MINIMAL CLASS OF BLOW-UP APPROXIMATE SELF-SIMILAR SOLUTIONS OF THE HEAT EQUATION

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To Professor A.A. Samarskii on his 80th birthday

Abstract

We study the asymptotic properties of blow-up solutions of the heat equation $u_t = u_{xx}, \quad x > 0, \; t \in (0,1)$, with given blowing up function on the boundary $u(0,t) = \psi(t) > 0$, $t \in (0,1)$, where $\psi(t) \to \infty$ as $t \to 1^-$, and a bounded initial data $u(x,0)$. We classify the asymptotic properties of the solutions near the blow-up time, $t \to 1^-$, in terms of the boundary data $\psi(t)$ and describe a class of singularities (boundary layers) of the heat equation. More precisely, we describe a minimal class I_1^- corresponding to the data $\psi(t)$ with a "slow" growth as $t \to 1^-$. It is proved that the asymptotic properties of all the solutions from I_1^- are given by a nonlinear transformation of a self-similar solution of the heat equation $u_*(x,t) = -\ln(1-t) + g(\xi)$, $\xi = x/(1-t)^{1/2}$, where g solves the ODE $g'' - g'\xi/2 - 1 = 0, \; \xi > 0$, $g(0) = 0$, and the boundary data take the form $\psi_*(t) = -\ln(1-t)$. We establish that this solution plays a role of the approximate self-similar solution for an infinite-dimensional set of other boundary data $\psi(t)$, e.g. $\psi(t) = \ln|\ln(1-t)|$ or $\psi(t) = \exp\{|\ln(1-t)|^{1/2}\}$. Exact rates of convergence and estimates on the corresponding final-time profile $u(x,1^-)$ are derived.

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1. Introduction

In this paper we consider some aspects of the asymptotic analysis of partial differential equations of parabolic type. An essential feature of the results is that we classify the asymptotic behaviour of the solutions to the heat equation in terms of the boundary conditions. We describe a minimal class of asymptotics (singular boundary layers) which was not studied in the literature.

We establish that the evolution singularity of the heat equation generated by a special logarithmic blowing up boundary function is structurally stable upon perturbations, and moreover in the rescaled variables can be viewed as a nonlinear perturbation of a linear flow governed by a linear ordinary differential operator. This approach exhibits some features of a general classification of the omega limits for a class of infinite-dimensional dynamical systems with variable coefficients.

1.1. On the plane of initial-boundary value problems. The asymptotic properties of different type of nonnegative solutions to the nonlinear heat-conduction equation (or the filtration equation) in one dimension:

$$u_t = (k(u)u_x)_x, \tag{1.1}$$

where $k(u) > 0$ for $u > 0$ is a given smooth function, have been studied in recent years in a number of well-known papers and monographs devoted to wide classes of uniformly parabolic or degenerate parabolic equations. Several important and delicate results on the asymptotic properties of solutions of equations (1.1), especially of those with extra symmetry properties corresponding to $k(u) = u^\sigma$ or $k(u) = e^u$, play an important role in understanding of the asymptotic behaviour depending on the nonlinear coefficient k as well as on the initial-boundary data.

Nevertheless, if we consider a wide (sufficiently "dense") infinite-dimensional subset of equations (1.1), then the asymptotic properties are far away from being entirely well-understood. Namely, this important problem of a description of different asymptotic classes of solutions to (1.1) is still open for a wide class of equations. Especially this is related to the question on the completeness of the results on the classification of the asymptotic behaviour for (1.1).

Let us first discuss the question on the "coordinates" in which the asymptotics of each individual solution can be considered. In order to specify the asymptotic classes, it is sufficient to consider a unique solution of the initial-boundary value problem (IBVP) for (1.1) in the half-space $\mathbf{R}_+ = (0, \infty)$ with the boundary function

$$u(0,t) = \psi(t) \to \infty \tag{1.2}$$

having a prescribed asymptotic behaviour. Then due to the standard parabolic theory, the initial data

$$u(x,0) = u_0(x) \ge 0, \quad x > 0, \tag{1.3}$$

which are assumed to be uniformly bounded, do not essentially affect the asymptotic behaviour as $\psi(t) \to \infty$. Therefore, in the first approximation, the omega-limits of specially rescaled orbits are expected to depend upon two functional parameters, k and ψ.

Given two functions, $k(u)$ and $\psi(t)$, we thus fix a family of solutions of the IBVP (1.1) - (1.3) and would like to study the asymptotic behaviour of this family. We thus arrive at a classification of the solutions on the infinite dimensional "plane of IBVPs" (1.1)-(1.3), $P_{IBVP} = \{\psi, k\}$. The term a "plane of IBVPs" was introduced by S.I. Pohozaev in 1981 when one of the authors gave a talk on the approximate self-similar solutions for quasilinear heat-conduction equation at the seminar on nonlinear equations of the Department of Mathematics of the Moscow Energy Institute. It turned out that under certain natural hypotheses on the functions $\{k\}$ and $\{\psi\}$ (such as smoothness, monotonicity and existence of some special limits, the assumptions which are always true for typical well-known coefficients), the total number of distinct asymptotic classes on the plane P_{IBVP} is expected to be *essentially finite.*

Moreover, the results of the general classification show that in the case of global solutions, when $\psi(t)$ is well-defined for all $t > 0$ and monotonically

$$\psi(t) \to \infty \quad \text{as } t \to \infty, \tag{1.4}$$

there exist about 10 distinct domains on P_{IBVP}, each one exhibits a specific type of asymptotic structure. See the last Section 12 in the paper [GS] which contains a summary of such a partial classification and a graphic description of the plane P_{IBVP}. Avoiding details, we just mention that in general each domain is supposed to contain a *generating (attractive) subset* (which can be a special "point" $P_0 = \{\psi_0, k_0\}$ or a special "line" γ_0) which via a nonlinear transformation determines the structure of asymptotics for all the points from the domain. These generating subsets are given either by exact self-similar (group-invariant) solutions of special equations (like the porous medium equation with $k(u) = u^\sigma$, or the equation with the exponential nonlinearity, $k(u) = e^u$) or by *approximate self-similar solutions* which are not group invariant, do not satisfy the equation but describe correctly the asymptotic behaviour. Several such examples can be found in [GS], in Chapter 6 of [SGK], see also references therein.

Nevertheless, a rigorous analysis of some of the asymptotic classes is very difficult and the asymptotic problems in some subsets of P_{IBVP} are open, see Figure on p. 134 in [GS]. The analysis becomes more delicate in the case of a singular boundary data given by a monotone *blow-up* function:

$$\psi(t) \to \infty \quad \text{as } t \to 1^-. \tag{1.5}$$

1.2. Classification on the axis A^+ for the heat equation. Consider now the heat equation

$$u_t = u_{xx}, \quad x > 0. \tag{HE}$$

In the case of global in time functions (1.4) such a classification of the axis $A^+ = \{k = 1\}$ on P^+_{IBVP} is already known [G]: there exists three classes (open "intervals" on the axis) I_1^+, I_2^+ and I_3^+ which are classified in terms of the limit which is supposed to exist:

$$a_\psi^+ = \lim_{t \to \infty} \left(\frac{\psi(t)}{\psi'(t)} \right)'. \tag{1.6}$$

The approximate self-similar solution takes the following standard form

$$u_a(x,t) = \psi(t)\theta_a(\xi), \quad \xi = x/\varphi(t), \tag{1.7}$$

where the propagation rate $\varphi(t)$ depends on the boundary function ψ from the corresponding classes and θ_a is a unique solutions of a linear ODE. Namely:

(1) $\psi \in I_1^+$ if $a_\psi = +\infty$, and then $\varphi(t) = t^{1/2}$.

(2) $\psi \in I_2^+$ if $a_\psi = 1/m \in (0,\infty)$, and then again $\varphi(t) = t^{1/2}$.

(3) $\psi \in I_3^+$ if $a_\psi = 0$, and then $\varphi(t) = [\psi(t)/\psi'(t)]^{1/2}$.

The corresponding stability theorems with an exact rate of convergence are presented in [G]. According to this classification, the classes of boundary functions $\psi(t)$ are naturally ordered with respect to the rate of growth as $t \to \infty$. For instance,

$$\psi(t) = \ln t \in I_1^+, \quad \psi(t) = t^2 \ln t \in I_2^+, \quad \psi(t) = \exp\{e^t\} \in I_3^+. \tag{1.8}$$

For all such functions the heat equation admits no nontrivial symmetries so that no explicit invariant solutions are available to describe the asymptotic behaviour. This classification implies certain completeness so that under the assumption on the existence of the limit a_ψ we have that in a certain natural sense $A^+ = \bar{I}_1^+ \cup \bar{I}_2^+ \cup \bar{I}_3^+$. Observe that the last class I_3^+ of supergrowing functions admits further decomposition into two subclasses $I_{31}^+ \cup I_{32}^+$ generated by self-similar solutions of a first-order Hamilton-Jacobi equations [G, p. 125], see also Chapter 6 in [SGK].

1.3. On the axis A^-. Similar to A^+, the axis A^- is supposed to be composed from similar classes I_1^-, I_2^- and I_3^- ordered according to the growth rate of $\psi(t)$ as $t \to 1^-$, so that I_1^- corresponds to slow growth functions like $\psi(t) = \ln|\ln(1-t)| \in I_1^-$, while the last class I_3^- contains the functions with any super growth, say, $\exp\{e^{(1-t)^{-1}}\} \in I_3^-$. The intermediate interval I_2^- is composed from almost power functions like $\psi(t) = (1-t)^n|\ln(1-t)|^\lambda$, $n < 0$.

It turns out that the results of [G] applies directly to the singular blow-up case (1.5) for the classes I_2^- and I_3^- which can be defined by means of the limit

$$a_\psi^- = \lim_{t \to 1^-} \left(\frac{\psi(t)}{\psi'(t)} \right)'. \tag{1.9}$$

Namely one can conclude exactly as in Theorem 2 and 3 in [G] that $\psi \in I_2^-$ if $a_\psi^- = 1/n \in (-\infty, 0)$ and $\psi \in I_3^-$ if $a_\psi^- = 0$. The corresponding approximate self-similar solutions are similar to those in the global case. Moreover, the class I_3^- is also decomposed into two subclasses, $I_3^- = I_{31}^- \cup I_{32}^-$. In particular, given the boundary function

$$\psi(t) = e^{(1-t)^n} \in I_{31}^-, \quad n < 0, \tag{1.10}$$

there exists a stable approximate self-similar solution satisfying a Hamilton-Jacobi equation, see details in [SGK, p. 170 and p. 375]. The structure of I_{32}^- can be also predicted, and its classification is based on the same generating solution with $\psi(t) = e^{e^t}$ as for I_{32}^+ in the global case, [G].

Nevertheless, the classification of the global axis A^+ and the singular one A^- are not completely similar. Namely, Theorem 1 in [G] does not apply to the class I_1^- of functions with a minimal growth as $t \to 1^-$. The reason is easy: the corresponding approximate self-similar solution does not exist, and a new approach is necessary.

1.3. New results: the minimal class I_1^-. In the present paper we consider the minimal class I_1^- for the heat equation which is the only one remaining open for the heat equation. We prove that the asymptotic analysis in I_1^- is entirely different from the similar one I_1^+ and hence the generating points ψ_1^- and ψ_1^+ are given by essentially distinct self-similar solutions. Observe that in the global case, ψ_1^+ is given by the elementary scaling-invariant solution

$$u_1^+(x,t) = g(\xi), \quad \xi = x/t^{1/2}, \tag{1.11}$$

see Table 1 in [G]. It is clear that a similar blow-up solution with the rescaled variable $\xi = x/(1-t)^{1/2}$ makes no sense and such a profile $g(\xi)$ does not exist.

We thus consider the IBVP for (HE) with the singular strictly monotone boundary data (1.2) satisfying (1.5). Without loss of generality, we take the trivial initial function:

$$u(x,0) \equiv 0. \tag{1.12}$$

We prove that the generating point in I_1^- is formed by a self-similar solution of the form:

$$u_1^-(t,x) = -\ln(1-t) + g(\xi), \quad \xi = x/(1-t)^{1/2}, \tag{1.13}$$

where the profile g is uniquely determined from a linear ODE. It essentially differs from that (1.11) in I_1^+.

Let us now describe the functional interval I_1^-. It is convenient to prescribe the boundary functions from the class using the representation

$$\psi(t) = F(-\ln(1-t)), \tag{1.14}$$

where F is a sufficiently smooth increasing function, $F'(s) > 0$ for $s \geq 0$ and $F(s) \to \infty$ as $s \to \infty$. We prove that $\psi \in I_1^-$ provided that

$$G(s) \equiv \frac{F''(s)}{F'(s)} \to 0 \quad \text{as} \quad s \to \infty. \tag{1.15}$$

For instance, $F(s) = \ln s$, $F(s) = s^\alpha$, $\alpha > 0$, $F(s) = \exp(s^\beta)$ with $\beta \in (0,1)$, etc. One can see that (1.15) is valid if

$$\left(\frac{\psi'(t)}{\psi''(t)}\right)' \to -1, \quad t \to 1^-. \tag{1.15'}$$

We thus define the minimal class I_1^- to be studied in this paper as follows

$$I_1^- = \{\psi : \ \ (1.15') \text{ holds}\}. \tag{1.16}$$

This is a more convenient definition of the functional interval I_1^-, where the self-similar solution (1.13) can be treated as an "attractor". We give a detailed description of the asymptotic behaviour of the solutions and derive an optimal rate of convergence to the transformed rescaled self-similar solution (1.13). The new condition (1.15') "almost everywhere" coincides with the known condition $a_\psi^- = -\infty$, and we prove that in Appendix.

In fact, we show that the minimal class I_1^- of singularities (blow-up boundary layers) for the heat equation is governed by the stable manifold of the following linear operator in $L^2_{\rho_0}(\mathbf{R}_+)$:

$$\mathbf{A} = \frac{1}{\rho}\frac{d}{d\xi}\left(\rho\frac{d}{d\xi}\right), \quad \rho(\xi) = e^{-\xi^2/4}.$$

This means that for any $\psi \in I_1^-$, in suitable rescaled variables, the boundary singularity is described by an asymptotically small perturbation of the stable hyperbolic equilibria of the linear equation $Y_s = \mathbf{A}Y$.

The convergence result makes it possible to derive one of the most important for applications asymptotic characteristics of the solutions, the so-called *semi-width,* which is the effective depth of penetration $x_{\rm eff}(t)$ of the thermal wave. Under the above monotonicity assumption, for any $t \in (0,1)$ it is uniquely defined from the equation

$$u(x_{\rm eff}(t), t) = \frac{1}{2}\psi(t). \tag{1.17}$$

We also derive the *finite-time profile* $u(x, 1^-)$ which is formed by the given blow-up regime on the boundary and plays an important role in the processes of heat localization effect, see Chapter 3 in [SGK].

Observe that the solution of the IBVP for the heat equation admits an integral representation as the heat potential of a double layer

$$u(t,x) = \frac{x}{2\sqrt{\pi}}\int_0^t \exp\left\{-\frac{x^2}{4(t-\tau)}\right\}(t-\tau)^{-3/2}\psi(\tau)d\tau.$$

Nevertheless, this singular integral has a complicated form, and a necessary derivation of the asymptotic behaviour is not that straightforward. Moreover, in any case we avoid the potential as it plays no role for the nonlinear equation (1.1) which can be studied by our approach.

1.4. On the domain S_{11}^-. In a forthcoming paper, [CG], we show that our asymptotic techniques apply to the quasilinear equation (1.1), so that (1.13) can be extended as the generating solution from the interval I_1^- to a certain domain $S_{11}^- \subset P_{IBVP}^-$. This means a structural stability of the particular self-similar solution (1.13) with respect to perturbations of the boundary function and also upon nonlinear perturbations of the heat equation which leave the point $\{\psi, k\}$ in S_{11}^-. The main such hypotheses are as follows: (i) (1.15') for the boundary function $\psi(t)$ and (ii) a joint condition on both coefficients $\psi(t)$ and $k(u)$:

$$\frac{k'(F(s))F'(s)}{k(F(s))} \to 0 \quad \text{as } s \to \infty, \tag{1.18}$$

where F is as in (1.14). In a certain formal sense, the limit (1.18) defines monotone "curves" on the plane P^-_{IBVP} as the border lines of the domain on which the function $(k'(F)F'/k(F))(s)$ "first time" ceases to vanish as $s \to \infty$. We emphasize a rather delicate non-rectangular structure of the minimal domain $S^-_{11} \subset P^-_{IBVP}$.

2. Approximate self-similar solution, main results.

We now study in detail the spatio-temporal structure of solutions of the problem (HE), (1.2), (1.5), (1.12) and the corresponding approximate self-similar solution (ap. s-s.s.). As we mentioned before, it will be done on the basis of the exact self-similar solution (1.13) of (HE), with the boundary condition $u(0,t) = -\ln(1-t)$, i.e., condition (1.14) with the identity transformation $F(s) \equiv s$.

In fact, we carry out the construction of ap. s-s.s.'s in the form

$$u_a(x,t) = F(-\ln(1-t) + g(\xi)). \tag{2.1}$$

As usual and according to the structure (2.1), we define the rescaled solution $\theta(\xi,\tau)$, the similarity representation of the solution of the problem, by

$$u(x,t) = F(-\ln(1-t) + \theta(\xi,\tau)) \tag{2.2}$$

or equivalently,

$$\theta(\xi,\tau) = F^{-1}(u(\xi(1-t)^{1/2},t)) + \ln(1-t). \tag{2.3}$$

Here the new time variable τ is defined by $\tau = -\ln(1-t)$ and satisfies $\tau \to \infty$ as $t \to 1^-$. By substituting (2.2) into (HE), we arrive at the rescaled equation

$$\theta_\tau = \theta_{\xi\xi} - \frac{1}{2}\xi\theta_\xi - 1 + \frac{F''(\tau+\theta)}{F'(\tau+\theta)}\,(\theta_\xi)^2, \tag{2.4}$$

and the corresponding boundary condition, taking into account (1.14), becomes $\theta(0,\tau) \equiv 0$ for every $\tau \geq 0$. Under the above assumptions, equation (2.4) for the rescaled function θ can be viewed as a small perturbation (as $\tau \to \infty$) of the linear autonomous equation

$$\theta_\tau = \theta_{\xi\xi} - \frac{1}{2}\xi\theta_\xi - 1. \tag{2.5}$$

In fact, it will be shown that under the given assumptions, the self-similar representation $\theta(\xi,\tau)$ converges as $\tau \to \infty$ to a unique decreasing stationary solution $g(\xi)$ of (2.5):

$$g'' - \frac{1}{2}\xi g' - 1 = 0, \quad \xi > 0; \quad g(0) = 0, \tag{2.6}$$

exhibiting a minimal behaviour as $\xi \to \infty$. More precisely, we establish the following result.

Theorem A. *Let $F \in C^3(\mathbf{R}_+)$ be a monotonically increasing function. Assume that (1.15) holds and that*

$$\frac{G'(s)}{G(s)} \to 0 \quad \text{as} \quad s \to \infty. \tag{2.7}$$

Then uniformly on compact subsets in ξ

$$|\theta(\xi, \tau) - g(\xi)| = O(|G(\tau)|) \to 0, \quad \tau \to \infty.$$

This convergence result makes it possible to solve equation (1.17) for the semi-width as $t \to 1^-$. It turns out that for any IBVP from S_{11}^- the semi-width tends to zero as $t \to 1^-$ which corresponds to single-point blow-up.

We note that if the function $G(s)$ does not satisfy condition (2.7) but this limit exists, then $F(s)$ is a very small perturbation of a linear function. Section 5 is devoted to this especial "almost linear" case. In Section 6 we also discuss some convergence results when the limit in (2.7) does not exist, which corresponds to the oscilatory case.

The second result deals with the blow-up behaviour of the solution in the original independent variables (x, t). We prove the following

Theorem B. *Under the hypotheses of Theorem A, there exists a constant C_0, such that the final-time profile near the origin has the form*

$$u(x, 1^-) = F(-2\ln x + C_0 + o(1)) \quad \textit{as} \quad x \to 0.$$

We now illustrate the main results.

Example 1. Consider the heat equation with the boundary function

$$u(0, t) = \ln^\lambda |\ln(1-t)|, \quad \lambda > 0.$$

It is slower than the logarithmic self-similar one in (1.13). Then $F(s) = \ln^\lambda s$, (2.7) holds and therefore by Theorem A as $t \to 1^-$

$$u(x, t) \approx \ln^\lambda(-\ln(1-t) + g(x/(1-t)^{1/2})),$$

with convergence on compact subsets in ξ. By Theorem B, we estimate the final-time profile as follows:

$$u(x, 1^-) = \ln^\lambda |2\ln x| + \frac{\lambda C_0}{2} \frac{\ln^{\lambda-1}|2\ln x|}{|\ln x|}(1 + o(1)), \quad x \to 0.$$

Example 2. Consider now the case of the boundary function with a faster growth than that in (1.13):

$$u(0, t) = e^{|\ln(1-t)|^\lambda}, \quad \lambda \in (0, 1).$$

Then $F(s) = e^{s^\lambda}$, (2.7) is true (and $a_\psi^- = -\infty$) if and only if $\lambda < 1$. By Theorem A we derive the following structure of the singular behaviour in the blow-up boundary layer as $t \to 1^-$:

$$u(x,t) \approx e^{(-\ln(1-t)+g(x/(1-t)^{1/2}))^\lambda}.$$

Finally, Theorem B gives the final-time profile

$$u(x,1^-) = e^{(-2\ln x + C_0 + o(1))^\lambda}, \quad x \to 0.$$

Observe that if $\lambda = 1$ then the boundary function is a power-like one, $u(0,t) = (1-t)^{-1}$, and therefore belongs to the second interval I_2^- of the axis A^- (cf. [G]), where the singular structure is described by a distinct self-similar solution of the form

$$u_2^-(x,t) = (1-t)^{-1}g(\xi), \quad \xi = x/(1-t)^{1/2}.$$

This self-similar solution is known to be stable upon perturbations of both the boundary functions and the equation, see Chapter 6 in [SGK] and [GS].

We begin with some properties of the ap. s.-s.s.

2.1. The rescaled stationary solution. Integrating (2.6) once, we obtain that the minimal profile $g(\xi)$ satisfies

$$g'(\xi) = e^{\xi^2/4}\left[\int_0^\xi e^{-s^2/4}ds + g'(0)\right].$$

Integrating again, we have that

$$g(\xi) = \int_0^\xi e^{s^2/4}\left[\int_0^s e^{-r^2/4}dr + g'(0)\right]ds.$$

It follows that the value $g'(0) = -\sqrt{\pi}$ is the only one which makes $g'(\xi)$ a negative function going to 0 as $\xi \to \infty$. Moreover, the asymptotic expansion of $g'(\xi)$ is given by

$$g'(\xi) = -\frac{2}{\xi} + \frac{4}{\xi^3} + o\left(\frac{1}{\xi^3}\right) \quad \text{as } \xi \to \infty \tag{2.8}$$

so that the minimal solution satisfies

$$g(\xi) = -2\ln\xi + C_0 + O\left(\frac{1}{\xi^2}\right) \quad \text{as } \xi \to \infty. \tag{2.9}$$

Finally, the second term in (2.9) is obtained by considering $\varepsilon(\xi) = g(\xi) + 2\ln(1+\xi) \to C_0$ as $\xi \to \infty$. After substituting and integrating twice as above, we obtain

$$C_0 = \int_0^\infty \left[2/(1+\xi) - e^{\xi^2/4}\int_\xi^\infty e^{-s^2/4}ds\right]d\xi.$$

We also use the following estimates of $g'(\xi)$, which can be easily deduced from the previous analysis and from the properties of the function $g(\xi)$.

Lemma 2.1. *There exists a positive constant m_0 such that for every $\xi \in \mathbf{R}$,*

$$(g'(\xi))^2 \le m_0 e^{g(\xi)}, \quad (g'(\xi))^2 \le m_0/(\xi+1)^2. \tag{2.10}$$

2.2. Extension of the function $F(s)$. Given $F(s)$ for $s \ge 0$, under the hypothesis of Theorem A, we define its extension for all $s \in \mathbf{R}$, namely an increasing sufficiently smooth function $\hat{F}(s)$ satisfying $\hat{F}(s) = F(s)$ for $s \ge 0$. An example is given by the formula

$$\hat{F}(s) = A(e^{as} - 1) + F(0) \quad \text{for} \quad s \le 0,$$

where $a = F''(0)/F'(0)$ and $A = F''(0)/a^2$ if $F''(0) \ne 0$. If $F''(0) = 0$, we take the linear function $\hat{F}(s) = F(0) + F'(0)s$. In all the cases, the associated extended function $\hat{G}(s)$ is given by $\hat{G}(s) = G(0)$, for every $s \le 0$. We point out that the only essential fact concerning $F(s)$, involved in the study of the blow-up problem, is its behaviour for s large enough. This fact will be used in order to obtain estimates of the solution and a similar extension of the function $F(s)$ will be done for $s \le s_0$ with $s_0 \gg 1$.

2.3. Perturbation analysis. We now study the behaviour of the perturbation term $|G(\tau+\theta)|(\theta_\xi)^2$ evaluated on certain special functions related to the profile $g(\xi)$. The result is obtained for arbitrary constants c_0, c_1 and D and the only restriction on C is that $C|G(\tau)| \le 1$. All of these constants will be defined in an appropriate way in the proof of Theorem A.

Lemma 2.2. *Given a fixed $\xi_0 > 0$, there exists a $\tau^* \gg 1$ such that for every $\tau \ge \tau^*$*

$$|G(\tau + g(\xi) \pm C|G(\tau)|[-g(\xi) + D])|\,[g'(\xi)]^2 \le m_0 e|G(\tau)| \tag{2.11}$$

if $\xi \le \xi_0$ and

$$|G(\tau + g(\xi) \pm C|G(\tau)|[c_0\xi + c_1 + D])|\,[g'(\xi)]^2 \le m_0 e|G(\tau)| + K m_0 c_0^2 (CG(\tau))^2 \tag{2.12}$$

if $\xi \ge \xi_0$, where K is the bound of $|G|$ and m_0 is defined in (2.10).

Proof. By hypothesis (2.7), there exists a $s_0 \gg 1$ such that for any $s \ge s_0$,

$$\frac{|G'(s)|}{|G(s)|} \le \varepsilon = \min\{1/4(-g(\xi_0) + D)\,, 1/4(1 + c_1 + D)\}, \tag{2.13}$$

which implies by integrating and using (2.10) that (2.11) follows for all $\tau > s_0 + A$ with $A = -2g(\xi_0) + D$. In order to get the second inequality we proceed as follows.

First of all, by a similar argument, we deduce that for every value of τ and ξ satisfying $c_0 C|G(\tau)|\xi \le 1$ and with $\tau + g(\xi) \pm C|G(\tau)|(c_0\xi + c_1 + D) \ge s_0$, there holds

$$|G(\tau + g(\xi) \pm C|G(\tau)|[c_0\xi + c_1 + D])|[g'(\xi)]^2 \le m_0 e|G(\tau)|.$$

If $c_0 C|G(\tau)|\xi \le 1$ but $\tau + g(\xi) \pm C|G(\tau)|(c_0\xi + c_1 + D) \le s_0$, then, taking into account the first inequality in (2.10), we obtain that

$$|G(\tau + g(\xi) \pm C|G(\tau)|[c_0\xi + c_1 + D])|\,[g'(\xi)]^2 \le K m_0 e^{-\tau + s_0 + 1 + c_1 + D} \le m_0 e|G(\tau)|,$$

if $\tau \ge \tau^*$ with $\tau^* > s_0 + A$ large enough by the asymptotic properties of $G(s)$. Finally, if $c_0 C|G(\tau)|\xi \ge 1$, we use the second estimate in (2.10) and the bound of $|G(s)|$ to get

$$|G(\tau + g(\xi) \pm C|G(\tau)|[c_0\xi + c_1 + D])|\,[g'(\xi)]^2 \le K m_0 c_0^2 (CG(\tau))^2,$$

and (2.12) follows. $\square$

3. Estimates and convergence

1. First estimates. In this section we construct bounds on the rescaled orbit satisfying equation (2.4). The first upper and lower estimates do not allow to derive directly the convergence of the function $\theta(\xi,\tau)$ to the profile $g(\xi)$, but they will be necessary in constructing new sub and supersolutions with the exact rate of convergence.

Lemma 3.1. *There exists a positive constant D, such that for every $\xi \geq 0$ and $\tau \geq 0$,*

$$2g(\xi) - D \leq \theta(\xi,\tau) \leq D. \tag{3.1}$$

Proof. Assume for the moment that $|G(\tau)| \leq K \leq 1/4\pi$. Then, the result is a consequence of the Maximum Principle. In fact, it is clear that the function $\overline{\theta}(\xi,\tau) \equiv \bar{D}$ is a supersolution and the initial and boundary data are ordered if $\bar{D} \gg 1$. In the same way, the function $\underline{\theta}(\xi,\tau) = 2g(\xi) - \underline{D}$ is a subsolution since by a straightforward computation

$$\underline{\theta}_\tau - [\underline{\theta}_{\xi\xi} - \frac{1}{2}\xi\underline{\theta}_\xi - 1 - G(\tau+\underline{\theta})(\underline{\theta}_\xi)^2]$$

$$= -1 - G(\tau+\underline{\theta})4(g')^2 \leq -1 - G(\tau+\underline{\theta})4\pi \leq 0$$

for every $\tau > 0$ and the initial and boundary data are ordered if $\underline{D}$ is large enough.

If $K > 1/4\pi$ we need to perform an "extra extension" of the function $F(s)$ before the rescaling. Hypothesis (1.15) implies that $|G(s)| < 1/4\pi$ for every $s \geq s_0$ if we choose s_0 large enough. We turn to the original problem (HE), (1.14) for $t \geq t_0 \approx 1^-$ such that $-\ln(1-t_0) > s_0$ with initial data $u(x,t_0)$. Then by (1.14) and the choice of t_0, the solution $u(x,t)$ for $t \geq t_0$ depends only on the behaviour of the boundary function $F(s)$ on $[s_0,\infty)$. Therefore we can modify this function for $s \leq s_0$ without changing the solution $u(x,t)$ for any $t \geq t_0$. Thus, we consider an extension of $F(s)$, namely $\hat{F}(s)$, similar to the one constructed in Subsection 2.2 for $s \leq s_0$ and define the rescaled function by

$$u(x,t) = \hat{F}(-\ln(1-t) + \theta(\xi,\tau))$$

for every $t \geq t_0$. Now the function $|\hat{G}(s)| \leq 1/4\pi$ for every $s \in \mathbf{R}$ and we can construct the same sub and supersolution as before (up to a fixed constant). $\square$

2. Proof of Theorem A. We now deal with the construction of special sub and supersolution leading to convergence. We perform the proof for $G(s) > 0$ for large values of s. The proof for $G(s)$ negative can be done by using similar arguments. Consider $H(\xi,\tau) = \theta(\xi,\tau) - g(\xi)$. It follows from (2.4), (2.5) that this function satisfies the equation

$$H_\tau = \mathbf{A}(H) \equiv H_{\xi\xi} - \frac{1}{2}\xi H_\xi + G(\tau + g(\xi) + H)(g' + H_\xi)^2,$$

and $H(0,\tau) \equiv 0$. Lemma 3.1 implies that for every $\tau \geq 0$,

$$g(\xi) - D \leq H(\xi,\tau) \leq -g(\xi) + D.$$

In order to construct appropriate sub and supersolutions, we take fixed values ξ_0 and τ_0 in the following way. On one hand, consider $\xi_0 \gg 1$ such that $c_0 = -g'(\xi_0)$ satisfies $c_0^2 < 1/8K(1+m_0 e)$ and $c_0\xi_0 > 1$. This choice is possible by the properties of $g(\xi)$. In particular, the asymptotic expansions (2.8) and (2.9) are used. Define also $c_1 = -g(\xi_0) - c_0\xi_0$.

On the other hand, consider τ_0 large enough such that $1/G(\tau_0) > 8m_0 e$ and such that the estimates (2.11)-(2.13) in Lemma 2.2 hold for $\tau \geq \tau_0$ with the prescribed choice of the constants. We can assume that $CG(\tau) \leq 1$ for every $\tau \geq \tau_0$ where $C = 1/G(\tau_0)$ (by the properties of $G(s)$ there exists a $\bar{\tau}_0 > \tau_0$ satisfying this property). We begin with the construction of the supersolution. Define

$$\bar{H}(\xi,\tau) = \begin{cases} CG(\tau)(-g(\xi)+D), & \xi < \xi_0, \\ CG(\tau)(c_0\xi + c_1 + D), & \xi \geq \xi_0. \end{cases}$$

Under the stated conditions and using the perturbation analysis in Lemma 2.2, we have in the region $\xi \leq \xi_0$, $\tau \geq \tau_0$,

$$\begin{aligned} \bar{H}_\tau - \mathbf{A}(\bar{H}) &\geq CG'(\tau)[-g(\xi_0)+D] + CG(\tau) \\ &- G(\tau + g(\xi) + \bar{H}(\xi,\tau))[1 - CG(\tau)]^2 (g')^2 \geq \frac{C}{2} G(\tau) - m_0 G(\tau) e > 0. \end{aligned}$$

On the other hand, in the region $\xi \geq \xi_0$,

$$\begin{aligned} \bar{H}_\tau - \mathbf{A}(\bar{H}) &\geq CG'(\tau)(c_0\xi + c_1 + D) + \frac{C}{2} G(\tau) c_0 \xi \\ &- G(\tau + g(\xi) + \bar{H}(\xi,\tau))[g' + c_0 CG(\tau)]^2 \geq \frac{C}{4} G(\tau) \\ &- m_0 e G(\tau) - K(m_0 e + 1) c_0^2 (CG(\tau))^2 \geq \left(\frac{C}{8} - m_0 e\right) G(\tau) \geq 0. \end{aligned}$$

Here we have used estimates (2.10) - (2.13) and therefore the term $(g')^2$ is involved in this part. Finally, using (3.1), the properties of g and the construction of $\bar{H}$, it follows that $\bar{H}(\tau_0,\xi) \geq H(\tau_0,\xi)$, and therefore, for every $\tau \geq \tau_0$,

$$\bar{H}(\tau,\xi) \geq H(\tau,\xi).$$

We proceed in a similar way in the construction of the subsolution. Define

$$\underline{H}(\xi,\tau) = \begin{cases} -CG(\tau)(-g(\xi)+D), & \xi < \xi_0, \\ -CG(\tau)(c_0\xi + c_1 + D), & \xi \geq \xi_0. \end{cases}$$

Then, we have for every $\tau \geq \tau_0$,

$$\underline{H}_\tau - \mathbf{A}(\underline{H}) \leq -CG'(\tau)[-g(\xi_0)+D] - CG(\tau) < -\frac{C}{2} G(\tau) < 0.$$

In the same way, we obtain for $\xi \geq \xi_0$ and $\tau \geq \tau_0$,

$$\underline{H}_\tau - \mathbf{A}(\underline{H}) \leq -CG'(\tau)(c_0\xi + c_1 + D) - \frac{C}{2} G(\tau) c_0 \xi \leq 0,$$

under the stated assumptions. Finally, by the choice of C, we conclude that the functions are ordered at $\tau=\tau_0$ and therefore for every $\tau>\tau_0$ by comparison. Then we have obtained that

$$-CG(\tau)(h(\xi)+D)\le\theta(\xi,\tau)-g(\xi)\le CG(\tau)(h(\xi)+D),$$

with $h(\xi)=-g(\xi)$ for $\xi\le\xi_0$ and $h(\xi)=c_0\xi+c_1$ for $\xi\ge\xi_0$.

In the case of negative $G(s)<0$, the proof is quite similar where we first choose $\xi_0\gg 1$ such that

$$c_0=-g'(\xi_0)<\min\left\{1,\ \frac{1}{8K(m_0e+1+2\pi)}\right\}.$$

The proof of Theorem A is completed. $\square$

4. Final-time profile

In this section we derive the behaviour of the final-time profile near the origin established in Theorem B. We follow the lines of the proof in [GV], Section 8, but in this case, the unboundedness of the profile $g(\xi)$ does not allow to apply the convergence result only in Theorem A. In order to obtain the result, we begin by studying the blow-up behaviour of the function $v(x,t)$ related to $u(x,t)$ by the formula

$$u(x,t)=F(v(x,t)).$$

By substituting into (HE), we obtain that v solves the boundary value problem

$$\left\{\begin{array}{l} v_t=v_{xx}+\frac{F''(v)}{F'(v)}(v_x)^2,\\ v(0,t)=-\ln(1-t)\end{array}\right. \tag{4.1}$$

in the half-space $\mathbf{R}_+$, $0\le t<1$, with bounded initial data $v_0(x)$. We establish the following result which is equivalent to Theorem B.

Theorem 4.1. *Assume that F satisfies the hypothesis of Theorem A. Then*

$$v(x,1^-)=-2\ln x+C_0+o(1)\quad as\quad x\to 0.$$

Proof. To begin with, using the structure of the rescaled function (2.2), we introduce the family of functions $\{v_\alpha,\ \alpha\in[0,1)\}$,

$$v_\alpha(x,t)=\ln(1-\alpha)+v((x+\xi)(1-\alpha)^{1/2},\alpha+(1-\alpha)t). \tag{4.2}$$

Here the variables x, ξ and t vary in the range

$$0\le x\le L,\quad \delta\le\xi\le M,\quad 0\le t\le 1, \tag{4.3}$$

where $\delta>0$ is a fixed constant and L and M can be fixed arbitrarily large. By using (2.2), one can see that

$$v_\alpha(x,t)=-\ln(1-t)+\theta(\bar\xi,\bar\tau), \tag{4.4}$$

where θ is the rescaled function corresponding to the variables

$$\bar\xi=\frac{x+\xi}{(1-t)^{1/2}},\quad \bar\tau=-\ln(1-\bar t), \tag{4.5}$$

and $\bar t$ is given by $\bar t=\alpha+(1-\alpha)t$, so that the limit $\bar t\to 1$ is equivalent to $\alpha\to 1$ or $t\to 1$.

In order to obtain uniform bounds for the family v_α as $\alpha \to 1$, on compact subsets of type (4.3), we need to establish new estimates of the function $\theta(\xi, \tau)$. We point out that the estimates performed in the previous sections, do not provide uniform bounds as $t \to 1$. However, they will be used in passing to the limit in the initial and boundary data $v_\alpha(x, 0)$ and $v_\alpha(0, t)$ respectively.

Similarly to Lemma 3.1, using the extension of $F(s)$ if necessary, we obtain the following

Lemma 4.2. *There exists a large enough constant $\underline{D}$ such that*

$$\theta(\xi, \tau) \geq -2\ln\xi - \underline{D}, \quad \text{whenever} \quad \tau \geq 0, \quad \xi \geq \delta.$$

Lemma 4.3. *If $\bar{D}$ is a large enough constant, then the rescaled function $\theta(\xi, \tau)$ satisfies for every $\xi \geq 0$ and $\tau \geq 0$*

$$\theta(\xi, \tau) \leq g(\xi) + (\xi - 1)e^{-\tau/2} + g'(\xi) + \bar{D}.$$

Proof. Since $g'(\xi) + g''(\xi) < 0$, we can define the following supersolution

$$\bar{\theta}(\xi, \tau) = g(\xi) + (\xi - 1)e^{-\tau/2} + g'(\xi) + \bar{D}.$$

It is clear that this function is an upper bound for the initial and boundary functions of $\theta(\xi, \tau)$ if the constant $\bar{D}$ is large enough. Using the extension approach, we again can assume that $|G(s)| \leq K = 1/4\pi$. Then, as above we easily deduce that

$$\bar{\theta}_\tau - \mathbf{A}(\bar{\theta}) \geq 0 \quad \text{for every} \quad \tau \geq 0,$$

and the result follows from the Maximum Principle. $\square$

End of the proof of Theorem 4.1. Translating now these results according to the change (4.4), (4.5), we obtain uniform bounds on compact subsets (4.3), for the family $\{v_\alpha\}$, and therefore, for $\{v_{\alpha x}\}$, as $\alpha \to 1$. In fact, it follows from Lemmas 4.2 and 4.3 that

$$-2\ln(x + \xi) - \underline{D} \leq v_\alpha(x, t) \leq -\ln(1 - t) + \bar{\theta}(\bar{\xi}, \bar{\tau}) \leq -2\ln(x + \xi) + \bar{D} + 1, \tag{4.6}$$

for $\alpha \approx 1$. Hence, the left-hand and the right-hand sides are uniformly bounded on compact subsets (4.3). Rewriting equation (4.1), we see that v_α solves the uniformly parabolic equation

$$(v_\alpha)_t = (v_\alpha)_{xx} + \frac{F''(v_\alpha - \ln(1 - \alpha))}{F'(v_\alpha - \ln(1 - \alpha))}(v_\alpha)_x^2. \tag{4.7}$$

On the other hand, by the results stated in previous sections (see Theorem A), the corresponding initial functions $v_\alpha(x, 0)$ satisfy

$$v_\alpha(x, 0) = \theta(x + \xi, -\ln(1 - \alpha)) \to g(x + \xi) \tag{4.8}$$

as $\alpha \to 1$ uniformly on compact subsets (4.3). Hence, using the parabolic regularity theory and standard compactness arguments in passing to the limit $\alpha \to 1$, we obtain that

$$v_\alpha \to \bar{v} \quad \text{in} \quad L^\infty([0,1] : C_{\rm loc}(\mathbf{R}_+)), \tag{4.9}$$

where $\bar{v}$ solves the heat equation

$$\bar{v}_t = \bar{v}_{xx} \quad \text{in} \quad \mathbf{R}_+ \times (0,1], \tag{4.10}$$

with the initial function

$$\bar{v}(x,0) = g(x+\xi). \tag{4.11}$$

Now, using again Theorem A, we see that $\bar{v}(0,t) = -\ln(1-t) + g(\xi/(1-t)^{1/2})$ for every $0 \le t < 1$. Therefore, by uniqueness for the IVBP for the heat equation, we conclude that $\bar{v}$ is the unique self-similar solution

$$\bar{v}(x,t) = -\ln(1-t) + g((x+\xi)/(1-t)^{1/2}). \tag{4.12}$$

Setting here $x = 0$ and $t = 1$, and using (2.9) we deduce that

$$\bar{v}(0,1) = -2\ln\xi + C_0 \quad \text{for} \quad \xi \ge \delta > 0.$$

Thus, setting $t = 1$ in (4.9) and using known regularity of $\bar{v}$, we deduce that for all $\alpha \approx 1$,

$$v_\alpha(0,1) \equiv \ln(1-\alpha) + v(\xi(1-\alpha)^{1/2}, 1) = -2\ln\xi + C_0 + o(1) \tag{4.13}$$

uniformly on any compact subset $\xi \in [\delta, M]$. Setting here $x = \xi(1-\alpha)^{1/2} \ll 1$ for a fixed $\alpha \approx 1$, we obtain that

$$v(x, 1^-) = -2\ln x + C_0 + o(1) \tag{4.14}$$

for small $x > 0$. This completes the proof. $\square$

5. Exponential perturbations

In this section we study the convergence problem when the function $G(s)$ does not satisfy (2.7). More precisely, we consider the IBVP with $F(s)$ satisfying (1.15) and such that

$$\lim_{s\to\infty} \frac{G'(s)}{G(s)} = -\gamma < 0. \tag{5.1}$$

First of all, we note that these conditions imply that the function $F(s)$ corresponds to an exponentially small perturbation of a linear function. In fact, it can be deduced that in this case

$$F(s) = Cs + D + f(s),$$

where $|f(s)| = O(e^{-(\gamma-\varepsilon)s})$ with $\varepsilon > 0$ arbitrarily small. According to this argument, in this case it is more convenient to define the following rescaled function:

$$u(x,t) = C(-\ln(1-t) + \theta(\xi,\tau)) + D,$$

with ξ and τ as above. The corresponding convergence result is as follows.

Theorem C. *Let $F \in C^3(\mathbf{R}_+)$ be a monotonically increasing function. Assume that (1.15) and (5.1) hold. Then, as $\tau \to \infty$ uniformly on compact subsets in ξ there holds*

$$|\theta(\xi,\tau) - g(\xi)| = O(e^{-\delta\tau}),$$

with $\delta = \min\{\gamma - \varepsilon, \frac{1}{2} - \varepsilon\}$ where $\varepsilon > 0$ is arbitrarily small.

The results stated in Theorem B about the blow-up behaviour of the solution in the original variables, apply also to this case and the proof can be repeated with some minor changes.

Proof. It is quite similar to the proof in Theorem A. Now the function $H(\xi,\tau) = \theta(\xi,\tau) - g(\xi)$ satisfies the equation

$$H_\tau = H_{\xi\xi} - \frac{1}{2}\xi H_\xi, \quad H(0,\tau) = f(\tau),$$

and $|H(\xi,0)| \leq -g(\xi) + D$. Following the arguments in the previous sections, it is possible to find values $\rho \ll 1$ and $\xi_0 \ll 1$ such that the function

$$\bar{H}_\rho(\xi,\tau) = \begin{cases} e^{-\rho\tau}(-g(\xi) + D), & \xi < \xi_0, \\ e^{-\rho\tau}(c_0\xi + c_1 + D), & \xi \geq \xi_0 \end{cases}$$

is a supersolution and $\underline{H}_\rho(\xi,\tau) = -\bar{H}_\rho(\xi,\tau)$ is a subsolution for every $\tau \geq 0$ and we obtain the convergence result via the Maximum Principle. This extra step is necessary to neglect the term corresponding to the constant D. Note that now we do not have the condition $G'(s) \ll G(s)$. The exact decay rate of convergence is obtained by considering a similar type of sub and supersolution (up to a constant) with $\rho = \gamma - \varepsilon$. $\square$

6. Oscillation problem

In this section we deal with some problems where the hypothesis related to the existence of the $\lim_{s\to\infty} G'/G$ is removed. First of all, since $G(s) \to 0$ as $s \to \infty$, it follows that

$$\liminf_{s\to\infty} (G'(s)/G(s)) = \beta \leq 0.$$

If $\limsup_{s\to\infty} G'/G = \gamma < 0$, then $F(s)$ is a small exponential perturbation of the identity (up to a multiplier) and we proceed as in Section 5 with $Ce^{(\beta-\varepsilon)s} \leq |f(s)| \leq Ce^{(\gamma+\varepsilon)s}$. Obviously, the rate of convergence in this case is given by $e^{(-\delta+\varepsilon)s}$ with $\delta = \min\{|\gamma|, 1/2\}$ and $\varepsilon > 0$ arbitrarily small.

If $\limsup_{s\to\infty} G'/G = 0$ and β is bounded, then we can follow the lines of the proof of Theorem A to obtain that

$$|\theta(\xi,\tau) - g(\xi)| = O(|G(\delta\tau)|) \to 0, \quad \tau \to \infty,$$

with $\delta = 1$ if $|\beta| \leq 1/2$ and $\delta = 1/(2\beta + \varepsilon)$ with $\varepsilon > 0$ arbitrarily small if $|\beta| > 1/2$.

Finally, we mention that the previous results apply also to functions of the form

$$F(s) = \hat{F}(s) + f(s),$$

where $\hat{F}(s)$ is a monotonically increasing function satisfying (1.15) and (2.7) or (5.1) holds for $\hat{G}(s)$ and the function $f(s)$ is continuous with $f(s) = o(\hat{F}(s))$. In this case, the corresponding rescaling and therefore the results are established in terms of $\hat{F}$. Several cases with bounded or unbounded oscillations of $G(s) = F''(s)/F'(s)$ can be solved in this way.

Appendix: equivalent definition of interval I_1^-

Proposition A.1. *Let $\psi(t) = F(-\ln(1-t))$ be a suitable monotone blow-up function such that the following limits (finite or infinite) exist*

$$a_\psi^- = \lim_{t\to 1}\left(\frac{\psi(t)}{\psi'(t)}\right)', \quad \kappa_\psi^- = \lim_{t\to 1}\left(\frac{\psi'(t)}{\psi''(t)}\right)'.$$

Then:

(i) (1.15) is equivalent to (1.15');

(ii) there holds

$$a_\psi^- = -\infty \iff \kappa_\psi^- = -1.$$

Proof. (i) Consider a suitable blowing up function $\psi(t)$ satisfying $\kappa_\psi^- = -1$. Under the hypotheses, one can see that $\lim_{t\to 1}\psi'(t)/\psi''(t) = 0$. Therefore, we can apply the L'Hospital rule to obtain that

$$\lim_{t\to 1}\frac{(1-t)\psi''(t)}{\psi'(t)} = 1. \tag{6.1}$$

Using that

$$\frac{F''(s)}{F'(s)} = \frac{(1-t)\psi''(t)}{\psi'(t)} - 1, \tag{6.2}$$

the result follows.

In the same way, if we assume condition (1.15), we get (6.1) from (6.2) and therefore (1.15') follows by the L'Hospital rule.

(ii) Let $\kappa_\psi^- = -1$. One can see that under the assumptions, $F'(s)/F(s) \to 0$ as $s \to \infty$, and therefore

$$\lim_{t\to 1}\frac{(1-t)\psi'(t)}{\psi(t)} = 0,$$

whence $a_\psi^- = -\infty$ by the L'Hospital rule.

On the other hand, let $a_\psi^- = -\infty$. This implies in particular that

$$\lim_{t\to 1}\frac{\psi(t)}{(1-t)^{-\alpha}} = 0 \tag{6.3}$$

for any arbitrary $\alpha > 0$. Now, assume for contradiction that $\kappa_\psi^- \neq -1$. Since this limit exists, using (6.2) it follows that

$$\lim_{t\to 1}\frac{(1-t)\psi''(t)}{\psi'(t)} = c \neq 1.$$

Then, integrating the function $(1-t)\psi''(t)/\psi'(t)$, we obtain a contradiction with (6.3) if $c > 1$ and with the property of blow-up of ψ if $c < 1$. This completes the proof. $\square$

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