Invariant solutions of two models of evolution of turbulent bursts

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Abstract

We consider two problems related to the b–l and b–ε models of propagation of turbulent bursts. We show that these equations admit some particular exact solutions which reduce to a finite-dimensional dynamical systems. This makes it possible to describe a singular effect of finite-time extinction, and, in particular, nonsymmetric solutions which do not exhibit symmetrization in the asymptotic behaviour. We show that in the multi-dimensional equation related to the b–l model the nonsymmetric extinction behaviour is governed by the first-order equation. For the b–ε model with $\alpha = \beta = 1$ and $\gamma < 1$, using such particular solutions, we establish that the ω-limit set of all the rescaled extinction orbits is essentially infinite-dimensional.

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1 Singular behaviour in a modified b–l model

We consider the following quasilinear parabolic equation with nonlocal, time-dependent coefficients

$$(1) \qquad b_t = l(t)\Delta b^{3/2} - k\frac{b^{1/2}}{l^2(t)} \quad \text{for } x \in \mathbf{R}^N, \ t > 0,$$

where $k > 0$ is a fixed constant and $l(t) > 0$ is the measure of the support of a nonnegative bounded solution $b(x,t)$,

$$(2) \qquad l(t) = \text{meas supp}\, b(\cdot,t) \equiv \text{meas}\,\{x \in \mathbf{R}^N : \ b(x,t) > 0\}.$$

Equation (1) belongs to the class of quasilinear nonlocal degenerate parabolic equations. The most well-known equation of such type is the b–l model of the propagation of turbulent bursts from a horizontally uniform layer [1], [2], p. 291:

$$(3) \qquad b_t = l(t)(b^{1/2}b_x)_x - k\frac{b^{3/2}}{l(t)}, \quad x \in \mathbf{R}, \ t > 0,$$

which is the equation for turbulent energy balance, in which $b(x,t) \geq 0$ represents the turbulent energy density. Existence results for (3) based on the known theory of the porous media equation (PME) can be found in [24], self-similar solutions were studied in [16] and [28]. The asymptotic behaviour as $t \to \infty$ of the finite mass solutions is then proved to be self-similar [24].

It is important that in the case of equation (3), self-similarity is of the *second kind* [5], [16], [28], i.e., the exponents α, β in the self-similar representation

$$b(x,t) = t^{-\alpha} f(xt^{-\beta})$$

cannot be found from a standard dimensional consideration. These exponents have to be determined from a nonlinear eigenvalue problem.

Observe that equation (3) admits no exact solutions except the single case $k = -k_* < 0$ [QV] where it admits a separation-of-variables solution $b_*(x,t) = (T-t)^{-2}\theta(x)$, so that $l(t) = l_* \equiv \text{meas supp}\,\theta$ for all $t \in (0,T)$. Therefore, $b_*(x,t)$ solves the quasilinear reaction-diffusion equation

$$b_t = l_*(b^{1/2}b_x)_x + \frac{k_*}{l_*}\,b^{3/2},$$

so that b_* is the localized solution exhibiting regional blow-up (the so-called S-regime of combustion) in the interval of the *fundamental* length l_*, see [29], Chapter 4.

We consider the model (1) where the lower-order term is proportional to $b^{1/2}/l^2(t)$, which also corresponds to a distinct physically relevant situation. Unlike (3), the model (1) admits a family of *exact (low-dimensional) solutions* to be constructed. The case $k > 0$ can be treated as a modified b–l model of turbulent bursts propagation, while for $k < 0$ (1) represents a version arising in the so-called *coal-coke problem.* In the latter model, let us present a short justification of the nonlocal term on the right-hand side of (1) which is due to G.I. Barenblatt [3]. Consider the heat wave propagation in a plane thin coal layer in $\mathbf{R}^2$ generated by the initial amount of heat Q. Let $T^* \geq 0$ be the critical ignition temperature of the coal-coke transformation. Assuming a nonlinear heat conduction, radially symmetric heat propagation is described by the reaction-duffusion equation

$$T_t = \Delta_r \phi(T) + \frac{Q}{\pi r_*^2(t)} f(T), \text{ if } T \geq T_*, \quad T_t = \Delta_r \phi(T), \text{ if } T \leq T_*,$$

where $r_*(t)$ is the radius of the cylindrical combustion front. Assuming the power-type conductivity coefficient $k(T) = \phi'(T) \sim T^{m-1}$, $m > 1$, and $f(T) \sim T^n$, and setting $T_* = 0$, after necessary rescaling, for the nonnegative compactly supported solutions we obtain the equation

$$u_t = \Delta u^m + \frac{\alpha}{l^2(t)} u^n, \quad u \geq 0,$$

with the nonlocal lower-order reaction-diffusion term. In the case $n = 2 - m$ this equation admits special exact solutions composed from a finite sum of functions (see comments below). As we show, the extra nonlocal multiplyer $l(t)$ in the quasilinear diffusion term in (1) does not change this invariant property of the model.

On the other hand, equations like (1) with the nonlinear absorption terms ($k < 0$) can describe a filtration-like process of seepage on a permeable bed [4].

For the nonlinear models under consideration we construct special exact solutions on linear subspaces invariant under nonlinear operators. The idea of such particular solutions composed from a finite number of terms and looking like a finite subset of an infinite series is quite fruitful in the study of many nonlinear evolutions equations. We refer to the papers [9]-[11], [12], [19], [20], [21], [22], [23], [25], [26], [27], [30]; see also references therein. In general, such solutions can be obtained via the method of a "nonlinear separation of variables" [9], [25], [14] or by means of linear subspaces invariant under nonlinear operators [10], [11]. See also the results on invariant sets on linear subspaces in [13].

1.1. Exact solutions on an invariant subspace. We set

$$v = 3b^{1/2}. \tag{4}$$

Then the "pressure" $v(x,t) \geq 0$ solves the following quasilinear parabolic equation:

$$v_t = \mathbf{A}(v) \equiv l(t) \left(\frac{1}{2} v \Delta v + |\nabla v|^2 \right) - \frac{3}{2} \frac{k}{l^2(t)} \tag{5}$$

in the positivity domain $Q_+[v] = \{\mathbf{R}^N \times \mathbf{R}_+\} \cap \{v > 0\}$. The lateral boundary of $Q_+[v]$ is a free boundary as in the PME theory [18]. The quadratic operator $\mathbf{A}$ admits the following invariant property, cf. [10]. Here and later on $\mathbf{L}\{\cdot\}$ denotes the linear span of the given functions.

Proposition 1 *The operator* $\mathbf{A}$ *in (5) admits a linear invariant subspace of the form*

$$W_{N+1} = \mathbf{L}\{1, x_1^2, ..., x_N^2\}, \tag{6}$$

i.e.,

$$\text{forany } \phi \in W_{N+1} \text{ thereholds } \mathbf{A}(\phi) \in W_{N+1}. \tag{7}$$

Proof. Substituting

$$\phi(x) = C_0 - C_1 x_1^2 - ... - C_N x_N^2 \in W_{N+1} \tag{8}$$

we have that

$$\mathbf{A}(\phi) = -lC_0 \left(\sum_{k=1}^{N} C_k \right) - \frac{3}{2}\frac{k}{l^2} + l \sum_{i=1}^{N} \left[\left(\sum_{k=1}^{N} C_k \right) C_i + 4C_i^2 \right] x_i^2, \tag{9}$$

so that $\mathbf{A}(W_{N+1}) \subseteq W_{N+1}$. ■

A detailed analysis of solutions on the invariant polynomial-like subspaces (6) of the porous medium equation can be found in [21]. For the reaction-diffusion-absorption equations with lower-order terms such solutions exhibiting essentially non-symmetric asymptotic behaviour were studied in [12].

We now consider the set of nonnegative functions

$$W_{N+1}^+ = \{(\phi)_+ : \quad \phi \in W_{N+1}\}, \tag{10}$$

where $(\cdot)_+$ denotes the positive part. The functions $(\phi)_+$ are locally Lipschitz continuous in x. Thus, we consider particular weak solutions of (5) of the form

$$V(x,t) = \left[C_0(t) - C_1(t)x_1^2 - ... - C_N(t)x_N^2 \right]_+ \tag{11}$$

of the problem

$$v_t = \mathbf{A}(v), \quad v(\cdot, t) \in W_{N+1}^+, \ t > 0. \tag{12}$$

Theorem 2 *The problem (12) is equivalent to the following* $(N+1)$*-dimensional dynamical system on the coefficients* $\{C_0, C_1, ..., C_N\}$*:*

$$C_0' = -lMC_0 - \frac{3}{2}\frac{k}{l^2}, \tag{13}$$

$$C_i' = l[MC_i + 4C_i^2], \quad i = 1, 2, ..., N, \quad t > 0, \tag{14}$$

where

$$M = \sum_{k=1}^{N} C_k \tag{15}$$

and

$$l(t) = \text{meas supp} V(\cdot, t) = \omega_N C_0^{N/2}(t) \left[\Pi_{k=1}^{N} C_k(t)\right]^{-1/2}, \tag{16}$$

ω_N *being the volume of the unit ball in* $\mathbf{R}^N$.

Proof is straightforward. We substitute (11) into (5) and using (9), equate the coefficients of the expansions in the subspace W_{N+1}. ■

Remark. The operator $\mathbf{A}$ in (5) also admits the invariant subspace

$$\tilde{W}_{N+1} = \mathbf{L}\{1, x_1, ..., x_l, x_{l+1}^2, ..., x_N^2\} \quad (1 \le l < N) \tag{17}$$

which gives exact solutions with noncompact supports of a paraboloidal form.

1.2. General solution. In order to solve the system (13), (14), we introduce the new time variable

$$\tau = \int_0^t l(\eta) M(\eta) d\eta, \tag{18}$$

so that the system becomes

$$\frac{d}{d\tau} C_0 = -C_0 - \frac{3}{2} \frac{k}{l^3 M}, \quad \frac{d}{d\tau} C_i = C_i + 4 \frac{C_i^2}{M}. \tag{19}$$

Now setting

$$C_0 = e^{-\tau} \alpha_0; \;\; C_i = e^{\tau} \alpha_i, \;\; i = 1, 2, ..., N, \tag{20}$$

we arrive at

$$\frac{d}{d\tau} \alpha_0 = -\frac{3}{2} \frac{k}{l^3 m}, \quad \frac{d}{d\tau} \alpha_i = 4 \frac{\alpha_i^2}{m}, \tag{21}$$

where

$$m = \sum_{k=1}^{N} \alpha_k. \tag{22}$$

By introducing again the new time variable

$$G = \int_0^\tau \frac{d\eta}{m(\eta)}, \tag{23}$$

we have from (15) that

$$\frac{d}{dG} \alpha_0 = -\beta_0 e^{3N\tau} \frac{(\Pi \alpha_k)^{3/2}}{\alpha_0^{3N/2}}, \;\; \beta_0 = \frac{3k}{2\omega_N^3}, \tag{24}$$

$$\frac{d}{dG}\alpha_i = 4\alpha_i^2, \quad i = 1, 2, ..., N. \tag{25}$$

Integrating equations (25) we obtain

$$\alpha_i(G) = C_i^0(1 - 4C_i^0 G)^{-1}, \quad i = 1, 2, ..., N, \tag{26}$$

where

$$C_i^0 = C_i(0), \quad i = 0, 1, ..., N. \tag{27}$$

It then follows from (23) that $d\tau = mdG$ whence by (26)

$$\tau = -\frac{1}{4}\log\left[\Pi_{k=1}^N (1 - 4C_k^0 G)\right]. \tag{28}$$

The coefficient α_0 is then calculated from (23):

$$\alpha_0^{3N/2}\frac{d\alpha_0}{dG} = -\beta_0(\Pi C_k^0)^{3/2}\left[\Pi(1 - 4C_k^0 G)\right]^{-3(N+2)/4}. \tag{29}$$

Therefore

$$\alpha_0^{(3N+2)/2} = (C_0^0)^{(3N+2)/2} - \frac{3N+2}{2}\beta_0(\Pi C_k^0)^{3/2}\int_0^G \left[\Pi(1 - 4C_k^0 z)\right]^{-3(N+2)/4} dz. \tag{30}$$

The original time variable t is then calculated from (18) and (23), which yields the ODE

$$\frac{dt}{dG} = \alpha_0^{-N/2}(G)(\Pi C_k^0)^{1/2}\left[\Pi(1 - 4C_k^0 G)\right]^{-(N+1)/4}. \tag{31}$$

1.3. Asymptotic behaviour of extinction. We assume that

$$C_i^0 > 0 \quad \text{forall } i = 0, 1, ..., N.$$

The exact solutions (11) exhibit finite-time extinction, which is easily seen from the original system (13), (14). It follows from (30) and (26) that there exists a finite

$$0 < G_* < (4 \max_k C_k^0)^{-1}$$

such that $\alpha_0(G_*) = 0$ and $\alpha_k(G_*) > 0, \ k = 1, ..., N$. Then (31) implies that there exists an extinction time T such that $C_0(t) > 0$ on $[0, T)$ and

$$C_0(T) = 0. \tag{32}$$

Moreover the limits $C_k(T)$ are finite:

$$C_k(t) \to \nu_i > 0 \quad \text{as } t \to T^-, \ k = 1, ..., N. \tag{33}$$

In view of (16), this means that as $t \to T^-$

$$l(t) = \omega_N C_0^{N/2} (\Pi\nu_k)^{-1/2}(1+o(1)), \tag{34}$$

and then equation (13) yields for $t \approx T^-$

$$C_0' = -\frac{3k}{2\omega_N^2}(\Pi\nu_k)C_0^{-N}(1+o(1)). \tag{35}$$

Integrating (35) we have

$$C_0(t) = a_0(T-t)_+^{1/(N+1)}(1+o(1)) \quad \text{as } t \to T^-, \tag{36}$$

where

$$a_0 = \left[\frac{3k(N+1)}{2}\,\frac{\Pi\nu_k}{\omega_N^2}\right]^{1/(N+1)}. \tag{37}$$

Finally, substituting (33) and (36) into the solution (11) we deduce that as $t \to T^-$

$$V(x,t) = \left[a_0(T-t)_+^{1/(N+1)}(1+o(1)) - \sum_{k=1}^{N} \nu_k x_k^2 (1+o(1))\right]_+. \tag{38}$$

In terms of the rescaled spatial variable

$$\eta = \frac{x}{(T-t)^{1/2(N+1)}} \in \mathbf{R}^N, \tag{39}$$

the asymptotic behaviour of the solution $V(x,t)$ is given by

$$V(x,t) = (T-t)^{1/(N+1)}\left(a_0 - \sum_{k=1}^{N}\nu_k\eta_k^2\right)_+(1+o(1)) \equiv V_*(x,t)(1+o(1)). \tag{40}$$

Thus, we have the following properties:

(i) Single point extinction where $V(x,t) \to 0$ and meas supp$V(\cdot,t) \to 0$ as $t \to T^-$.

(ii) In the rescaled variable (39) the behaviour (40) as $t \to T^-$ in general is not symmetric: the limit rescaled profile

$$g_0(\eta) = \left(a_0 - \sum_{k=1}^{N}\nu_k\eta_k^2\right)_+$$

is essentially non-symmetric. We do not see a symmetrization effect in the limit which is valid for a wide class of (eventually) uniformly parabolic equations.

(iii) It can be shown that the limit rescaled function V_* presented in (40) solves the equation without the diffusion term which is negligible on the asymptotic stage. This ordinary differential equation takes the following asymptotic form:

$$V_t = -k\frac{V^{1/2}}{l^2(t)} \approx -\alpha(T-t)^{-N/(N+1)}, \quad t \to T^-,$$

where $\alpha > 0$ is a constant. Using the self-similar rescaling given by (40) and setting $V_*(x,t) = (T-t)^{1/(N+1)} g(\eta,\tau)$, where η is the spatial variable in (39) and $\tau = -\log(T-t) \to \infty$ as $t \to T^-$ is the new time variable, we conclude that the rescaled profile $g(\eta,\tau)$ satisfies the first-order Hamilton-Jacobi equation

$$g_\tau = -\frac{1}{2(N+1)} \nabla g \cdot \eta + \frac{1}{N+1} g - \alpha, \quad \tau \gg 1.$$

The compactly supported ellipsoidal function $g_0(\eta)$ is a stationary solution of the Hamilton-Jacobi equation. Thus, the porous medium operator in (5) forms as $t \to T^-$ a *singular perturbation.* This makes it possible to get a nonsymmetric function in the limit.

This kind of behaviour with convergence to Hamilton-Jacobi profiles was studied in [GV] for the one-dimensional PME with absorption like

$$u_t = \Delta u^{3/2} - u^{1/2}. \tag{41}$$

A nonsymmetric behaviour as $t \to T^-$ in $\mathbf{R}^N$, $N > 1$, remains an open problem for both local (41) and non-local equation (1). Exact nonsymmetric solutions of (41) on an invariant subspace can be found in [12].

Remark: invariant extension of the model. The same invariant subspace W_{N+1} and the corresponding invariant solutions exist for the equation

$$b_t = F(l(t)) \Delta b^m - k \frac{b^{2-m}}{G(l(t))}, \tag{42}$$

where $m > 1$ and F and G are arbitrary nonnegative functions of the support measure $l \geq 0$.

2 Extinction in the b–ε model

We consider the b–ε model of turbulence motion based on ideas of A.N. Kolmogorov and L. Prandtl:

$$b_t = \alpha \left(\frac{b^2}{\varepsilon} b_x \right)_x - \varepsilon, \quad \varepsilon_t = \beta \left(\frac{b^2}{\varepsilon} \varepsilon_x \right)_x - \gamma \frac{\varepsilon^2}{b}, \quad x \in \mathbf{R}, \; t > 0, \tag{43}$$

with nonnegative bounded initial data $b_0(x)$ and $\varepsilon_0(x)$, where α, β and γ are positive constants. Here $b(x,t) \geq 0$ stays for the turbulent energy density and $\varepsilon(x,t) \geq 0$ is the dissipation rate of turbulent energy. See [5], [1], Section 10, [6] and references therein.

We consider the case of the equal diffusivity coefficients: $\alpha = \beta = 1$. The case of unequal coefficients, $\alpha \neq \beta$, is much harder and is studied less. We would like to mention a perturbation result [17] on the existence of the self-similar solutions (see also [1], Section 10).

Several important results for this model have been proved in recent years. In particular first exact solutions were constructed in [5] for $\gamma > 3/2$. More general exact solutions as well as existence and regularity results are presented in [7] and [8]. The asymptotic behaviour as $t \to \infty$ is shown there to be self-similar if $\gamma \geq 3/2$. For $1 \leq \gamma < 3/2$ the behaviour as $t \to \infty$ is of a separate-variable form [7]. On the other hand if

$$\gamma \in (0,1), \tag{44}$$

then solutions are proved to exhibit extinction in finite time [BPK2], so that there exists a time $T >$ such that

$$b(x,T) \equiv \varepsilon(x,T) \equiv 0. \tag{45}$$

We focus our attention on the extinction behaviour in the less studied case (44). We first derive certain invariant properties of the b–ε model.

2.1. Invariant subspace. One can see that the set ($\gamma \neq 1$)

$$M(t) = \{b \geq 0,\ \varepsilon \geq 0 : \quad b = (\gamma-1)t\varepsilon \ \text{ for } x \in \mathbf{R}\} \tag{46}$$

is an *invariant* of the system (43), i.e., if $(b(\cdot,t_0),\varepsilon(\cdot,t_0)) \in M(t_0)$ for some $t_0 \neq 0$ then $(b(\cdot,t),\varepsilon(\cdot,t)) \in M(t)$ for $t > t_0$. Then (43) admits particular solutions

$$b(x,t) = V(x,t), \quad \varepsilon(x,t) = V(x,t)/(\gamma-1)t, \tag{47}$$

where $V(x,t) \geq 0$ solves a single quasilinear parabolic equation

$$V_t = \mathbf{A}_*(V) \equiv (\gamma-1)t(VV_x)_x - \frac{V}{(\gamma-1)t}, \quad x \in \mathbf{R},\ \ t < 0, \tag{48}$$

which is the system (43) restricted to the invariant set $M \times \mathbf{R}_+$. If $\gamma = 1$ then $M = \{\beta = \varepsilon = V\}$ and V solves $V_t = (VV_x)_x - V$.

The quadratic operator in (48) satisfies the following property.

Proposition 3 $\mathbf{A}_*$ *admits the invariant subspace* $W_2 = \mathbf{L}\{1, x^2\}$.

Hence equation (48) on W_2 reduces to a two-dimensional dynamical system on the coefficients $\{C_0, C_1\}$ of the expansion

$$V_*(x,t) = [C_0(t) + C_1(t)x^2]_+. \tag{49}$$

It has the form

$$C_0' = 2(\gamma-1)tC_0C_1 - \frac{C_0}{(\gamma-1)t}, \quad C_1' = 6(\gamma-1)tC_1^2 - \frac{C_1}{(\gamma-1)t}. \tag{50}$$

This system gives the exact stable self-similar solutions and other ones presented in [5], [7] and [8].

2.2. On the ω-limit set for $\gamma < 1$. Let us consider the corresponding exact solutions (49) on W_2 in the singular extinction case

$$V_* = (-t)^\delta \left\{ A[1 - B(-t)^{(3-2\gamma)\delta}]^{-1/3} - \frac{3-2\gamma}{6(1-\gamma)^2} B[1 - B(-t)^{(3-2\gamma)\delta}]^{-1} x^2 \right\}_+ , \tag{51}$$

where $\delta = 1/(1-\gamma) > 0$ and A, B are arbitrary constants. If both are positive then $T = 0$ is the extinction time in the sense of (45).

It follows from (51) that as $t \to 0^-$ the asymptotic extinction behaviour takes a standard self-similar (separated-variable) form

$$V_*(x,t) = (-t)^\delta \left\{ A - \frac{3-2\gamma}{6(1-\gamma)^2} B x^2 + o(1) \right\}_+ . \tag{52}$$

This means that the ω-limit set of all the rescaled exact solutions defined as

$$\theta(x,t) \equiv (-t)^{-\delta} V(x,t) \tag{53}$$

is at least two-dimensional. Namely, the ω-limit set $\omega(E_*)$ of bell-shaped solutions on W_2 with a fixed extinction time:

$$E_* = \{(V_*)_+ ,\ V_* \in W_2 \ \text{ has } T = 0\}, \tag{54}$$

satisfies

$$\omega(E_*) = \{(a - bx^2)_+ ,\ a > 0,\ b > 0\}. \tag{55}$$

Low-dimensional properties of the ω-limits (attractors) of the sets of solutions with finite-time singularities like extinction or blow-up are a typical features for nonlinear parabolic equations, see [29], [15] and references therein.

Nevertheless, let us show that in general the extinction behaviour is not self-similar so that unlike the range $\gamma \geq 3/2$ the solutions on W_2 *are not attractive* as $t \to T^-$. We prove that the ω-limit set $\omega(E)$ of the rescaled solutions to equation (48) in the class E of bounded continuous compactly supported initial data $V_0(x) \geq 0$ is actually infinite-dimensional. Hence the same is true for the b–ε model.

In order to prove such a result and to establish unstability of the self-similarity, we need to study the model beyond the invariant subspace. It turns out that it is suffice to consider the system (43) on the invariant set $M \times \mathbf{R}_+$.

Theorem 4 *Let (44) hold. Given an arbitrary $g \in E$, for any $\varepsilon > 0$ there exists $V_0 \in E$ such that $\omega(V_0) = \{f\}$ where*

$$\sup_{x \in \mathbf{R}} |f(x) - g(x)| < \varepsilon.$$

Proof. Consider (48), the equation on the invariant set. Setting in (48)

$$V(x,t) = (-t)^{\delta} v(x,\tau), \quad \tau = -\frac{(1-\gamma)^2}{3-2\gamma}(-t)^{(3-2\gamma)\delta}, \tag{56}$$

we arrive at the PME

$$v_\tau = (v v_x)_x, \quad x \in \mathbf{R}, \ \tau \in (\tau_0, 0), \tag{57}$$

with initial data $v(x,\tau_0) = (-t_0)^{-\delta} V_0(x)$, $\tau_0 = \tau(t_0) < 0$. We then obtain the result due to the continuity of weak solutions of the PME [18]. ■

Let us note that the result $\dim \omega(E) = \infty$ is proved in a similar way for the range $\gamma \in [1, 3/2)$ in the large-time behaviour as $t \to \infty$, cf. [7], Theorem 4.1.

2.3. Solutions in $\mathbf{R}^N$. Consider the N-dimensional version of (43):

$$b_t = \nabla \cdot \left(\frac{b^2}{\varepsilon} \nabla b \right) - \varepsilon, \quad \varepsilon_t = \nabla \cdot \left(\frac{b^2}{\varepsilon} \nabla \varepsilon \right) - \gamma \frac{\varepsilon^2}{b}. \tag{58}$$

The invariant set (46) stays the same and substituting (47) into (2.16) we obtain the equation on $M \times \mathbf{R}_+$

$$V_t = \mathbf{A}_*(V) \equiv (\gamma - 1)t\, \nabla \cdot (V \nabla V) - \frac{V}{(\gamma-1)t}, \quad x \in \mathbf{R}^N, \tag{59}$$

with the known invariant property of the quadratic operator $\mathbf{A}_*$. Using the transformation (56), we arrive at

$$v_\tau = \mathbf{a}_*(v) \equiv \nabla \cdot (v \nabla v). \tag{60}$$

Using the invariant subspace (6) of $\mathbf{a}_*$, one constructs nonsymmetric compactly supported solutions which for $\gamma < 1$ exhibit finite-time extinction with eventually non-symmetric ellipsoidal support. On the other hand, taking the coefficients $\{C_0(0), ..., C_N(0)\}$ in (11) changing sign, we obtain solutions with hyperboloid-like unbounded supports. Finally, on the invariant subspace (17) the supports have paraboloid-like forms.

One can see that 4 is true for $N > 1$.

Remark 1: invariant extension of the model. The system in $\mathbf{R}^N$

$$b_t = \nabla \cdot \left(\frac{b^2}{\varepsilon} \nabla b \right) - \varepsilon^m b^{1-m}, \quad \varepsilon_t = \nabla \cdot \left(\frac{b^2}{\varepsilon} \nabla \varepsilon \right) - \gamma \varepsilon^n b^{1-n}, \tag{61}$$

where m and n are arbitrary exponents ($m = 1$, $n = 2$ and $N = 1$ for the b–ε model), admits the invariant set $M(t) = \{\varepsilon = \psi(t) b\}$ where ψ is a solution to the ODE

$$\psi' = \psi^{m+1} - \gamma \psi^n. \tag{62}$$

The corresponding solutions in $M(t)$ are $b = V$, $\varepsilon = \psi(t)V$ where V solves the nonautonomous parabolic equation

$$V_t = \mathbf{A}_*(V) \equiv \frac{1}{\psi(t)} \nabla \cdot (V \nabla V) - \psi^m(t) V. \tag{63}$$

The operator $\mathbf{A}_*$ admits the invariant subspace (6) so that (63) in W_{N+1} is an easily derived $(N+1)$-dimensional dynamical system. In the subspace (17) we obtain exact solutions with unbounded supports.

Remark 2: finite-dimensional attractor. Consider equation (63). Let us show that in the range

$$n \in (0,1), \quad m+1 > n, \tag{64}$$

4 is not true. We take a vanishing solution $\psi(t) \to 0$ as $t \to 0^-$ of (62) with the behaviour

$$\psi(t) = c_0(-t)^{1/(1-n)}(1+o(1)), \quad t \to 0^-, \quad c_0 = [\gamma(1-n)]^{1/(1-n)}. \tag{65}$$

Then setting in (63)

$$V(x,t) = A(t)v(x,\tau), \tag{66}$$

where

$$A' = -\psi^m A \quad \text{and} \quad d\tau = \frac{A}{\psi}\, dt, \tag{67}$$

we arrive at the PME

$$v_\tau = \nabla \cdot (v \nabla v), \quad x \in \mathbf{R}^N. \tag{68}$$

It follows from (67) and (65) that $A(t)$ satisfies

$$\log A(t) = -\int^t \psi^m(s)ds = -c_0^m \int^t (-s)^{m/(1-n)}(1+o(1))ds,$$

and since in the range (64) the integral converges, we can choose $A(t)$ such that

$$A(t) \to 1, \quad t \to 0^-.$$

Then it follows from (67) that the new time variable satisfies

$$\tau = b_0(-t)^{-n/(1-n)}(1+o(1)) \to \infty, \quad t \to 0^-, \tag{69}$$

where $b_0 = (1-n)/c_0 n$. Therefore we arrive at the PME (60) in $\mathbf{R} \times \mathbf{R}_+$. The asymptotic behaviour of finite-mass solutions to the PME as $\tau \to \infty$ is well-known to be described by Zel'dovich-Kompaneetz-Barenblatt solutions, see [2], p. 77. Substituting τ from (69), we finally obtain the following extinction behaviour as $t \to 0^-$ for an *arbitrary* finite-mass solution of (63):

$$V(x,t) \approx b_0^{-N/(N+2)}(-t)^{N\delta}[A_0(a^2 - b_0^{-2/(N+2)}(-t)^{2\delta}|x|^2)]_+, \tag{70}$$

where $\delta = n/(N+2)(1-n)$, $A_0 = 1/2(N+2)$, and $a > 0$ is the unique parameter (the total mass of L^1-solutions of the PME). After necessary rescaling this gives a one-dimensional attractor of equation (63), (65) in the class of finite-mass solutions.

3 Conclusion

We have shown that the above two nonlinear models related to the well-known models of turbulence exhibit special “nonlinear” invariant properties: their operators admit linear invariant subspaces. This makes it possible to explain the known exact solutions and derive some new ones describing an interesting asymptotic behaviour, which is hard to study by standard mathematical methods. We also present some extensions of these models which also exhibit similar invariant properties.

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