denote free constants of integration.

Substituting (2.9) into the first equation and integrating once we get the first order ODE

$$\psi' = B_0 + \frac{\alpha A_0}{\beta - 2}\psi^{2-\beta}, \quad \beta \neq 2, \tag{2.10}$$

$$\psi' = B_0 - \alpha A_0 \log \psi, \quad \beta = 2. \tag{2.11}$$

The system (2.7), (2.9)-(2.11) defines a multi-dimensional set of equations (1.2) which admit sign-invariant (2.1). The structure of the corresponding exact solutions is calculated from the identity $u_t = \psi(u)$ so that it takes the form

$$u(x,t) = \Phi^{-1}(t + \rho(x)), \tag{2.12}$$

where Φ^{-1} is the inverse function to

$$\Phi(u) = \int_1^u dz/\psi(z), \tag{2.13}$$

and $\rho(x)$ is a smooth function to be determined later by substituting (2.12) into equation (1.2). We then arrive at the following result.

Theorem 2.3. *(i) If $\beta \neq 2$ in (2.7) then the equation*

$$u_t = \frac{(2-\beta)\,\Delta u}{C_0|\nabla u|^\beta + \alpha |\nabla u|^2} + A_0 \psi^{1-\beta}(u) + \psi(u), \tag{2.14}$$

where ψ solves (2.10), admits sign-invariant (2.1) and the exact solution (2.12) where ρ solves an elliptic equation of the form

$$\Delta \rho + B_0 |\nabla \rho|^2 + \frac{A_0 C_0}{2-\beta} |\nabla \rho|^\beta = 0. \tag{2.15}$$

(ii) If $\beta = 2$ then the parabolic equation takes the form

$$u_t = \frac{\Delta u}{(C_0 + \alpha \log|\nabla u|)|\nabla u|^2} + \frac{A_0}{\psi(u)} + \psi(u) \tag{2.16}$$

with ψ satisfying (2.11), and ρ in (2.12) solves the semilinear elliptic equation

$$\Delta \rho + (\mu + \nu \log|\nabla \rho|)|\nabla \rho|^2 = 0, \quad \mu = B_0 + A_0 C_0, \ \nu = \alpha A_0. \tag{2.17}$$

We now begin to study properties of particular simpler solutions of such type.

Example 2.1. We consider the case $\alpha = 2$ and $\beta = 0$ in (2.7), where h solves

$$\frac{h'}{h} p = -2hp^2. \tag{2.18}$$

Then $h(p) = (1+p^2)^{-1}$ and (2.2) reads $\mathcal{F}(u,p) \equiv f'\psi - f\psi' + h(p)p^2[\psi'' - 2(\psi - f)\psi'] = 0$. This yields the system of the ODE's (cf. (2.8))

$$f'\psi - f\psi' = 0, \quad \psi'' - 2(\psi - f)\psi' = 0.$$

The general solution of this system is $f(u) = \lambda\psi(u)$, $\lambda \in \mathbf{R}$, where ψ solves the ODE

$$\psi'' - 2(1-\lambda)\psi\psi' = 0.$$

This representation includes the equation with $f \equiv 0$, i.e. $\lambda = 0$, see a remark below.

We now consider the solution with $\lambda = 1/2$ so that $\psi(u) = 2f(u)$ where f solves $f'' - 2ff' = 0$ and hence

$$f'(u) - f^2(u) = \gamma, \tag{2.19}$$

γ being a free constant. We consider three cases.

(i) If $\gamma = 0$ in (2.19) then $f(u) = -1/u$. Integrating the ordinary differential equation

$$u_t = \psi(u) \equiv -\frac{2}{u},$$

we obtain the explicit solution (cf. (2.12))

$$u_*(x,t) = (\tilde{\rho}(x) - 4t)^{1/2}, \tag{2.20}$$

Substituting (2.20) into the corresponding parabolic equation (which coincide with (1.3) in the one-dimensional case)

$$u_t = \frac{\Delta u}{1+|\nabla u|^2} - \frac{1}{u} \tag{2.21}$$

we arrive at the the following linear elliptic equation for $\tilde{\rho}$:

$$\Delta\tilde{\rho} + 2 = 0, \quad x \in \mathbf{R}^N. \tag{2.22}$$

Let us discuss a few properties of such symmetric explicit solutions, $u_* = u_*(r,t)$, $r = |x|$. Then (2.22) is easily integrated:

$$\begin{aligned} \tilde{\rho}(r) &= -\frac{1}{N}r^2 + \frac{c_0}{2-N}r^{2-N} + c_1, \quad N \geq 3, \\ \tilde{\rho}(r) &= -\frac{r^2}{2} + c_0 \log r + c_1, \quad N = 2, \end{aligned} \tag{2.23}$$

where c_0 and c_1 are arbitrary constants. In the one-dimensional case $N = 1$, (2.20) is obviously the exact self-similar solution ($c_0 = c_1 = 0$ by translations in x and t)

$$u_*(x,t) = 2\sqrt{-t}\left(1 - \frac{\xi^2}{4}\right)^{1/2}, \quad \xi = \frac{x}{\sqrt{-t}}, \tag{2.24}$$

describing the disappearance (extinction) of the solution at the origin $x = 0$ at the finite time $t = 0$. One can see that (2.24) is defined in the shrinking domain $\{|x| < 2\sqrt{-t},\ t < 0\}$ and it is not a weak solution of the free boundary problem to (2.21) for $x \in \mathbf{R}$ in the standard sense of integration by parts.

On the other hand, if $N \neq 1$, then solution (2.20), (2.23) is not self-similar and has the form ($c_1 = 0$ by translation in t and $c_0 = 1$)

$$
\begin{aligned}
u_*(r,t) &= \left(-\frac{r^2}{N} - \frac{r^{2-N}}{N-2} - 4t\right)^{1/2}, \quad N \geq 3, \\
u_*(r,t) &= \left(-\frac{r^2}{2} + \log r - 4t\right)^{1/2}, \quad N = 2.
\end{aligned}
\tag{2.25}
$$

It describes extinction of the solution to the IBVP on the sphere $\{|x| \equiv r_0 = (N/2)^{1/N}\}$ at the moment

$$
T = -\frac{1}{4(N-2)}\left(\frac{N}{2}\right)^{2/N}, \quad N \geq 3; \qquad T = -\frac{1}{8}, \quad N = 2. \tag{2.26}
$$

The behaviour as $t \to T^-$, $r \to r_0$, is asymptotically self-similar.

(ii) If $\gamma = 1$ in (2.19) then $f(u) = \tan u$. By integrating the ODE $u_t = 2\tan u$ we deduce that

$$
u_*(x,t) = \arcsin(\tilde{\rho}(x)e^{2t}), \quad \tilde{\rho} = e^{2\rho}, \tag{2.27}
$$

where ρ solves the elliptic equation (2.15) with $\beta = 0$, $A_0 = -1/2$, $B_0 = C_0 = 2$. By substituting into equation

$$
u_t = \frac{\Delta u}{1 + |\nabla u|^2} + \tan u, \tag{2.28}
$$

we obtain that $\tilde{\rho}(x)$ solves the elliptic equation

$$
\Delta\tilde{\rho} - \tilde{\rho} = 0, \quad x \in \mathbf{R}^N. \tag{2.29}
$$

(iii) Consider now the last case $\gamma = -1$. Then, the parabolic equation takes the form

$$
u_t = \frac{\Delta u}{1 + |\nabla u|^2} + \frac{1 + e^{2u}}{1 - e^{2u}}, \tag{2.30}
$$

which admits the exact solution

$$
u(x,t) = \pm \log\left[\sqrt{e^{-4t}\tilde{\rho}^2(x) - 1} + e^{-2t}\tilde{\rho}(x)\right], \quad \tilde{\rho} = \frac{1}{2}\,e^{-2\rho}, \tag{2.31}
$$

where ρ satisfies (2.15) and $\tilde{\rho}(x)$ solves the elliptic equation

$$
\Delta\tilde{\rho} + \tilde{\rho} = 0. \tag{2.32}
$$

Similarly, the equation

$$
u_t = \frac{\Delta u}{1 + |\nabla u|^2} + \frac{1 - e^{2u}}{1 + e^{2u}}
$$

admits the exact solution

$$u(x,t) = \pm \log\left[\sqrt{e^{-4t}\tilde{\rho}^2(x)+1} + e^{-2t}\tilde{\rho}(x)\right], \quad \tilde{\rho} = \frac{1}{2}\, e^{-2\rho},$$

and $\tilde{\rho}$ satisfies (2.32). In particular, in the one-dimensional case we deduce the following 2π-periodic solution

$$u(x,t) = \pm \log\left[\sqrt{e^{-4t}\sin^2 x+1} + e^{-2t}\sin x\right].$$

Remark 1. In the case of equation (1.1), $N=1$, where $h'(p)p/h(p) = \sigma$, we obtain the following two independent ODE's

$$\psi'' = 0, \quad f'\psi - f\psi' + \sigma(\psi - f)\psi' = 0.$$

This yields the linear sign invariant $u_t - (au+b)$ in the case $f(u) = c(au+b)^{\sigma+1} + (au+b)$. The corresponding heat equation (1.1) is well-known to admit an explicit solution in separate variables describing for $\sigma > 0$ standing regionally localized blow-up patterns (the so-called S-regime of localization), see [12].

Remark 2. It follows from (2.8) that in the case of the pure gradient-diffusivity equation with $f \equiv 0$, there exists a nontrivial sign-invariant (2.1) provided that $\beta = 0$. Then the system (2.8) reduces to a single ODE of the form $\psi'' = \alpha\psi\psi'$. In particular, in the case (2.18), with $\alpha = 2$, $\beta = 0$ in (2.7), we obtain $\psi'' - 2\psi\psi' = 0$ whence

$$\psi' - \psi^2 = \gamma, \quad \gamma \in \mathbf{R}.$$

We then deduce that the equation

$$u_t = \frac{\Delta u}{1+|\nabla u|^2}$$

admits the exact solution (2.12) where $\rho(x)$ solves the elliptic equation (cf. (2.15))

$$\Delta\rho + \gamma|\nabla\rho|^2 - 1 = 0.$$

In particular, there exist all the exact solutions (2.20), (2.24), (2.25) and (2.31) where t has to be replaced by $t/2$.

Case 3: dim $W = 1$. It is easy to see that we can restrict out attention to the function $h(p) = p^{-2}$. Then identity (2.2) reduces to a single ODE of the form

$$(\psi' + f\psi - \psi^2)' = 0. \tag{2.33}$$

Integrating it once we obtain

$$\psi' + f\psi - \psi^2 = \gamma. \tag{2.34}$$

Given a smooth function $f(u)$ we fix an arbitrary solution $\psi(u)$ to (2.34). Then we obtain the structure (2.12) of the explicit solutions. Substituting into the parabolic equation

$$u_t = \frac{\Delta u}{|\nabla u|^2} + f(u), \tag{2.35}$$

and using (2.34) we deduce that $\rho(x)$ satisfies the following semilinear elliptic equation

$$\Delta \rho + \gamma |\nabla \rho|^2 = 0, \quad x \in \mathbf{R}^N, \tag{2.36}$$

which reduces to the Laplace equation $\Delta p = 0$ in $\mathbf{R}^N$ by setting $p(x) = e^{\gamma \rho(x)}$. Recall that due to the single equation (2.33) on the function ψ the above explicit solutions to (2.35) exist for arbitrary lower-order terms $f(u)$, i.e. the set of such quasilinear parabolic equations is *infinite-dimensional.* On the other hand, for $N = 1$ via the well-known transformation [13] $x = \psi(y,t)$ where ψ is such that $u(\psi(y,t),t) \equiv y$, the one-dimensional equation (2.35) reduces to the linear heat equation with convection $\psi_t = \psi_{yy} - f(y)\psi_y$. In terms of ψ this implies a certain linear superposition property of solutions.

This completes the analysis of identity (2.2) on the subspace (2.4).

III. Non-autonomous stationary sign-invariant

In this section we consider sign-invariants of the stationary structure which is non-autonomous in the spatial variable $x \in \mathbf{R}^N$. We restrict our attention to a class of radial symmetric solutions $u = u(r,t)$, $r = |x| \geq 0$. In this class the first-order operator has the form

$$H_*(u) \equiv u_r - g(r)F(u) \quad \text{for } t > 0, \tag{3.1}$$

with yet unknown smooth nonnegative functions $g(r) \not\equiv \text{const}$ and $F(u) \neq 0$ a.e. The idea of such sign-invariants goes to the gradient bounds introduced [14] for semilinear parabolic equations. Generalizations to the quasilinear gradient-dependent operators can be found in [15]. We apply operator (3.1) to equation (1.1) written in the equivalent radially symmetric form (we suppose for convenience that $u_r > 0$)

$$u_t = (u_r)^\sigma \left[(\sigma+1)u_{rr} + \frac{N-1}{r}\, u_r \right] + f(u), \tag{3.2}$$

with $\sigma \neq -1$.

Proposition 3.1. *Operator (3.1) is a sign-invariant of (3.2) if and only if the functions $g(r)$ and $F(u)$ satisfy the identity*

$$\begin{aligned} \frac{\mathcal{F}(r,u)}{(\sigma+1)F^{\sigma+1}} \equiv\, & (g^\sigma g')'(r) + 2(\sigma+1)(g^{\sigma+1}g')(r)F' \\ & + g^{\sigma+3}(r)[FF'' + \sigma(F')^2] + g(r)\left(\frac{f}{F}\right)' \frac{F^{1-\sigma}}{(\sigma+1)} \\ & + g^\sigma(r)\frac{N-1}{r}\left[g'(r) + g^2(r)\frac{\sigma}{\sigma+1}F' - \frac{g(r)}{(\sigma+1)r} \right] = 0. \end{aligned} \tag{3.3}$$

Proof. The proof is straightforward, see the previous one in II. Indeed, direct computations lead to the parabolic equation (2.3) for the function $J = H_*(u)$ with an elliptic operator $\mathcal{L}$ and the lower order term $\mathcal{F}$ given in (3.3). □

We now give a description of the *maximal* sign-invariant $\mathcal{H}_{\max}$, i.e., the whole set of sign-invariants of the form (3.1) with nontrivial (see above) functions g and F.

3.1. Dimension $N = 1$. Here the Greek letters $\alpha,\ \beta,\ \gamma$... denote arbitrary constants.

Theorem 3.2. *Let $N = 1$. Then $\mathcal{H}_{\max} = \mathcal{H}_1 \cup \mathcal{H}_2$ where the sets $\mathcal{H}_{1,2}$ are defined as follows: (i) $H_* \in \mathcal{H}_1$ if g satisfies the ODE*

$$(g^\sigma g')' = \beta g^{\sigma+1} g' + \gamma g^{\sigma+3} + \delta g, \quad \gamma = -\sigma \left[\frac{\beta}{2(\sigma+1)}\right]^2, \tag{3.4}$$

and F is a linear function

$$F(u) = -\frac{\beta u}{2(\sigma+1)}. \tag{3.5}$$

(ii) $H \in \mathcal{H}_2$ if g and F satisfy the ODE's

$$g^\sigma g' = \gamma g^{\sigma+2} + \delta, \tag{3.6}$$

$$(F^\sigma F')' + 2\gamma(\sigma+1)F^{\sigma-1}F' + \gamma^2(\sigma+2)F^{\sigma-1} = 0. \tag{3.7}$$

The corresponding function $f(u)$ is then given by

$$\text{(i)} \quad f(u) = \frac{2\delta(\sigma+1)^2}{\sigma\beta}\left(-\frac{\beta u}{2(\sigma+1)}\right)^{\sigma+1}, \tag{3.8}$$

$$\text{(ii)} \quad f(u) = \frac{\delta(\sigma+1)}{\gamma} F^{\sigma+1}F' + \rho F. \tag{3.9}$$

Proof of Theorem 3.2. For $N = 1$, the right-hand side of (3.3), as a function of the space variable r, is a linear combination (with coefficients depending on u) of four terms which should be linearly dependent (otherwise no nontrivial sign-invariant exists). Therefore we assume that

$$\alpha_1(g^\sigma g')' + \beta_1 g^{\sigma+1}g' + \gamma_1 g^{\sigma+3} + \delta_1 g = 0. \tag{3.10}$$

We need to consider two cases.

Case 1: $\alpha_1 \neq 0$. Let $\alpha_1 = -1$, i.e.

$$(g^\sigma g')' = \beta_1 g^{\sigma+1}g' + \gamma_1 g^{\sigma+3} + \delta_1 g. \tag{3.11}$$

Substituting (3.11) into (3.3) we obtain a linear combination of the form

$$I_1(u)g^{\sigma+1}g' + I_2(u)g^{\sigma+3} + I_3(u)g = 0 \tag{3.12}$$

on the linear space $\mathcal{W} = \mathcal{L}\{g^{\sigma+1}g',\ g^{\sigma+3},\ g\}$. Then we have: (i) If $\dim \mathcal{W} = 3$ then we get the system of ODE's for F and f

$$\begin{gathered} I_1 \equiv \beta_1 + 2(\sigma+1)F' = 0, \quad I_2 \equiv \gamma_1 + FF'' + \sigma(F')^2 = 0, \\ I_3 \equiv \delta_1 + \left(\frac{f}{F}\right)' \frac{1}{(\sigma+1)F^{\sigma-1}} = 0, \end{gathered} \tag{3.13}$$

which corresponds to the case $H_* \in \mathcal{H}_1$.

(ii) If $\dim \mathcal{W} = 2$ then there exist constants γ and δ such that

$$g^{\sigma+1}g' = \gamma g^{\sigma+3} + \delta g. \tag{3.14}$$

Dividing (3.14) over $g \not\equiv 0$, differentiating once and comparing with (3.11) we conclude that such a function g exists if and only if

$$\gamma_1 = \gamma[\gamma(\sigma+2) - \beta_1], \quad \delta_1 = \delta[\gamma(\sigma+2) - \beta_1]. \tag{3.15}$$

One can see that this corresponds to the case $H_* \in \mathcal{H}_2$. Any less dimensional reduction of $\mathcal{W}$ is impossible.

Case 2: $\alpha_1 = 0$. This reduces to (3.14) and to the same analysis. $\square$

The sign-invariants from $\mathcal{H}_1$ and the corresponding exact solutions in separate variables are already known and studied in detail [12,16]. Observe that the ODE (3.7) always admits a linear solution $F(u) = Cu$ with either $C = -\gamma$ or $C = -\gamma(\sigma+2)/\sigma$, which corresponds, after the transformations $u = e^{\rho C t}v$, to stationary or separate-variable solutions.

We now consider new exact solutions generated by sign-invariants from $\mathcal{H}_2$. These solutions are new even for the semilinear heat equation (1.1) with $\sigma = 0$.

Theorem 3.3. *The one-dimensional equation*

$$u_t = (\sigma+1)(u_x)^\sigma u_{xx} + f(u), \tag{3.16}$$

where $f(u)$ is given by (3.9) with F satisfying (3.7), admits the exact solution $u(x,t)$ given by

$$\Phi(u) \equiv \int_1^u dz/F(z) = \int_0^x g(y)dy + C_0(t) \tag{3.17}$$

with $g(y)$ being a solution to (3.6) with the boundary conditions

$$g(0) = \alpha > 0. \tag{3.18}$$

The function C_0 solves the ODE

$$\frac{dC_0}{dt} = (\sigma+1)\left(\alpha^{\sigma+2} + \frac{\delta}{\gamma}\right) F^\sigma(\Phi^{-1}(C_0))(F'(\Phi^{-1}(C_0)) + \gamma) + \rho, \quad t > 0. \tag{3.19}$$

Proof. Substituting (3.17) into (3.16) and using (3.14) we obtain the identity

$$\frac{dC_0}{dt} = (\sigma+1)\left(g^{\sigma+2} + \frac{\delta}{\gamma}\right) F^\sigma(F' + \gamma) + \rho \equiv G(x,t). \tag{3.20}$$

One can check that under the hypotheses of the theorem $G_x \equiv 0$. In order to prove this identity one can differentiate $G(x,\cdot)$ and substitute $g^{\sigma+1}g'$ from (3.14). Finally, (3.7) yields the result.

Hence (3.20) is equivalent to the equation $C_0' = G(0,t)$ and (3.19) follows. $\square$

3.2. Dimensions $N > 1$. If $N \neq 1$ then, in general, (3.3) with $\sigma \neq 0$ is a linear combination in a five-dimensional space of the form

$$W_\sigma = \mathcal{L}\left\{\varphi,\ g^{\sigma+1}g',\ g,\ g^{\sigma+3},\ \frac{g^{\sigma+2}}{r}\right\}, \quad \varphi = \nabla\cdot(g^\sigma \nabla g) - \frac{N-1}{\sigma+1}\frac{g^{\sigma+1}}{r^2}. \tag{3.21}$$

If $\sigma = 0$ then the space is four-dimensional

$$W_0 = \mathcal{L}\{\varphi,\ gg',\ g,\ g^3\}, \quad \varphi = \Delta g - (N-1)\frac{g}{r^2}. \tag{3.22}$$

The Semilinear Case $\sigma = 0$. We now prove the main result on completeness for the semilinear equation with $\sigma = 0$.

Theorem 3.4. *Let $\sigma = 0$. A sign-invariant of the form (3.1) exists in the two cases: (i) for the equation with a special logarithmic nonlinearity*

$$u_t = \Delta u + Au\log u + Bu, \tag{3.23}$$

and (ii) for a class of equations when H_* contains the special functions $g(r) = c/r$ or $g(r) = cr$.

Proof. Consider the space W_0. In view of (3.3) with $\sigma = 0$ we have to have that (see (3.22))

$$\varphi = \alpha gg' + \beta g + \gamma g^3. \tag{3.24}$$

If we assume that

$$gg',\ g \text{ and } g^3 \text{ are linearly independent}, \tag{3.25}$$

then substituting (3.24) into (3.3) we arrive at the system

$$2F' + \alpha = 0, \quad FF'' + \gamma = 0, \quad F(f/F)' + \beta = 0,$$

whence $\gamma = 0$ and F is a linear function. After translation in u and necessary rescaling in the independent variables we get (3.23).

If (3.25) does not hold then g satisfies

$$g' = \varepsilon g^2 + \delta, \quad \text{so that } g'' = 2\varepsilon^2 g^3 + 2\varepsilon\delta g. \tag{3.26}$$

Substituting (3.26) into (3.24) we arrive at a cubic algebraic equation for g

$$(2\varepsilon^2 - \alpha\varepsilon - \gamma)g^3 + \frac{\varepsilon(N-1)}{r}g^2 + \left(2\varepsilon\delta - \frac{N-1}{r^2} - \alpha\delta - \beta\right)g + \frac{\delta(N-1)}{r} = 0.$$

One can see that it matches with (3.26) if and only if $\varepsilon\delta = 0$. Indeed, if $\varepsilon\delta \neq 0$ then by (3.26) the function $g(r)$ is given either by an exponential-like function or by the tan. Both functions do not satisfy the cubic equation. Therefore either $g' = \delta$ ($\varepsilon = 0$) or $g' = \varepsilon g^2$ ($\delta = 0$) completing the proof. □

In the case (ii) nontrivial exact solutions can be found in Section 3 in [8].

Equation (3.23) is well-known to admit separable solutions $u = M(t)\psi(x)$ and others. See p. 93 in [17]. Exact solutions corresponding to the functions $g(r) = r$ or $1/r$ were constructed in [8]. Observe that Theorem 3.4 proves a kind of completeness of the results of Section 3 in [8].

The corresponding maximal sign-invariant is presented in the proof of the theorem.

THE QUASILINEAR CASE, $\sigma \neq 0$. The analysis presented below gives a complete study of the sign-invariants (3.1) in the quasilinear case. We stress our attention to some particular situations, which correspond to new types of exact solutions, and omit details of the study of the known types or trivial solutions.

Consider the space W_σ. First of all it follows from (3.3) that the function φ given in (3.21) must satisfy

$$\varphi = \alpha g^{\sigma+1} g' + \beta g + \gamma g^{\sigma+3} + \delta \frac{g^{\sigma+2}}{r}. \tag{3.27}$$

(i) Similar to the case $\sigma = 0$ one can show that the assumption (cf. (3.25))

$$g^{\sigma+1} g', \; g, \quad g^{\sigma+3} \text{ and } g^{\sigma+2}/r \text{ are linearly independent}, \tag{3.28}$$

leads to the equation

$$u_t = \operatorname{div}\left(|\nabla u|^\sigma \nabla u\right) + A u^{\sigma+1} + B u, \tag{3.29}$$

which admits separable solutions [12].

(ii) Assume now that (3.28) is false so that

$$g^\sigma g' = \varepsilon g^{\sigma+2} + \nu + \mu \frac{g^{\sigma+1}}{r}. \tag{3.30}$$

We study compatibility conditions for the system (3.27), (3.30). Differentiating (3.30) and substituting $(g^\sigma g')'$ into (3.27), as in the semilinear case, we arrive at the algebraic equation

$$A g^{\sigma+3} + B g + C \frac{g^{\sigma+2}}{r} + D \frac{g^{\sigma+1}}{r^2} + E \frac{1}{r} = 0, \tag{3.31}$$

with the coefficients

$$\begin{aligned}
&A = \varepsilon^2(\sigma+2) - \alpha\varepsilon - \gamma, \quad B = \varepsilon\nu(\sigma+2) - \alpha\nu - \beta, \\
&C = \varepsilon[\mu(2\sigma+3) + N - 1] - \alpha\mu - \delta, \\
&D = (\sigma+1)\left(\mu - \frac{1}{\sigma+1}\right)\left(\mu - \frac{1-N}{\sigma+1}\right), \\
&E = \nu(\sigma+1)\left(\mu - \frac{1-N}{\sigma+1}\right).
\end{aligned} \tag{3.32}$$

Setting $w = r g(r)$ we obtain the system

$$r^{\sigma+2}(Bw + E) = -(A w^{\sigma+3} + C w^{\sigma+2} + D w^{\sigma+1}), \tag{3.33}$$

$$r w^\sigma w' = \varepsilon w^{\sigma+2} + (\mu+1) w^{\sigma+1} + \nu r^{\sigma+2}. \tag{3.34}$$

It follows from (3.33) that $\sigma = -2$ plays a special role.

A. Let $\sigma \neq -2$. Then we need to consider two possibilities.

A.1. It follows from (3.33), (3.34) that under a suitable choice of the parameters there exists a special solution $w = \text{const}$, i.e. the function

$$g(r) = \frac{1}{r}, \tag{3.35}$$

which will be considered in detail in Section IV. Observe that the function (3.35) gives a consistent type of sign-invariants (3.1) for a wide class of quasilinear parabolic equations, see Section 3 in [8].

A.2. We now look for a nontrivial solution

$$w \not\equiv \text{const}. \tag{3.36}$$

Consider first the special case of (3.33):

$$B = E = 0 \quad \text{and } A = C = D = 0. \tag{3.37}$$

Then from (3.32) we get two cases: either $\mu = 1/(\sigma+1)$ or $\mu = (1-N)/(\sigma+1)$.

If $\mu = 1/(\sigma+1)$ $(D=0)$ and $\nu = 0$ $(E=0)$ then (3.34) is solved explicitly:

$$g(r) = \left[-\frac{\varepsilon(\sigma+1)}{\sigma+2}\, r + k r^{-1/(\sigma+1)} \right]^{-1}, \quad k = \text{const}. \tag{3.38}$$

The identity (3.3) reads

$$\begin{aligned} & g^{\sigma+3}[\alpha\varepsilon + \gamma + 2(\sigma+1)\varepsilon F' + FF'' + \sigma(F')^2] \\ & + \frac{g^{\sigma+2}}{r} \left[\alpha\mu + \delta + \left(2\mu(\sigma+1) + \frac{(N-1)\sigma}{\sigma+1} \right) F' \right] \\ & + g \left[\alpha\nu + \beta + 2(\sigma+1)\nu F' + \left(\frac{f}{F} \right)' \frac{F^{1-\sigma}}{\sigma+1} \right] \equiv 0. \end{aligned} \tag{3.39}$$

Then similarly to the case $N = 1$, from (3.39) we obtain the parabolic equation (3.2) with a linear term $f(u) = cu$ (and $F(u) = cu$), which admits a separable solution.

If $\mu = (1-N)/(\sigma+1)$ $(D = E = 0)$ then (3.3) leads to $F(u) = -\varepsilon u$ and $f(u) = -(\sigma+1)\nu(-\varepsilon u)^{\sigma+1}$. Moreover, the resulting exact solution is stationary.

Assume now that $BE \neq 0$. Then substituting $r^{\sigma+2}$ from (3.33):

$$r^{\sigma+2} = -\frac{w^\sigma(Aw^3 + Cw^2 + Dw)}{Bw + E},$$

into equation (3.34) rewritten in the form

$$\frac{dr^{\sigma+2}}{dw} = \frac{(\sigma+2) r^{\sigma+2} w^\sigma}{\varepsilon w^{\sigma+2} + (\mu+1) w^{\sigma+1} + \nu r^{\sigma+2}},$$

we arrive at a fifth-order polynomial of the independent variable $w \in \mathbf{R}$ with the coefficients depending on the parameters introduced in (3.27) and in (3.30), which has to vanish identically. This gives extra five equations on the parameters. The sixth coefficient of $w^0 = 1$ is $DE[E(\mu - 1/(\sigma+1)) - D\nu] = 0$ by (3.32). It can be shown that this case provides no new solutions.

B. Let $\sigma = -2$. Then (3.33) and (3.34) take the form

$$(A+B)w + (C+E) + Dw^{-1} = 0, \tag{3.40}$$

$$rw^{-2}w' = \varepsilon + (\mu+1)w^{-1} + \nu. \tag{3.41}$$

B.1. There exists a special solution $w = \text{const}$, to be considered in IV.

B.2. Assume (3.36) holds. Then (3.40) implies that

$$A+B=0, \quad C+E=0, \quad D=0.$$

Since $D = -(\mu+1)(\mu-N+1)$ by (3.32), we consider two possibilities.

B.2.1. Let $\mu = N-1$. Then $D = E = 0$ and hence $C = 0$. From (3.30) we then get the function

$$g(r) = \Big(-\frac{\rho}{N}r + kr^{1-N}\Big)^{-1}, \quad \rho = \varepsilon + \nu. \tag{3.42}$$

In this case the coefficients of the second term $1/r$ on the right-hand side of (3.39) vanish identically, $g^{\sigma+3} \equiv g$ for $\sigma = -2$, and (3.39) reduces to a single equation for two functions f and F:

$$FF'' - 2(F')^2 - 2\rho F' - \Big(\frac{f}{F}\Big)' F^3 = 0. \tag{3.43}$$

Integrating it once we obtain

$$f(u) = \frac{1}{F(u)}[F'(u) + \rho] + MF(u), \quad M = \text{const}. \tag{3.44}$$

B.2.2. If $\mu = -1$ then (3.30) takes the form

$$g^{-2}g' = \rho - \frac{1}{gr}, \quad \rho = \varepsilon + \nu,$$

and hence

$$g(r) = (kr - \rho r \log r)^{-1} \quad (\rho \neq 0). \tag{3.45}$$

Substituting the parameters into (3.39) we obtain a linear combination on the subspace $\mathcal{L}\{g, r^{-1}\}$ and hence two linearly independent differential equations on f and F which give separate-variable solutions.

IV. Sign-invariants and new solutions for $N > 1$

Using the results on the maximal sign-invariants in Section III we now construct new exact solutions of the quasilinear equation (3.2) with $N > 1$. It follows from Theorem 3.4 ($\sigma = 0$) and further results for $\sigma \neq 0$ that nontrivial solutions exist in two cases only: either (3.35) or (3.42), $\sigma = -2$.

4.1. The case $g = 1/r$. For convenience and for further applications we first perform calculations of the sign-invariants for a power-like function

$$g(r) = r^l. \tag{4.1}$$

Then from (3.3) we arrive at the four term identity

$$\mathcal{F} \equiv r^{l(\sigma+3)} I_1(u) + r^{l(\sigma+2)-1} I_2(u) + r^{l(\sigma+1)-2} I_3(u) + r^l I_4(u) \tag{4.2}$$

with the coefficients (we suppose that $F > 0$)

$$I_1 = F^2 (F^{\sigma+1})'', \quad I_2 = [2l(\sigma+1)^2 + \sigma(N-1)] F^{\sigma+1} F', \tag{4.3}$$

$$I_3 = [l(\sigma+1) - 1][N - 1 + (\sigma+1)l] F^{\sigma+1}, \quad I_4 = f'F - fF'. \tag{4.4}$$

Therefore, identity $\mathcal{F} \equiv 0$ yields four differential equations on the two functions f and F which is inconsistent in general.

From (4.2) it is clear that the case $l = -1$ plays a special role (other critical exponents l will be discussed in the final remarks of this section). Namely, if $l = -1$ and $\sigma \neq -2$ then (4.2) reduces to a two term identity on the span $\mathcal{L}\{r^{-(\sigma+3)}, r^{-1}\}$ which determines a system of two ODE's

$$\begin{gathered} F^2 (F^{\sigma+1})'' + aF^{\sigma+1} F' + bF^{\sigma+1} = 0, \\ a = \sigma(N-1) - 2(\sigma+1)^2, \quad b = (\sigma+2)(\sigma+2-N), \end{gathered} \tag{4.5}$$

$$f'F - fF' = 0. \tag{4.6}$$

The last equation yields

$$f(u) = \lambda F(u), \quad \lambda = \text{const}. \tag{4.7}$$

Thus, the quasilinear equation with gradient diffusivity

$$u_t = |u_r|^\sigma \left[(\sigma+1) u_{rr} + \frac{N-1}{r}\, u_r \right] + \lambda F(u), \tag{4.8}$$

where $F(u)$ is an arbitrary solution of the ODE (4.5) admits the sign-invariant

$$H(u) = u_r - \frac{1}{r}\, F(u). \tag{4.9}$$

We now construct the corresponding exact solution.

Theorem 4.1. *The parabolic equation (4.8), (4.5) admits the solution*

$$v(r,t) \equiv \int_1^{u(r,t)} \frac{dz}{F(z)} = \log r + C_0(t), \tag{4.10}$$

where

$$C_0(t) = -\frac{1}{\sigma+2} \log\left(1 + C_1 e^{(\sigma+2)\lambda t}\right) + \lambda t + C_2 \tag{4.11}$$

and C_1, C_2 are arbitrary constants.

Proof. The function v satisfies the evolution equation

$$v_t = F^\sigma (v_r)^\sigma \left[(\sigma+1)(v_{rr} + (v_r)^2 F') + \frac{N-1}{r} v_r\right] + \lambda. \tag{4.12}$$

Substituting (4.10), which directly follows from (4.9) ($H(u) = 0$), we get the equation

$$C_0' = \frac{1}{r^{\sigma+2}} \Phi(v) + \lambda, \tag{4.13}$$

where

$$\Phi(v) = F^\sigma(u)[(\sigma+1)(F'(u) - 1) + N - 1].$$

One can see from the ODE (4.5) that $\Phi' = (\sigma+2)\Phi$ whence $\Phi(v) = de^{(\sigma+2)v}$, with $d = \Phi(0)$. Finally, substituting $\Phi(v)$ into (4.13) and using (4.10) we arrive at the ODE $C_0' = de^{(\sigma+2)C_0}, \quad t > 0$, which gives (4.11). $\square$

Example 4.1. The exact solution (4.10) depends on the ODE (4.5). It is integrated two times if $\sigma = 1$ when it takes the form

$$(F^2)'' + aF' + b = 0, \quad a = N - 9, \;\; b = 3(3 - N).$$

If $N = 3$ then $b = 0$ and the general solution is

$$F(u) - C\log(F(u) + C) = 3u, \quad C = \text{const}.$$

If $N \neq 3$ then we get the following algebraic equation for F

$$(F(u) - 3u)^3 \left(F(u) + \frac{N-3}{2} u\right)^{(N-3)/2} = C.$$

4.2. The case $\sigma = -2$, $\mu = N - 1$. Recall that the function (3.42) satisfies equation (3.30):

$$g^{-2} g' = \rho + \frac{N-1}{r g(r)}. \tag{4.14}$$

Exact solutions, which follow from the zero-invariance of the operator (3.1) are given in the following statement, which gives exact solutions to an *infinite-dimensional* family of quasilinear solutions.

Theorem 4.2. *The quasilinear equation*

$$u_t = \frac{1}{(u_r)^2}\left(-u_{rr} + \frac{N-1}{r}\,u_r\right) + \frac{F'(u)+\rho}{F(u)} + MF(u), \tag{4.15}$$

where $F \not\equiv 0$ is an arbitrary smooth function and $M = const$, admits the exact solution $u(r,t)$ given by

$$\Phi(u) \equiv \int_1^u \frac{dz}{F(z)} = -\frac{1}{\rho}\log\left(1 - \frac{\rho}{kN}\,r^N\right) + Mt. \tag{4.16}$$

Observe that (4.15) is a parabolic equation backward in time.

Proof. Note that $f(u)$ in (4.15) is given by (3.44) and the solution (4.16) takes the form

$$\Phi(u) = \int_0^r g(y)dy + C_0(t), \tag{4.16$'$}$$

where g is specified in (3.42). Therefore as in the proof of Theorem 3.3, from (4.16') we deduce that $u_t = FC_0'$, $u_r = Fg$, $u_{rr} = FF'g^2 + Fg'$, and substituting these expressions into (4.15) we obtain the equation

$$C_0' = \frac{1}{F^2}\left(-g^{-2}g' + \rho + \frac{N-1}{rg(r)}\right) + M.$$

In view of (4.14) we get

$$C_0' = M, \tag{4.17}$$

and we arrive at the solution (4.16). $\square$

Example 4.2. Setting e.g. $F(u) = u^2$ we obtain that the quasilinear equation

$$u_t = \frac{1}{(u_r)^2}\left(-u_{rr} + \frac{N-1}{r}\,u_r\right) + \frac{Mu^4 + 2u + \rho}{u^2}$$

admits the solution

$$u(r,t) = \left[\frac{1}{\rho}\log\left(1 - \frac{\rho}{kN}\,r^N\right) - Mt\right]^{-1}.$$

Final Remarks. 1. As far as we know the sign-invariant (3.1), (3.42), (3.43) for equation (4.15) with $N > 1$ is a unique example of such a sign-invariant with a non-power function $g(r)$ given in (3.42) with $\rho k \neq 0$. If we set $k = 0$ in (3.42) then $g(r) \sim 1/r$ coincides with (4.1) for $l = -1$. On the other hand, solution (4.16') then coincides with (4.10) (cf. (4.11), $\sigma = -2$).

2. It follows from identity (4.2) that the case $l = N-1$, $\sigma = -2$ is also a special one for the existence of a sign-invariant. It corresponds to $\rho = 0$ in (3.42) and in (4.14).

3. The last special exponent is $l = 1/(\sigma+1)$. Note that the corresponding differential inequality with the negative sign, $u_r + r^{1/(\sigma+1)}F(u) \le 0$, plays an important part in the analysis of blow-up of radial solutions of the quasilinear equation (3.2) with $f(u) \sim u^p$, $p > 1$. See [15]. Substituting $l = 1/(\sigma+1)$ into (4.2) we get the following ODE's on the functions f and F

$$(F^{\sigma+1})'' = 0, \quad f'F - fF' + F^{\sigma+1}F'[2+\sigma(N+1)] = 0. \tag{4.18}$$

This yields the functions

$$F(u) = (au+b)^{1/(\sigma+1)}, \quad f(u) = -\frac{2+\sigma(N+1)}{\sigma}(au+b) + \alpha(au+b)^{1/(\sigma+1)}. \tag{4.19}$$

In this case operator (3.1) with $g(r) = r^{1/(\sigma+1)}$ is also zero-invariant of equation (1.1) and exact solutions exist. However, more general solutions can be constructed by another invariant approach [16]. For instance, using the transformation $u = v^{(\sigma+1)/\sigma}$ in the equation

$$u_t = \operatorname{div}(|\nabla u|^\sigma \nabla u) + \alpha u^{1/(\sigma+1)} + \beta u$$

we obtain the equation

$$v_t = \mathcal{B}(v) \equiv \mu^{-\sigma}|v_r|^\sigma \left[(\sigma+1)(vv_{rr} + \frac{1}{\sigma}(v_r)^2) + \frac{N-1}{r}vv_r\right] + \mu(\alpha+\beta v),$$

where $\mu = \sigma/(\sigma+1)$. The operator $\mathcal{B}$ admits the two-dimensional invariant subspace $W_2 = \mathcal{L}\{1, |x|^\gamma\}$, $\gamma = (\sigma+2)/(\sigma+1)$, i.e., $\mathcal{B}(W_2) \subseteq W_2$. Thus, there exists exact solutions of the form $v_*(x,t) = C_0(t) + C_1(t)|x|^\gamma \in W_2$, where the coefficients $\{C_0(t), C_1(t)\}$ solve a nonlinear dynamical system. See details and generalizations in [16].

4. The structure of the exact solution (4.16) (as well as (2.12)) shows that it can be treated as a solution on a one-dimensional subspace or on a set invariant under a nonlinear operator of the equation. See different examples in [8,9,18,19].

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