CONCAVITY AND B-CONCAVITY OF SOLUTIONS OF QUASILINEAR FILTRATION EQUATIONS

BY

Victor A. Galaktionov

Abstract

We study spatial concavity properties of nonnegative weak solutions of the filtration equations with absorption $u_t = (\varphi(u))_{xx} - \psi(u)$ in $Q = \mathbf{R} \times (0, \infty)$, $\varphi' \geq 0$, $\psi \geq 0$. Under certain assumptions on the coefficients φ, ψ we prove that concavity of the pressure function is a consequence of a "weak" convexity of travelling wave solutions of the form $V(x,t) = \theta(x - \lambda t + a)$. We establish that the global structure of a so-called proper set $\mathcal{B} = \{V\}$ of such particular solutions determines a property of B-concavity for more general solutions which is preserved in time. For the filtration equation $u_t = (\varphi(u))_{xx}$ we prove a semiconcavity estimate for the pressure, $v_{xx} \leq (t+\tau)^{-1}\theta''(\xi)$, due to the B-concavity of the solution to the subset $\mathcal{B}$ of the explicit self-similar solutions $\theta(x/\sqrt{t+\tau})$.

The analysis is based on the intersection comparison based on the Sturmian argument of the general solution $u(x,t)$ with subsets $\mathcal{B}$ of particular solutions. We also study other aspects of the B-concavity/convexity with respect to different subsets of explicit solutions.

AMS (MOS) Subject Classification. 35K55, 35K65

Keywords and Phrases. Filtration equation with absorption, concavity, Maximum Principle, explicit solutions, intersection comparison.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

In this paper we study the spatial concavity property for the weak compactly supported nonnegative solutions $u = u(x,t)$ of the one-dimensional quasilinear degenerate parabolic equations. We establish that this property is preserved in time for general solutions provided that there exists a *complete* subset of *convex* particular solutions with vanishing convexity at the finite interfaces. As a basic example, we consider the well-known nonlinear filtration equations with absorption

$$u_t = (\varphi(u))_{xx} - \psi(u) \quad \text{in } Q = \mathbf{R} \times (0,\infty). \tag{1.1}$$

Here $\varphi \in C^2((0,\infty)) \cap C^1([0,\infty))$, $\psi \in C^2((0,\infty)) \cap C([0,\infty))$ are given functions satisfying

$$\varphi'(u) > 0, \ \psi(u) > 0 \quad \text{for } u > 0, \ \varphi'(0) = 0. \tag{1.2}$$

A typical example of (1.1) arising in mechanics and plasma physics is the *porous medium equation* (PME) with absorption

$$u_t = (u^m)_{xx} - \gamma\, u^p, \quad m > 1, \ p > 0, \ \gamma \geq 0. \tag{1.3}$$

We also consider a gradient diffusivity equation (or the p-Laplacian equation) with absorption

$$u_t = (|u_x|^\sigma u_x)_x - \gamma\, u^p, \quad \sigma > 0, \ p > 0, \ \gamma \geq 0, \tag{1.4}$$

which is known in the theory of turbulent diffusion and of non-Newtonian liquids. Cf. the survey paper [**14**].

Questions of spatial concavity or the logarithmic concavity of solutions of linear, semilinear or quasilinear elliptic and parabolic equations are popular in the mathematical literature. We refer to the papers [**5**], [**6**], [**7**], [**15**], [**16**] where an extensive list of further references can be found. It is known that special aspects of the concavity/convexity analysis occur for the quasilinear degenerate equations like (1.1) with weak solutions having finite interfaces.

We study a particular one-dimensional aspect of the general concavity problem. Namely, we give a treatment of geometrical properties of solutions in terms if various proper subsets $\mathcal{B}$ of particular solutions which define special B-concavity/convexity properties preserved in time. In fact, we discuss an "optimal" notion of concavity/convexity which is suitable for a fixed class of solutions to the given quasilinear evolution equation (1.1). This paper continues the study of geometrical properties of solutions of nonlinear parabolic equations began in [**13**], see also earlier results in [**8**], [**10**], [**11**], [**12**] and in [**18**, Chapter 7].

1.1. First results for the PME. We begin with geometrical properties of concavity/convexity in x of the nonnegative weak solutions with compact support. Such investigation for the PME has been started in the papers [**2**], [**4**] and [**19**]. The results are stated in terms of the *pressure* function which for equation (1.3) is defined as

$$v = \frac{m}{m-1}\, u^{m-1}. \tag{1.5}$$

Then $v(x,t)$ solves the degenerate parabolic equation

$$v_t = \mathcal{P}_0(v) \equiv (m-1)vv_{xx} + (v_x)^2 - \rho v^\alpha, \tag{1.6}$$

where the constants are given by

$$\alpha = \frac{p+m-2}{m-1}, \quad \rho = \gamma m^{-(p-1)/(m-1)}(m-1)^\alpha \ge 0.$$

Let us state some well-known concavity-type results for equation (1.6) which we will treat below from a geometrical point of view. For given bounded, nonnegative, continuous, compactly supported initial data $v_0(x) = v(x,0) \ge 0$ in $\mathbf{R}$ there hold:

(i) [**4**] If $\rho = 0$ (i.e., $\gamma = 0$) and

$$v_0'' \le 0 \quad \text{in the connected support supp } v_0 = \{x: \ v_0(x) > 0\}, \tag{1.7}$$

then for all $t > 0$

$$v_{xx}(x,t) \le 0 \quad \text{in supp}\, v(\cdot,t) = \{x: \ v(x,t) > 0\}. \tag{1.8}$$

Thus, *concavity* of the solution is preserved in time.

(ii) [**2**] If $\rho = 0$ and (1.7) does not hold, then (1.8) is valid for all large t, which is the *eventual concavity* of the solution.

(iii) [**19**] If $\rho > 0$ and $p > 1$, then (1.7) implies (1.8), i.e. concavity is preserved in time.

Let us begin with some known geometric ideas for studying different concavity/convexity properties of the solutions of the one-dimensional parabolic equations via intersection comparison with subsets of particular exact solutions.

1.2. Concavity and convexity via solutions on linear invariant subspaces. It is shown in [**13**] that the results (i) and (ii) are a straightforward consequence of the fact that equation (1.6) with $\rho = 0$ admits a two-dimensional set $\mathcal{B}$ of the travelling wave (TW) solutions of the form $\{V(x,t) = (\lambda^2 t + \lambda x + a)_+, \ \lambda, \ a \in \mathbf{R}\}$, which are piecewise linear in space. The concavity result is then proved by using the intersection comparison technique based on the Sturmian argument for linear parabolic equations. Recall that the proof in [**4**] is based on an iterative technique applied to semigroups corresponding to splitting the nonlinear operator on the right-hand side of (1.6) ($\rho = 0$) and using a Trotter-Lie formula. (A similar approach has been used in [**19**] for equation (1.6) with $\rho > 0, \ p > 1$).

A different *proper* (see the definition in [**13**]) subset $\mathcal{B} = \{V\}$ of piecewise linear solutions exists also in the cases $\alpha = 1$ ($p = 1$) and $\alpha = 0$ ($p = 2 - m$), and (i) is again proved to be valid. In fact in all these cases the nonlinear operator $\mathcal{P}_0$ in (1.6) admits the two-dimensional *invariant* subspace

$$\mathcal{W}_2 = \mathcal{L}\{1, x\}, \tag{1.9}$$

where $\mathcal{L}\{\cdot\}$ denotes the linear span of the given functions, i.e. $\mathcal{P}_0(\mathcal{W}_2) \subseteq \mathcal{W}_2$. This yields the set $\mathcal{B} = \{V(\cdot,t) \in \mathcal{W}_2$ for $t \ge 0\}$ of explicit solutions, whence concavity results via the intersection comparison.

Another kind of concavity/convexity results is shown in [**13**] to exist if $\mathcal{P}_0$ admits a different invariant subspace. This holds, for instance, for the PME with $\rho = 0$. Then $\mathcal{P}_0$ admits the following *three-dimensional* invariant subspace

$$\mathcal{W}_3 = \mathcal{L}\{1, x, x^2\}. \tag{1.10}$$

Thus, choosing an arbitrary proper subset $\mathcal{B} \subset \mathcal{W}_3$ of the explicit Zel'dovich-Kompaneetz-Barenblatt (ZKB) solutions ($\mathcal{B}$ is assumed to satisfy the properties of completeness, continuity and monotonicity upon the parameters), we obtain a so-called B-concavity/convexity of the solution with respect to the given functional subset $\mathcal{B}$. These properties are proved to be preserved in time.

1.3. Main new results: concavity/convexity via fully nonlinear sets. The present paper is devoted to concavity/convexity properties in a more general situation where the set $\mathcal{B}$ of explicit solutions is not induced by a linear invariant subspace. We consider equation (1.1) with rather arbitrary nonlinearities. Therefore the corresponding operator $\mathcal{P}_*$ in the pressure equation (see (2.3)) does not admit any invariant subspace of the type (1.9) or (1.10). Nevertheless, we show that in certain cases the existence of a "fully nonlinear" set $\mathcal{B}$ of particular solutions can guarantee the usual concavity property. In particular, in Section 2 we prove that (iii) is true for the pressure corresponding to (1.1). Equation (1.4) is treated in Section 3. The proof is based on a local property of "small" convexity of the travelling wave solutions. On the other hand, in Section 4 we prove that itself the above set $\mathcal{B}$ of travelling wave solutions is proper and derive estimates of B-concavity preserved in time.

In Section 5 we consider the general filtration equation

$$u_t = (\varphi(u))_{xx} \tag{1.11}$$

with bounded compactly supported initial data $u_0(x)$. It is well-known that the solutions of the PME in the pressure form (1.6), $\rho = 0$, satisfy the semiconvexity estimate [**1**] (see also related properties in [**2**])

$$v_{xx} \geq -[(m+1)(t+\tau)]^{-1}, \tag{1.12}$$

where the constant τ depends on v_0: $\tau = -[(m+1)\,(\inf\, v_0'')]^{-1}$. In [**13**] it is proved that (1.12) and the same *"semiconcavity"* bound with the opposite inequality sign, $\leq$, are a straightforward consequence of the B-concavity/convexity with respect to a subset $\mathcal{B}$ of the exact ZKB-solutions of the PME which belong to the invariant subspace (1.10). The general filtration equation (1.11) does not admit such solutions, so that the only known result is the convexity/concavity of the type (i) proved in [**13**, Sections 2, 3] via intersection comparison with the TW-solutions on subspace (1.9). Observe that this estimate is not optimal in the class of bounded compactly supported solutions.

Nevertheless, we show that equation (1.11) admits a new, sharper than (i), estimate of B-concavity of the form (cf. (1.12) with the opposite sign)

$$v_{xx} \leq (t+\tau)^{-1}\, \theta''_{\xi\xi}(\xi) \tag{1.13}$$

due to a different subset of explicit self-similar solutions of the form $\{V = \theta(x/\sqrt{t+\tau})\}$ which were invented many years ago, see e.g. [**17**] and carefully studied later in [**3**], see also references in [**18**]. The derivative $\theta''(\xi)$ in (1.13) is shown to be uniquely determined from the tangency assumption, so that in general it depends, on v, v_x and t.

We also study the case where the set $\mathcal{B}$ is not complete and show that this yields a *conditional* concavity/convexity properties, Section 6. Here we use various subsets of stationary solutions of (1.3). Finally, in Section 7 we prove that there exists the property of *partial eventual* B-concavity with respect to the complete set $\mathcal{B}$ of stationary solutions.

Combining the results of the present paper and those from [**13**] makes it possible to say that any proper (complete, continuous and monotone) two-dimensional subset $\mathcal{B}$ of particular solutions of a quasilinear parabolic equation of the form (1.1) generates a certain *geometric space* with properties of B-concavity/convexity (in other words, B-curvature). In general, under necessary restrictions these properties are preserved in time for a wide infinite-dimensional subsets of general solutions. For other classes of solutions which are not B-concave/convex at $t = 0$, these properties can occur *eventually* in time and also *partially* in space. Therefore, in several cases various sets of explicit solutions play a fundamental role and describe a sharp (asymptotic) geometry of wide classes of more general solutions under consideration. Since our geometric approach is based on the intersections comparison with different subsets of explicit solutions, several results can be treated from the point of view of a certain "superposition principle" for nonlinear parabolic equations. Indeed, we translate (via B-concavity/convexity) some common properties of proper sets of *particular (explicit) solutions* to a wide class of *general solutions.*

Let us finally mention that in this paper we solve a *forward* problem: to determine B-convexity/concavity properties preserved in time via a given proper subset $\mathcal{B}$ of particular solutions. We show that such properties define a differential operator $\mathcal{H}$ which preserves signs on the orbits generated by the evolution equation (1.1). $\mathcal{H}$ is then called *sign-invariant* of (1.1) [**9**]. The *backward* problem: to reconstruct a subset of particular solutions $\mathcal{B}$ via the known structure of sign-invariants, has been studied in [**9**], see also references therein. Let us note that this makes it possible to construct new types of exact solutions and to establish different *infinite-dimensional* classes of equations (1.1) which admit exact solutions [**9**]. The backward problem is directly related to several well established aspects of classical and non-classical symmetries and the group-invariant analysis of nonlinear evolution equations.

2. Local concavity analysis via TW-solutions

2.1. Assumptions on the coefficients. In this section we study the concavity property of nonnegative solutions of (1.1). We assume that the coefficient $\varphi(u)$ satisfies the condition of finite speed of propagation, i.e.,

$$\Phi(u) = \int_0^u \frac{\varphi'(s)}{s}\, ds < \infty \quad \text{for all } u > 0, \tag{2.1}$$

cf. a survey paper [**14**]. We introduce the pressure variable

$$v = \Phi(u) \geq 0, \tag{2.2}$$

which satisfies the equation

$$v_t = \mathcal{P}_*(v) \equiv F(v)v_{xx} + (v_x)^2 - q(v), \tag{2.3}$$

where $F(v) = \varphi'(\Phi^{-1}(v))$ and

$$q(v) = \frac{\psi(u)\varphi'(u)}{u} \quad \text{with } u = \Phi^{-1}(v). \tag{2.4}$$

Let us impose necessary restriction on the coefficients which reflect properties of equation (1.3). We assume that

$$q'(s) \geq 0,\ q''(s) \geq 0,\ F''(s) \leq 0 \quad \text{for } s > 0; \tag{2.5}$$

$$\frac{q(s)}{F(s)} \to 0 \quad \text{as } s \to 0; \quad q(0) = F(0) = 0,\ q(\infty) = \infty; \tag{2.6}$$

$$\int^{\infty} \frac{q(s)}{F(s)}\, ds = \infty. \tag{2.7}$$

For convenience, we now impose a condition under which equation (1.1) does not admit a nontrivial stationary solution with a finite interface, so that no space localization of compactly supported solutions occurs. (See comments below how to get rid of this assumption). This means that the problem

$$(\varphi(S))'' - \psi(S) = 0 \quad \text{for } x > 0,\ S(0) = S'(0) = 0, \tag{2.8}$$

does not have a solution $S > 0$ for $x > 0$. One can see that this is true if the following integral diverges (cf. [**14**])

$$\int_0 \left(\int_0^s \psi(\Phi^{-1}(\eta))\, d\eta \right)^{-1/2} ds = \infty. \tag{2.9}$$

All the above assumptions are valid for equation (1.3) with $p \geq m$. The case corresponding to $1 < p < m$ will be discussed later.

2.2. The set of travelling-wave solutions $\mathcal{B}_*$. Equation (2.3), which is autonomous with respect to the independent variables x, t, admits the two-dimensional set $\mathcal{B}_* = \{V(x,t)\}$ of TW-solutions of the form

$$V(x,t) = \theta(\xi), \quad \xi = x - \lambda t + a, \tag{2.10}$$

where $\lambda, a \in \mathbf{R}$ are the parameters and the function $\theta(\xi) \geq 0$ has a finite interface, say, at $\xi = 0$. Substituting (2.10) into (2.3) we get an ordinary differential equation for the function θ

$$F(\theta)\theta'' + (\theta')^2 + \lambda\,\theta' - q(\theta) = 0 \tag{2.11}$$

with the conditions at the origin

$$\theta(0) = 0, \quad [\varphi(\Phi^{-1}(\theta))]'(0) = 0. \tag{2.12}$$

The second condition is the continuity of the heat flux $-(\varphi(u))_x$ of the solution $u(x,t)$ with the zero continuation either for $x > 0$ or $x < 0$. Setting $\theta' = P$ in (2.11), we have

$$\theta'' = P\frac{dP}{d\theta}\,, \tag{2.13}$$

and we arrive at the first-order equation

$$\frac{dP}{d\theta} = \frac{1}{PF(\theta)}\,[q(\theta) - P(P+\lambda)]. \tag{2.14}$$

Assuming now without loss of generality that $\lambda > 0$ we study equation (2.14) in the quarter of the plane $\{\theta > 0,\ P < 0\}$. It is easy to check that both the conditions (2.12) are valid for the trajectory $P = P_*(\theta)$ satisfying

$$P_*(0) = -\lambda\ . \tag{2.15}$$

(Observe that $P_*(\theta) \equiv -\lambda$ is a solution if $q \equiv 0$). Note that under hypothesis (2.9) the above problem with $\lambda = 0$ does not have a nontrivial (stationary) solution.

Proposition 2.1. *Under the above hypotheses on the coefficients, for any given $\lambda > 0$ problem (2.14), (2.15) has a unique solution $P_*(\theta)$ such that*

$$P_*(\theta) < -\lambda, \quad \frac{dP_*}{d\theta}(\theta) < 0 \quad \textit{for } \theta > 0, \tag{2.16}$$

$$P_*(\theta) \to\ -\infty \quad \textit{as } \theta \to \infty, \tag{2.17}$$

$$P_*(\theta) \to 0 \quad \textit{as } \lambda \to 0 \textit{ uniformly on compact subsets.} \tag{2.18}$$

Proof. It follows from (2.1) that the function F satisfies

$$\int_0^1 \frac{d\theta}{F(\theta)} = \infty.$$

Then using a standard local analysis of the ODE (2.14), (2.15) we conclude that a unique solution exists and for small $\theta > 0$ it has the expansion

$$P_*(\theta) = -\lambda - \frac{1}{\lambda}G(\theta)\int_0^\theta \frac{q(z)dz}{F(z)G(z)} + ..., \quad G(\theta) = \exp\left(\int_\theta^1 \frac{dz}{F(z)}\right).$$

The trajectory P_* is the separatrix. Monotonicity of $P_*(\theta)$ follows from the monotonicity of the function $q(s)$. Indeed, we have from (2.14) that the zero-cline

$$P_0(\theta) = -\frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + q(\theta)} < -\lambda \quad \text{for } \theta > 0 \tag{2.19}$$

is monotone decreasing for $\theta > 0$, and since by construction

$$P_*(\theta) > P_0(\theta) \quad \text{for } \theta > 0, \quad P_*(0) = P_0(0) = -\lambda,$$

we arrive at (2.16). Unboundedness (2.17) easily follows from (2.7). (2.18) is a straightforward consequence of (2.9). $\square$

In terms of the function $\theta(\xi)$ we have the following properties.

Corollary 2.2. *For any $\lambda > 0$ problem (2.11), (2.12) has a unique solution $\theta(\xi) > 0$ for $\xi < 0$ satisfying*

$$\theta''_{\xi\xi}(\xi) \equiv V_{xx} \geq 0 \quad \textit{whenever } \theta > 0, \tag{2.20}$$

$$\theta''_{\xi\xi} \to 0 \quad \textit{as } \theta \to 0, \tag{2.21}$$

$$\theta'_{\xi}(0) = -\lambda, \;\; \theta'_{\xi}(\xi) \to -\infty \quad \textit{as } \theta \to \infty. \tag{2.22}$$

Proof. (2.20) follows from (2.13) and (2.16). Since $P_*(\theta) < -\lambda$, from (2.14) we also conclude that

$$\theta''_{\xi\xi} \equiv P\frac{dP}{d\theta} \leq \frac{q(\theta)}{F(\theta)} \to 0 \quad \text{as } \theta \to 0 \tag{2.23}$$

by assumption (2.6). $\square$

If the function $\theta(\xi)$ becomes unbounded (blows up) as $\xi \to \bar{\xi} + 0$, for convenience we set $\theta = \infty$ for $\xi \leq \bar{\xi}$. Finally, we notice that for $\lambda < 0$ by means of the obvious invariant transformation in (2.14), $\lambda \mapsto -\lambda$, $P \mapsto -P$, we obtain a similar trajectory $-P_*(\theta) > 0$ such that properties (2.20) and (2.21) again hold.

2.3. Main result on concavity. We now consider the Cauchy problem for equation (2.3) with initial data

$$v(x, 0) = v_0(x) \geq 0 \quad \text{in } \mathbf{R}, \tag{2.24}$$

where $v_0 \in C(\mathbf{R})$ is bounded and smooth whenever positive. A global weak solution $v(x,t)$ exists, unique and smooth in $\{v > 0\}$ [**14**].

Theorem 2.3. *Let all the assumptions on the coefficient of equation (2.3) be valid. Let the support of v_0 be bounded and connected and also v_0 is positively concave:*

$$v_0'' \leq 0 \quad \textit{in } \{v_0 > 0\}. \tag{2.25}$$

Then for every $t > 0$ $v(\cdot, t)$ is positively concave:

$$v_{xx}(x,t) \leq 0 \quad \textit{in } \{v(x,t) > 0\}. \tag{2.26}$$

Proof. Let the bounded interval $l(t) = (\zeta_-(t), \zeta_+(t))$ be the support of the solution $v(x,t)$. The second derivative $z = v_{xx}$ solves in $\Omega = l(t) \times \mathbf{R}_+$ the following linear parabolic equation

$$z_t = F\; z_{xx} + 2v_x(F' + 1)z_x + [(F' + 2)z + F''(v_x)^2 - q']z - q''(v_x)^2. \tag{2.27}$$

In view of assumptions (2.5) we have that in $\Omega \cap \{z > 0\}$

$$z_t \leq F z_{xx} + 2v_x(F' + 1)z_x + (F' + 2)z^2. \tag{2.28}$$

Remark. Observe that the hypothesis $F'' \leq 0$ in (2.5) can be weakened if we take into account a Bernstein-type estimate on the first derivative, $|v_x| \leq C$, see [**14**].

Fix an arbitrary $T > 0$ such that $v(x,t) \not\equiv 0$ for all $t \in (0,T)$. By the Maximum Principle we have that

$$v \le M_0 = \sup\, v_0 \quad \text{in } \Omega_T = \Omega \cap \{t < T\}. \tag{2.29}$$

We now consider a key problem of the behaviour of the function $z(x,t)$ near the interfaces. Fix a small $\varepsilon > 0$ and a $\delta = \delta(\varepsilon) > 0$ such that

$$v \le \varepsilon \quad \text{in } S_\varepsilon = \Omega_T \setminus \Omega_T^\delta,\ \Omega_T^\delta = (\zeta_-(t)+\delta, \zeta_+(t)-\delta) \times (0,T). \tag{2.30}$$

We first prove that v_{xx} cannot be large positive near the interfaces. The analysis is based on a certain local study of the convexity of the travelling wave solutions.

Proposition 2.4. *There holds*

$$v_{xx} \le m_0(\varepsilon) = \sup_{s \in (0,\varepsilon)} \frac{q(s)}{F(s)} \quad \text{in } S_\varepsilon. \tag{2.31}$$

Proof. We use the same technique as in the proof of Proposition 3.2 in [**13**]. For a fixed $t \in (0,T)$ denote by $J(t;V)$ the number of intersections of the solution $v(x,t)$ and a given travelling wave solution $V(x,t) = \theta(\xi) \in \mathcal{B}_*$, see Section 2 in [**13**]. It follows from (2.25) and (2.20) that

$$J(0;V) \le 2 \quad \text{for any } V \in \mathcal{B}_*. \tag{2.32}$$

Since $J(t;V)$ does not increase in time by the Sturmian Theorem (see lists of references in [**13**] and [**18**, Chapter 4]) we have the following basic result of intersection comparison with the set $\mathcal{B}$

$$J(t;V) \le 2,\ t \ge 0,\ \text{for any } V \in \mathcal{B}_*\ . \tag{2.33}$$

We now prove (2.26) arguing by contradiction. Assume that there exists a point $(x_1,t_1) \in S_\varepsilon$ such that

$$v_{xx}(x_1,t_1) > m_0(\varepsilon). \tag{2.34}$$

Set $v(x_1,t_1) = \mu \in (0,\varepsilon)$ and without loss of generality assume that $\nu = v_x(x_1,t_1) < 0$. (If $\nu = 0$, in a similar way we compare with the flat solutions $V(t)$ which being independent of x are indeed convex functions of x and satisfy the ODE $V' = -q(V)$, see Section 4). Notice that by the Maximum Principle, in view of assumption (2.25), for any $t \in (0,T)$ the function $v(x,t)$ has a unique positive maxima where $v_x = 0$ and $v_{xx} \le 0$, so that $\nu \ne 0$, say $\nu < 0$. Let $V(x,t) \in \mathcal{B}_*$ be a *tangent solution* at the point (x_1,t_1) [**13**, Section 6], i.e.

$$V(x_1,t_1) = \mu,\ V_x(x_1,t_1) = \nu. \tag{2.35}$$

It follows from (2.18) and (2.22) that such a solution exists, and let

$$V(x,t) = \theta(x - \lambda t + a), \tag{2.36}$$

with constants λ, a determined from (2.35). From (2.23), (2.32) and (2.34) we then conclude that at the point (x_1, t_1)

$$v_{xx} > V_{xx} \quad (\text{and } v = V,\ v_x = V_x). \tag{2.37}$$

Therefore, since $v(x, t_1)$ is uniformly bounded and $V(x, t_1)$ is unbounded as $x \to -\infty$ by (2.22), we have that there exists at least one intersection in $\{x < x_1\}$ of the profiles $v(x, t_1)$ and $V(x, t_1)$ so that

$$J(t_1; V) \geq 1. \tag{2.38}$$

We now introduce another solution $V'(x, t)$ which is $V(x, t)$ shifted to the right. Namely, we choose $a' < a,\ a - a' \ll 1$, and set

$$V'(x, t) = \theta(x - \lambda t + a'). \tag{2.39}$$

One can see from (2.37) and (2.38) that

$$J(t_1; V') \geq 3 \tag{2.40}$$

by construction. This contradicts (2.33) completing the proof of Proposition 2.4. $\square$

End of the proof of Theorem 2.3. The rest of the proof is based on a standard application of the Maximum Principle. We introduce a supersolution $\bar{z}(t)$ of the equation (2.27) in Ω_T^δ. Set

$$b_F = \sup_{s \in (0, M_0)} F'(s), \tag{2.41}$$

and denote by $\bar{z}(t)$ a unique solution of the ODE

$$\bar{z}' = (b_F + 2)\bar{z}^2 \quad \text{for } t > 0,\ \ \bar{z}(0) = m_0(\varepsilon). \tag{2.42}$$

From (2.6) we deduce that for any small $\varepsilon > 0$ the function $\bar{z}(t)$ exists on $[0, T]$ and also

$$\bar{z}(t) \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly on } [0, T]. \tag{2.43}$$

It follows from (2.28) and (2.5) that $\overline{z}(t)$ is a supersolution of equation (2.27), and (2.25) implies that $z(x, 0) \leq \overline{z}(0)$ in $(\zeta_-(0) + \delta, \zeta_+(0) - \delta)$. Moreover, $z \leq \overline{z}$ on the lateral boundary of Ω_T^δ by (2.31). Therefore by comparison

$$z \equiv v_{xx} \leq \overline{z}(t) \quad \text{in } \Omega_T^\delta. \tag{2.44}$$

Passing to the limit $\varepsilon \to 0$ in (2.44) and using (2.43) we conclude that $z \leq 0$ in Ω_T. Since T is arbitrary we arrive at (2.26) which completes the proof. $\square$

2.4. A GENERALIZATION. If (2.9) does not hold, then (2.18) is not valid. Therefore a tangent solution $V \in \mathcal{B}_*$ does not exist provided that $|\nu|$ is small enough. In terms of [**13**] this means that the subset $\mathcal{B}_*$ is not *complete*. In this case a complete subset $\mathcal{B}_*^s = \mathcal{B}_* \cup \mathcal{B}_s$ is formed by adding to $\mathcal{B}_*$ a subset of *stationary* solutions $\mathcal{B}_s = \{V,\ \lambda = 0\}$ including the stationary solution $V_0 = \Phi(S)$ defined from (2.8) and all others which are *strictly positive*. It is easily seen that on the $\{\theta, P\}$-plane with $\lambda = 0$ the subset $\mathcal{B}_s$ consists of the nontrivial separatrix P_* (corresponding to V_0) and all other orbits which lie *above* the separatrix which thus correspond to convex stationary solutions. One can check that (2.35) defines a tangent solution $V \in \mathcal{B}_*^s$ for all $\mu > 0$, $\nu \in \mathbf{R}$ so that $\mathcal{B}_*^s$ is a complete functional subset. The end of the proof of Theorem 2.3 stays the same. See Section 7 for further details.

Combining this analysis with the Maximum Principle applied to the differential inequality (2.28), under necessary natural assumptions we can state the following general conclusion about concavity of weak solutions with finite interfaces: *initially positively concave solution preserves its concavity in time if the equation admits a complete subset $\mathcal{B} = \{V\}$ of particular solutions which are convex in the positivity domain and $V_{xx} = 0$ on the free boundaries.*

3. Concavity for p-Laplacian equation with absorption

In this section we briefly show that a similar concavity result is valid for equation (1.4) with

$$\sigma > 0,\ p \geq \sigma + 1,\ \gamma > 0. \tag{3.1}$$

The “pressure” function is (see Section 5 in [**13**])

$$v = u^{\sigma/(\sigma+1)}, \tag{3.2}$$

and the corresponding equation has the form

$$v_t = \mathcal{P}_1(v) \equiv \mu^\sigma |v_x|^\sigma [(\sigma+1) v\ v_{xx} + \mu (v_x)^2] - \frac{\gamma}{\mu}\ v^\alpha, \tag{3.3}$$

where $\mu = (\sigma+1)/\sigma$ and

$$\alpha = \frac{p(\sigma+1) - 1}{\sigma} > 1. \tag{3.4}$$

By the regularity results [**14**] we may assume that for $t > 0$ the derivative v_x exists in $\{v > 0\}$. Since in general v_{xx} does not exist at a positive maxima or minima, we use a standard notion of concavity and convexity. Namely, a nonnegative function $v_0 \in C^1$ is said to be positively *concave* (*convex*) if for any $x_0 \in \{v_0 > 0\}$ there holds $v_0(x) \leq [l(x)]_+$ (resp. $\geq [l(x)]_+$) in $\mathbf{R}$, where $l(x)$ is the tangent line to the function $v_0(x)$ at $x = x_0$. $[\cdot]_+$ denotes the positive part. Then we can state the concavity result which is similar to Theorem 2.3.

Theorem 3.1. *Assume that (3.1) holds and the initial data $v_0 \in C^1$ have bounded, connected support. Then the condition*

$$v_0 \quad \textit{is positively concave} \tag{3.5}$$

implies that the solution $v(x,t)$ of equation (1.4) satisfies

$$v(\cdot,t) \quad \textit{is positively concave for all } \; t>0. \tag{3.6}$$

Proof. As in the proof of Theorem 2.3 we need to check certain properties of the set $\mathcal{B}_*$ of the travelling wave solutions (2.10). Here θ satisfies the equation

$$\mu^\sigma|\theta'|^\sigma[(\sigma+1)\theta\;\theta''+\mu(\theta')^2]+\lambda\theta'-\frac{\gamma}{\mu}\theta^\alpha=0 \tag{3.7}$$

with the boundary conditions

$$\theta(0)=0, \quad (\theta^{1/\sigma}\theta')(0)=0. \tag{3.8}$$

Setting $\theta'=P$ yields the equation (cf. (2.14))

$$\frac{dP}{d\theta}=\frac{1}{(\sigma+1)\theta P}\left[\frac{1}{\mu^\sigma|P|^\sigma}\left(\frac{\gamma}{\mu}\theta^\alpha-\lambda P\right)-\mu P^2\right]. \tag{3.9}$$

Let $\lambda>0$. Then it is easy to check that in the quarter $\{\theta>0, P<0\}$ equation (3.9) admits a unique trajectory $P=P_*(\theta)$ such that

$$P_*(\theta)\to-\rho=-\frac{1}{\mu}\lambda^{1/(\sigma+1)}, \;\; \theta\to 0; \quad P_*(\theta)<-\rho, \quad \frac{dP_*}{d\theta}(\theta)<0 \;\text{ for } \theta>0, \tag{3.10}$$

and also (2.17) holds. In addition, (2.18) is valid if $p\ge\sigma+1$. In particular by (2.13) all the solutions $V(x,t)\equiv\theta(\xi)$ are convex functions and (2.20) is valid. Using (3.10) in (3.9) we also deduce that as $\theta\to 0$ (cf. (2.23))

$$\theta''_{\xi\xi}\equiv P\frac{dP}{d\theta}\le\frac{\gamma}{(\sigma+1)\mu^{\sigma+1}}\frac{1}{|P|^\sigma}\theta^{\alpha-1}\le\frac{2\gamma}{(\sigma+1)\mu^{\sigma+1}\rho^\sigma}\theta^{\alpha-1}\to 0\,. \tag{3.11}$$

The end of the proof based on the intersection comparison argument is similar. The only difference is as follows. Since equation (3.3) is double degenerated, the second derivative $z=v_{xx}$ satisfies the equation which is uniformly parabolic on any compact subset of $Q\cap\{v_x\neq 0\}$, where $v\in C^\infty$ [**14**]. Therefore we study first the behaviour of z near the curves where $v_x=0$. Then as in the proof of Proposition 2.4 the function z is small positive via (3.11) in a δ-neighbourhood of the interfaces. On the other hand, near the other curves with $v_x=0$ inside the support of $v(x,t)$, from (3.5) we easily deduce that z takes nonpositive values since v attains maxima on these curves. $\square$

Remark. In the case $1<p<\sigma+1$ Theorem 3.1 is still true. The proof is based on comparison with the set $\mathcal{B}_*^s$ constructed exactly in the same way as in Section 2.

4. B-concavity with respect to travelling wave solutions

We again consider equation (1.1) in the pressure representation defined in Section 2

$$v_t = \mathcal{P}_*(v) \equiv F(v)v_{xx} + (v_x)^2 - q(v) \quad \text{in } Q. \tag{4.1}$$

We assume that all the hypotheses on the coefficients of (4.1) stated in Section 2 are valid. This makes it possible to introduce the set $\mathcal{B}_*$ of travelling wave solutions which are convex (in x) functions with finite interfaces.

Denote

$$\mathcal{B}_{**} = \mathcal{B}_* \cup \mathcal{B}_\infty,$$

where $\mathcal{B}_\infty = \{V = V(t) : \ V' = -q(V), t > 0\}$ is the set of all the nontrivial solutions of (4.1) which are independent of x. $\mathcal{B}_{**}$ is a complete supplement of $\mathcal{B}_*$, see below. Observe that the functions $V \in \mathcal{B}_\infty$ can be treated as a limit case $\lambda = \infty$ of the TW-solutions $V = \theta(\eta)$ with the variable $\eta = x/\lambda - t$. Roughly speaking, the set $\mathcal{B}_{**}$ is an extension of $\mathcal{B}_*$ by continuity. One can see that $\mathcal{B}_\infty$ is a one-dimensional family $\{\tilde{V}(t+\alpha), \ \alpha \in \mathbf{R}\}$, where $\tilde{V}(t)$ is a fixed solution of the ODE.

The functions $V \in \mathcal{B}_{**}$ are not piecewise linear in x. Therefore as in [**13**, Sections 6-10] we will study the generalized B-concavity of the solution $v(x,t)$ with respect to the functional set $\mathcal{B}_{**}$. We assume that initial data $v_0 \geq 0$ is bounded, compactly supported and smooth whenever positive.

1. Properties of the set of TW-solutions. In order to use the intersection comparison argument, the set $\mathcal{B}_{**}$ must satisfy the following three properties, cf. [**13**, Sections 2 and 6]. Then $\mathcal{B}_{**}$ is said to be a *proper* one.

(i) Completeness of $\mathcal{B}_*$. This means that for an arbitrary $(x_0, t_0) \in \bar{Q}$ such that $v(x_0, t_0) > 0$ there exists a *unique* function $V \in \mathcal{B}_{**}$, called the *tangent solution* to $v(x,t)$ at the point (x_0, t_0), such that

$$V(x_0, t_0) = v(x_0, t_0) = \mu > 0, \ \ V_x(x_0, t_0) = v_x(x_0, t_0) = \nu, \tag{4.2}$$

and $V(x,t)$ exists for all $t \in [0, t_0]$.

Proposition 4.1. *The set $\mathcal{B}_{**}$ is complete.*

Proof. Existence of such a solution $V \equiv \theta(\xi) \in \mathcal{B}_*$ satisfying (4.2) with $\nu \neq 0$ follows from the properties of the function $\theta(\xi)$ with different $\lambda \neq 0$ stated in Section 2, see Proposition 2.1 and Corollary 2.2. $V(x,t)$ exists for all $t \in \mathbf{R}$. If $\nu = 0$ then the tangent solution $V(t) \in \mathcal{B}_*$ is defined from the equation $V' = -q(V), \ t > t_0; \ V(t_0) = \mu$. By the Strong Maximum Principle in the positivity domain we then conclude that the number of intersections $J(t;V)$ satisfies $J(t;V) \geq 2$ for all $t \in [0, t_0)$, so that the tangent solutions $V(t)$ is bounded and can be extended on the interval $[0, t_0]$.

We now prove uniqueness. For $\nu = 0$ it is obvious. Assume that $\nu < 0$, then $\lambda > 0$. We show that the derivative $V_x \equiv \theta'_\xi$ on a fixed level $V \equiv \theta = \mu$ is strictly increasing in

$\lambda > 0$. Fix arbitrary $\lambda_2 > \lambda_1$ and let $P_*^{(1)}(\theta)$ and $P_*^{(2)}(\theta)$ be the corresponding solutions to equation (2.14) with the boundary conditions

$$P_*^{(\nu)}(0) = -\lambda_\nu, \ \ \nu = 1, 2.$$

Then $P_*^{(2)}(\theta) < P_*^{(1)}(\theta)$ for small $\theta > 0$. Assume that the trajectory $P_*^{(1)}$ intersects first time $P_*^{(2)}$ at some $\bar{\theta} > 0$. Then at $\theta = \bar{\theta}$

$$P_*^{(1)} = P_*^{(2)} < 0, \ \ \frac{dP_*^{(1)}}{d\theta} \le \frac{dP_*^{(2)}}{d\theta}. \tag{4.3}$$

It follows from equation (2.14) for both functions $P_*^{(\nu)}$ that at $\theta = \bar{\theta}$

$$F(\bar{\theta})P_*^{(1)}(\bar{\theta}) \left[\frac{dP_*^{(1)}}{d\theta} - \frac{dP_*^{(2)}}{d\theta} \right] = P_*^{(1)}(\bar{\theta})(\lambda_2 - \lambda_1) < 0, \tag{4.4}$$

which indeed contradicts (4.3). Therefore

$$P_*(\theta) \text{ is strictly decreasing in } \lambda > 0 \text{ for all } \theta \ge 0. \tag{4.5}$$

This yields uniqueness of the tangent solution which completes the proof. □

(ii) Continuity. According to the technique in [**13**], any function $V \in \mathcal{B}_*$ and the derivative V_x must depend continuously on the parameters (λ, a) on compact subsets of the domain of positivity $\{v > 0\}$. Indeed, it is true in our case. Obviously, $\tilde{V}(t + \alpha)$ depends continuously on α in the necessary sense.

(iii) Monotonicity. A general definition of the property of monotonicity of a given set is stated in Section 6 of [**13**]. In the case of the travelling wave solutions (2.10) where the second parameter a plays a role of the translation in space, the monotonicity is guaranteed by the obvious property $V(x, t) \equiv \theta(\xi) \to \infty$ as $a \to -\infty$ and $V \to 0$ as $a \to \infty$ monotonically uniformly on compact subsets of Q. For the set $\mathcal{B}_\infty$, monotonicity with respect to α is trivial. See also [**13**, Section 2].

2. B-CONCAVITY, DIFFERENTIAL INEQUALITIES AND SIGN-INVARIANTS. Having the above properties of the set $\mathcal{B}_{**}$, we introduce the notion of B-concavity, cf. Definition 6.1 in [**13**].

Given a $t_0 \ge 0$, a solution $v(x, t) \in C^1(\{v(x, t) > 0\})$ is said to be B-concave at $t = t_0$ with respect to the set $\mathcal{B}_{**}$ if for any $x_0 \in \{v(x, t_0) > 0\}$ there holds

$$v(x, t_0) \le V(x, t_0) \quad \text{in } \mathbf{R}, \tag{4.6}$$

where $V \in \mathcal{B}_{**}$ is a unique tangent solution at the point (x_0, t_0).

Note that due to the regularity in the positivity domain, (4.6) also yields the inequality

$$v_{xx}(x_0, t_0) \le V_{xx}(x_0, t_0). \tag{4.7}$$

In the case of piecewise linear functions V such that $V_{xx} = 0$ a.e., (4.7) implies the usual notion of a positively concave function: $v_{xx} \le 0$ in $\{v > 0\}$.

It was proved in [**13**, Section 6] that if the set $\mathcal{B}_{**}$ is *proper* then the following result holds.

Theorem 4.2. *Under the given assumptions we have that*

$$v_0(x) \quad \textit{is } B\textit{-concave} \tag{4.8}$$

implies

$$v(x,t) \quad \textit{is } B\textit{-concave for all } t>0. \tag{4.9}$$

Finally, we show that, as usual, B-concavity of the solution yields a certain nonlinear differential inequality on the solutions. In order to rewrite (4.7) in terms of a differential inequality we first translate the *geometrical coordinates* $\{v, v_x\}$ of the solution $v(x,t)$ at a given point (x_0, t_0) into the *parametric coordinates* $\{\lambda, a\}$ (or α) of the corresponding tangent solution $V \in \mathcal{B}_{**}$. Assume first that $\nu \neq 0$. In view of (2.10) and (4.2) this yields the following algebraic system of equations of the above *change of variables*

$$\theta(x_0 - \lambda t_0 + a) = \mu > 0, \quad \theta'(x_0 - \lambda t_0 + a) = \nu \in \mathbf{R} \setminus \{0\}. \tag{4.10}$$

We have proved in Proposition 4.1 that these equations admit a unique solution which we denote

$$\lambda = \lambda(\mu, \nu, t), \quad a = a(\mu, \nu, x_0, t_0). \tag{4.11}$$

In general, this solution cannot be written in an explicit form. If $\nu = 0$ then $V \in \mathcal{B}_\infty$ with $V_x = V_{xx} \equiv 0$.

We now formulate the desired estimate.

Proposition 4.3. *If $v(x,t)$ is B-concave for $t \geq 0$ then in $Q \cap \{v>0,\ v_x \neq 0\}$*

$$v_{xx} \leq \frac{1}{F(v)} \left\{ q(v) - v_x[v_x + \lambda(v, v_x, t)] \right\}, \tag{4.12}$$

where $\lambda(\mu, \nu, t)$ is the unique root of algebraic system (4.10).

Proof. Let $V \equiv \theta(\xi) \in \mathcal{B}_*$ be the unique tangent solution at (x_0, t_0). Then by (2.11)

$$V_{xx} \equiv \theta''_{\xi\xi} = \frac{1}{F(\theta)}[q(\theta) - \theta'_\xi(\theta'_\xi + \lambda)]. \tag{4.13}$$

Hence, since at a tangency point $V \equiv \theta = v,\ V_x \equiv \theta'_\xi = v_x$, estimate (4.12) follows from (4.7) and (4.13). $\square$

Observe that if $v_x = 0$ then $V = V(t) \in \mathcal{B}_\infty$ and (4.7) implies the obvious inequality $v_{xx} \leq 0$.

Using equation (4.1) estimate (4.12) can be rewritten in the form

$$\mathcal{H}(v) \equiv v_t + \lambda(v, v_x, t) v_x \leq 0 \quad \text{in } Q \cap \{v>0,\ v_x \neq 0\}, \tag{4.14}$$

which is the first-order *differential inequality* of the Hamilton-Jacobi type.

Recall that B-convexity for another class of solutions is also preserved in time [**13**], and is equivalent to the inequality

$$\mathcal{H}(v) \geq 0 \quad \text{in } Q \cap \{v > 0,\ v_x \neq 0\}. \tag{4.15}$$

Therefore, the Hamilton-Jacobi operator $\mathcal{H}$ is a *sign-invariant* [**9**] of the parabolic equation (2.3) and it preserves both signs, $\leq$ and $\geq$, in the positivity domains for classes of suitable solutions. It is worth mentioning that due to our construction the operator $\mathcal{H}$ is generated by the proper subset $\mathcal{B}_*$ of TW-solutions. On the other hand, as the backward approach, under certain circumstances a given sign-invariant can reproduce the corresponding (proper) subset of exact solutions. Such backward problems were treated in [**9**] for several types of sign-invariants.

5. Semiconcavity for the filtration equation

We now consider equation (1.11) which in the pressure representation (2.2) has the form

$$v_t = F(v)v_{xx} + (v_x)^2 \quad (F(v) = \varphi'(\Phi^{-1}(v))). \tag{5.1}$$

Initial data $v_0(x)$ is assumed to be bounded and smooth in the positivity interval $\{v_0 > 0\}$.

5.1. Subset of parabolic solutions is proper. We calculate the inequality corresponding to the B-concavity to a subset $\mathcal{B}_p$ of the self-similar (*parabolic*) solutions

$$V(x,t) = \theta(\xi), \quad \xi = x/\sqrt{t+\tau}, \tag{5.2}$$

where $\tau \geq 0$ is a fixed constant. Substituting (5.2) into (5.1) we obtain the ODE

$$F(\theta)\theta'' + (\theta')^2 + \frac{1}{2}\,\theta'\xi = 0. \tag{5.3}$$

We are interested in the profiles $\theta = \theta(\xi; b)$ satisfying the interface conditions at a fixed finite $\xi = b > 0$:

$$\theta(b) = 0, \quad \theta'(b^-) = -b/2, \tag{5.4}$$

so that $\theta(\xi) \equiv 0$ for $\xi > b$. This corresponds to the continuity of the heat flux $-\varphi'(u)u_x$ on the interface. We have $\theta'(\xi) < 0$ for $\xi < b$. Let us impose the condition

$$\int_0^\infty \frac{\varphi'(\eta)}{\eta}\, d\eta = \infty, \tag{5.5}$$

which implies that the solution to (5.3), (5.4) can be continued for all $\xi < b$, and $\theta' < 0$, see [**3**]. In fact, this plays no important role since we deal with bounded solutions $v(x,t) \leq \sup v_0$.

Fix a $\tau \geq 0$. We define the functional subset

$$\mathcal{B}_p = \{V = \theta(\pm(x-a)/\sqrt{t+\tau}; b),\ a \in \mathbf{R},\ b > 0\} \cup \{V \equiv c,\ c > 0\}, \tag{5.6}$$

which is a "continuous" supplement of the family of nontrivial self-similar solutions (5.2).

Proposition 5.1. *Subset (5.6) is proper.*

Proof. Consider equation (4.2) for the tangent solution. If $\nu = 0$ then there exists the unique tangent solution $V \equiv \mu$. Let $\nu < 0$. It follows from (5.4) and other obvious properties of profiles θ that by continuity, a tangent solution exists. Assume for contradiction that it is not unique, and there exists two profiles $\theta_s = \theta(\xi; b_s)$, $s = 1, 2$, such that $b_1 < b_2$ and

$$\begin{aligned} &\theta_2'(\xi) < \theta_1'(\xi) \quad \text{on any level } \theta = \mu \in (0, \bar{\mu}), \\ &\theta_2'(\xi) = \theta_1'(\xi) \quad \text{on the level } \theta = \bar{\mu}. \end{aligned} \tag{5.7}$$

Let $\theta_1(\xi_1) = \theta_2(\xi_2) = \bar{\mu}$, $\xi_1 < \xi_2$. We apply an energy argument. Consider the functions $f_s(\xi) = \Phi^{-1}(\theta_s(\xi))$, which define the self-similar solutions $u_s(x,t) = f_s(x/\sqrt{t+\tau})$ of the filtration equation (1.11). It follows from (5.7) that

$$\int_{\xi_1}^{b_1} f_1(\eta) d\eta > \int_{\xi_2}^{b_2} f_2(\eta) d\eta. \tag{5.8}$$

Then integrating equation (1.11) with $u = u_s$ over $(\xi_s\sqrt{t+\tau}, b_s\sqrt{t+\tau})$ and taking into account (5.7) we arrive at a contradiction with (5.8) for $t \gg 1$. Thus for any $b_1 < b_2$ we have that

$$\theta_2'(\xi) < \theta_1'(\xi) \quad \text{on any level } \theta = \mu \in (0, \sup \theta_2). \tag{5.9}$$

Continuity and monotonicity are straightforward, see Section 4 and [**13**]. $\square$

5.2. Semiconvexity estimate and sign-invariant. It follows from Proposition 5.1 and [**13**, Section 6] that Theorem 4.2 holds where B-concavity is defined by means of the subset (5.6). We now derive the corresponding B-convexity estimate. It follows from (4.7) and (5.6) that it has the form

$$v_{xx} \le (t+\tau)^{-1}\, \theta''_{\xi\xi}(\xi; b) \quad \text{in } \{v > 0\}, \tag{5.10}$$

where by Proposition 5.1 the second derivative $\theta''(\xi; b)$ at the tangency point ξ is uniquely determined from the tangency equations (4.2) which for the subset (5.6) are

$$\theta(\xi; b) = \mu = v, \quad (t+\tau)^{-1/2}\, \theta'_\xi(\xi; b) = \nu = v_x. \tag{5.11}$$

This is an algebraic system for unknown a, b or c. Here we suppose that $\nu = v_x < 0$. If $\nu > 0$ then we fix the profiles $\theta(-\xi; b)$ and if $\nu = 0$ then by (5.6) $\theta(\xi) \equiv \mu$ and hence $\theta'' = 0$. Moreover, from (5.3) one can derive the following equivalent form of (5.10)

$$F(v) v_{xx} \le -(v_x)^2 - \frac{1}{2}(t+\tau)^{-1/2} v_x \xi, \tag{5.12}$$

where ξ (and b) is uniquely determined from the system (5.11).

Theorem 5.2. *If for a fixed $\tau \geq 0$ inequality (5.10), (5.11) holds for $t = 0$, then it is true for $t > 0$.*

Observe that (5.10) provides us with an upper bound of the second derivative v_{xx} on the interfaces, which together with the standard semiconvexity estimate gives control of v_{xx} in the positivity domain via the Maximum Principle applied to equation (2.27) with $q \equiv 0$.

Indeed, there exists the corresponding B-convexity estimate with the opposite sign $\geq$ in (5.10). In view of the evident properties of the subset (5.6) this result is true for v_0 which is supported from one side. Therefore, (5.12) describes the first-order sign-invariant which by equation (5.1) takes the form

$$\mathcal{H}(v) = v_t + \frac{1}{2}(t+\tau)^{-1/2} v_x \xi, \tag{5.13}$$

and preserves both signs on the corresponding solutions of (5.1).

Finally, let us note that there exist infinitely many other proper subsets $\mathcal{B}$ belonging to the set (5.2), each one corresponds to a suitable non-constant function $\tau = \tau(b)$, see similar examples in [**13**, Section 7]. Therefore, the set of particular solutions (5.2) determines an infinite number of different notions of B-concavity/convexity preserved in time or, under certain assumptions (see [**13**, Section 3]), occurring eventually in time. Each proper subset generates the corresponding sign-invariant.

6. Concavity to incomplete functional subsets

We now discuss the problem on the concavity/convexity analysis with respect to a subset $\mathcal{B}$ which *is not complete* in the sense of the definition in Section 4. We prove that if $\mathcal{B}$ is also continuous and monotone then the intersection comparison can be performed in a similar way. To show that we begin with a particular example of such an incomplete subset.

1. Subset of stationary solutions. We consider the PME with absorption

$$u_t = (u^m)_{xx} - u^p \quad \text{in } Q \tag{6.1}$$

where

$$m > 1, \;\; 0 < p < m, \;\; p \neq m-2. \tag{6.2}$$

Equation (6.1) admits a subset $\mathcal{B}_0$ of the following stationary solutions having a unique finite interface

$$U(x) = \left\{ \frac{m-p}{\sqrt{2m(m+p)}} [\pm(x-a)]_+ \right\}^{2/(m-p)}, \tag{6.3}$$

where $a \in \mathbf{R}$ is an arbitrary parameter. In fact, the subset $\mathcal{B}_0 = \{V\}$ is one-dimensional, and hence it is not complete. Nevertheless, we now introduce the notion of B-concavity.

2. B-CONCAVITY. We assume that initial data u_0 is bounded, compactly supported and smooth in $\{u_0 > 0\}$.

We say that for a given $(x_0, t_0) \in \bar{Q}$ such that $\mu = u(x_0, t_0) > 0$, $\nu = u_x(x_0, t_0) \in \mathbf{R}$, the point with the geometrical coordinates (μ, ν) is *admissible* to the subset $\mathcal{B} = \{U(x,t)\}$ if there exists a unique function $U \in \mathcal{B}$ such that

$$U(x_0, t_0) = \mu, \quad U_x(x_0, t_0) = \nu, \tag{6.4}$$

and U exists for all $t \in [0, t_0]$.

In the above notations, we introduce the B-concavity property which is similar to that in Section 4. Given $t_0 \geq 0$, the solution $u(x,t) \in C^2(\{u > 0\})$ is said to be B-concave with respect to the subset $\mathcal{B}$ if for any $x_0 \in \{u(x, t_0) > 0\}$ such that $(u(x_0, t_0), u_x(x_0, t_0))$ is admissible to $\mathcal{B}$ there holds

$$u(x, t_0) \leq U(x, t_0) \quad \text{in } \mathbf{R}, \tag{6.5}$$

where $U \in \mathcal{B}_0$ is the unique tangent solution at (x_0, t_0). In particular, (6.5) implies that

$$u_{xx}(x_0, t_0) \leq U_{xx}(x_0, t_0). \tag{6.6}$$

Let now $\mathcal{B} = \mathcal{B}_0$ defined in (6.3). One can see that $\mathcal{B}_0$ is continuous and also it is monotone with respect to the translational parameter $a \in \mathbf{R}$. Therefore, using the same technique as in Section 2 and Sections 6, 7 of [**13**], we arrive at the fact that B-concavity of the solution $u(x,t)$ with respect to the subset $\mathcal{B}_0$ is preserved in time (cf. Theorem 4.2).

Theorem 6.1. *In the given notations, there holds*

$$u_0 \ \textit{is B-concave} \implies u(\cdot, t) \ \textit{is B-concave for } t > 0. \tag{6.7}$$

It is easy to calculate that B-concavity of $u(x,t)$ means that at any point $(u(x,t), u_x(x,t))$ admissible to $\mathcal{B}_0$ there holds

$$u_{xx} \leq \frac{2-m+p}{m(m+p)}\, u^{1+p-m}. \tag{6.8}$$

Using (6.3) and definition (6.4), inequality (6.8) becomes

$$u_{xx} \leq C|u_x|^{2(1+p-m)/(2-m+p)}, \tag{6.9}$$

where $C = C(m,p)$ is a constant. In view of equation (6.1) we deduce that (6.8) is equivalent to the following inequality of the Hamilton-Jacobi type

$$u_t \leq m(m-1)u^{m-2}(u_x)^2 - \frac{2(m-1)}{m+p}\, u^p. \tag{6.10}$$

7. Partial eventual B-concavity: an application

We consider an application of the property of eventual B-concavity to the study of positive solutions of the PME with absorption

$$u_t = (u^m)_{xx} - u^p \quad \text{in } Q \tag{7.1}$$

with the parameters (cf. (6.2))

$$m > 1, \quad p \in \mathbf{R}. \tag{7.2}$$

Equation (7.1) with $p < 1$ is known to exhibit the phenomenon of extinction [**14**], when because of the *strong* absorption ($p \in [0, 1)$) or the *singular* one ($p < 0$) any bounded solution vanishes identically in finite time. We assume that initial data,

$$u(x, 0) = u_0(x) \quad \text{in } \mathbf{R}, \tag{7.3}$$

is strictly positive and sufficiently smooth:

$$0 < c_0 \le u_0(x) \le C_0, \quad [(u_0^m)']^2 \le C_1. \tag{7.4}$$

Denote by $T = T(u_0) \in (0, \infty]$ the extinction time:

$$T = \sup\{\tau > 0 : u(x, t) > 0 \text{ in } \mathbf{R} \times (0, \tau)\}. \tag{7.5}$$

Then $T = \infty$ if $p \ge 1$ and $T < \infty$ if $p < 1$, see [**14**]. We consider the solution $u(x, t)$ in $Q_T = Q \cap \{t < T\}$, so that by the regularity of the coefficients of the equation, $u \in C^\infty(Q_{T'})$ with any $T' \in (0, T)$.

We show that under certain necessary assumptions on initial data the geometry of the complete set $\mathcal{B}$ of stationary solutions generates the property of *eventual* $\mathcal{B}$-concavity having an interesting application. Remind that a simpler property of eventual concavity is natural for a general filtration equation, see Section 3 in [**13**]. Nevertheless, the fact that the property of eventual B-concavity with respect to a nonlinear complete set $\mathcal{B}$ exists is not trivial. Let us note that an "envelope analysis" in the method of stationary states [**8**], [**10**] as well as the monotonicity property of large solutions of quasilinear heat equations with source, see [**11**], [**12**], can be treated as a partial eventual B-convexity result.

7.1. Subsets of stationary solutions. Consider the stationary equation

$$(U^m)_{xx} - U^p = 0\,. \tag{7.6}$$

Let us list necessary stationary subsets.

(i) First, we notice that if $-m < p < m$ then (7.6) admits the one-dimensional subset $\mathcal{B}_0$ of stationary solutions (6.3).

(ii) For arbitrary exponents satisfying (7.2) there exist strictly positive solutions $U(x; \lambda)$ satisfying at a minima point, say at $x = 0$, the conditions

$$U(0; \lambda) = \lambda > 0, \quad U_x'(0; \lambda) = 0. \tag{7.7}$$

Multiplying (7.6) by $(U^m)_x$ and integrating over $(0, x)$ with conditions (7.7) we obtain that

$$[(U^m)_x]^2 = \frac{2m}{p+m}\left(U^{p+m} - \lambda^{p+m}\right) \quad \text{if } p \neq -m, \tag{7.8}$$

$$[(U^m)_x]^2 = m \log \frac{U}{\lambda} \quad \text{if } p = -m. \tag{7.9}$$

Denote by $U_*(-x) \equiv U_*(x)$ the unique solution of problem (7.6), (7.7) with $\lambda = 1$. Then by symmetry we have that the full subset $\mathcal{B}_1$ of such stationary solutions shifted in space is given by the formula

$$\mathcal{B}_1 = \{U(x) = \lambda\, U_*((x+a)\lambda^{(p-m)/2}),\ \ \lambda > 0,\ \ a \in \mathbf{R}\}. \tag{7.10}$$

One can see from (7.8), (7.9) that if $p \leq m$ then the functions U are well defined in $\mathbf{R}$. On the other hand, if $p > m$ then $U(x)$ blows up as $x \to a \pm x_0(\lambda) \mp 0$, where

$$x_0(\lambda) = x_* \lambda^{-(p-m)/2}, \tag{7.11}$$

with $x_* > 0$ being the blow-up point of U_*. It follows from (7.8), (7.9) that as $\lambda \to 0$ on any level set $\{x:\ \ U = c > 0\}$

$$\begin{aligned} &|(U^m)_x| \to \frac{2m}{p+m}\, c^{p+m} \quad \text{if } p > -m, \\ &|(U^m)_x| \to \infty \quad \text{if } p \leq -m. \end{aligned} \tag{7.12}$$

(iii) The last subset $\mathcal{B}_2$ of stationary solutions exists for $p > -m$. The solution $U \in \mathcal{B}_2$ vanishes at some point, say $x = 0$, where it satisfies the conditions

$$U(0) = 0, \quad (U^m)_x(0) = \lambda \neq 0.$$

Integrating (7.6) with these boundary conditions we arrive at the formula (cf. (7.8))

$$[(U^m)_x]^2 = \lambda^2 + \frac{2m}{p+m}\, U^{p+m}. \tag{7.13}$$

Notice that $\mathcal{B}_2 = \{U(x+a), a, \lambda \in \mathbf{R}\}$ is a two-dimensional subset as well as $\mathcal{B}_1$.

7.2. Completeness and proper subsets. First, the properties of continuity with respect to the parameters (λ, a) and monotonicity with respect to a or λ of all subsets $\mathcal{B}_0, \mathcal{B}_1$ and $\mathcal{B}_2$ are obvious. The property of completeness depends on m and p.

Proposition 7.1. *The following sets are proper:*

$$\begin{aligned} &\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \quad \textit{if } p \geq m, \\ &\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \quad \textit{if } -m < p < m, \\ &\mathcal{B} = \mathcal{B}_1 \quad \textit{if } p \leq -m. \end{aligned} \tag{7.14}$$

Proof. We prove completeness. Existence and uniqueness of a tangent solution $U(x)$ satisfying for arbitrary given $x_0 \in \mathbf{R},\ \mu > 0,\ \nu \in \mathbf{R}$, the system

$$U(x_0) = \mu, \quad U_x'(x_0) = \nu, \tag{7.15}$$

follow from the corresponding results for the ODE (7.6). Then using possible continuations of the function $U(x)$ into the domain $\{0 < U \ll 1\}$ (see (7.8), (7.9) and (7.13)) we obtain the result. Observe that completeness of $\mathcal{B}_1$ for $p \leq -m$ is easily seen from (7.12). $\square$

We thus conclude that the existence of the above proper set $\mathcal{B}$ implies B-convexity which is preserved in time. We now study a more delicate property of the eventual B-convexity of solutions.

7.3. Partial eventual B-concavity. This property means that the solution becomes eventually B-convex in a domain where it is small enough, i.e. *partially* in Q_T. Namely, we have the following natural generalization of the concept of the eventual convexity/concavity of solutions which is efficient for the problem (7.1), (7.3), cf. Section 6 in [**13**]. Fix an arbitrary proper subset $\mathcal{B}$ from (7.14).

Definition 7.2. A solution $u(x,t) > 0$ in Q_T is said to be *partially eventually B-concave* (with respect to the fixed proper subset $\mathcal{B}$ of stationary solutions) if there exists a constant $\delta_0 = \delta_0(u_0) > 0$ such that

$$u(\cdot,t) \quad \text{is B-concave in } Q_T \cap \{u < \delta_0\}. \tag{7.16}$$

Theorem 7.3. *Let (7.2) and (7.4) hold. Then:*

(i) The solution $u(x,t)$ is partially eventually B-concave in the case $p \le -m$ with

$$\delta_0 = m_0 e^{-C_1/m} \quad \textit{if } p = -m, \tag{7.17}$$

$$\delta_0^{p+m} = \frac{|p+m|}{2m}\, C_1 + c_0^{p+m} \quad \textit{if } p < -m. \tag{7.18}$$

(ii) If $p > -m$ then (7.16) is also true under the assumption

$$C_1 < \frac{2m}{p+m}\, c_0^{p+m}\ , \tag{7.19}$$

with

$$\delta_0^{p+m} = c_0^{p+m} - \frac{p+m}{2m}\, C_1. \tag{7.20}$$

Proof. We show that for any $(x_0,t_0) \in Q_T \cap \{u < \delta_0\}$ there holds

$$u(x,t_0) \le U(x) \quad \text{in } \mathbf{R}, \tag{7.21}$$

where $U \in \mathcal{B}$ is a tangent solution at (x_0,t_0). Indeed, (7.21) implies B-concavity in $\{0 < u < \delta_0\}$. According to the general scheme of the concavity/convexity analysis with respect to a given proper subset $\mathcal{B}$ (see Section 4 and [**13**]), (7.21) is valid if for any $\mu \in (0,\delta_0)$ and $\nu \in \mathbf{R}$, the number of intersections $J(0;U)$ of the initial function u_0 and a tangent profile $U \in \mathcal{B}$ satisfying (7.15) is such that

$$J(0;U) \le 2 \quad \text{for any } x_0 \in \mathbf{R}. \tag{7.22}$$

Then since new intersections cannot appear at $x = \pm\infty$ (recall that $u(x,t)$ is uniformly bounded unlike the stationary solutions) we have

$$J(t;U) \le 2 \quad \text{for all } t \in (0,T). \tag{7.23}$$

Therefore, if the solution u takes at a point (x_0, t_0) given values $u(x_0, t_0) = \mu,\ u_x(x_0, t_0) = \nu$, then (7.23) with $t = t_0$ immediately gives (7.21).

In view of assumptions (7.4), one can see that (7.22) is valid if $|(U^m)_x| \geq |(u_0^m)'|$ on any level set $\{x \in \mathbf{R} : u_0(x) = c\}$ with an arbitrary $c \in [c_0, C_0]$. With the choice of the values of δ_0 given in (7.17), (7.18) and (7.20), this follows from (7.8), (7.9) or (7.14). This completes the proof. $\square$

Remark. Let $p \geq m$ and $u_0(x) \geq 0$ be compactly supported. Then $T = \infty$ and meas supp $u(\cdot, t) \to \infty$ as $t \to \infty$ [**14**]. In this case one can also prove a partial eventual B-concavity result for the solutions where the "partiality" property includes not only the restriction $u < \delta_0$ as in (7.14) but also $x \in K$ where K is a certain compact subset from supp u_0. Actually, a suitable notion of partial eventual B-concavity depends on the properties of the solution under consideration.

7.4. Generalizations to arbitrary nonlinearities. Theorem 7.3 is easily translated to a class of equations (1.1) with general nonlinearities including the case of singular absorption, $\psi(u) \to \infty$ as $u \to 0$. Let us briefly discuss such a generalization. The initial function (7.3) satisfies

$$c_0 \leq u_0(x) \leq C_0, \quad [(\varphi(u_0))']^2 \leq C_1. \tag{7.24}$$

From the corresponding stationary equation

$$(\varphi(U))_{xx} - \psi(U) = 0 \tag{7.25}$$

we easily derive for $U \in \mathcal{B}_1$ (cf. (7.8), (7.9))

$$[(\varphi(U))_x]^2 = 2\int_\lambda^U \varphi'(s)\psi(s)ds. \tag{7.26}$$

If

$$\int_0^1 \varphi'(s)\psi(s)ds < \infty, \tag{7.27}$$

then there exists also the set $\mathcal{B}_2$ such that (cf. (7.14))

$$[(\varphi(U))_x]^2 = \lambda^2 + 2\int_0^U \varphi'(s)\psi(s)ds, \quad \lambda \neq 0. \tag{7.28}$$

A necessary and sufficient condition of existence of $\mathcal{B}_0$ derived from (7.28) with $\lambda = 0$ is the convergence of the integral in (2.9). Identities (7.26) and (7.28) are enough to check the necessary condition (7.22). Namely we have:

(i) If

$$\int_0^1 \varphi'(s)\psi(s)ds = \infty \tag{7.29}$$

(this corresponds to $p \le -m$ in (7.1)), then the solution $u(x,t)$ becomes partially eventually B-concave.

(ii) If (7.27) holds, then the same is true, i.e. there exists $\delta_0 > 0$ such that (7.16) holds, if (cf. (7.19))

$$C_1 < 2\int_0^{c_0} \varphi'(s)\psi(s)ds. \tag{7.30}$$

In this case $\delta_0 > 0$ is taken from the following algebraic equation (cf. (7.20))

$$2\int_{\delta_0}^{c_0} \varphi'(s)\psi(s)ds = C_1 \ . \tag{7.31}$$

7.5. An application. Finally, we derive a differential inequality on the solutions, which is equivalent to (7.16).

Corollary 7.4. *If $u(x,t)$ is partially B-concave then*

$$u_t < 0 \quad \textit{in } Q_T \cap \{u < \delta_0\}. \tag{7.32}$$

Proof. Inequality (7.21) with an arbitrary $(x_0,t_0) \in Q_T \cap \{u < \delta_0\}$ implies that

$$u_{xx}(x_0,t_0) \le U_{xx}(x_0). \tag{7.33}$$

In view of (7.1) and (7.6), since $u = U,\ u_x = U_x$ at the tangency point (x_0,t_0) we have that

$$u_t \le 0 \quad \text{in } Q_T \cap \{u < \delta_0\}. \tag{7.34}$$

The strict inequality in (7.32) follows from (7.32) by the Strong Maximum Principle applied to the linear parabolic equation

$$z_t = m(u^{m-1}z)_{xx} - p\,u^{p-1}z \tag{7.35}$$

satisfied by the derivative $z = u_t \not\equiv 0$. $\square$

Thus, (7.32) shows that under above hypotheses near the extinction point any suitable solution always *decreases monotonically in time.* This is a general property of nonlinear equation (7.1) which says that no oscillations in time can appear near extinction.

Acknowledgments

The author would like to thank J.L. Vazquez for useful discussions. This paper was written during a visit of the first author Departamento de Matemáticas, Universidad Autónoma de Madrid (UAM), supported by a sabbatical grant from Ministerio de Educación y Ciencia (MEC), Spain. The first author is grateful to both institutions. This work has been partially supported by EEC Grant SC1-0019-C(TT) and DGICYT Project PB90-0218 (Spain).

References

[**1**] D.G. Aronson and Ph. Bénilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbf{R}^N$, C. R. Acad. Sci. Paris., Ser. I **288** (1979), 103-105.

[**2**] D.G. Aronson and J.L. Vazquez, *Eventual* C^∞*-regularity and concavity for flows in one-dimensional porous media*, Arch. Ration. Mech. Anal. **99** (1987), 329-348.

[**3**] F.V. Atkinson and L.A. Peletier, *Similarity solutions of the nonlinear diffusion equation*, Arch. Ration. Mech. Anal. **54** (1974), 373-392.

[**4**] Ph. Bénilan and J.L. Vazquez, *Concavity of solutions of the porous medium equation*, Trans. Amer. Math. Soc. **299** (1987), 81-93.

[**5**] H.J. Brascamp and E.H. Lieb, *On extension of the Brunn-Minkowski and Prekoja-Leindler theorems, including inequalities for* log *concave functions, and with application to diffusion equation*, J. Funct. Anal. **22** (1976), 366-389.

[**6**] L.A. Caffarelli and A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. **52** (1985), 431-456.

[**7**] L.A. Caffarelli and J. Spruck, *Convexity properties of solutions of some classical variational problems*, Commun. Partial Differ. Equat. **7** (1982), 1337-1379.

[**8**] V.A. Galaktionov, *On localization conditions for unbounded solutions of quasilinear parabolic equations*, Soviet Math. Dokl. **25** (1982), 775-780.

[**9**] V.A. Galaktionov, *Quasilinear heat equations with first-order sign-invariants and new explicit solutions*, Nonlinear Anal., Theory, Meth. and Appl. **23** (1994), 1595-1621.

[**10**] V.A. Galaktionov, S.P. Kurdyumov, and A.A. Samarskii, *On the method of stationary states for quasilinear parabolic equations*, Math. USSR Sbornik **67** (1990), 449-471.

[**11**] V.A. Galaktionov and S.A. Posashkov, *Applications of new comparison theory in the investigation of unbounded solutions of nonlinear parabolic equations*, Differ. Equat. **22** (1986), 809-815.

[**12**] V.A. Galaktionov and S.A. Posashkov, *Any large solution of nonlinear heat conduction equation becomes monotone in time*, Proc. Roy. Soc. Edinburgh **118A** (1991), 13-20.

[**13**] V.A. Galaktionov and J.L. Vazquez, *Geometrical properties of the solutions of one-dimensional nonlinear parabolic equations*, Math. Ann. **303** (1995), 741-769.

[**14**] A.S. Kalashnikov, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations*, Russian Math. Surveys **42** (1987), 169-222.

[**15**] N.J. Korevaar, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **32** (1983), 603-614.

[**16**] P.L. Lions, *Two geometrical properties of solutions of semilinear problems*, Appl. Anal. **12** (1981), 267-272.

[**17**] P.Ya. Polubarinova-Kochina, *On a nonlinear differential equation encountered in the theory of filtration*, Dokl. Akad. Nauk SSSR **63** (1948), 623-627.

[**18**] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, "Blow- up in Quasilinear Parabolic Equations", Nauka, Moscow, 1987 (in Russian). English translation: Walter de Gruyter, Berlin/New York, 1995.

[**19**] J.L. Vazquez, *Convexity properties of the solutions of nonlinear heat equations*, in "Contributions to Nonlinear Partial Differential Equations", vol. II, J.I. Díaz and P.L. Lions Eds., Pitman Research Notes in Math., Longman, 1987, 267-275.

Victor A. Galaktionov
School of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
and Keldysh Inst. Appl. Math., Miusskaya Sq. 4, 125047 Moscow, RUSSIA
E-mail: vag@maths.bath.ac.uk