ORDERED INVARIANT SETS FOR NONLINEAR EVOLUTION EQUATIONS OF KdV-TYPE

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Abstract

We consider a quasilinear third-order evolution equation

$$u_t = u_{xxx} + \psi(u)u_x, \tag{EE}$$

where $\psi(u)$ is a sufficiently smooth function. For $\psi(u) = u$ the equation becomes the Korteveg-de Vries equation.

It is proved that there exists a class of equations (EE), with functions ψ satisfying a nonlinear fourth-order ordinary differential equation, which admit exact solutions belonging to a so-called ordered invariant set S_0. These solutions are new and are not invariant under Lie groups of transformations, but nevertheless a connection with group-invariant solutions is discussed. Namely, for any such ψ, the equation (EE) on S_0 becomes a nonlinear first-order ordinary differential equation. The solutions $u(\cdot, t) \in S_0$ are ordered in the sense that they satisfy the standard comparison theorem. The KdV equation is shown to be a limit case of such equations admitting invariant sets.

Some asymptotic properties of the invariant solutions are presented. A further generalization to quasilinear equations is discussed.

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Key Words. Korteveg-de Vries-type equations, exact solution, invariant set, comparison, blow-up.

1. Introduction

We construct invariant sets and new exact solutions of the generalized Korteveg-de Vries (KdV) equation which is a semilinear third-order evolution equation of the form

$$u_t = u_{xxx} + \psi(u)u_x, \quad (x,t) \in \mathbf{R} \times \mathbf{R}, \tag{EE}$$

where $\psi(u)$ is a sufficiently smooth function. If $\psi(u) = u$ then (EE) becomes the KdV equation from the theory of water waves (a model for long wave propagation in a channel), while for $\psi(u) = u^2$ it is the modified KdV equation. The KdV-type equations and their generalizations are well known in the mathematical literature concerning soliton theory, inverse scattering, symmetries of evolution equations and application of group theory (see e.g. [BK], [I]) as well as in problems of well-posedness of (EE) with sufficiently arbitrary functions $\psi(u)$ in suitable functional classes (see e.g. [BS], [CM], [KF] and references therein).

The KdV and the modified KdV equation are special integrable equations and admit a countable set of exact solutions called solitons. The soliton solutions are known to play a fundamental role in the description of nonlinear interaction of structures in the nonlinear media under consideration. Much less is known about exact solutions of the generalized KdV-type equations (EE) with a general nonlinear smooth non-power coefficient ψ.

In this paper we show that there exists a class of equations (EE), with a specially chosen nonlinear coefficient $\psi(u)$, and any such equation admits a one-dimensional *invariant* set S_0. These new exact solutions are presented in Theorem 2.1. The solutions in S_0 are not invariant under Lie groups of transformations. Nevertheless, they can be treated as those constructed by means of a certain nonlinear generalization of the classical approaches based on groups of scaling transformation. See the last section.

The special generalized scaling structure of these solutions essentially differs from the soliton structure for the KdV equations. Moreover, a curious features of S_0 is that the evolution equation (EE) restricted to S_0 generates an order preserving group, where the order is understood in the standard comparison sense. It is well-known that the solutions of the third-order equation (EE) do not satisfy the usual comparison with respect to initial data. The Comparison Principle (CP), which is a natural property of general first-order ordinary differential equations (ODE) (as well as of the second-order parabolic equations), is not true for higher-order equations.

We call such invariant sets S_0 *ordered.* Equation (EE) on S_0 becomes a first-order in time t *nonlinear ODE*, and the spatial variable x then plays the role of a parameter. Under natural assumptions, the solutions on S_0 satisfy the standard comparison theorem: if $u_1(\cdot,t)$ and $u_2(\cdot,t)$ are local in time solutions of (EE) on S_0 then

$$u_1(\cdot,0) \le u_2(\cdot,0) \implies u_1(\cdot,t) \le u_2(\cdot,t) \quad \text{for } t > 0. \tag{1.1}$$

Moreover, we show that it is possible to compare different solutions on $S_0 = S_0(\psi)$ corresponding to different nonlinear coefficients ψ, thus establishing an order solution property of a family of nonlinear equations (EE).

It is convenient to determine the invariant set via a first-order differential constraint of the form

$$S_0 = \left\{ u = u(x,t): \quad u_x = \frac{1}{x} F(u) \right\}, \tag{1.2}$$

with an unknown smooth function $F(u) \not\equiv 0$ depending on the nonlinear coefficient of the equation under consideration. The differential structure of the constraint makes it possible to reduce the spatial differentiation on S_0 to a simple algebraic procedure (the algebraic differentiation on S_0), and finally reduce (EE) to an ODE.

We thus apply this algebraic technique to the third-order evolution equation. As for the second-order ones, this constraint was proved in [G2, Sect. 3] to be efficient for a general quasilinear second-order parabolic equation of the form

$$u_t = (k(u)u_x)_x + F(u), \tag{1.3}$$

and for the corresponding N-dimensional equations in the radial spatial geometry. It gives a class of new exact solutions of (1.3), which are not of a group-invariant nature. Such new exact solutions exist even for the corresponding *semilinear* heat equation with $k(u) \equiv 1$

$$u_t = u_{xx} + F(u), \tag{1.4}$$

which contains a single nonlinear coefficient F. These exact solutions are constructed for $F(u)$ satisfying the ODE

$$FF'' - 2F' + 2 = 0$$

[G2, Example 3.2]. Moreover, for the parabolic equation (1.3) the corresponding first-order operator

$$\mathcal{H}(u) = u_x - \frac{1}{x} F(u) \tag{1.5}$$

is proved [G2, Prop. 3.3] to be a *sign-invariant*, i.e. $\mathcal{H}(u)$ preserves signs on the evolution orbits of (1.3) in the sense of a formal application of the Maximum Principle (MP) for linear parabolic equations. See details in [G2, Sect. 1, 2].

For the higher-order equation (EE) the MP is not valid, and the operator like (1.5) cannot be a sign-invariant (a sign preserving operator). Nevertheless, we show that due to the special structure of the constraint (1.2) we can get a certain order preserving property. In Section 2 we consider the KdV-type equation (EE). Section 3 is devoted to some generalizations of the method. In particular we show that an ordered invariant set and the corresponding exact solutions are constructed for a quasilinear equation of the form

$$u_t = \phi(u)u_{xxx} + \psi(u)u_x, \tag{1.6}$$

which contains an extra arbitrary smooth nonlinear coefficient $\phi(u)$. It turns out that the invariant set can be constructed for arbitrary fixed ϕ with a special choice of ψ depending on ϕ. Therefore we construct invariant sets for an *infinite-dimensional* family of equations (1.6).

2. Main result

1. Invariant set and exact solutions. We state the main result of the paper.

Theorem 2.1. *Let $F(u)$ be a solution of the following third-order ordinary differential equation*

$$\mathbf{D}(F) \equiv F(FF')'' - 6FF'' - 3(F')^2 + 9F' - 6 = 0. \tag{2.1}$$

We set in (EE)

$$\psi(u) = \exp\left\{\int_1^u \frac{dz}{F(z)}\right\}. \tag{2.2}$$

Then: (i) Equation (EE) with ψ given by (2.2) admits the invariant set S_0, (1.2), of exact solutions

$$\int_1^{u(x,t)} \frac{dz}{F(z)} = \log x + H(t), \tag{2.3}$$

where the function H solves the ordinary differential equation

$$H' = de^{3H} + e^H, \quad t \in \mathbf{R}, \tag{2.4}$$

$d \in \mathbf{R}$ being the constant

$$d = [(FF')' - 3F' + 2](1). \tag{2.5}$$

(ii) (EE) on S_0 is equivalent to a nonlinear first-order ODE for the function $u = u(\cdot, t)$

$$u_t = \frac{1}{x^3}[F(FF')' - 3FF' + F](u) + \frac{1}{x}(\psi F)(u). \tag{2.6}$$

The ODE (2.6) is the evolution equation (EE) on the invariant set S_0. We will show that the KdV equation represents a particular trivial case of S_0.

2. Proof of Theorem 2.1. (i) Let us show that under the given hypotheses the set of smooth functions

$$S_0 = \left\{u(x,t): \quad \mathcal{H}(u) \equiv u_x - \frac{1}{x}F(u) = 0\right\} \tag{2.7}$$

is invariant under the flow generated by the equation (EE). Indeed, calculating the time derivative we obtain that

$$\mathcal{H}_t|_{\mathcal{H}=0} = u_{tx} - \frac{1}{x}F'(u)u_t = u_{xxxx} + (\psi u_x)_x - \frac{1}{x}F'(u_{xxx} + \psi u_x). \tag{2.8}$$

Using the following algebraic rule of differentiation on S_0:

$$u_{xx} = \frac{F}{x^2}(F' - 1), \quad u_{xxx} = \frac{F}{x^3}[(FF')' - 3F' + 2], \tag{2.9}$$

and

$$u_{xxxx} = \frac{F}{x^4}[(F(FF')')' - 6(FF')' + 11F' - 6]$$

and substituting into (2.8) we conclude that the invariance condition

$$\mathcal{H}_t|_{\mathcal{H}=0} \equiv \frac{F}{x^2} I(u) + \frac{F}{x^4}\mathbf{D}(F) = 0, \tag{2.10}$$

is true provided that the coefficients of the linearly independent functions x^{-2} and x^{-4} vanish identically:

$$I(u) \equiv F\psi' - \psi = 0 \quad \text{and} \quad \mathbf{D}(u) = 0. \tag{2.11}$$

This dynamical system on the coefficients coincides with (2.2) and (2.1). Substituting $F = \psi'/\psi$ into (2.1) we obtain a fourth-order ODE for the coefficient ψ of (EE). (2.11) is the criterion of the invariance of the whole set (2.7) containing nonstationary solutions $u(x,t)$ which essentially depends on the time variable t.

In order to construct the exact solutions (2.3), we integrate in x equation $\mathcal{H}(u) = 0$ and conclude that the function

$$v(x,t) \equiv \int_1^u \frac{dz}{F(z)} = \log x + H(t) \tag{2.12}$$

satisfies

$$v_t \equiv \frac{u_t}{F(u)} = H', \quad v_x \equiv \frac{u_x}{F(u)} = \frac{1}{x}.$$

Substituting $u_t = Fv_t$, $u_x = F/x$ and u_{xxx} from (2.9) into (EE) and using that $\psi(u) = e^v$ by (2.2), (2.12), we arrive at the following equation for the function v:

$$v_t = \frac{1}{x^3}[(FF')' - 3F' + 2] + \frac{1}{x}e^v. \tag{2.13}$$

Substituting (2.12) into (2.13) we obtain

$$H'(t) = \frac{1}{x^3}\Phi(v) + e^H, \quad \text{with } \Phi(v) = (FF')'(u) - 3F'(u) + 2. \tag{2.14}$$

The function Φ satisfies

$$\Phi'_v \equiv \Phi'_u u'_v \equiv [(FF')'' - 3F'']F = 3\Phi, \tag{2.15}$$

which is exactly the ODE (2.1). Integrating (2.15) we obtain $\Phi(v) = de^{3v}$ where d is calculated from (2.14) by setting in the last formula $v = 0$, i.e. $u = 1$. Substituting $\Phi(v)$ into (2.14) we arrive at the ODE (2.4) for the function $H(t)$.

(ii) Let $u(\cdot,t) \in S_0$. Substituting the third derivative from (2.9) and $u_x = F/x$ into (EE) we obtain the ODE (2.6). This completes the proof of Theorem 2.1. $\square$

Remark: a parabolic representation. Using the differentiation rule on S_0 given by (2.9), one can derive that $u \in S_0$ also solves the second-order evolution equation

$$u_t = Au_{xx} + Bu_x, \tag{2.16}$$

with the coefficients

$$A = \frac{1}{x}F', \quad B = \frac{1}{x^2}[FF'' - 2(F'-1)] + \psi, \tag{2.17}$$

which is of parabolic type provided that $A \geq 0$. Then the comparison principle follows from the application of the MP for linear parabolic equations. Obviously, this parabolic representation of (EE) on S_0 is not unique.

3. Order properties. Let us discuss order preserving properties of the evolution groups restricted to the invariant sets. For a fixed function F, i.e. the coefficient ψ, the order (comparison) property of different solutions on S_0 in terms of the function $v(x,t)$ presented in (2.12) follows from the first-order ODE (2.4) which admits monotone trajectories only. On the other hand, consider two different functions $F_1(u)$ and $F_2(u)$ which solve (2.1) so that the corresponding exact solutions $u_i \in S_0(\psi_i)$, $i = 1, 2$, satisfy different equations (EE). Assume now that the functions F_i are chosen so that the constants d_i given by (2.5) satisfy $d_1 > d_2$. Then by standard comparison results from the ODE's theory we conclude that $v_1(\cdot,t) \leq v_2(\cdot,t)$ for $t > 0$ provided that this inequality holds at $t = 0$. One can translate that to the comparison of the solutions u_i under a suitable hypotheses on the functions F_i.

4. KdV as the limit equation. We first construct trivial invariant set for the KdV equation. Let us find a linear solution $F(u) = Cu$ of the ODE (2.1). Substituting we arrive at the quadratic equation $C^2 - 3C + 2 = 0$ whence two linear solutions

$$F_1(u) = u \quad \text{and} \quad F_2(u) = 2u. \tag{2.18}$$

The first one gives $\psi_1(u) = u$ and corresponds to the KdV equation

$$u_t = u_{xxx} + uu_x. \tag{2.19}$$

It then follows from (2.3) that $u(x,t) = e^{H(t)}x \equiv h(t)x$ on S_0. Substituting into (2.19) we get $h' = h^2$ whence the separate-variables solution $u(x,t) = -x/(t+c)$. It is clear that these solutions are ordered.

Similarly, $\psi_2(u) = \sqrt{u}$ and the equation takes the form

$$u_t = u_{xxx} + \sqrt{u}u_x, \tag{2.20}$$

with the invariant set consisting of the solutions $u(x,t) = x^2/(c-t)^2$.

Let us now show that the linear solutions (2.18) are the limit cases of more general solution of the ODE (2.1). By a standard local analysis of the ODE at $u = \infty$ it can be shown that it admits other solutions satisfying up to higher-order terms

$$F(u) = u + O(e^{-6u}), \quad \text{and} \quad F(u) = 2u + O(e^{-3u}) \quad \text{as } u \to \infty. \tag{2.21}$$

The first asymptotic expansion means that the KdV equation (2.19) plays a role of the limit state (as $u \to \infty$ that means rapidly growing solutions) of the family of KdV-type equations (EE) with ordered invariant sets under a special choice of the coefficients $\psi(u)$.

5. On the blow-up structure of the maximal solution. First of all, let us mention that the solutions of the ODE (2.4) are essentially local in time. Let $d > 0$. Then for any initial value $H(0)$ the function $H(t)$ blows up in finite time $T < \infty$ and as $t \to T^-$

$$H(t) = -\frac{1}{3} \log(3d(T-t))(1 + o(1)) \to \infty. \tag{2.22}$$

Second, it can be shown that there exists a solution of (2.1), which exhibits the *maximal* growth as $u \to \infty$, with the following expansion for $u \gg 1$:

$$F_m(u) = \frac{3}{2} u \log u (1 + o(1)). \tag{2.23}$$

Substituting both expansions (2.22) and (2.23) into (2.3), we arrive at the following asymptotic blow-up behaviour of the solutions on $S_0(\psi_m)$:

$$u(x,t) \sim \exp\{x^{3/2}/\sqrt{3d(T-t)}\}. \tag{2.24}$$

Observe that this blow-up pattern is an asymptotic solution of the equation

$$u_t = u_{xxx} + |\log u|^{2/3} u_x. \tag{2.25}$$

Remark: solutions on an invariant subspace. Let us present another example of exact solutions to (EE) with a logarithmic nonlinearity

$$u_t = u_{xxx} + \log u \, u_x. \tag{2.26}$$

Setting $u = e^v$ we obtain the equation

$$v_t = \mathbf{B}(v) \equiv v_{xxx} + 3v_x v_{xx} + (v_x)^3 + v v_x. \tag{2.27}$$

One can see that the linear span $W_2 = \mathrm{Span}\{1, x\}$ is invariant under the cubic operator $\mathbf{B}$: $\mathbf{B}(W_2) \subseteq W_2$. Therefore the evolution equation (2.27) on W_2 reduces to a two-dimensional dynamical system for the coefficients of the expansion $v(x,t) = C_0(t) + C_1(t)x \in W_2$, which has the form

$$C_0' = C_0 C_1 + C_1^3, \quad C_1' = C_1^2,$$

and is easily integrated. This gives the following exact solution of (2.26):

$$u(x,t) = \exp\{(1 + At - xt)t^{-2}\}, \tag{2.28}$$

where A is an arbitrary constant. This solution blows up as $t \to 0^-$. Several examples of linear subspaces and sets invariant under nonlinear operators of the third and higher orders can be found in [G1] and [GPS].

Remark: on an operator representation. It can be shown that the set of solutions (2.3), (2.4) can be treated as an one-dimensional subspace invariant under a certain nonlinear operator. The concept of linear subspaces admitted by nonlinear operators is a useful one in the study of the one-dimensional and multi-dimensional nonlinear evolution equations, see [G1], [G2], [G4], [GPS] and references therein. It was shown that given a nonlinear operator, the problem of the determination of linear invariant subspaces reduces to an extremely difficult nonlinear eigenvalue problem. On the other hand, for an ODE operators of finite order, given a linear subspace, the class of operators admitted it can be described in terms of Lie-Bäcklund symmetries of the linear ODE describing the subspace, [S]. In the present paper we thus give a new *representation* of nonlinear operators admitting one-dimensional invariant subspaces of a special structure.

3. Quasilinear KdV-type equations

In this section we show that the technique from Section 2 applies to quasilinear equations of the KdV-type.

1. We begin with the equation

$$u_t = \phi(u)u_{xxx} + \psi(u)u_x, \tag{3.1}$$

where both $\phi(u)$ and $\psi(u)$ are arbitrary smooth functions. Let us briefly discuss some new features in the application of the same technique.

(i) The coefficient ψ is still given by the expression (2.2) where F solves the following governing ODE

$$\mathbf{D}(F) + \frac{\phi'}{\phi} F[(FF')' - 3F' + 2] = 0, \tag{3.2}$$

with the operator $\mathbf{D}$ from (2.1). The exact solution is given by (2.3) where $H(t)$ solves the ODE (2.4) with the constant

$$d = \{\phi[(FF')' - 3F' + 2]\}(1).$$

(ii) (3.1) on S_0 is equivalent to the ODE

$$u_t = \frac{F}{x^3}\phi[(FF')' - 3F' + 2] + \frac{F}{x}\psi. \tag{3.3}$$

The proof goes along the same lines. In order to derive from (3.3) the ODE for the function $H(t)$ we again note that in the left-hand side $u_t = F(u)H'$, and on the right-hand side the function

$$\Phi(v) = \phi(u)[(FF')' - 3F' + 2](u) \tag{3.4}$$

is the exponential one, $\Phi(v) = de^{3v}$, which follows from the equation $\Phi'_v = 3\Phi$ or in view of (3.4) that

$$\{\phi[(FF')' - 3F' + 2]\}'F = 3\phi[(FF')' - 3F' + 2]. \tag{3.5}$$

This is the ODE (3.2).

Observe that the invariant set and the exact solutions exist for arbitrary nonlinear coefficient $\phi(u)$. On the other hand, this means that for an arbitrary fixed second coefficient $\psi(u)$, which determines the function F by (2.2), there exists the first coefficient $\phi(u)$ such that (3.1) admits the invariant set.

2. Let us finally show how the "scaling order" of the nonlinear operator affects the ODE structure of the invariant set S_0. If we add to the equation a quasilinear second-order term

$$u_t = \phi(u)u_{xxx} + \rho(u)u_{xx} + \psi(u)u_x, \tag{3.6}$$

then it follows from (2.9) that the corresponding ODE on S_0 takes the form (cf. (3.3))

$$u_t \equiv FH' = F\left[\frac{1}{x^3}\phi[(FF')' - 3F' + 2] + \frac{1}{x^2}\rho(F'-1) + \frac{1}{x}\psi\right]. \tag{3.7}$$

Therefore in order to get the exact solution (2.11) we need to introduce an extra equation in addition to (2.2) and (3.2). Namely, we deduce that the function $\tilde{\Phi}(v) = \rho(u)(F'(u)-1)$ must satisfy $\tilde{\Phi}'_v = 2\Phi$, i.e. we obtain the third equation

$$[\rho(F'-1)]'F = 2\rho(F'-1),$$

which is equivalent to

$$FF'' - 2F' + 2 + \frac{\rho'}{\rho}F(F'-1) = 0. \tag{3.8}$$

From (3.7) we then obtain the ODE

$$H' = de^{3H} + d_1 e^{2H} + e^H, \tag{3.9}$$

where

$$d_1 = \tilde{\Phi}(0) = [\rho(F'-1)](1).$$

3. Comparison with solutions invariant under a group of scalings. The number of distinct exponential terms on the right-hand side of the ODE (3.9) (which is (3.6) on S_0) thus depends on the total number of the distinct scaling orders of the operators in (3.6), which contains three operators of the scaling orders $3, 2$ and 1. For instance, in the model equation

$$u_t = \phi(u)u_{xxx} + \mu(u)(u_x)^3 + \nu(u)u_x u_{xx}, \tag{3.10}$$

the scaling orders of all the three operators is equal to 3. Then the ODE on S_0 takes the simplest one-term form

$$H' = de^{3H},$$

whence the solution $H(t) = -\log(3d(-t))/3$. Substituting this into the exact solution from (2.3) we obtain the self-similar solution of the form

$$u(x,t) = \theta(\xi), \quad \xi = \frac{x}{t^{1/3}}, \tag{3.11}$$

which is invariant under a Lie group of scaling transformations with the infinitesimal generator

$$X = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}.$$

Substituting (3.11) into (3.1) one obtains the ODE for the function θ

$$\phi(\theta)\theta''' + \mu(\theta)(\theta')^3 + \nu(\theta)\theta'\theta'' + \frac{1}{3}\theta'\xi = 0.$$

Thus in the case of a single scaling order operator, the ordered invariant set S_0 is generated by the invariant of a Lie group of scalings.

The idea of the first order differential constraint (1.2) applies to more general evolution equations with rather arbitrary algebraic scaling structure. The corresponding results will appear in a forthcoming paper, [G3].

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