HYPERKÄHLER POTENTIALS IN COHOMOGENEITY TWO

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Abstract. A hyperKähler potential is a function ρ that is a Kähler potential for each complex structure compatible with the hyperKähler structure. Nilpotent orbits in a complex simple Lie algebra are known to carry hyperKähler metrics admitting such potentials. In this paper, we explicitly calculate the hyperKähler potential when the orbit is of cohomogeneity two. In some cases, we find that this structure lies in a one-parameter family of hyperKähler metrics with Kähler potentials, generalising the Eguchi-Hanson metrics in dimension four.

1. Introduction

HyperKähler metrics are special Ricci-flat structures that are known to arise in many physical theories. For example, moduli spaces of magnetic monopoles often carry such metrics. For good choices of boundary conditions, these moduli spaces can be identified with more familiar mathematical objects. In this way, hyperKähler metrics have been shown to exist on the adjoint orbits of a complex semi-simple Lie group $G^{\mathbb{C}}$ [21, 20, 4, 19]. In [11], it was shown that these examples include all hyperKähler metrics of cohomogeneity one.

Some of the earliest examples of hyperKähler metrics were found by Calabi [8]. His method was to take a complex symplectic manifold, such as the cotangent bundle $T^*\mathbb{C}\mathrm{P}(n)$, and find a potential for a Kähler structure that would combine with the complex symplectic structure to give a hyperKähler metric. This approach has been applied to certain semi-simple nilpotent orbits by a number of authors. Biquard & Gauduchon [5] gave a beautiful construction for a potential on those semi-simple orbits that are the cotangent bundle of a Hermitian symmetric space. At the other extreme, Hitchin [14] used spectral theory to describe a potential for the biggest semi-simple orbit in $\mathfrak{sl}(n,\mathbb{C})$ in terms of theta functions (the special case of $n = 2$ may be found in [22]).

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Much attention has been paid to the semi-simple orbits, because one can show that they are the only orbits to admit hyperKähler metrics that are complete. However, the incomplete metrics on nilpotent orbits still have much interest. One reason, is that each such orbit admits a *hyperKähler potential*, a function that is a Kähler potential for each complex structure compatible with the hyperKähler structure, and so these metrics on nilpotent orbits determine quaternionic Kähler metrics of positive scalar curvature on a certain quotient manifold [23, 24].

The structures considered on coadjoint orbits are invariant under the action of the compact group G. For nilpotent orbits, there is a natural partial order given by inclusions of closures. When G is simple, the smallest non-trivial orbits in this order are unique and they are distinguished by being of cohomogeneity one under the action of G. In [12], it was shown that the nilpotent orbits of cohomogeneity two also fit nicely in to the partial order: except when $G = SU(3)$, they are exactly the next-to-minimal orbits. Given that the nilpotent orbits of cohomogeneity one are understood [11] (see also [17]), it is natural to look at those of cohomogeneity two.

In this paper, we consider cohomogeneity-two nilpotent orbits and find all compatible G-invariant hyperKähler metrics on them that admit Kähler potentials. Our approach is that of Calabi's and we obtain the hyperKähler potentials explicitly. The hyperKähler potentials are unique, but in a few cases we find that they lie in a one-parameter family of hyperKähler metrics with Kähler potential. These families may be regarded as generalisations of the Eguchi-Hanson metrics in dimension four.

Combining our results with [16], means that hyperKähler potentials are now known for all next-to-minimal orbits. One feature of the cohomogeneity-two case that makes the calculations possible, is that each element of the orbit lies in a small rank 2 real subalgebra which determines much of the hyperKähler structure. In fact, unless G is the exceptional Lie group G_2, that subalgebra is $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ and the geometry is the product of the structures from each factor.

For some cohomogeneity-two orbits the hyperKähler potential may also be obtained by one of three other methods: a hyperKähler quotient construction, a finite-cover by a minimal orbit for another group, or a limit of a family of semi-simple orbits. The first two methods will be described elsewhere; the first only succeeds if the hyperKähler quotient is sufficiently simple and the second only covers orbits on the list of "shared orbits" of Brylinski & Kostant [7]. The third is contained in Biquard & Gauduchon's work [5]. However, there are orbits for which the approach of this paper is the only one known to give the result and our approach is uniform for all orbits of cohomogeneity two.

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2. Preliminaries

2.1. **HyperKähler Structures.** Let M be a manifold with endomorphisms I, J and K of the tangent bundle TM satisfying the quaternion identities

$$I^2 = J^2 = -1 \quad \text{and} \quad IJ = K = -JI.$$

This gives T_xM the structure of an $\mathbb{H}$-module and so implies that the dimension of M is a multiple of 4. If g is a Riemannian metric on M preserved by I, J and K, in the sense that $g(IX, IY) = g(X, Y)$, etc., for all tangent vectors X, Y, then we can define two-forms ω_I, ω_J and ω_K by

$$\omega_I(X, Y) = g(X, IY), \qquad \text{etc.}$$

If these three two-forms are closed, the structure (M, g, I, J, K) is said to be *hyperKähler*.

Hitchin [13] showed that on a hyperKähler manifold, the almost complex structures I, J and K are integrable, and thus (M, g) is a Kähler manifold in three distinct ways. The restricted holonomy group Hol_g of (M, g) is then contained in $Sp(n)$. As $Sp(n)$ is a subgroup of $SU(2n)$, this implies that any hyperKähler metric g is Ricci-flat.

A function $\rho\colon M \to \mathbb{R}$ is a Kähler potential for the complex structure I if $\omega_I = -i\partial_I\overline{\partial_I}\rho$. This may be reformulated as

$$\begin{aligned} \omega_I &= -i\partial_I\overline{\partial_I}\rho = -id\overline{\partial_I}\rho = -\tfrac{i}{2}d(d - iId)\rho \\ &= -\tfrac{1}{2}dId\rho. \end{aligned} \tag{2.1}$$

The function ρ is a *hyperKähler potential* if it is simultaneously a Kähler potential for I, J and K. HyperKähler potentials are defined up to an additive constant. The existence of a hyperKähler potential implies strong restrictions on the geometry of M [23]: the metric g and potential ρ satisfy $\nabla^2\rho = g$; the manifold M admits an infinitesimal action of $\mathbb{H}^*$, with $Sp(1) \leqslant \mathbb{H}^*$ preserving g and permuting I, J and K; the $\mathbb{H}^*$-orbits are flat and totally geodesic; locally M fibres over a quaternionic Kähler orbifold of positive scalar curvature.

We will be considering hyperKähler structures that are invariant under the action of a compact group G. It is therefore worth noting that if we have a Kähler potential then this may be taken to be G-invariant. Indeed, if ρ is any Kähler potential, then since the G-action preserves I, the expression $\partial_I\overline{\partial_I}\rho$ is equivariant for the action of G. However, $\omega_I = -i\partial_I\overline{\partial_I}\rho$, is assumed to be G-invariant, so averaging ρ over the group action produces an invariant Kähler potential.

2.2. **Lie Algebras and Orbits.** On the semi-simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, let $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be the negative of the Killing form and let σ be a real structure giving a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$.

At a point X of a nilpotent orbit $\mathcal{O}$, the vector field generated by A in $\mathfrak{g}^{\mathbb{C}}$ is $\xi_A = [A, X]$. These vector fields satisfy $[\xi_A, \xi_B] = \xi_{-[A,B]}$, for all $A, B \in \mathfrak{g}^{\mathbb{C}}$.

The orbit $\mathcal{O}$ carries a complex structure I defined by

$$I\xi_A = i\xi_A = \xi_{iA}. \tag{2.2}$$

There is also a complex symplectic form, known as the Kirillov-Kostant-Souriau form, on $\mathcal{O}$ which we take to be given by

$$\omega_c^{\mathcal{O}}(\xi_A, \xi_B)_X = \langle X, [A, B]\rangle = -\langle \xi_A, B\rangle\,. \tag{2.3}$$

We will be looking for hyperKähler structures on $\mathcal{O}$ with I given by (2.2) and $\omega_J + i\omega_K = \omega_c^{\mathcal{O}}$. We will call these *compatible hyper-Kähler structures* on $\mathcal{O}$.

3. Potentials Depending on Two Invariants

Consider the following two functions on a nilpotent orbit $\mathcal{O}$:

$$\eta_1(X) = \langle X, \sigma X\rangle \quad\text{and}\quad \eta_2(X) = -\langle [X, \sigma X], [X, \sigma X]\rangle\,.$$

Note that $\eta_2(X) = \langle Y, \sigma Y\rangle$ with $Y = [X, \sigma X]$, so is positive, and that both η_1 and η_2 are invariant under the action of the compact group G. Suppose ρ is a Kähler potential for I depending only on η_1 and η_2, i.e.,

$$\rho = \rho(\eta_1, \eta_2). \tag{3.1}$$

Lemma 3.1. *At $X \in \mathcal{O}$, the two-form ω_I defined by ρ in formula* (2.1) *is*

$$\begin{aligned}
\omega_I(\xi_A, \xi_B)_X = 2\rho_1 \operatorname{Im}\langle \xi_A, \sigma\xi_B\rangle &\\
- 4\rho_2 \operatorname{Im}\langle \xi_A, [\sigma\xi_B, [X, \sigma X]] + [\sigma X, [X, \sigma\xi_B]]\rangle &\\
+ 2\rho_{11} \operatorname{Im}\big(\langle \xi_A, \sigma X\rangle \langle \sigma\xi_B, X\rangle\big) &\\
- 4\rho_{12} \operatorname{Im}\big(\langle \xi_A, [\sigma X, [X, \sigma X]]\rangle \langle \sigma\xi_B, X\rangle &\\
+ \langle \xi_A, \sigma X\rangle \langle \sigma\xi_B, [X, [\sigma X, X]]\rangle\big) &\\
+ 8\rho_{22} \operatorname{Im}\big(\langle \xi_A, [\sigma X, [X, \sigma X]]\rangle \langle \sigma\xi_B, [X, [\sigma X, X]]\rangle\big), &
\end{aligned} \tag{3.2}$$

where $\rho_i = \partial\rho/\partial\eta_i$*, etc.*

Proof. Expanding (2.1), we have

$$\begin{aligned} -2\omega_I = \rho_1\, dId\eta_1 + \rho_2\, dId\eta_2 + \rho_{11}\, d\eta_1 \wedge Id\eta_1 \\ + \rho_{12}(d\eta_2 \wedge Id\eta_1 + d\eta_1 \wedge Id\eta_2) + \rho_{22} d\eta_2 \wedge Id\eta_2. \end{aligned} \tag{3.3}$$

The exterior derivative of η_1 is given by

$$d\eta_1(\xi_A)_X = \langle [A,X], \sigma X\rangle + \langle X, \sigma[A,X]\rangle = 2\operatorname{Re}\langle \xi_A, \sigma X\rangle .$$

Hence $Id\eta_1(\xi_A) = 2\operatorname{Im}\langle \xi_A, \sigma X\rangle$ and $dId\eta(\xi_A,\xi_B) = -4\operatorname{Im}\langle \xi_A, \sigma\xi_B\rangle$, at $X \in \mathcal{O}$.

For η_2, the initial computation is similar and gives

$$d\eta_2(\xi_A)_X = -4\operatorname{Re}\langle \xi_A, [\sigma X, [X, \sigma X]]\rangle .$$

The second derivative, however, is slightly more involved:

$$\begin{aligned} &dId\eta_2(\xi_A,\xi_B)_X \\ &\quad = \xi_A(Id\eta_2(\xi_B)) - \xi_B(Id\eta_2(\xi_A)) - Id\eta_2([\xi_A,\xi_B]) \\ &\quad = -4\operatorname{Im}\big\{ \langle \xi_B, [\sigma\xi_A, [X,\sigma X]]\rangle + \langle \xi_B, [\sigma X, [\xi_A, \sigma X]]\rangle \\ &\qquad\qquad + \langle \xi_B, [\sigma X, [X, \sigma\xi_A]]\rangle + \langle [B,\xi_A], [\sigma X, [X,\sigma X]]\rangle \\ &\qquad\qquad - \langle \xi_A, [\sigma\xi_B, [X,\sigma X]]\rangle - \langle \xi_A, [\sigma X, [\xi_B, \sigma X]]\rangle \\ &\qquad\qquad - \langle \xi_A, [\sigma X, [X, \sigma\xi_B]]\rangle - \langle [A,\xi_B], [\sigma X, [X,\sigma X]]\rangle \\ &\qquad\qquad + \langle [[A,B],X], [\sigma X, [X,\sigma X]]\rangle \big\} \\ &\quad = -4\operatorname{Im}\big\{ -\langle [\xi_A, \sigma\xi_B], [X,\sigma X]\rangle + \langle [\xi_B, \sigma\xi_A], [X,\sigma X]\rangle \\ &\qquad\qquad + \langle [\sigma X, \xi_A], [X, \sigma\xi_B]\rangle - \langle [X, \sigma\xi_A], [\sigma X, \xi_B]\rangle \big\} \\ &\quad = 8\operatorname{Im}\langle \xi_A, [\sigma\xi_B, [X,\sigma X]] + [\sigma X, [X, \sigma\xi_B]]\rangle . \end{aligned}$$

Combining these formulæ gives the claimed result. □

The two-form ω_I is our candidate for a Kähler form on $\mathcal{O}$.

Remark 3.2. The corresponding symmetric bilinear form is given by $g(\xi_A,\xi_B) = \omega_I(I\xi_A,\xi_B)$ and is simply the right-hand side of equation (3.2) with 'Im' replaced by 'Re' throughout.

We will eventually require g to be positive definite. However for now simply assume that g is non-degenerate and define an endomorphism J of $T_X\mathcal{O}$ by

$$g(\xi_A,\xi_B) = \operatorname{Re}\omega_c^{\mathcal{O}}(J\xi_A,\xi_B). \tag{3.4}$$

Lemma 3.3. *The endomorphism J of $T_X\mathcal{O}$ is given by*

$$
\begin{aligned}
J\xi_A = {}& -2\rho_1[X,\sigma\xi_A] \\
&+ 4\rho_2(2[X,[\sigma X,[X,\sigma\xi_A]]] - [X,[X,[\sigma X,\sigma\xi_A]]]) \\
&- 2\rho_{11}\left\langle \sigma\xi_A, X\right\rangle [X,\sigma X] \\
&+ 4\rho_{12}\big(\left\langle \sigma\xi_A,[X,[\sigma X,X]]\right\rangle [X,\sigma X] \\
&\qquad + \left\langle \sigma\xi_A, X\right\rangle [X,[\sigma X,[X,\sigma X]]]\big) \\
&- 8\rho_{22}\left\langle \sigma\xi_A,[X,[\sigma X,X]]\right\rangle [X,[\sigma X,[X,\sigma X]]].
\end{aligned}
\tag{3.5}
$$

Proof. Equation (3.4) implies $g(\xi_A,\xi_B) = -\operatorname{Re}\left\langle J\xi_A, B\right\rangle$, and then (3.2) gives the above formula for J, except that the coefficient of ρ_2 is

$$4\big([X,[\sigma\xi_A,[X,\sigma X]]] + [X,[\sigma X,[X,\sigma\xi_A]]]\big). \tag{3.6}$$

Using the Jacobi identity, we have

$$[\sigma\xi_A,[X,\sigma X]] = -[X,[\sigma X,\sigma\xi_A]] + [\sigma X,[X,\sigma\xi_A]].$$

Applying this to the first term in (3.6) gives the result. □

At this stage there is no guarantee that $J^2 = -1$. It is imposing this condition that severely restricts the possibilities for ρ.

4. Small Nilpotent Orbits and Real Subalgebras

The nilpotent orbits in $\mathfrak{g}^{\mathbb{C}}$ are partially ordered by saying $\mathcal{O}_1 \preceq \mathcal{O}_2$ if and only if $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. When $\mathfrak{g}^{\mathbb{C}}$ is simple, there is a unique non-zero orbit $\mathcal{O}_{\text{min}}$ which is minimal for this partial order. This orbit is of cohomogeneity one with respect to the action of the compact group G, and for each $X \in \mathcal{O}_{\text{min}}$, the subalgebra spanned by $\{X, \sigma X\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and is the complexification of an $\mathfrak{su}(2)$-subalgebra of $\mathfrak{g}$. Note that $\mathcal{O}_{\text{min}}$ is the orbit of a root vector for the longest root.

In general, the Jacobsen-Morosov Theorem says that each nilpotent element X lies in an $\mathfrak{sl}(2,\mathbb{C})$-subalgebra (see e.g. [9]). However, in general this subalgebra is not σ-invariant. The following result is usually attributed to Borel [6].

Proposition 4.1 (Borel)**.** *Each nilpotent orbit $\mathcal{O}$ contains an element X such that the linear span of $\{X, \sigma X, [X,\sigma X]\}$ is a real subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. Fix X' in $\mathcal{O}$ and take any $\mathfrak{sl}(2,\mathbb{C})$ containing X. There are H and Y in $\mathfrak{sl}(2,\mathbb{C})$ such that $[H,X'] = 2X'$, $[X',Y] = H$ and $[H,Y] = -2Y$. The element H is thus semi-simple in $\mathfrak{sl}(2,\mathbb{C})$ and hence in $\mathfrak{g}^{\mathbb{C}}$, so we find a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}^{\mathbb{C}}$ containing H and choose a system of positive roots Δ^+ so that X lies in a sum of positive root spaces. The pair $(\mathfrak{t}, \Delta^+)$ has an associated real structure σ', which maps Δ^+ to Δ^- and defines a compact real form of $\mathfrak{g}^{\mathbb{C}}$. Now all compact real forms of $\mathfrak{g}^{\mathbb{C}}$ are conjugate, so there is a $g \in G^{\mathbb{C}}$ such that $\operatorname{Ad}_g(\sigma' A) = \sigma \operatorname{Ad}_g A$, for all $A \in \mathfrak{g}^{\mathbb{C}}$. Taking $X = \operatorname{Ad}_g X'$ gives an element of $\mathcal{O}$ of the desired type. $\square$

Let us recall the Morse theory picture of the nilpotent variety described in [24] (see also [16, 21]). Each nilpotent orbit $\mathcal{O}$ admits a certain free action of $\mathbb{H}^*/\{\pm 1\}$. The quotient $\mathfrak{M}(\mathcal{O}) = \mathcal{O}/\mathbb{H}^*$ may be described as a submanifold of the Grassmannian $\widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ of oriented three-planes in the real Lie algebra $\mathfrak{g}$. One defines a functional $\psi\colon \widetilde{\mathrm{Gr}}_3(\mathfrak{g}) \to \mathbb{R}$ by $\psi(V) = \langle e_1, [e_2, e_3]\rangle$, where $\{e_1, e_2, e_3\}$ is an oriented orthonormal basis for V. Away from zero, ψ is a non-degenerate G-equivariant Morse function in the sense of Bott. The points on the non-zero critical sets correspond to subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$. The set of real $\mathfrak{su}(2)$-subalgebras associated to $\mathcal{O}$ via Proposition 4.1, oriented so that ψ is positive, forms a non-zero critical manifold $\mathcal{C}(\mathcal{O})$. The manifold $\mathfrak{M}(\mathcal{O})$ is the stable manifold attached to $\mathcal{C}(\mathcal{O})$. The partial order on stable manifolds for the gradient flow induces the partial order $\preceq$ on nilpotent orbits. In particular, the maximum of ψ is achieved on $\mathfrak{M}(\mathcal{O}_{\mathrm{min}})$.

We are interested in orbits of cohomogeneity two. These were computed in [12] and are the orbits listed in Table 1. We say that a nilpotent orbit $\mathcal{O}$ is *next-to-minimal* if $\mathcal{O} \succneqq \mathcal{O}_{\mathrm{min}}$ and there is no orbit $\mathcal{O}'$ with $\mathcal{O} \succneqq \mathcal{O}' \succneqq \mathcal{O}_{\mathrm{min}}$. It is pleasing to note that the orbits listed in Table 1 are precisely the next-to-minimal orbits in the given algebras. The only next-to-minimal orbit that does not occur is that in $\mathfrak{sl}(3, \mathbb{C})$, which is cohomogeneity four.

Recall that according to Proposition 5.1 elements of cohomogeneity-one nilpotent orbits lie in a real $\mathfrak{sl}(2, \mathbb{C})$, i.e., in a σ-invariant rank one Lie algebra. It is remarkable that the elements of cohomogeneity-two orbits lie in σ-invariant rank two Lie algebras. The following can be thought of as a cohomogeneity-two version of Borel's result.

Theorem 4.2. *Suppose G is a compact simple Lie group and that $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}^{\mathbb{C}}$ of cohomogeneity two. Suppose X is an element of $\mathcal{O}$ that does not lie in a real $\mathfrak{sl}(2, \mathbb{C})$-subalgebra.*

Let $\mathfrak{h}_X^{\mathbb{C}}$ be the subalgebra of $\mathfrak{g}^{\mathbb{C}}$ generated by X and σX. Then $\mathfrak{h}_X^{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(4, \mathbb{C})$, unless $\mathfrak{g} = \mathfrak{g}_2$, in which case $\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{g}_2^{\mathbb{C}}$. In all cases, the embedding $\mathfrak{h}^{\mathbb{C}} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ is a homothety with respect to the Killing forms.

Type	Orbit
A_n	$(2^2 1^{n-3})$
$B_{(n-1)/2}$, $D_{n/2}$	(31^{n-3})
	$(2^4 1^{n-8})$
C_n	$(2^2 1^{2n-4})$

Type	Orbit
G_2	$0 \Rrightarrow 1$
F_4	$00 \Rightarrow 01$
E_6	$10\overset{0}{0}01$
E_7	$010\overset{0}{0}00$
E_8	$0000\overset{0}{0}01$

TABLE 1. Orbits of cohomogeneity two in simple Lie algebras. Orbits in classical algebras are specified by partitions and n is to be taken large enough so that the partition can occur. The orbits in exceptional algebras are given by their weighted Dynkin diagram (see e.g. [9]). Note that for type D_{2m}, the partition $(2^4 1^{4m-8})$ describes two orbits; their union is one orbit under the action of $O(2m)$.

Proof. Consider the Morse theory picture. Firstly, in $\mathfrak{g}^{\mathbb{C}}$, the closure of $\mathcal{O}$ is $\mathcal{O} \cup \mathcal{O}_{\min} \cup \{0\}$. In $\widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ we have $\overline{\mathfrak{M}(\mathcal{O})} = \mathfrak{M}(\mathcal{O}) \cup \mathfrak{M}(\mathcal{O}_{\min})$. For the orbits of cohomogeneity two, $\mathfrak{M}(\mathcal{O})$ is a manifold of cohomogeneity one; the usual scaling by $\mathbb{R}_{>0}$, which is also part of the $\mathbb{H}^*/\{\pm 1\}$-action, is transverse to the G-orbits on $\mathcal{O}$. Suppose ℓ is a curve in $\overline{\mathfrak{M}(\mathcal{O})}$ joining a point of $\mathcal{C}(\mathcal{O})$ to a point of $\mathfrak{M}(\mathcal{O}_{\min})$. Then the fact that $\mathfrak{M}(\mathcal{O})$ is the stable manifold for the gradient flow of the G-invariant functional ψ, implies that the image of ℓ in $\mathfrak{M}(\mathcal{O})/G$ is the whole (one-dimensional) quotient space.

Now to parameterise $\overline{\mathcal{O}}/G$ it is enough to find a two-dimensional family of elements which is invariant under scaling and contains an element lying over $\mathcal{C}(\mathcal{O})$ and an element of $\mathcal{O}_{\min}$. When $\mathfrak{g} \neq \mathfrak{g}_2$, we will find such a family lying in a σ-invariant $\mathfrak{so}(4,\mathbb{C})$-subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

The Lie algebra $\mathfrak{so}(4,\mathbb{C})$ splits as $\mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_-$. It contains three non-trivial nilpotent orbits: $\mathcal{O}_\pm$, the non-trivial nilpotent orbits in the factor $\mathfrak{sl}(2,\mathbb{C})_\pm$; and $\mathcal{O}_\Delta = \mathcal{O}_+ \times \mathcal{O}_-$. The orbits $\mathcal{O}_\pm$ are cohomogeneity one and $\mathcal{O}_\Delta$ is cohomogeneity two. Our orbit $\mathcal{O}$ will meet $\mathfrak{so}(4,\mathbb{C})$ in $\mathcal{O}_\Delta$ and $\mathcal{O}_+ \cup \mathcal{O}_-$ will be the intersection $\mathcal{O}_{\min} \cap \mathfrak{so}(4,\mathbb{C})$.

For the classical groups, we use the Jordan normal forms for elements of the orbits. For type A_n, the Jordan normal form is $(2^2 1^{n-3})$ and the matrices

$$X_{s,t} = \begin{pmatrix} \begin{smallmatrix} 0 & s \\ 0 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix} & \\ & & 0 \end{pmatrix}$$

lie in the orbit unless s or t is zero. They also lie in the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-subalgebra contained in the first two (2×2) diagonal blocks. The matrix $X_{1,1}$ lies in a real $\mathfrak{sl}(2,\mathbb{C})$-subalgebra and $X_{1,0}$ is in $\mathcal{O}_{\mathrm{min}}$. So this two parameter family is as required. Exactly the same technique works for C_n.

For types B and D, we are looking at matrices in $\mathfrak{so}(n,\mathbb{C})$. It is convenient to take $\mathfrak{so}(n,\mathbb{C})$ to be the set of complex $(n\times n)$ matrices A such that $A^tB+BA=0$, where B is the matrix with 1's down the anti-diagonal and 0's elsewhere. For Jordan form (31^{n-3}), we just take an $\mathfrak{so}(4,\mathbb{C})$-subalgebra containing the Jordan block (3). When the Jordan type is (2^41^{n-8}), and the Lie algebra type is not D_{2n}, we have the same situation as for A_n, but now the blocks come in pairs. Thus the two families one considers are

$$\begin{pmatrix} \begin{smallmatrix} 0 & s \\ 0 & 0 \end{smallmatrix} & & & & \\ & \begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix} & & & \\ & & 0 & & \\ & & & \begin{smallmatrix} 0 & -t \\ 0 & 0 \end{smallmatrix} & \\ & & & & \begin{smallmatrix} 0 & -s \\ 0 & 0 \end{smallmatrix} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & 0 \\ & \begin{smallmatrix} 0 & s & t & 0 \\ & 0 & 0 & -t \\ & & 0 & -s \\ & & & 0 \end{smallmatrix} & \\ 0 & & 0 \end{pmatrix}. \tag{4.1}$$

For D_{2n}, the matrices of Jordan type (2^41^{n-8}) form a single $O(n,\mathbb{C})$-orbit but split into two orbits $(2^41^{n-8})_\pm$ under the action of $SO(n,\mathbb{C})$. We thus obtain $(2^41^{n-8})_-$ from $(2^41^{n-8})_+$ by conjugating by an element W of determinant -1 in $O(n,\mathbb{C})$. One now considers three representative matrices, two as in (4.1) and

$$\begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & & & \\ & \begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix} & & & \\ & & 0 & & \\ & & & \begin{smallmatrix} 0 & -t \\ 0 & 0 \end{smallmatrix} & \\ \begin{smallmatrix} s & 0 \\ 0 & -s \end{smallmatrix} & & & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}, \quad \text{obtained with } W = \begin{pmatrix} 0 & & & & -1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -1 & & & & 0 \end{pmatrix}.$$

In all cases, the matrix lies over $\mathcal{C}(\mathcal{O})$ when $s=t\neq 0$ and is in $\mathcal{O}_{\mathrm{min}}$ when $t=0$ and $s\neq 0$.

For the exceptional Lie algebras we use the Beauville bundle $N(\mathcal{O})$ [1] as a tool for computation. This bundle is defined as follows. Find a real $\mathfrak{sl}(2,\mathbb{C})$-subalgebra associated to $\mathcal{O}$ and let $\{e,f,h\}$ be a basis for this subalgebra, with $f=-\sigma e$, $h=[e,f]$ and $[h,e]=2e$. The eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}^{\mathbb{C}}$ are known to be integers (see [9]). Let $\mathfrak{g}(i)$ be the i-eigenspace of $\operatorname{ad} h$. Put

$$\mathfrak{p} = \bigoplus_{i\geqslant 0} \mathfrak{g}(i) \quad \text{and} \quad \mathfrak{n} = \bigoplus_{i\geqslant 2} \mathfrak{g}(i).$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and the corresponding homogeneous space $\mathcal{F} = G^{\mathbb{C}}/P$ is a flag manifold. The subalgebra $\mathfrak{n}$ is preserved by the adjoint action of P and the Beauville bundle $N(\mathcal{O})$ is

Type	$\mathfrak{k}$	$\mathfrak{n}$	α	β
F_4	$\mathfrak{so}(2)+\mathfrak{so}(7)$	$\mathbb{R}^2\otimes\mathbb{R}^7$	$23\Rightarrow 42$	$01\Rightarrow 22$
E_6	$2\,\mathfrak{so}(2)+\mathfrak{so}(8)$	$\mathbb{R}^2\otimes\mathbb{R}^8$	$12\overset{2}{3}21$	$11\overset{1}{2}21$
E_7	$\mathfrak{so}(2)+\mathfrak{su}(2)+\mathfrak{so}(10)$	$\mathbb{R}^2\otimes\mathbb{R}^{10}$	$123\overset{2}{4}32$	$122\overset{2}{3}21$
E_8	$\mathfrak{so}(2)+\mathfrak{so}(14)$	$\mathbb{R}^2\otimes\mathbb{R}^{14}$	$2345\overset{3}{6}42$	$0123\overset{2}{4}32$

TABLE 2. The data for $\mathfrak{so}(4)$-subalgebras corresponding to next-to-minimal orbits in four exceptional algebras.

defined to be the bundle over $\mathcal{F}$ associated to $\mathfrak{n}$, i.e.,

$$N(\mathcal{O}) = G^{\mathbb{C}} \times_P \mathfrak{n}.$$

The important property of $N(\mathcal{O})$ is that it contains $\mathcal{O}$ as an open dense $G^{\mathbb{C}}$-orbit.

Now each flag manifold is a homogeneous manifold for the action of the compact group. So $\mathcal{F} = G/K$ for some compact subgroup K of G. (In fact, the Lie algebra $\mathfrak{k}$ of K is given by $\mathfrak{k}^{\mathbb{C}} = \mathfrak{g}(0)$.) The Beauville bundle is then $G \times_K \mathfrak{n}$ and the cohomogeneity of $\mathcal{O}$ is the cohomogeneity of the action of K on $\mathfrak{n}$. Choose a Cartan subalgebra in $\mathfrak{g}(0)$ and a root system for $\mathfrak{g}(0)$ with all root spaces in $\mathfrak{p}$. Note that, by definition, the weighted Dynkin diagram for $\mathcal{O}$ gives the eigenvalues of $\operatorname{ad} h$ on the positive simple root spaces, from which all the other eigenvalues are easily computed.

In the case of cohomogeneity-two orbits not in $\mathfrak{g}_2$, we find that $\mathfrak{n} \cong \mathbb{R}^2 \otimes V$ as a representation of $K = SO(2)L$, with V irreducible and L acting two-point transitively on the unit sphere in V. Thus under the action of K, we can move any nilpotent element in $\mathcal{O}$ into any complex two-dimensional subspace (for the complex structure induced by the action of $SO(2)$). We then find root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ contained in $\mathfrak{n}$ with α and β orthogonal long roots such that $\alpha \pm \beta$ is not a root. The σ-invariant subalgebra containing these root spaces is then the required $\mathfrak{so}(4,\mathbb{C})$. For the relevant four exceptional algebras, this information is given in Table 2.

For G_2, the isotropy group for $\mathcal{F}$ is $K = U(1)\,SU(2)$. We have $\mathfrak{n} = \mathfrak{g}(2) + \mathfrak{g}(3)$ with $\mathfrak{g}(2) \cong L^2$ and $\mathfrak{g}(3) \cong L^3S^1$, where $L = \mathbb{C}$ and $S^1 = \mathbb{C}^2$ are the fundamental representations of $U(1)$ and $SU(2)$, respectively. The orbit $\mathcal{O}$ in this case is the orbit of short root vectors and the subalgebra generated by X and σX contains both short and long roots, so is all of $\mathfrak{g}_2$. $\square$

5. Two Models

In this section we compute Kähler potentials for hyperKähler structures on two particular nilpotent orbits: one in $\mathfrak{sl}(2,\mathbb{C})$ and the other in $\mathfrak{so}(4,\mathbb{C})$. These results will be used in the next section to derive the hyperKähler potentials for cohomogeneity-two orbits. In view of Theorem 4.2, we consider these cases with inner products that are multiples of that given by the Killing form.

We start by considering $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ with inner product $k^2 \langle\cdot,\cdot\rangle_{\mathfrak{sl}(2)}$, where $k > 0$ is constant and $\langle\cdot,\cdot\rangle_{\mathfrak{sl}(2)}$ is negative of the Killing form. This Lie algebra contains only one non-trivial nilpotent orbit $\mathcal{O}$ consisting of the (2×2) matrices X such that $X^2 = 0$ and $X \neq 0$. The orbit is the minimal nilpotent orbit in $\mathfrak{sl}(2,\mathbb{C})$ and is of cohomogeneity one under the adjoint action of $SU(2)$. In fact two elements of the orbit have the same norm if and only if they are $SU(2)$-conjugate. Thus any $SU(2)$-invariant Kähler potential ρ on $\mathcal{O}$ is a function of just $\eta = k^2 \langle X, \sigma X\rangle_{\mathfrak{sl}(2)}$. Write

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1}$$

Then $\{e,f,h\}$ is an $\mathfrak{sl}(2,\mathbb{C})$ triple, with $f = -\sigma e$ and $h = -\sigma h$. Using the action of $SU(2)$, we may assume that $X = t\,e$, for some $t > 0$. The tangent space $T_X\mathcal{O} = [X, \mathfrak{sl}(2,\mathbb{C})]$ is spanned by e and h. If we consider the complex symplectic form $k^2\omega_C^{\mathcal{O}}$, we may perform the same calculations as in (3.5) and get

$$J_X e = 2t(\rho' + \eta\rho'')h \quad \text{and} \quad J_X h = -4t\rho' e,$$

where $\rho' = d\rho/d\eta$, etc. As $\eta(X) = 4k^2t^2$, the condition that $J^2 = -1$ is equivalent to

$$2\eta\rho'(\rho' + \eta\rho'') = k^2. \tag{5.2}$$

The left-hand side is simply the derivative of $(\eta\rho')^2$ with respect to η, so

$${\rho'}^2 = (k^2\eta + c)/\eta^2, \tag{5.3}$$

for some constant c. In order to have the potential defined on the whole orbit we need $c \geqslant 0$. The corresponding metric may be calculated as in Remark 3.2 and is given by

$$\begin{aligned} g(\xi_A,\xi_B) = \frac{2k^4}{\eta}\operatorname{Re}\Big(\rho'\big(\langle\xi_A,\sigma\xi_B\rangle\langle X,\sigma X\rangle - \langle\xi_A,\sigma X\rangle\langle X,\sigma\xi_B\rangle\big) \\ + \frac{k^2}{2\eta\rho'}\langle\xi_A,\sigma X\rangle\langle X,\sigma\xi_B\rangle\Big), \end{aligned} \tag{5.4}$$

which is positive definite provided we take the positive square root in (5.3). Now (5.3) determines ρ up to an additive constant, and this is enough to fix the metric structure.

Proposition 5.1. *For fixed k, the nilpotent orbit in $\mathfrak{sl}(2,\mathbb{C})$ has a one-parameter family of $SU(2)$-invariant hyperKähler metrics with a Kähler potential and with $k^2\omega_c^{\mathcal{O}}$ as the complex symplectic form. The G-invariant Kähler potential ρ is given by*

$$\rho' = \frac{1}{\eta}\sqrt{k^2\eta + c}, \tag{5.5}$$

where $c \geqslant 0$ is a constant and $\eta(X) = k^2 \langle X, \sigma X\rangle_{\mathfrak{sl}(2)}$. □

Note that if we rewrite everything in terms of the variable t, we get

$$\frac{d\rho}{dt}(te) = \sqrt{(2k)^4 + \frac{4c}{t^2}}. \tag{5.6}$$

Proposition 5.2. *The Kähler potential ρ of Proposition 5.1 is a hyperKähler potential if and only if $c = 0$. In this case, $\rho = 2k\sqrt{\eta}$ and $\rho(te) = 4k^2 t$.*

Proof. Let $Y = (d\rho)^\sharp$ be the vector field dual to $d\rho$. If ρ is a hyperKähler potential then IY is an isometry preserving I and ρ is the corresponding moment map [23, Proposition 5.5]. Now $d\rho = \rho' d\eta = 2k^2\rho' \operatorname{Re}\langle \cdot, \sigma X\rangle$, whereas, by Remark 3.2,

$$g(Y, \xi_A) = 2\operatorname{Re}(\rho' k^2 \langle \xi_A, \sigma Y\rangle + \rho'' k^4 \langle \xi_A, \sigma X\rangle \langle X, \sigma Y\rangle).$$

So we have

$$\rho' X = \rho' Y + \rho'' k^2 \langle Y, \sigma X\rangle X,$$

which implies that $Y = \lambda X$ with

$$\lambda = \frac{\rho'}{\rho' + \eta\rho''} = \frac{2\eta\rho'^2}{k^2} = 2 + \frac{2c}{k^2\eta},$$

using (5.2). The vector field X is generated by scaling in the nilpotent orbit, so IX preserves the complex structure I. Now

$$\begin{aligned} (L_{IY}I)(Z) &= [\lambda IX, IZ] - I[\lambda IX, Z] \\ &= \lambda(L_{IX}I)Z - ((IZ)\lambda)IX + (Z\lambda)X, \end{aligned}$$

so with $L_{IX}I = 0$, we have $L_{IY}I = 0$ only if λ is constant. But this is exactly the requirement that $c = 0$. □

Remark 5.3. The substitution $k^2\eta + c = (\frac{r}{2})^4$ in equation 5.4 shows that these are the Eguchi-Hanson metrics (cf. [10]). See [17] for details.

Let us now turn to the regular nilpotent orbit $\mathcal{O}_\Delta$ in $\mathfrak{so}(4,\mathbb{C})$. As in the proof of Theorem 4.2, we write $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_-$ and note that $\mathcal{O}_\Delta = \mathcal{O}_+ \times \mathcal{O}_-$ where $\mathcal{O}_\pm$ is the nilpotent orbit in $\mathfrak{sl}(2,\mathbb{C})_\pm$. Let $\{e_\pm, f_\pm, h_\pm\}$ be bases for $\mathfrak{sl}(2,\mathbb{C})_\pm$ as in (5.1). Again we will use the inner product which is $k^2 \langle \cdot,\cdot \rangle_{\mathfrak{so}(4)}$.

Using the action of $SO(4)$, we may take our representative element X of $\mathcal{O}_\Delta$ to be $X = X_+ + X_- = s\,e_+ + t\,e_-$ with $s,t > 0$. We have one invariant for each $\mathfrak{sl}(2,\mathbb{C})$: we write $\eta_\pm = k^2 \langle X_\pm, \sigma X_\pm \rangle_{\mathfrak{sl}(2)}$, so $\eta_+ = 4k^2 s^2$, etc. Let $\rho_+ = \partial\rho/\partial\eta_+$, etc. Then we may calculate the Kähler form ω_I and the candidate almost complex structure J as in §3. For the Kähler form we get

$$\begin{aligned}\omega_I(\xi_A,\xi_B) = 2k^2 \operatorname{Im}\Bigl(&\rho_+ \langle \xi_A^+, \sigma\xi_B^+ \rangle + \rho_- \langle \xi_A^-, \sigma\xi_B^- \rangle \\ &+ \rho_{++} k^2 \langle \xi_A^+, \sigma X_+ \rangle \langle \sigma\xi_B^+, X_+ \rangle \\ &+ \rho_{+-} k^2 \bigl(\langle \xi_A^+, \sigma X_+ \rangle \langle \sigma\xi_B^-, X_- \rangle \\ &\qquad + \langle \xi_A^-, \sigma X_- \rangle \langle \sigma\xi_B^+, X_+ \rangle \bigr) \\ &+ \rho_{--} k^2 \langle \xi_A^-, \sigma X_- \rangle \langle \sigma\xi_B^-, X_- \rangle \Bigr),\end{aligned}$$

where $\xi_A^+ = [A, X_+]$, etc. The endomorphism J is given by

$$\begin{aligned}J_X(\xi_A^+) = &-2\rho_+[X_+, \sigma\xi_A^+] \\ &- 2k^2 \langle \sigma\xi_A^+, X_+ \rangle \bigl(\rho_{++}[X_+, \sigma X_+] + \rho_{+-}[X_-, \sigma X_-] \bigr)\end{aligned}$$

and a similar expression for ξ_A^-. In particular, $J_X h_+ = -4s\,\rho_+ e_+$ and

$$J_X e_+ = 2s(\rho_+ + \eta_+\rho_{++})h_+ + 2\frac{t^2}{s}\eta_+\rho_{+-}h_-.$$

Thus the $\mathfrak{sl}(2,\mathbb{C})_-$-component of $J_X^2 h_+$ is a constant times $\eta_+\eta_- \times \rho_+\rho_{+-}h_-$. For J_X^2 to be -1, we need $\rho_+\rho_{+-} = 0$, which implies $\partial(\rho_+^2)/\partial\rho_- = 0$ and hence $\rho_{+-} = 0$. Thus J_X preserves the $\mathfrak{sl}(2,\mathbb{C})$-summands of $\mathfrak{so}(4,\mathbb{C})$.

Proposition 5.4. *Any hyperKähler structure on the regular orbit $\mathcal{O}_\Delta = \mathcal{O}_+ \times \mathcal{O}_-$ of $\mathfrak{so}(4,\mathbb{C})$ which is $SO(4)$-invariant, admits a Kähler potential and has complex-symplectic form $k^2\omega_c^{\mathcal{O}_\Delta}$, is a product of $SU(2)$-invariant structures on the factors $\mathcal{O}_\pm$, and these are given by Proposition 5.1.* ☐

6. Potentials for Next-to-Minimal Orbits

We now come to the main result of this paper. We consider next-to-minimal orbits with compatible G-invariant hyperKähler metrics,

Type	A_n, C_n	B_n, D_n	F_4	E_6	E_7	E_8
k^2	$\frac{1}{2}(n+1)$	$\frac{1}{2}(n-2)$	$\frac{9}{2}$	6	9	$\frac{35}{2}$

TABLE 3. The constant k^2 in the potentials of Theorem 6.1.

except for $G = SU(3)$. We show that such metrics admitting a hyperKähler potential are unique, and we calculate the potential.

If we assume that the potential is only Kähler, we still have uniqueness in some cases, but we get a list of exceptions: orbits which admit a one-parameter family of hyperKähler metrics. These can be thought of as a generalisation of the Eguchi-Hanson metric (cf. Remark 5.3).

Theorem 6.1. *Suppose G is a compact simple Lie group and $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}^{\mathbb{C}}$ of cohomogeneity two.*

(i) *$\mathcal{O}$ admits a unique G-invariant compatible hyperKähler metric with hyperKähler potential. This potential is given by*

$$\rho = 2k\sqrt{\eta_1 + 2\sqrt{\tfrac{1}{2}\eta_1^2 - k^2\eta_2}} \tag{6.1}$$

for $\mathfrak{g} \neq \mathfrak{g}_2$, where the constant k is given in Table 3, and, for $\mathfrak{g}_2$,

$$\rho = \sqrt{8}\sqrt{\eta_1 + \sqrt{6}\sqrt{\eta_1^2 - 4\eta_2}}. \tag{6.2}$$

(ii) *The above metric on $\mathcal{O}$ is in fact a unique G-invariant compatible hyperKähler metric with a* Kähler *potential unless $\mathfrak{g}$ is one of $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, $\mathfrak{so}(8)$ or $\mathcal{O}$ is of Jordan type (31^{n-3}) in $\mathfrak{so}(n)$. In these cases, the metric lies in a one-parameter family of hyperKähler metrics with Kähler potentials.*

Remark 6.2. Note that the Theorem provides hyperKähler potentials for all next-to-minimal orbits, except when $\mathfrak{g} = \mathfrak{su}(3)$. However, the potential in this remaining case was computed in [16], see also [18].

We divide the proof of the Theorem into three parts.

6.1. **The General Case.** This is when $\mathfrak{g}$ is neither $\mathfrak{su}(3)$ nor $\mathfrak{g}_2$.

Let X be a generic element of $\mathcal{O}$. By Theorem 4.2, X lies in the regular orbit $\mathcal{O}_\Delta$ of a real $\mathfrak{so}(4,\mathbb{C})$-subalgebra.

For $\rho(\eta_1,\eta_2)$ to be a hyperKähler potential for $\mathcal{O}$ it is necessary that ρ is a Kähler potential for an invariant hyperKähler structure on $\mathcal{O}_\Delta$. To see this, first note that equation (2.1) is invariant by pull-back under the inclusion map $\mathcal{O}_\Delta \hookrightarrow \mathcal{O}$. Now equation (3.5) shows that

$J\xi_A$ remains in the subalgebra generated by A, X and σX. Thus if $A \in \mathfrak{so}(4,\mathbb{C})$, so is JA and thus $\mathcal{O}_\Delta$ is a hyperKähler submanifold of $\mathcal{O}$.

As in §5, write $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})_+ \oplus \mathfrak{sl}(2,\mathbb{C})_-$, $\mathcal{O}_\Delta = \mathcal{O}_+ \times \mathcal{O}_-$ and $X = X_+ + X_- = se_+ + te_-$. Our two invariants on $\mathcal{O}$ are given by

$$\begin{aligned}\eta_1(X) &= \langle X, \sigma X\rangle_{\mathfrak{g}} = -\langle se_+ + te_-, sf_+ + tf_-\rangle_{\mathfrak{g}}\\ &= -(s^2+t^2)k^2\langle e, f\rangle_{\mathfrak{su}(2)} = 4k^2(s^2+t^2),\end{aligned}$$

and a similar computation gives

$$\eta_2(X) = 8k^2(s^4+t^4),$$

where k^2 is the constant such that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}|_{\mathfrak{so}(4,\mathbb{C})} = k^2\langle\cdot,\cdot\rangle_{\mathfrak{so}(4)}$. Now

$$\begin{aligned}d\rho &= \rho_1 d\eta_1 + \rho_2 d\eta_2\\ &= 8k^2\left(s(\rho_1 + 4s^2\rho_2)ds + t(\rho_1 + 4t^2\rho_2)dt\right),\end{aligned}$$

so $\rho_s := \partial\rho/\partial s = 8k^2 s(\rho_1 + 4s^2\rho_2)$, etc., and solving for ρ_1 and ρ_2 we get

$$\rho_1 = -\frac{t^3\rho_s - s^3\rho_t}{8k^2 st(s^2-t^2)}, \qquad \rho_2 = \frac{t\rho_s - s\rho_t}{32k^2 st(s^2-t^2)}.$$

Note that, by Proposition 5.4 and (5.6), ${\rho_s}^2 = 16k^4 + c_+/s^2$ and ${\rho_t}^2 = 16k^4 + c_-/t^2$, for some constants $c_\pm$.

The elements X_+ and X_- lie in the closure of $\mathcal{O}_\Delta$ and hence of $\mathcal{O}$; so $X_\pm$ lie in the minimal nilpotent orbit of $\mathfrak{g}^{\mathbb{C}}$. We deduce that $M_+ := G/N(SU(2)_+)$ is a Wolf space and hence, since $SU(2)_+$ corresponds to a highest root [25],

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{k}_+ + S^1_+ \otimes E_+, \tag{6.3}$$

where $\mathfrak{k}_+$ commutes with $\mathfrak{sl}(2,\mathbb{C})$, E_+ is a non-trivial representation of $\mathfrak{k}_+$ and $S^1_+ \cong \mathbb{C}^2$ is the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})_+$. On the other hand, we have a similar decomposition of $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\mathfrak{sl}(2,\mathbb{C})_-$. As $\mathfrak{sl}(2,\mathbb{C})_+$ and $\mathfrak{sl}(2,\mathbb{C})_-$ commute with each other, we deduce that $\mathfrak{sl}(2,\mathbb{C})_- \subset \mathfrak{k}_+$ and that $E_+ \supset S^1_-$. So as an $\mathfrak{so}(4,\mathbb{C})$-module, $\mathfrak{g}^{\mathbb{C}}$ always contains a copy of $S^1_+ \otimes S^1_-$.

On the orthogonal complement to $\mathfrak{so}(4,\mathbb{C})$, we have, from (3.5),

$$\begin{aligned}J_X\xi_A &= -2\rho_1[X, \sigma\xi_A]\\ &\quad + 4\rho_2\left(2[X,[\sigma X,[X,\sigma\xi_A]]] - [X,[X,[\sigma X,\sigma\xi_A]]]\right).\end{aligned} \tag{6.4}$$

First suppose that E_+ contains a trivial $\mathfrak{sl}(2,\mathbb{C})_-$-module $\mathbb{C}^r$; take r maximal. The real structure σ preserves the module $S^1_+ \otimes \mathbb{C}^r$ and acts on $S^1_+ \cong \mathbb{H}$ as j, so $\mathbb{C}^r$ has a quaternionic structure $\mathfrak{j}$ and is even-dimensional. Choose a basis for S^1_+ so that $\operatorname{ad} e_+$ acts as $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. Then any tangent vector $\xi_A \in S^1_+ \otimes \mathbb{C}^r$ has the form $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes v$ and we have

$$J_X \xi_A = -2s(\rho_1 + 4s^2\rho_2)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \mathrm{j}v = -\tfrac{1}{4k^2}\rho_s \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \mathrm{j}v.$$

Thus $J^2 = -1$ on $S^1_+ \otimes \mathbb{C}^r$ if and only if ${\rho_s}^2 = 16k^4$. This implies that the constant c_+ is zero if E_+ has an trivial $\mathfrak{sl}(2,\mathbb{C})_-$-submodule.

The existence of an trivial $\mathfrak{sl}(2,\mathbb{C})_-$-submodule in E_+ is not guaranteed. However, we do always have an S^1_--summand, so we now consider the case when ξ_A lies in an $\mathfrak{so}(4)$-module $S^1_+ \otimes S^1_-$. This is Killing orthogonal to $\mathfrak{so}(4,\mathbb{C})$. We choose bases so that $\operatorname{ad} X$ acts as

$$s\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \otimes \mathrm{Id} + t\,\mathrm{Id} \otimes \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$$

and $\sigma = j \otimes j$ for j the standard quaternionic structure on $S^1 \cong \mathbb{H}$. The image of $\operatorname{ad} X$ is two-dimensional and spanned by

$$\xi_1 := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \quad \text{and} \quad \xi_2 := s\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) + t\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right).$$

These satisfy

$$\begin{gathered}[X,\xi_1] = 0, \qquad [\sigma X, \xi_1] = \sigma\xi_2,\\ [X,\xi_2] = 2st\xi_1 \quad \text{and} \quad [\sigma X, \xi_2] = -(s^2+t^2)\sigma\xi_1.\end{gathered}$$

So, equation (6.4) gives

$$\begin{gathered}J\xi_1 = -2(\rho_1 + 4\rho_2(s^2+t^2))\xi_2,\\ J\xi_1 = 2(\rho_1(s^2+t^2) + 4\rho_2(s^4+t^4))\xi_1.\end{gathered}$$

Substituting for ρ_1 and ρ_2 in terms of ρ_s and ρ_t, gives

$$J^2\xi_1 = -\frac{t^2{\rho_t}^2 - s^2{\rho_s}^2}{16k^4(t^2-s^2)}\xi_1.$$

So $J^2 = -1$ on $S^1_+ \otimes S^1_-$ if and only if $t^2{\rho_t}^2 - s^2{\rho_s}^2 = 16k^4(t^2-s^2)$. But ${\rho_s}^2 = 16k^4 + c_+/s^2$, etc., so $c_+ = c_-$.

We conclude that if E_+ contains a trivial $\mathfrak{sl}(2,\mathbb{C})_-$-summand, then $c_+ = c_- = 0$. This gives $\rho_s = 4k^2$ and $\rho_t = 4k^2$, so $\rho(s,t) = 4k^2(s+t)$. Rewriting this in terms of η_1 and η_2 gives the potential in the Theorem. If E_+ does not have a trivial summand, we get a one-parameter family of potentials and hyperKähler metrics with $c_+ = c_-$.

It remains to determine the constant k and when E_+ contains a trivial $SU(2)_-$-module. The decomposition (6.3) gives the action of $\operatorname{ad} e_+$ and hence the Killing inner product $\langle e_+, \sigma e_+\rangle_{\mathfrak{g}}$ is $4 + \dim_{\mathbb{C}} E_+$, since $\langle e_+, \sigma e_+\rangle_{\mathfrak{su}(2)_+} = 4$. So $k^2 = (4 + \dim_{\mathbb{C}} E_+)/4$. Moreover, $S^1_+ \otimes E_+ = TM_+ \otimes \mathbb{C}$, so $\dim_{\mathbb{C}} E_+$ is half the real dimension of the Wolf space M_+, which may be found in, e.g., Besse [3, p. 409], or read-off from the discussion below. This leads to Table 3.

Finally, we determine the decompositions of E_+ under the action of $\mathfrak{sl}(2,\mathbb{C})_-$.

If $G = SU(n)$, then $\mathfrak{k}_+ \cong \mathfrak{u}(n-2)$, and $E_+ = \mathbb{C}^{n-2}$ is the fundamental representation twisted by a representation of the central $\mathfrak{u}(1)$. Now $\mathfrak{sl}(2,\mathbb{C})_-$ corresponds to a highest root vector in $\mathfrak{k}_+$, so $E_+ = S^1_- + \mathbb{C}^{n-4}$ as a $\mathfrak{sl}(2,\mathbb{C})_-$-module. So for $n = 4$, we have a one-parameter family of potentials $c_+ = c_-$, and for $n > 4$, the potential is unique.

For $G = Sp(n)$, $\mathfrak{k}_+ \cong \mathfrak{sp}(n-1,\mathbb{C})$ and $E_+ \cong \mathbb{C}^{2n-2} \cong \mathbb{H}^{n-1}$ is the fundamental representation. Under the highest root $\mathfrak{sl}(2,\mathbb{C})$, this representation splits as $S^1_- + \mathbb{C}^{2n-4}$, so for $n > 1$, we have a unique potential.

In the case $G = SO(n)$, there are two orbit types to consider. The centraliser $\mathfrak{k}_+ = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{so}(n-4,\mathbb{C})$ and there are two choices for $\mathfrak{sl}(2,\mathbb{C})_-$, one in each summand of $\mathfrak{k}_+$. When $\mathfrak{sl}(2,\mathbb{C})_- = \mathfrak{sl}(2,\mathbb{C})$, we get $E_+ \cong S^1_- \otimes \mathbb{R}^{n-4}$, and there is a one-parameter family of potentials. On the other hand, if $\mathfrak{sl}(2,\mathbb{C})_-$ lies in the summand $\mathfrak{so}(n-4,\mathbb{C})$, then $E_+ \cong \mathbb{C}^2 \otimes (S^1_- + \mathbb{R}^{n-8})$. For $n > 8$, this gives a unique potential, but for $n = 8$, we again get a family.

We now come to the four exceptional cases. Firstly, if $G = F_4$, then $\mathfrak{k}_+ \cong \mathfrak{sp}(3,\mathbb{C})$ and if $E = \mathbb{H}^3$ is the fundamental representation, then $E_+ \cong \Lambda^3_0 E = \Lambda^3 E - E$, is a 14-dimensional irreducible representation. For a highest root $\mathfrak{sl}(2,\mathbb{C})_-$ in $\mathfrak{sp}(3,\mathbb{C})$, we have $E \cong S^1_- + \mathbb{C}^4$ and hence $E_+ \cong 5S^1_- + \mathbb{C}^4$. So E_+ has a trivial summand and hence the potential is unique.

For $G = E_6$, $\mathfrak{k}_+ \cong \mathfrak{sl}(6,\mathbb{C})$ and $E_+ \cong \Lambda^{3,0}\mathbb{C}^6$. Under a highest root $\mathfrak{sl}(2,\mathbb{C})_-$, we have $\Lambda^{1,0}\mathbb{C}^6 \cong S^1_- + \mathbb{C}^4$ and hence $E_+ = 4S^1_- + \mathbb{C}^8$, giving a unique potential.

When $G = E_7$, $\mathfrak{k}_+ \cong \mathfrak{so}(12,\mathbb{C})$ and $E_+ \cong \Delta^{12}_+$, the positive spin representation. For a highest root $\mathfrak{sl}(2,\mathbb{C})_-$, the normaliser in $\mathfrak{so}(12,\mathbb{C})$ is $\mathfrak{sl}(2,\mathbb{C})_- + \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{so}(8,\mathbb{C})$ and the fundamental representation of $SO(12)$ decomposes as $\mathbb{C}^{12} \cong S^1_- \otimes S^1 + V$, where $V \cong \mathbb{C}^8$ is the fundamental representation of $\mathfrak{so}(8,\mathbb{C})$. The spin representation splits as $\Delta^{12}_+ \cong S^1_- \otimes \Delta^8_+ + S^1 \otimes \Delta^8_-$, and so $E_+ \cong 8S^1_- + \mathbb{C}^{16}$ has a trivial summand.

Finally, for $G = E_8$, $\mathfrak{k}_+ \cong \mathfrak{e}_7^{\mathbb{C}}$ and $E_+ \cong \begin{smallmatrix} &&&0&& \\ 1&0&0&0&0&0 \end{smallmatrix}$. A highest root $\mathfrak{sl}(2,\mathbb{C})_-$ in $\mathfrak{e}_7^{\mathbb{C}}$ has centraliser $\mathfrak{so}(12,\mathbb{C})$ and $E_+ \cong 12S^1_- + \mathbb{C}^{32}$, where $\mathbb{C}^{32} \cong \Delta^{12}_+$. So again we get a unique potential.

6.2. **The Exceptional Case G_2.** The Dynkin diagram for the next-to-minimal orbit $\mathcal{O}$ in G_2 is $0 \Rrightarrow 1$. This says that there is a basis $\{\alpha,\beta\}$ for the simple positive roots, with α short and β long, such that $\operatorname{ad} h$ acts on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ with eigenvalues 1 and 0 respectively. We thus have

$\mathfrak{g}(2) = \mathfrak{g}_{\beta+2\alpha}$ and $\mathfrak{g}(3) = \mathfrak{g}_{\beta+3\alpha} \oplus \mathfrak{g}_{2\beta+3\alpha}$. From the discussion in §4, the isotropy group $SU(2)\,U(1)$ of the Beauville bundle acts transitively on the unit sphere in $\mathfrak{g}(3)$, so using the action of the compact group G_2, we can move a typical element of $\mathcal{O}$ to $X \in \mathfrak{g}_{\beta+2\alpha} \oplus \mathfrak{g}_{2\beta+3\alpha}$. We may thus write $X = sE_{\beta+2\alpha} + tE_{2\beta+3\alpha}$, with $s, t > 0$, where E_i are such that for $F_i := -\sigma E_i$ and $H_i = [E_i, F_i]$ we have $[H_i, E_i] = 2E_i$.

At X, our two invariants are

$$\eta_1(X) = 8(s^2 + 3t^2) \quad \text{and} \quad \eta_2(X) = 16(s^4 + 6s^2t^2 + 3t^4).$$

As in the previous section, we compute J^2 on particular tangent vectors using (3.5) and then rewrite the equations in terms of s and t. This is quite hard work to do by hand, and so we used Maple to do the following computations. The code for this is described in [15].

On $\mathfrak{g}_{\alpha+\beta}$, one finds that $J^2 = -1$ only if

$$\frac{1}{64s}\rho_s(s\rho_s + t\rho_t) = 1, \tag{6.5}$$

where ρ_s is $\partial\rho/\partial s$, etc. Now $X = [X, H_\beta - H_\alpha] = [X, 3H_{2\beta+3\alpha} - 5H_{\beta+2\alpha}]$, so X is tangent to the orbit $\mathcal{O}$. The condition $J^2X = -X$, gives the following three equations

$$\begin{aligned} s(2s\rho_s + t\rho_t)\rho_{ss} + t(t\rho_t + 3s\rho_s)\rho_{st} + t^2\rho_s\rho_{tt}& \\ +2(t\rho_t + s\rho_s)\rho_s = 128s&, \end{aligned} \tag{6.6a}$$

$$9s\rho_s\rho_{ss} + (9t\rho_s + s\rho_t)\rho_{st} + t\rho_t\rho_{tt} + 9\rho_s^2 + \rho_t^2 = 576 \tag{6.6b}$$

$$\begin{aligned} &3st(9t\rho_s + s\rho_t)\rho_{ss} - st(s\rho_t - 3t\rho_s)\rho_{tt} \\ &\quad +(3t(s^2 + 9t^2)\rho_s + s(3t^2 - s^2)\rho_t)\rho_{st} = (s\rho_t - 9t\rho_s)(s\rho_t + 3t\rho_s) \end{aligned} \tag{6.6c}$$

by considering the components in $\mathfrak{g}_{\beta+2\alpha}$, $\mathfrak{g}_{2\beta+3\alpha}$ and $\mathfrak{g}_{2\beta+\alpha}$. Considering $9s\rho_s$ times (6.6a) minus $s(2\rho_s s + t\rho_t)$ times (6.6b) gives a new equation not involving ρ_{ss}. In a similar way, we may eliminate ρ_{ss} form the pair of equations (6.6a) and (6.6c). Eliminating ρ_{st} from these two new equations not involving ρ_{ss}, we get the following equation which does not involve ρ_{tt}:

$$s^3t(2s\rho_s + t\rho_t)(s\rho_t - 9t\rho_s)^2 = 0.$$

Thus either

$$\text{(i)} \quad \rho_t = -2\frac{s}{t}\rho_s, \qquad \text{or} \qquad \text{(ii)} \quad \rho_t = 9\frac{t}{s}\rho_s. \tag{6.7}$$

In case (i), substituting into (6.5) one gets $\rho_s^2 = -64$, which has no (real) solutions. In case (ii), we have

$$\rho_s = \varepsilon \frac{8s}{\sqrt{s^2+9t^2}}, \qquad \rho_t = \varepsilon \frac{72t}{\sqrt{s^2+9t^2}},$$

where $\varepsilon \in \{\pm 1\}$. Integrating we find that

$$\rho = \varepsilon 8\sqrt{s^2+9t^2}. \tag{6.8}$$

To get a positive-definite metric, take $\varepsilon = +1$. Rewriting (6.8) in terms of η_1 and η_2 gives the claimed result. One may check directly that the resulting J satisfies $J^2 = -1$ on the whole tangent space.

6.3. **Uniqueness of HyperKähler Potentials.** The only statement left to verify in the proof of Theorem 6.1, is that equations (6.1) and (6.2) give the unique compatible hyperKähler potentials on the orbits. In the cases, when the Kähler potential is unique there is nothing to prove, because the general theory [23] gives the existence of such a potential. We may therefore assume we are in the general case and that our generic element X lies in a real $\mathfrak{so}(4,\mathbb{C})$-subalgebra. Now $\mathcal{O}_\Delta$ is a hyperKähler submanifold of $\mathcal{O}$, and so by (2.1), ρ is a hyperKähler potential for $\mathcal{O}$ only if it restricts to a hyperKähler potential for $\mathcal{O}_\Delta$. However, the hyperKähler structure on $\mathcal{O}_\Delta$ is the product of two hyperKähler structures on $\mathfrak{sl}(2,\mathbb{C})$-orbits and on each of these factors the hyperKähler potential is unique by Proposition 5.2. Thus there is only one hyperKähler potential compatible with the structure of $\mathcal{O}$. □

Remark 6.3. The hyperKähler metrics constructed in Theorem 6.1 have an extra $U(1)$-symmetry given by $X \mapsto e^{i\theta}X$ which preserves the complex structure I but moves J. In the case of $\mathfrak{sl}(2,\mathbb{C})$, the metrics are of Bianchi type IX and it is known, e.g., from [2], that there are triaxial hyperKähler metrics that do not have $U(2)$-symmetry. Thus concentrating on metrics admitting a Kähler potential is a genuine restriction.

Remark 6.4. The one-parameter families in Theorem 6.1 occur exactly when $E_+ \cong \mathbb{C}^s \otimes S^1_-$. Considering the weights of the action of a semisimple element in the diagonal $\mathfrak{sl}(2,\mathbb{C})$-subalgebra of $\mathfrak{so}(4,\mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$, we see that this exactly the case when $\mathfrak{g}(1) = 0$. This says that the Beauville bundle coïncides with the cotangent bundle $T^*\mathcal{F}$, rather than being a proper subbundle. In the case of the one-parameter families $\mathcal{F} = \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ and for $c \neq 0$, the Kähler potentials extend to give non-singular metrics on $T^*\mathcal{F}$, generalising the Eguchi-Hanson metrics on $T^*\,\mathbb{C}\mathrm{P}(1)$. As $\mathcal{F}$ is Hermitian symmetric, this is one of the cases considered by Biquard & Gauduchon [5].

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