GEOMETRIC THEORY OF ONE-DIMENSIONAL NONLINEAR PARABOLIC EQUATIONS I. SINGULAR INTERFACES

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Abstract. We present the concepts of a geometric analysis of the free-boundary problems for a nonlinear one-dimensional parabolic PDE with strong singularities

$$u_t = F(u, u_x, u_{xx}), \quad x \in \mathbf{R},\ t > 0; \quad u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbf{R},$$

where the function $F(p, q, r)$ is smooth for $p > 0$, satisfies the parabolicity condition $F_r(p, q, r) > 0$ for $p > 0$ and, in general, is not defined (singular) at $p = 0$. A proper maximal solution $u(x, t) \geq 0$ is constructed as the limit of a monotone decreasing sequence of smooth strictly positive solutions of the regularized problems, which are assumed to be well-posed. The solutions exhibit finite interface propagation on the singular level $\{u = 0\}$.

The geometric theory consists of the study of the PDE via a family of the ODEs. We show that existence, Bernstein-type gradient estimates, first interface regularity, the interface equation, etc. are essentially the questions of the standard theory of a family of ODEs associated with the nonlinear PDE. For the autonomous PDEs we utilise a complete set $B = \{V\}$ of the particular travelling wave solutions $V(x, t) = f(x - \lambda t)$, where $f(\xi)$ solves a nonlinear ODE with a parameter $\lambda \in \mathbf{R}$. The proofs are based on the Sturmian argument and intersection comparison techniques of the solutions $u(x, t)$ with the family B. We study the regularity properties and derive the interface equations for several types of the quasilinear and fully nonlinear equations from combustion, filtration and detonation theory.

1. Introduction: singularity and geometric analysis

In this paper we present the concepts of a geometric analysis of the existence, differential properties and free-boundary propagation for nonlinear one-dimensional second-order parabolic equations with strong singularities. The class of equations under consideration includes those with the degenerate porous medium (PM) operator, the p-Laplacian one and other types of quasilinear degenerate diffusion-type operators with extra lower-order terms of reaction-absorption-convection type. We also consider classes of fully nonlinear equations from filtration and detonation theory.

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The main subject of the geometric theory is to study a nonlinear singular parabolic PDE via a one-parameter family of the associated ODEs.

1.1. **Singular free-boundary problems.** In order to describe the geometric approach, we consider solutions $u(x,t) \ge 0$ of a fully nonlinear parabolic PDE

$$u_t = \mathbf{F}(u) \equiv F(u, u_x, u_{xx}), \quad x \in \mathbf{R}, \; t > 0, \tag{1}$$

where $F(p,q,r)$ is a given function being sufficiently smooth for $p > 0$ and satisfying the *parabolicity condition*

$$F_r(p,q,r) > 0, \quad p > 0, \; q, r \in \mathbf{R}. \tag{2}$$

We consider for (1) the Cauchy problem with continuous initial data

$$u(x,0) = u_0(x) \ge 0, \quad x \in \mathbf{R}. \tag{3}$$

A common feature of highly nonlinear PDEs to be considered is as follows. They admit *weak* (for divergent equations) or, in general, *maximal proper* solutions which describe propagation on a *singular* level $\{u = 0\}$, where the operators $\mathbf{F}$ are not well-defined (say either $F(0,q,r)$ or $F_p(0,q,r)$ are not finite) or, at least, are not uniformly parabolic. We consider the case where the solutions, constructed by monotone approximation, are the maximal ones and the equation of such singular free-boundary interfaces is not given *a priori*. Due to the construction, an extra free-boundary condition on the singular set makes the problem overdetermined. We call such free boundaries the *singular interfaces* unlike the free boundaries in the nonlinear Stefan-like problems where the interface equation is prescribed in the problem statement. A classical example of such singular interfaces is given by the porous medium equation (PME), [12].

We assume that the singular level is the zero one, $\{u = 0\}$, and the solutions $u = u(x,t) \ge 0$ are assumed to be sufficiently smooth away from the singularity. A natural typical example of such a finite (singular) interface propagation is presented by the equation with doubly nonlinear diffusion and an absorption term

$$u_t = [|(u^m)_x|^\sigma (u^m)_x]_x - u^p, \tag{4}$$

where $m > 0$, $\sigma > -1$ and $p > -m$ are fixed constants (for $p \le -m$ no nontrivial solutions exist). Equations like that are known in the theory of filtration of non-Newtonian liquids. The level $\{u = 0\}$ can be singular for both diffusion and absorption operators, especially when $p \in (0,1)$ (strong absorption) or even $p \le 0$ (singular absorption). The diffusion operator in (4) includes the PM one of $\sigma = 0$ and the p-Laplacian if $m = 1$. A singular free-boundary propagation occurs for more general equations in the divergent form

$$u_t = [\varphi(u, u_x)]_x + \psi(u, u_x), \quad \varphi_q(p,q) > 0, \; p > 0. \tag{5}$$

Another important example of *very* singular propagation is presented by *blow-up* solutions $u(x,t) > 0$ of nonlinear parabolic equations from combustion theory, like (5), with a reaction (source) term $\psi = \psi(u) > 0$ which is a superlinear function for $u \gg 1$. Then for (5) the singular level is the blow-up one $\{u = +\infty\}$ where the solution is unbounded. The free boundaries then are the singular blow-up interfaces which bound the regions with infinite values of the solution ("burnt zones"). Obviously, using the transformation $u \mapsto 1/u$, we reduce the blow-up propagation to the propagation on the zero level $\{u = 0\}$ for the quasilinear non-divergent equation with absorption

$$u_t = -u^2[\varphi(1/u, -u_x/u^2)]_x - u^2\psi(1/u, -u_x/u^2). \tag{6}$$

In particular, it follows that finite blow-up interface propagation occurs for the reaction-diffusion equation

$$v_t = [|(v^m)_x|^\sigma (v^m)_x]_x + v^p, \tag{7}$$

in the parameter range

$$p \in (1, p_c], \quad p_c = \frac{\sigma+2}{\sigma+1} - m > 1 \quad \left(m < \frac{1}{\sigma+1}\right), \tag{8}$$

and we will derive the interface equation. If $p > p_c$, the blow-up propagation exhibits infinite speed, and this means *complete blow-up*. We will consider fully nonlinear equations like

$$u_t = g(u_{xx}) \pm \psi(u), \quad g'(r) > 0, \;\; r > 0. \tag{9}$$

The power-like function $g(u_{xx}) = |u_{xx}|^{m-1}u_{xx}$, $m > 0$, corresponds to the *dual* porous medium operator, see [4]. Finite interfaces also occur in the fully nonlinear equation with absorption

$$u_t = g(uu_{xx}) - (u_x)^2 - \psi(u) \tag{10}$$

from detonation theory. See [5] and references in [23]. As a natural generalization of equations (4) and (7), we can study fully nonlinear equations (cf. [44])

$$|u_t|^{\gamma-1}u_t = [|(u^m)_x|^\sigma (u^m)_x]_x \pm u^p, \quad \gamma > 0. \tag{11}$$

1.2. **Subject: PDE via ODEs.** It is well understood that the main difficulty in the analysis of such singular free-boundary problems consists in the study of differential properties of the solutions near the interfaces as well as of the regularity of the interfaces. The a priori estimates in special functional classes play a fundamental role in the theory of nonlinear parabolic equations and this is the subject of the well-known mathematical literature. There are many examples where these crucial a priori estimates of the regularized solutions are essentially refined even for some particular semilinear or quasilinear equations with the PM or the p-Laplacian operators having power-type nonlinearities. This means that similar necessary estimates for equations with more general (and even arbitrary) nonlinearities are supposed to be much more delicate and the corresponding computations for the general equations are expected to be extremely difficult.

In a sufficient generality, we would like to study the following standard questions of the theory of nonlinear one-dimensional parabolic equations:

(i) existence or nonexistence of a nontrivial proper solution,
(ii) optimal two-sided Bernstein gradient bounds near the singular interfaces,
(iii) first or second-order estimates on the interfaces,

(iv) the interface equation and the interface regularity, etc.

We show that for a wide class of nonlinear one-dimensional parabolic equations (1) these and related questions *are essentially the questions of the ODEs theory* in the sense that the answers are based on the properties of a "proper" set B of particular singular solutions generated by a family of the second-order ODEs associated with the nonlinear parabolic PDE. The geometric analysis translating properties of the family of the ODEs to the PDE is the *intersection comparison technique* based on the *Sturmian argument* on the zero-set analysis for linear parabolic equations.

We prove that a suitable complete proper set B of particular solutions of the nonlinear PDE can generate an existence-regularity theory in the so-called functional B-classes. We choose the case of the finite interface propagation with a priori unknown interface equation. It turns out that in the case of finite propagation for the autonomous PDEs, the set B can be composed from the simplest *travelling wave* (TW) solutions, $V(x,t) = f_\lambda(x-\lambda t)$ ($\lambda \in \mathbf{R}$ is an arbitrary parameter), which propagate along the straight lines on the $\{x,t\}$-plane. On the other hand, some other sets B can be also used. The geometric approach based on TWs can be treated as a version of a "nonlinear methods of characteristics" for the nonlinear second-order parabolic PDEs. The set B generates a kind of the *Rankin-Hugoniot condition* of the TW-propagation of weak discontinuities at $\{u = 0\}$ which is shown to be true for general proper solutions of the parabolic PDE. Unlike the first-order equations [51], for the parabolic PDEs the Rankin-Hugoniot condition can be of the second-order.

In other words, we show that a complete set B of particular solutions (a kind of heteroclinic connections with the singular level) gives a crucial existence-regularity results for (1). This means a certain *duality* between the nonlinear parabolic PDE and a family of ODEs defining the set B.

It is worth mentioning that once we deal with singular free-boundary problems in a maximal generality, we have to avoid any kind of "essentially computational approaches" to derive necessary bounds on the solutions. Several well-established computational approaches using the divergent or other special structure of the equation and based on, sometimes, cumbersome algebraic and differential manipulations with the PDEs, cannot be efficient for general fully nonlinear equations with arbitrary nonlinearities. Therefore, we follow the idea that the main results including existence and the regularity properties should be extracted from the family of nonlinear ODEs by the geometric intersection comparison approach. Sometimes we are allowed to apply an elementary Maximum Principle analysis which is assumed to be true for the general parabolic PDEs.

1.3. **Main concepts of the geometric analysis.** We consider the Cauchy problem (1), (3) and assume that the initial function u_0 is essentially singular: $\operatorname{supp} u_0 \equiv \{x \in \mathbf{R} : u_0(x) > 0\} \neq \mathbf{R}$. In the most cases we assume that $u_0(x)$ is monotone increasing. Then a solution (if any) $u(x,t) \geq 0$ is monotone increasing in x for $t > 0$ by the Maximum Principle, and hence the singular propagation of finite interfaces starts at the initial moment $t = 0$, the interface $x = s(t)$ is unique, and we arrive at a (singular) free-boundary problem.

We assume that for non-singular initial data u_0 uniformly bounded away from $u = 0$, the Cauchy problem admits a unique sufficiently smooth solution which exists locally in time, and standard results from the parabolic theory such as uniqueness, interior regularity, comparison and the Maximum Principle apply to such classical solutions. For such non-singular initial data one can study the occurrence of the singularity in finite time. We now present the main steps of our analysis.

1. Set B of proper TWs (Section 2). We first describe the necessary hypotheses we need to impose on a set $B = \{V\}$ of special particular solutions of equation (1). For all the autonomous equations mentioned above, there exists a set of the *travelling wave* (TW) solutions depending on the single independent variable:

$$V(x,t) = f_\lambda(\xi), \quad \xi = x - \lambda t - a, \tag{12}$$

where $\lambda \in \mathbf{R}$ is the constant speed of the TW and $a \in \mathbf{R}$ is the translational parameter. Continuous functions f_λ satisfy a nonlinear second-order ODE with a parameter $\lambda \in \mathbf{R}$

$$F(f, f', f'') + \lambda f' = 0. \tag{13}$$

We always assume that it can be studied by standard ODEs methods. We consider proper TW profiles $f_\lambda(\xi)$, $\lambda \in \Delta \subseteq \mathbf{R}$, which play a role of heteroclinic connections with the singular level $u = 0$ in the sense that their ranges satisfy

$$0 \in \overline{\mathrm{Im}}\, f_\lambda(\cdot), \quad \lambda \in \Delta.$$

We next study singular TWs with finite propagation on the singular level where $\{\xi \in \mathbf{R} : \ f_\lambda(\xi) = 0\} \neq \emptyset$, provided that $\lambda \in \Delta_0 \subseteq \Delta$. We choose continuous branches of the *proper* TW solutions f_λ, $\lambda \in \Delta_0$, which can be constructed as the limit of a sequence of non-singular solutions of the ODE (13). Moreover, they are strictly monotone decreasing upon λ. This comparison-like property is also governed by the Maximum Principle for uniformly parabolic equations.

Let B be the two-parametric set of all the singular proper TW solutions $\{V = f_\lambda(x - \lambda t - a),\ \lambda \in \Delta_0,\ a \in \mathbf{R}\}$ with the straight interfaces $s_\lambda(t) = \lambda t + a$. We assume that B is a *complete*, continuous and monotone set. The set $\{f_\lambda(\xi),\ \lambda \in \Delta_0\}$ where all the functions f_λ have the interfaces at the origin $\xi = 0$, is the proper TW-bundle (or B-bundle). Using the complete structure of the TW-bundle, we introduce the following basic notions of our analysis:

(i) the pressure variable,

(ii) the interface operators $\mathbf{M}_i(u)$ and $\mathbf{N}_i(u)$, $i = 1, 2$, and the interface slopes S of the first or second order,

(iii) the TW-diagram $\lambda = A_0^{-1}(S) \in \Delta_0$, which is the Rankin-Hugoniot condition of the singular propagation for the parabolic PDEs,

(iv) the gradient function (G-function) which characterizes optimal Bernstein-type estimates in special functional B-classes, etc.

2. Proper maximal solution. Existence if $\Delta \neq \emptyset$ (Section 3). A unique proper maximal solution $u(x,t)$ in $S_T = \mathbf{R} \times (0,T)$, $T \leq \infty$, is constructed as the limit of a monotone sequence of smooth non-singular solutions $\{u_n\}$ of the regularized equation with regularized initial data. The regularized Cauchy problem is supposed to be well-posed and the classical results for fully nonlinear parabolic equations apply. The existence of a nontrivial limit $u = \lim u_n \not\equiv 0$ in the case $\Delta \neq \emptyset$ (which means *incomplete singularity* for (1) at $u = 0$) follows by comparison with the TW solutions f_λ, $\lambda \in \Delta$. If $\Delta_0 \neq \emptyset$ then the proper solutions u exhibit finite propagation on the singular level $\{u = 0\}$.

3. Sturmian argument and intersection comparison with B, nonexistence if $\Delta = \emptyset$, geometric notions (Sections 4 and 5). The main geometric tool of the analysis is the intersection comparison of a singular proper solution $u(x,t)$ with the complete set B of the TW solutions. The intersection comparison is based on the classical Sturmian argument of zero-set analysis for the linear parabolic equations. As the first result, we show that in the case $\Delta = \emptyset$ the proper solutions are trivial, $u = \lim u_n \equiv 0$ for $t > 0$, which means that at $\{u = 0\}$ there occurs a *complete* singularity. We also show how the complete set B defines the property of the B-concavity preserved in time, the corresponding sign-invariants and B-number.

4. Optimal Bernstein estimates in B-classes (Section 6). A complete set B of the TW solutions makes it possible to introduce two functional classes $B^+_{\rm loc}(c)$ and $B^-_{\rm loc}(c)$ where $c \in \Delta_0$ is a parameter. The local classes $B^\pm_{\rm loc}(c)$, which actually mimic spatial shape of the TWs $f_{\pm c}(x)$, are defined according to a single intersection (from above or below) near the interface of the solution profile $u(\cdot,t)$ with TWs $f_c(x-a)$, where $a \approx +0$ or $a \approx -0$ is the translational parameter. The intersection comparison then shows that

$$u_0 \in B^\pm_{\rm loc}(c) \;\Rightarrow\; u(t) \in B^\pm_{\rm loc}(c) \;\text{ for small } t > 0. \tag{14}$$

Under extra assumptions, this analysis establishes the *instantaneous smoothing effect* for arbitrary initial data:

$$\text{for any } t_0 \in (0,T) \text{ there exists a large } c(t_0) \in \Delta_0 \text{ such that } u(t_0) \in B^\pm_{\rm loc}(\mp c). \tag{15}$$

These classes are shown to provide us with optimal *gradient bounds* of Bernstein type on the singular solutions near the singular level and a bound on the interfaces. Observe that on the interface these bounds are not necessarily of the first order but can be of the second order. The global B-classes, $B^\pm(c)$, describe the behaviour in x of the proper solutions everywhere in $\mathbf{R}$.

5. Interface operators and interface equation (Section 7). The above estimates in B-classes extended up to the singular interface

$$s(t) = \inf\{x \in \mathbf{R} :\; u(x,t) > 0\}, \quad t \in [0,T),$$

give uniform bounds on the *interface operators* defined by the TWs set B. As a consequence, we obtain uniform bounds on the right-hand and left-hand derivatives $D^\pm s(t)$ on the interface and derive the *first-order interface equation*

$$D^+ s(t) = \mathbf{N}_1(u(\cdot,t)), \tag{16}$$

or the *second-order interface equation* (a system of two equations)

$$\mathbf{M}_1(u(\cdot,t)) = A_0 = \text{const}, \quad D^+ s(t) = \mathbf{N}_2(u(\cdot,t)), \tag{17}$$

where the interface operator $\mathbf{N}_1$ and $\mathbf{M}_1$ are of the first order while $\mathbf{N}_2$ is of the second order. Equation (16) and system (17) are the Rankin-Hugoniot conditions (derived first for the TWs) of finite singular propagation for *general* solutions.

6. Applications (Section 8). Optimal estimates and the interface equation can be established in the maximal generality provided that some detailed properties of the whole set of TW profiles $B = \{f\}$ satisfying the ODE (13) are established. In Section 8 we consider several applications of the method to quasilinear and fully nonlinear equations.

1.4. **Discussion and generalizations.** Some aspects of the proposed geometric analysis have been separately discussed and applied to particular quasilinear parabolic equations in a series of papers, [15] - [20], [22], [24] - [26], [28] - [34], [48]. In particular, we use some ideas and results on the TW-analysis and on the intersection comparison with the families of TW-solutions presented in [14], [22], [28], [29], [31] - [34].

The geometric analysis can be based not necessarily on the set of TWs. For some special nonlinear parabolic equations other sets B of group-invariant (self-similar) solutions or solutions on linear subspaces invariant under nonlinear operators can be used. See examples in [31], [22] and the new families of exact solutions in [20], [21] and [27]. In these cases the particular solutions from B are given by reasonably simple nonlinear ODEs so that their properties can be studied by standard ODEs methods. Introducing an appropriate set $B = \{V\}$ of proper particular solutions we always take into account that the general proper solutions u under consideration belong to the corresponding functional class where the singular behaviour can be described by the solutions from B.

Therefore, the geometric theory can be developed on the basis of any suitable complete set B of particular solutions to the nonlinear PDEs. In particular, the same geometric concepts apply in the case of infinite propagation where $s(t) \equiv -\infty$. This means that $\Delta \neq \emptyset$ (the TW-bundle is not empty) but $\Delta_0 = \emptyset$. Then if a singularity does not occur at finite points (but at $x = -\infty$), the unknown existence-regularity properties and interface propagation are replaced by similar questions on the evolution of another infinite "interface" parameter again denoted by $s(t)$. It describes the motion of the general evolution orbit of the PDE in the complete set of particular orbits B. Then the equation for $s(t)$ of the motion of an orbit $u(\cdot, t)$ in B (the B-motion equation which replaces the natural interface equation in the case of infinite propagation) and existence-regularity questions in the B-classes are studied by the intersection comparison with the solutions from B.

The geometric approach exhibits certain aspects of a kind of a "nonlinear superposition principle" for the second-order parabolic PDEs, where several properties (like existence, differential properties, concavity, etc.) of suitable sets $B = \{V\}$ of *particular* solutions can be translated to the *general* solution.

The geometric analysis of the free-boundary propagation, based on the intersections comparison with a complete set B of the TWs or other particular solutions, applies to other types of singular and non-singular free-boundary problems. For instance, to singularities of uniformly bounded solutions $u(\cdot, t)$ with blowing up derivatives $u_x(\cdot, t)$ or

$u_{xx}(\cdot,t)$. This kind of singularities are typical for generalized mean curvature or curve shortening flows. On the other hand, we can consider generalized *Stefan-like* problems for the equation (1) with the free-boundary conditions

$$u(s(t),t) = 0, \quad D^+ s(t) = \Phi(u, u_x, u_{xx})(s(t),t), \ \ t > 0, \tag{18}$$

and a given function Φ. These results will be published in forthcoming papers.

2. The set B of singular TWs

2.1. **Properties of proper and improper TWs.** The essential feature of the geometric analysis is based on the use of a set $B = \{V\}$ of reasonably simple particular solutions of the equation exhibiting the free-boundary propagation on the singular level. Usually, we take the simplest two-dimensional set of the TW solutions

$$V(x,t;\lambda,a) = f(\xi;\lambda), \quad \xi = x - \lambda t - a, \tag{19}$$

where $\lambda, a \in \mathbf{R}$ are arbitrary parameters, λ is the speed and a is the translational one. For the general equation (1), we obtain the nonlinear ODE in the non-empty positivity domain of f with the "singularity condition"

$$F(f, f', f'') + \lambda f' = 0, \quad \xi > 0; \quad 0 \in \overline{\mathrm{Im}}\, f(\cdot). \tag{20}$$

Setting $f' = P$ in this autonomous ODE, we obtain the first-order ODE

$$F(f, P, PdP/df) = -\lambda P.$$

By the parabolicity condition, it is rewritten as follows

$$\frac{dP}{df} = \frac{1}{P}\, F^{-1}(f, P, -\lambda P), \quad f > 0, \ \ P > 0, \tag{21}$$

where F^{-1} is the inverse relative to the third variable in $F(p,q,r)$. The problem (20) is equivalent to the existence of a *singular* orbit (a heteroclinic connection of zero) of the first-order ODE (21) $P = P(f)$, which is defined for all small $f > 0$. Denote

$$\Delta = \{\lambda \in \mathbf{R}: \ \ \text{there exists a singular orbit of (21)}\}. \tag{22}$$

In what follows we always assume that $\Delta \neq \emptyset$, otherwise the PDE (1) does not admit a nontrivial solution (see Section 4).

Consider next the problem of finite propagation for the TW solutions. This means that equation (20) admits a continuous solution $f(\xi)$ satisfying the singular condition at a finite point, say, at the origin:

$$f(0) = 0. \tag{23}$$

Denote

$$\Delta_0 = \{\lambda \in \Delta: \ \ (20), \ (23) \ \text{has a continuous solution}\}. \tag{24}$$

We are assuming that this singular ODE problem admits, at least, a local in ξ continuous solution $f = f_\lambda(\xi) \not\equiv 0$ in a small neighbourhood of the origin $\xi = 0$ for a subset of the TW speeds $\lambda \in \Delta_0 \subseteq \mathbf{R}$. The continuous TW solution $V(x,t) = f_\lambda(x - \lambda t - a)$ with the straight interface $s_\lambda(t) = \lambda t + a$ then gives an orbit explaining a possible connection of the singular level $\{u = 0\}$ with the space of regular orbits above. In particular, a detailed TW-analysis for quasilinear reaction-diffusion-convection equations like

$$u_t = (a(u))_{xx} + (b(u))_x + c(u), \tag{25}$$

can be found in [35] and [36] (see also references therein), where necessary and sufficient conditions of $\Delta_0 \neq \emptyset$ for (25) are available.

We impose the main hypothesis:

(**H**) There exists a subset $\Delta_0 \subseteq \mathbf{R}$, $\Delta_0 \neq \emptyset$, such that for any $\lambda \in \Delta_0$ there exists a local in $\xi > 0$ solution $f_\lambda(\xi)$ of (20), (23), which is continuous on the maximal interval $[0, \xi_\lambda)$ and strictly positive at the interior points. If $\xi_\lambda < \infty$ then as $\xi \to \xi_\lambda - 0$ either $f_\lambda(\xi) \to 0$ or $f_\lambda(\xi) \to \infty$.

In the most cases we assume that the TW profiles $f_\lambda(\xi)$ are strictly monotone increasing which corresponds to the important case of the quasilinear heat conduction with absorption. The set B of such TW profiles $\{f_\lambda(\xi),\, \lambda \in \Delta_0\}$, then is supposed to describe some special TW-propagation along the singular level. We now choose special *proper* profiles satisfying a *monotonicity* property via the Maximum Principle due to the construction of the proper (generalized) solution of the PDE in Section 3. For the ODEs the corresponding definition can be stated as follows.

Definition 1. *A singular profile f_λ, $\lambda \in \Delta_0$, is said to be a proper one if on any interval $[0, \xi_*]$, $\xi_* < \xi_\lambda$, it can be constructed as the limit of a sequence $\{f_\lambda^n\}$ of strictly positive non-singular smooth solutions of the ODE (20).*

Without loss of generality we may suppose that the functions f_λ^n satisfy the boundary conditions

$$f_\lambda^n(0) = \frac{2}{n}, \quad (f_\lambda^n)'(0) = 0. \tag{26}$$

In general, for a fixed $\lambda \in \Delta_0$, the singular ODE problem may admit several solutions. It is not difficult to distinguish the proper profiles. Under certain natural restrictions on F, one can prove that continuous branches of proper solutions are *minimal* and *decreasing* in λ. These results follow from the standard phase-plane analysis of the ODE.

Proposition 1. *Fix a $\lambda \in \Delta_0$. Let the set of the singular solutions of (20), (23) consist of a discrete ordered subset $\{f_1, f_2, ..., f_k\}$ of solutions and f_1 is the minimal one: $f_1(\xi) < f_j(\xi)$ for small $\xi > 0$ for all $j = 2, ..., k$. Then f_1 is the proper solution and f_j, $j \geq 2$, are not.*

Proof. The singular solutions $f_1, ..., f_k$ define the continuous orbits $\{P_1, ..., P_k\}$ on the $\{f, P\}$-plane, $P_1(f) \leq P_j(f)$, $P_1 \not\equiv P_j(f)$ for small $f > 0$, which are extended up to $f = 0$. Let $\{P^n\}$ be the corresponding monotone increasing sequence of strictly positive orbits defined for $f \geq 2/n$ and satisfying (cf. (26))

$$P^n(2/n) = 0. \tag{27}$$

Since $P_1(f)$ is the minimal orbit, it follows from the phase-plane that the sequences $\{P^n\}$ cannot converge to P_j, $j \geq 2$, and converges to the minimal orbit P_1. $\square$

We thus conclude that $P = \lim P_n$ is the minimal positive orbit of the ODE (21) defined for all small $f > 0$. We denote it $P_{\min}(f)$. The proper TW profiles $f(\xi)$ are given by the ODE

$$\frac{df}{d\xi} = P_{\min}(f) > 0, \quad \xi > 0. \tag{28}$$

If $\Delta_0 = \emptyset$ ($\Delta \neq \emptyset$), it gives strictly positive solutions with the infinite interface at $\xi = -\infty$.

Choose all the λ-branches of the solutions of (20), (23). We characterize the following property of the proper TW profiles.

Proposition 2. *The set B of proper solutions $\{f_\lambda(\xi),\ \lambda \in \Delta_0\}$ is strictly decreasing in λ.*

Proof. Fix arbitrary $\lambda_1, \lambda_2 \in (a, b)$, $\lambda_1 < \lambda_2$, and let f_1 and f_2 be the corresponding singular TW profiles. Let $\{f_1^n\}$ and $\{f_2^n\}$ be the corresponding sequences of non-singular solutions satisfying (26). Then by the parabolicity condition on the operator we have

$$\frac{dP}{df} = \frac{1}{P} F^{-1}(f, P, -\lambda_1 P) > \frac{1}{P} F^{-1}(f, P, -\lambda_2 P),$$

so that by the standard comparison for the ODEs with smooth coefficients we have that $P_1^n(f) \geq P_2^n(f)$ for small $f \geq 2/n$. Passing to the limit, we get that the minimal orbits satisfy $P_1(f) \geq P_2(f)$ and integrating we obtain $f_1(\xi) \geq f_2(\xi)$ for small $\xi \geq 0$. $\square$

Increasing in λ *improper* branches of TW profiles generate TW solutions of the PDE which do not satisfy the comparison principle. Indeed, assume that there exist $\lambda_1 < \lambda_2$ such that $f_1(\xi) \leq f_2(\xi)$ so that at $t = 0$

$$V_1(x, 0) \equiv f_1(x) \leq V_2(x, 0) \equiv f_2(x). \tag{29}$$

Since the singular interfaces of $V_1(x, t)$ and $V_2(x, t)$ are $s_1(t) = \lambda_1 t < s_2(t) = \lambda_2 t$ for $t > 0$, the solutions V_1, V_2 are not ordered for $t > 0$ as they were in (29) at $t = 0$. They intersect each other for any small $t > 0$. In Section 3 we show that the proper singular solutions of the PDE satisfy the Maximum Principle which is a PDE selection principle of proper solutions directly related to the above ODE -principle.

According to these results, we denote by $B = \{f_\lambda,\ \lambda \in \Delta_0\}$ the set of the proper TW profiles. For any $\lambda \in \Delta_0$ the profile $f_\lambda \not\equiv 0$ is unique (the minimal one). By $\bar{B} = \{f_\lambda,\ \lambda \in \bar{\Delta}_0\}$ we denote the set of all the *improper* TW profiles, where $\bar{\Delta}_0 \subseteq \Delta_0$.

Finally, fix a $\lambda \notin \Delta \neq \mathbf{R}$. This means that the minimal orbit $P_{\min}(f)$ does not exist and for any $\lambda \in \mathbf{R} \setminus \Delta$ equation (20) does not admit a heteroclinic connection of the regular space with the singular set $\{f = 0\}$. In terms of approximation (26), using the natural monotonicity for the ODE (21), we mainly distinguish two types of nonexistence of a proper TW profile motivated by important applications in Section 8:

(i) "Steep limit": $P^n(f)$ is defined for $f > 2/n$ and for any $f > 0$ $P^n(f) \to \infty$, $n \to \infty$, so that $f^n(\xi)$, $\xi > 0$, becomes arbitrarily steep as $n \to \infty$.

(ii) "Flat limit": $P^n(f)$ is defined for $f < 2/n$ and $P^n(f) \to 0$ uniformly as $n \to \infty$, so that $f^n(\xi) \to 0$.

There are a few straightforward relations between existence and two types of nonexistence for different values of $\lambda \in \mathbf{R}$. See e.g. [32], Theorem 2.1. In particular, there holds:

Proposition 3. *Let $\lambda_1 \notin \Delta$ correspond to the steep limit. Then $(-\infty, \lambda_1) \cap \Delta = \emptyset$ and any $\lambda < \lambda_1$ exhibits the steep limit.*

Proof. Arguing by contradiction, without loss of generality, we may assume that there exists $f_\lambda \in B$ with a $\lambda < \lambda_1$. Then we have that the steep approximation satisfies $f^n_{\lambda_1}(x) \geq f_\lambda(x-a)$ for some a, so that $f^n_{\lambda_1}(x-\lambda_1 t) \geq f_\lambda(x-\lambda t - a)$ for $t > 0$ by local comparison. This obviously contradicts the assumption $\lambda < \lambda_1$. $\square$

Example 1. Consider the equation with a separate Hamilton-Jacobi operator

$$u_t = \varphi(u, u_x, u_{xx}) + H(u_x), \tag{30}$$

where φ and H smooth functions, $\varphi_r(p,q,r) > 0$ for $p > 0$ and $\varphi(p,q,0) \equiv 0$. The TW equation

$$\varphi(f, f', f'') + H(f') + \lambda f' = 0 \tag{31}$$

has the linear solutions

$$f_\lambda(\xi) = S\xi, \tag{32}$$

where $S = S(\lambda) > 0$, the spatial *slope* of the solution on the interface, satisfies the algebraic equation depending on the first-order term H

$$-\frac{H(S)}{S} = \lambda. \tag{33}$$

Observe that due to the specific structure of the second-order PDE, this is exactly the Rankin-Hugoniot condition of the weak discontinuity propagation at $\{u = 0\}$ for the first-order Hamilton-Jacobi equation

$$u_t = H(u_x).$$

In the case of the PME, where u stays for the pressure variable, $H(S) = S^2$, we have

$$u_t = (m-1)uu_{xx} + (u_x)^2, \quad m > 1, \tag{34}$$

and obtain the unique positive solution with $S = -\lambda > 0$ for any $\lambda \in (-\infty, 0) = \Delta_0$. This branch is monotone decreasing in λ so that these are the proper TW solutions. The same conclusion is true for the general equation (30) with $H(q) = q^2$ provided that (31) admits no other solutions but the linear ones (32) or these are the minimal ones. Any $\lambda \geq 0$ corresponds to the flat limit where $P^n(f) \to 0$, $n \to \infty$.

The propagation of blow-up interfaces on the singular level $\{v = \infty\}$ in the equation with a superlinear reaction term and fast diffusion [34]

$$v_t = (v^m)_{xx} + v^{2-m}, \quad m \in (0,1), \tag{35}$$

is described by the singular zero propagation for the "pressure" $u = mv^{m-1}/(1-m)$ satisfying

$$u_t = (1-m)uu_{xx} - (u_x)^2 - m, \quad H(S) = -S^2 - m. \tag{36}$$

Then for $\lambda > 0$ we obtain the equation

$$-\frac{H(S)}{S} \equiv S + \frac{m}{S} = \lambda.$$

The function $-H(S)/S = S + m/S$ has two continuous monotone branches so that the decreasing one for $S \in (0, \sqrt{m}]$ is the proper one with the unique solution

$$f_\lambda(\xi) = \left(\frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - m}\right), \quad \lambda \in \Delta_0 = [2\sqrt{m}, \infty). \tag{37}$$

The set of improper TW profiles $\bar{B}$ consists of

$$f_\lambda(\xi) = \left(\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - m}\right), \quad \lambda \in \bar{\Delta}_0 = (2\sqrt{m}, \infty). \tag{38}$$

No proper or improper TW profile exists for $\lambda < 2\sqrt{m}$, where we have the steep limit: $P^n(f) \to \infty$ as $n \to \infty$, see Theorem 3.1 in [34]. In particular, such a discontinuity of the proper branch at $\lambda = 2\sqrt{m}$ is responsible for the fact that the blow-up interfaces $x = s(t)$ are not C^2 functions (only $C^{1,1}$) and this kind of discontinuity occurs when the interface reaches the minimal speed: $D^+s(t) = 2\sqrt{m}$. See Section 5 in [34]. For a general H equation (31) can admit several continuous decreasing branches of proper TW profiles depending on the character of non-monotonicity of $-H(S)/S$.

2.2. **Pressure, interface operators, slopes and TW-diagram.** In general, the first interface equation governing the propagation of the TWs follows from the ODE (20)

$$\lambda = -\lim_{f \to +0} \frac{F(f, f', f'')}{f'}. \tag{39}$$

It expresses the interface velocity in terms of the asymptotic behaviour of the TW profiles $\{f_\lambda, \ \lambda \in \Delta_0\}$, near the interface at $\xi = 0$. Nevertheless, for the general solutions this formulae assumes much differential regularity in x on $u(x,t)$ which often is difficult to prove. We present a classification of the interface equations using the algebraic expansion of the TW profiles near the interface. Later on this classification applies to the general solutions of the parabolic PDE.

We assume there exists a smooth strictly monotone increasing function $Q : [0, \infty) \to [0, \infty)$, $Q(0) = 0$, $Q'(s) > 0$ for $s > 0$, such that $Q(f(\xi))$ exhibits almost linear in ξ behaviour near singularity:

$$Q(f(\xi)) = A_0(\lambda)\xi(1 + \rho(\xi, \lambda)), \quad \xi \to 0, \tag{40}$$

where $\rho(\xi, \cdot) = o(1)$. For linear solutions (32) $Q(f) \equiv f$. We assume that on different TW-branches such a function Q is independent of λ. Sometimes Q has to be defined separately on a different continuous branches of the proper profiles from B. See examples in Section 8. Thus, there exists a finite limit

$$\mathbf{M}_1(f) \equiv \lim_{\xi \to +0} \frac{1}{\xi} Q(f(\xi)) = A_0 > 0. \tag{41}$$

Using the terminology from the PME theory, we call $v = Q(u)$ the *pressure variable.* For the PME, $u_t = (u^m)_{xx}$, $m > 1$, the pressure $v \sim u^{m-1}$ is known to be Lipschitz continuous in x and after the waiting time it exhibits a linear behaviour near the interfaces, [40]. By the L'Hospital rule we obtain that for the TW profiles $\mathbf{M}_1(f) = (Q(f))'(0)$, so that $\mathbf{M}_1$ is the first-order differential operator. We distinguish two cases.

(I) The first-order TW interfaces: $A_0(\lambda)$ essentially depends on $\lambda \in \Delta_0$, is strictly monotone decreasing and there exists A_0^{-1}. Then we introduce the *first-order interface operator*

$$\lambda = \mathbf{N}_1(f) \equiv A_0^{-1}\left(\frac{1}{\xi} Q(f(\xi))(0)\right) \;\; \text{or} \;\; \mathbf{N}_1(f) = A_0^{-1}((Q(f))'(0)). \tag{42}$$

Denoting $S = \mathbf{M}_1(f)$, the *first-order slope* of the TW on the interface, the graph

$$\lambda = A_0^{-1}(S) \in \Delta_0 \quad (S > 0) \tag{43}$$

is called the (proper) *TW-diagram* on the $\{S, \lambda\}$-plane. It can be treated as a standard (first-order) Rankin-Hugoniot condition of the singular propagation (well-known in the theory of conservation laws, [51]) for nonlinear parabolic PDEs. The TW-diagram is strictly monotone decreasing in its domain of definition. The TW-diagram can be composed from several continuous monotone decreasing branches of the proper TW solutions. See Example 1 above with an essentially non-monotone function $-H(S)/S$.

(II) The second-order TW interfaces: $A_0 > 0$ does not depend on λ on a fixed proper TW-branch. In the general case, denoting by $\rho^{-1}(s, \cdot)$ the inverse function to $\rho(\xi, \cdot)$, we directly define the *second-order operator*

$$\mathbf{N}_2(f) \equiv \lim_{\xi \to +0} \rho^{-1}\left(\frac{Q(f(\xi))}{\xi} - A_0, \lambda\right) = \lambda, \tag{44}$$

which contains two indeterminancies in the limit. If the higher-order term $\rho(\xi, \lambda)$ of the expansion (40) is a power-like function, as it happens in several important applications in Section 7, or another suitable function, we can perform a further analysis (in fact, similar computations work in the most general case). We assume that there exists a continuous function $R : \mathbf{R} \to \mathbf{R}$, $R(0) = 0$, such that there exists

$$\mathbf{M}_2(f) \equiv \lim_{\xi \to +0} \frac{1}{\xi}\left[R\left(\frac{Q(f(\xi))}{\xi} - A_0\right)\right] = B_0(\lambda), \tag{45}$$

$B_0(\lambda)$ is continuous monotone decreasing and B_0^{-1} is well-defined. In the applications in Section 7, R is independent of λ. In addition, we introduce the second-order interface operator

$$\lambda = \mathbf{N}_2(f) \equiv B_0^{-1}\left(\frac{1}{\xi}\left[R\left(\frac{Q(f(\xi))}{\xi} - A_0\right)\right]\right)(0) \tag{46}$$

with two indeterminancies as $\xi \to +0$. Since $\xi = Q(f(\xi))(1+o(1))/A_0$, it can be written down in the form

$$\mathbf{N}_2(f) \equiv B_0^{-1}\left(\frac{A_0}{Q(f(\xi))}\left[R\left(\frac{Q(f(\xi))}{\xi} - A_0\right)\right]\right)(0).$$

Using the L'Hospital rule reduces (46) to a second-order differential operator defined at the interface: for the TW profiles

$$\mathbf{N}_2(f) = B_0^{-1}([R((Q(f(\xi)))' - A_0)]')(0).$$

The proper TW-diagram (the second-order Rankin-Hugoniot condition of the singular propagation) is then given by strictly monotone decreasing graph

$$\lambda = B_0^{-1}(S) \in \Delta_0 \quad (S > 0), \tag{47}$$

where S is the *second-order slope* of the TW solution on the interface. Actually, in this case the TW interface equation is a system of two equations, stationary and the dynamical one:

$$\mathbf{M}_1(f) = A_0, \quad \lambda = \mathbf{N}_2(f). \tag{48}$$

We will show that these interface operators are well-defined not only on the TW solutions but also on the general proper solutions of the PDE.

2.3. **Gradient function.** Given a proper TW profile $f_\lambda(\xi)$, $\lambda \in \Delta_0$, for a fixed small $\varepsilon > 0$, we denote by $G(\lambda, \varepsilon)$, the *gradient function* (*G-function*):

$$G(\lambda, \varepsilon) = \frac{d}{d\xi}\, Q(f_\lambda(\xi))|_{Q(f_\lambda(\xi))=\varepsilon}, \tag{49}$$

where the derivative $d/d\xi$ is calculated at the first (minimal) intersection between $Q(f_\lambda(\xi))$ and the ε-level $\{Q(f_\lambda) = \varepsilon\}$. The G-function (49) plays an important role in the derivation of optimal gradient bounds on proper solutions. We assume that $G(\lambda, \varepsilon)$ depends continuously upon the parameters $\lambda \in \Delta_0$ and $\varepsilon > 0$. We need the following monotonicity property of the G-function.

Lemma 4. *Given an $\varepsilon > 0$, $G(\lambda, \varepsilon)$ is strictly monotone decreasing for all $\lambda \in \Delta_0$ for which G is defined.*

Proof. We follow the lines of the analysis in [22], where TW-profiles with first-order interfaces were studied. Introducing in (21) the pressure variable, $g = Q(f) > 0$, we obtain the ODE

$$\psi(g) P \frac{dP}{dg} = F^{-1}(Q^{-1}(g), P, -\lambda P), \quad \psi(g) = Q'(Q^{-1}(g)). \tag{50}$$

Consider the TW-orbits on the $\{g, P\}$-plane, where $g > 0$ and $P > 0$. Fix arbitrary $\lambda_2 < \lambda_1$, $\lambda_{1,2} \in \Delta_0$. Let $P_1(g)$ and $P_2(g)$ be the corresponding minimal orbits. We have to prove the strict inequality

$$P_1(g) < P_2(g) \quad \text{for } g > 0. \tag{51}$$

By the approximation using the strict monotonicity of $F^{-1}(p, q, r)$ in $r > 0$, we have that $P_2^n(g) > P_1^n(g)$ for small $g - 2/n > 0$. Assume that $P_1^n(g)$ intersects $P_2^n(g)$ first time at a finite $g = \bar{g}$ so that $P_1^n(\bar{g}) = P_2^n(\bar{g}) = \bar{P} > 0$ and $P_1^n(g) < P_2^n(g)$ for $g \in (2/n, \bar{g})$. Subtracting equations (50) for the functions P_2^n and P_1^n we obtain at $g = \bar{g}$: $-\lambda_2 \bar{P} > -\lambda_1 \bar{P}$ so that by parabolicity

$$\psi(\bar{g})\bar{P}\left(\frac{dP_2^n}{dg} - \frac{dP_1^n}{dg}\right) = F^{-1}(Q^{-1}(\bar{g}), \bar{P}, -\lambda_2\bar{P}) - F^{-1}(Q^{-1}(\bar{g}), \bar{P}, -\lambda_1\bar{P}) > 0.$$

This gives a contradiction since the left-hand side is nonpositive. Thus, $P_2^n(g) > P_1^n(g)$ for $g > 2/n$. Passing to the limit $n \to \infty$, we first obtain the inequality $\leq$ in (51) and the strict inequality follows in the same way by the strict monotonicity of F^{-1}. $\square$

3. Existence of a nontrivial proper solution

3.1. **The limit semigroup and maximal solutions.** We assume that a continuous local in time solution $u(x, t) \geq 0$ of (1), (2) can be constructed in $S_T = (0, T) \times \mathbf{R}$ by approximation as the limit of a sequence $\{u_n \geq 1/n\}$ of uniformly positive sufficiently smooth solutions to regularized (truncated) equations with regularized initial data bounded away from the singular level:

$$u_t = F_n(u, u_x, u_{xx}) \quad \text{in } S_T; \quad u(x, 0) = u_{0n}(x) \geq 2/n, \;\; x \in \mathbf{R}, \tag{52}$$

where $u_{0n} \to u_0$ uniformly on compact subsets and the monotone decreasing sequence of smooth functions $\{F_n(p, q, r)\}$ converges to $F(p, q, r)$ uniformly on any compact subset from $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$. Moreover, in general, we may assume that

$$F_n(p, q, r) \equiv F(p, q, r) \quad \text{in } \{p \geq 2/n\} \times \mathbf{R} \times \mathbf{R}. \tag{53}$$

In this case we suppose that the operator $\mathbf{F}(u)$ in $\{u \geq 2/n\}$ exhibits no essential singularities. This defines a class of solutions under consideration. Assumption (53) means that we do not involve in the analysis any kind of interaction between $\{u = 0\}$ and other singularities. This can create new types of singular patterns which are not described by the straight line characteristic propagation of TWs.

The derivative $z = (u_n)_x$ satisfies a uniformly parabolic equation with regular coefficients:

$$z_t = F_r z_{xx} + F_q z_x + F_p z, \tag{54}$$

and we assume that in the appropriate functional class the Strong Maximum Principle holds for it. Under the hypothesis that $u_0(x)$ is increasing, choosing strictly increasing $u_{0n}(x)$ from the corresponding class, we can conclude that $(u_n)_x \leq 0$. Hence, $u(x, t)$ is increasing in x for all $t > 0$. Moreover, we assume that the Strong Maximum Principle applies in the positivity domain so that $u_x < 0$ in the positivity set $\{u > 0\}$.

Approximation techniques are well known to work for several quasilinear equations given above and in the case of the divergent differential operators this gives the weak solutions defined via integration by parts in the differential operator. See the survey paper [40] and the books [10], [12], [48]. In the most singular cases (say, for blow-up solutions in the L^∞-sense), a weak formulation beyond blow-up time is often not available. Then such an approximation makes it possible to construct a unique *proper minimal* solution. We refer to the extended semigroup theory of monotone order-preserving semigroups presented in [33], see also references therein. We thus always assume that the truncated problems are non-singular and can be solved by known methods. Necessary results for fully nonlinear parabolic equations are presented in [47], [42], [11], [4], [45], [46], [53], [54], etc.

Let us specify the main assumptions which are necessary to obtain a proper solution, see Section 2 in [33]. We impose some monotonicity assumptions on the approximation. We assume that the sequence of initial data $\{u_{0n}\}$ is monotone decreasing in n as well as the sequence of functions $\{F_n(p,q,r)\}$ for $p \geq 1/n$. The semigroups $S_n(t)$ of the regularized equation on the class of non-singular solutions $\{u \geq 1/n\}$ are assumed to be well-defined, order-preserving and continuous with respect to monotone decreasing convergence. Then (see [33], p. 12) the proper solutions are unique and do not depend on the type of the truncations of both the equation and the initial data provided that these are monotone decreasing.

By comparison the sequence $\{u_n\}$ is monotone decreasing and there exists the limit

$$u(t) \equiv T_*(t)u_0 = \lim S_n(t)u_{0n}, \tag{55}$$

which is called the *proper solution* and T_* is the *limit semigroup*. The proper solutions are *maximal* solutions in a natural way. Due to the construction, the proper solutions satisfy the standard comparison principle and the Maximum Principle. This construction exhibits some features of viscosity solutions of the second-order PDEs [9], [53], [54] which seems cannot cover the whole set of the nonlinear equations under consideration. Actually, in the general case, the proper solutions are maximal viscosity solutions.

The above definition of the proper solution assumes no further regularity and the a priory bounds on the monotone sequence $\{u_n\}$ in order to pass to the limit in (55) which always exists and is finite. This concept of the maximal solution is the weakest one, and the differential properties and other regularity results on the solution $u(x,t)$ are to be studied a posteriori. We first consider the question on the existence of a nontrivial solution, i.e., we derive bounds on u. By definition (55), there exist two possibilities: either the proper solution is nontrivial: $u(x,t) \not\equiv 0$ for small $t > 0$ (*incomplete singularity*) or $u(x,t) \equiv 0$ for any small $t > 0$ (*complete singularity*). We answer the first question.

3.2. **Usual comparison: incomplete singularity (existence).**

Theorem 5. *Let $\Delta_0 \neq \emptyset$. Then there exists the nontrivial proper solution $u(x,t) \not\equiv 0$ for small $t > 0$ with finite propagation.*

Proof. First of all, $u(x,t) \leq u_1(x,t)$ so that u is bounded above. Let us construct a lower bound. Fix a $\lambda \in \Delta_0$. If $u_0(x) \geq f_\lambda(x)$ in $\mathbf{R}$ then by approximation and comparison in view of (53) and by the monotonicity of $\{F_n\}$ we have $u(x,t) \ge f_\lambda(x-\lambda t)$. Hence $u(x,t) \not\equiv 0$.

Assume now that the continuous monotone function $u_0(x) > 0$ for $x > 0$, $u_0(x) \equiv 0$ for $x \le 0$, intersects $f_\lambda(x-a)$, $a \gg 1$ at $x = x_1 \gg 1$. Then $u_0(x_1)$ is bounded away from zero, say, $u_0(x_1) \ge 2$. The further analysis is based on the standard local comparison. Without loss of generality we suppose that $f_\lambda(\xi)$ is a monotone increasing functions and the intersection is unique. For convenience, we set $\xi = x - \lambda t$ so that $f_\lambda(\xi - a)$ becomes a stationary solution of the equation for $v(\xi,t) \equiv u(\xi + \lambda t, t)$

$$v_t = F(v, v_\xi, v_{\xi\xi}) + \lambda v_\xi. \tag{56}$$

Assume now that given u_0 the singularity is complete so that $v(\xi,t) \equiv 0$ for arbitrarily small $t > 0$. Compare in the domain $\{v \ge 2/n\}$ two solutions $v_n(\xi,t)$, $n \gg 1$, and $f_\lambda(\xi-a)$ of the regularized equation (56) under the assumption (53). There exists $\xi_1 \approx x_1$ such that $v_{0n}(\xi_1) = f_\lambda(\xi_1 - a)$. Choose ξ_2 such that $f_\lambda(\xi_2 - a) = 1$. By the assumption on the complete singularity, there exists a positive $t_n \approx +0$ such that $v_n(\xi_2, t_n) = f_\lambda(\xi_2 - a) = 1$ and $v_n(\xi, t_n) \le f_\lambda(\xi - a)$ for $\xi \ge \xi_2$. Thus we obtain that $v_n(\xi, 0) > f_\lambda(\xi - a)$ on $[\xi_2, \xi_1]$ and there holds $v_n(\xi, t_n) \le f_\lambda(\xi - a)$ on $[\xi_2, \xi_1]$, $v_n(\xi_2, t_n) = f_\lambda(\xi_2 - a)$. Obviously, this contradicts the Strong Maximum Principle for the linear parabolic equation for the difference $w = v_n - f_\lambda$ in the non-singular domain $[\xi_2, \xi_1] \times (0, t_n)$, where $v \ge 1/2$. In other words, such a behaviour in the interior domain is possible if there exists another singularity above $u = 0$. Finite propagation follows by the standard local comparison with the TW solutions. □

Thus, under the assumption $\Delta_0 \ne \emptyset$ we always may assume that the proper solution $u(x,t) \not\equiv 0$ is a continuous function in S_T and, by the standard interior regularity results for uniformly parabolic equations, u is sufficiently smooth on any subset where the solution is uniformly bounded away from zero. By the standard local comparison with the TWs, the interface $x = s(t)$ is continuous. By the same arguments, we also prove

Corollary 6. *Let $\Delta \ne 0$. Then there exists the proper solution $u(x,t) \not\equiv 0$ for small $t > 0$.*

In this case we cannot guarantee that the singular interface is finite, see conclusions below.

4. Intersection comparison and nonexistence

4.1. **Sturmian argument and intersection comparison.** An essential feature of the results on the complete or incomplete singularities and other properties for quasilinear degenerate heat equations already mentioned is that their proof is based on the use of the Sturmian argument. The main ideas goes to back to the original paper by C. Sturm [52]. In particular, the Sturmian argument says that given a smooth solution $w(x,t)$ of a linear parabolic equation with smooth coefficients

$$w_t = a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w \ \text{ in } S_T, \quad a > 0, \tag{57}$$

where w does not change sign on the parabolic boundary of S_T, the time-evolution of the number of zeros, $I(t;w)$ of $w(\cdot,t)$ (or the number of sign changes in x of the solution), satisfies the monotonicity property:

$$I(t;w) \ \text{does not increase in time.} \tag{58}$$

Moreover, the multiplicity of zeros also can be taken into account. The original result by Sturm was proved for the analytic in x solutions of the linear heat equation where the zero set is disjoint and the number of zeros is always finite for $t > 0$. Further generalizations [1], [43] showed that the same results are true without analyticity assumptions on the coefficients of the equation and are valid for equations with sufficiently smooth coefficients satisfying the Strong Maximum Principle. An extensive list of references is available in Chapter 4 in [48]. The Sturmian property is essentially used in the analysis of transversality for infinite-dimensional Morse-Smale systems [37], [2] and in the construction of the Floquet bundles (global in time solutions with a fixed number of zeros) for linear one-dimensional parabolic equations [8]. There are other important applications, especially in the stability theory. We refer to the well-known pioneer KPP-paper [41], which also seems to be the first where the convergence to the TW was proved by the intersection comparison (of the type $I(t) = 1$) of the solution with step-like initial data with a family of smooth TWs.

Given two smooth positive solutions $u_1, u_2 \geq \delta > 0$ of equation (1) with smooth right-hand side, one can see that the difference $w = u_1 - u_2$ satisfies the linear equation (57) with the coefficients $a = F_r$, $b = F_q$, $c = F_p$ calculated at the corresponding average points. Under the natural assumptions of the smoothness of the solutions beyond the singular level $\{u = 0\}$, we conclude that the number of zeros of the difference $w(\cdot,t)$ obeys the Sturmian property (58). By approximation, the Sturmian property is true also for the generalized proper solutions $u(x,t) = \lim u_n(x,t)$ since it is true for any pair of sufficiently smooth regularized solutions $u_{1n}(x,t)$ and $u_{2n}(x,t)$ and the result is obtained by the passage to the limit. Moreover, since we are assuming that in the domain where the solutions are uniformly bounded from zero the Strong Maximum Principle for the uniformly parabolic equations applies to the equation for the difference $w = u_1 - u_2$, for any $t \in (0,T)$ the intersections are always points and they form continuous intersection curves in the $\{x,t\}$-plane. New intersection curves cannot appear at a $t_0 > 0$ so that each one is originated at the initial moment $t = 0$.

When a whole *complete* set of special solutions $B = \{V\}$ is involved into the analysis, i.e., we compare $u(x,t)$ with many functions V at the same time, and for a fixed $t_0 \in [0,T)$, we *intersect* the solutions profile $u(x,t_0)$ with an (infinite) number of the profiles $V(x,t_0)$ and calculate the corresponding number of intersections $I(t_0;V)$ (the number of sign changes of the difference $w(x,t_0) = u(x,t_0) - V(x,t_0)$) non-increasing in time due to (58), we arrive at the ideas and techniques of the *intersection comparison*. These ideas are especially effective in the study of singular (say, blow-up) solutions of nonlinear parabolic equations. If we compare two solutions with the same blow-up times, then there occurs a possibility to control not only an upper bound on intersections (by (58) but also a lower bound. It can be done due to the following observation: under some hypotheses two non-intersecting solutions cannot have equal blow-up times. Estimates of $I(t;V)$ from both below and above enclose the analysis and produce and effective way to compare the time-evolution of spatial geometric forms of $u(\cdot,t)$ and particular solutions $\{V(\cdot,t)\}$. The art of intersection comparison essentially depends on the geometric ideas of describing the evolution of the solution curve $\{u(\cdot,t),\ t>0\}$ on the continuous dense set $\{V(\cdot,t),\ t>0;\ V\in B\}$. See the first results on blow-up solutions in [15], [16], [13], [17], [25], [7], more recent papers [3], [18], [19], [30], [32] - [34], and Sections 4 and 7 in [48]. In the present paper we use some of these intersection comparison ideas and techniques.

Below we study the evolution of the proper curve $\{u(\cdot,t),\ t>0\}$ on the complete set of the proper TW solutions $\{V=f_\lambda(x-\lambda t-a), \lambda\in\Delta_0, a\in\mathbf{R}\}$. In fact, in order to establish the main results we will use simple intersection comparison estimates like $I(t;V)\le 1$ or not more than $I(t;V)\le 2$.

4.2. **Complete singularity (nonexistence).** In the case

$$\Delta=\emptyset, \tag{59}$$

the TWs do not connect the singular level with the regular orbits. This is a strong indication that in the corresponding functional class the singularity is *complete*.

Theorem 7. *Let $\Delta=\emptyset$ and any $\lambda\in\mathbf{R}$ is in the steep limit. Then $u(x,t)\equiv 0$ for any small $t>0$.*

This means that the proper maximal solutions $u(t)\equiv 0$ is entirely singular and does not satisfy the initial condition. Therefore, we refer to this case as to the nonexistence of a local solution.

Proof. Fix an arbitrary large $\lambda\gg 1$. Under the assumptions, the solution $f_\lambda^n(\xi)$ with the conditions (26) does not have a finite limit as $n\to\infty$. For $n\gg 1$ and any fixed small $f>0$ there holds: $P^n(f)\to\infty$ as $n\to\infty$. As we know, this means that the TW profiles become very steep on any finite level for $n\gg 1$. Therefore, given arbitrary data $u_0(x)>0$ for $x>0$, $u_0(x)\equiv 0$ for $x\le 0$, choosing an approximation $u_{0n}(x)\le 2/n$ for $x\le 0$ we obtain that for any $n\gg 1$ $u_{0n}(x)\le f_\lambda^n(x+1)$ in an arbitrarily large neighbourhood of $x=0$. Then $u_n(x,t)\le f_n(x-\lambda t+1)$ for $t>0$ by comparison. Since $\lambda\gg 1$ is arbitrary, this means that for any $t>0$ any positive level propagation of $u(x,t)$ occurs with infinite speed. Hence $u(x,t)\equiv 0$ for $t>0$. $\square$

We now change for a while the properties of the initial function $u_0(x)$ and assume that it is strictly positive, say, $u_0\ge 1$ and has the inverse bell-shaped form. Then we may assume that the solution $u(x,t)$ is non-singular and sufficiently smooth for all $t\in(0,T)$ and has a bell-shaped form. Let $T<\infty$ be the first moment of time when the solution touches the zero-level at a finite point. We assume that

$$u(0,T)=0. \tag{60}$$

We then arrive at a similar global nonexistence-like result.

Proposition 8. *Under the hypotheses of Theorem 7, the proper (maximal) continuation of the solution for $t > T$ is trivial: $u(x,t) \equiv 0$.*

Proof. It is based on the intersection comparison with the family of steep TW solutions $\{f_\lambda^n\}$ and is essentially the same as in the TW-construction with $|\lambda| \gg 1$ in [32], see the proof of Theorem 4.1, where the completeness of the singularity is proved for a class of the quasilinear reaction-diffusion equation of the form

$$u_t = (\varphi(u))_{xx} + f(u). \tag{61}$$

It applied also to a fully nonlinear equation from detonation theory [23]. $\square$

4.3. **The pathological class of PDEs.** We now briefly discuss the special case

$$\Delta \neq \emptyset, \quad \Delta_0 = \emptyset. \tag{62}$$

This means that proper singular TW solutions exist (so that the proper solution $u(x,t)$ is nontrivial) but all of them exhibit infinite propagation: for any $\lambda \in \Delta$ $f_\lambda(\xi) > 0$ and $f_\lambda(\xi) \to 0$ as $\xi \to -\infty$. Assuming for simplicity that $f_\lambda'(\xi) \to 0$, we have that the minimal orbits satisfy

$$P_{\min}(f) \to 0, \quad f \to 0, \tag{63}$$

so that for any $\lambda \in \Delta$ there exists an unstable (remind that $P = f' > 0$) manifold of the origin for the ODE (21).

1. Infinite propagation. In general, this implies infinite singular propagation for the parabolic PDE, a result which was first known for the filtration equations (61), $f \equiv 0$, see references in the survey [40]. Such a general conclusion entirely based on the TW-analysis can be found in [36], Theorem 2, for the quasilinear equation (25).

For the fully nonlinear equation (1), one can show a similar result. This analysis is based on a suitable approximation of the minimal orbit $P_{\min}(f)$ from above and also uses a natural monotonicity for the first-order ODEs. For instance, assume that there exists a sequence $\{\lambda_k\} \subset \Delta$, $\lambda_k \to -\infty$, such that for any $k \gg 1$, the positive profile $f^k \equiv f_{\lambda_k}$ can be approached from below by a sequence $\{f_j^k(\xi) \geq 0\}$ of functions vanishing at a finite $\xi_{kj} \to -\infty$ as $j \to \infty$. Then by (63) we get

$$f_j^k(\xi + \xi_{kj}) \to 0, \quad j \to \infty, \tag{64}$$

uniformly on compact subsets. Let us prove inifinite propagation. Given initial data $u_0(x) > 0$ for $x > 0$, $s(0) = 0$, we choose $k \gg 1$ such that by (64) for any $n \gg 1$

$$u_{0n}(x) \geq f_j^k(x + \xi_{kj} - 1).$$

Therefore, passing to the limit $n \to \infty$, by the usual comparison, the maximal solution satisfies for small $t > 0$

$$u(x,t) \geq f_j^k(x - \lambda_k t + \xi_{kj} - 1),$$

so that the interface is such that $s(t) \leq \lambda_k t + 1$. Passing to the limit $k \to \infty$ ($\lambda_k \to -\infty$) we deduce that $s(t) = -\infty$ for arbitrarily small $t > 0$.

2. Pathological behaviour. Let us now consider the "pathological" property of solutions in the case (62), which corresponds to the occurrence of a singularity in finite time for initially strictly positive solution, $u_0(x) > 0$. Remind that the level $\{u = 0\}$ can correspond to an extremely strong singularity like blow-up or extinction. The simplest type of such singularities are exhibited by flat (in x) solutions $u = V(t)$ satisfying the ODE

$$V' = F(V, 0, 0), \;\; t > 0; \quad V(0) = c > 0. \tag{65}$$

Assume that for any small $c > 0$ the unique solution $V(t)$ becomes singular in finite time $T_c < \infty$: $V(t) > 0$ on $(0, T_c)$ and $V(T_c^-) = 0$.

On the other hand, consider the question of the singular behaviour for nontrivial spatially inhomogeneous initial data $u_0(x) > 0$ satisfying $u_0(x) \ge f_\lambda(x - a)$, $x \in \mathbf{R}$ for some TW speed $\lambda \in \Delta$ and $a \gg 1$ (we assume that $f_\lambda > 0$ in $\mathbf{R}$). Then

$$u(x,t) \ge f_\lambda(x - \lambda t - a) > 0 \tag{66}$$

in $\mathbf{R}$ for $t > 0$ by usual comparison of positive sufficiently smooth solutions. This is the pathological property of the PDEs satisfying (62): *while all the flat solutions (65) become singular in finite time, the spatially inhomogeneous (inverse bell-shaped) solutions never create a singularity.* Existence of pathologocal quasilinear heat equations (61) with blow-up, is established in [32], Section 3.

5. Some geometric notions

5.1. **Completeness of proper B-bundle.** The set B is supposed to satisfy a special geometric property of *completeness* which means that it is dense in the sense of construction of tangent solutions which is the basic idea of intersections comparison. Such a set is known to generate a special property of B-concavity/convexity for general solutions which are preserved in time in the appropriate functional solution classes [22], [31]. Loosely speaking, B-concavity of $u(x, t_0)$ means that it is concave with respect to the functions $\{V(x, t_0)\}$. For convenience, we add to the set B the flat solution $V_*(t)$ satisfying the ODE (65). This solution is invariant under translation $t \mapsto T_0 + t$ and can be formally treated as a special limit of the TW-profiles $f_\lambda(\xi - a)$. Denote such an extended TW-set $B_* = B \cup \{V_*\}$.

Consider the question of *completeness* ([31], Section 2). Fix an increasing in x solution $u(x, t) \ge 0$ in S_T so that $u_x \ge 0$. Completeness of B_* means that for any $P_0 = (x_0, t_0) \in \bar{S}_T$ such that $u(x_0, t_0) > 0$ there exists a *unique* function $V \in B_*$, called the *tangent solution* to u at P_0, such that

$$V(x_0, t_0) = u(x_0, t_0) = \mu, \quad V_x(x_0, t_0) = u_x(x_0, t_0) = \nu \tag{67}$$

for arbitrary $\mu > 0$, $\nu \ge 0$, and $V(x, t)$ is well defined for all $t \in [0, t_0]$. The tangential system (67) can be stated for the pressure variable $v = Q(u)$, $\operatorname{Im} Q = [0, \infty)$, on the corresponding branch of the TW-diagram. We have the following typical result, cf. [22].

Proposition 9. *Let the G-function (49), $G(\lambda, \varepsilon)$, be continuous and for any $\varepsilon > 0$*

$$\operatorname{Im}\{G(\lambda, \varepsilon), \;\; \lambda \in \Delta_0\} = (0, \infty).$$

Then the set B_* is complete.

Proof. The unique solvability of the tangential system (67) for $v = Q(u)$ reduces to the existence and uniqueness of $\lambda \in \Delta_0$ such that a unique TW-profile $f_\lambda(x)$ satisfies for some $x_0 > 0$

$$Q(f_\lambda)(x_0) = Q(\mu), \quad (Q(f_\lambda))'(x_0) \equiv G(\lambda, Q(\mu)) = Q'(\mu)\nu. \tag{68}$$

Completeness is a straightforward consequence of the continuity and the monotonicity property of G established in Proposition 2. If $\nu \neq 0$ then $\lambda \in \mathbf{R}$, if $\nu = 0$ then the unique tangent solution is given by V_*. □

If B_* is complete in the class of increasing in x solutions, we also say that the TW set B is complete (in the class of strictly increasing solutions) in the sense that (68) admits a unique solution for any $\mu > 0$ and $\nu > 0$. We then identify the proper set $B = \{V\}$ with the proper *B-bundle* (the *TW-bundle*) $B = \{f_\lambda(\xi),\ \lambda \in \Delta_0\}$ of the proper TW profiles with the interfaces at the origin $\xi = 0$. The completeness of B implies that for any $\xi_0 > 0,\ \mu > 0$ there exists a unique $f_\lambda(\xi) \in B$ such that

$$Q(f_\lambda(\xi_0)) = Q(\mu), \tag{69}$$

i.e., through any point (ξ_0, μ) on the $\{\xi, f\}$-plane there passes a unique curve $f_\lambda(\xi)$ from the TW-bundle. Moreover, the complete set $\{f_\lambda\}$ is ordered (strictly monotone decreasing in λ) for small $\xi > 0$ and is supposed to be continuous relative to λ.

5.2. ***B*-concavity.** Without loss of generality, we assume that given a $t_0 \in [0, T)$, there holds $u(x, t_0) \in C^1_+$ ($u(x, t_0)$ is differentiable in x in the positivity domain). We will use the following definition, see [31], [22].

Definition 2. *The solution profile $u(x, t_0)$ is said to be B-concave (resp. B-convex) if for any $x_0 \in \{x :\ u(x, t_0) > 0\}$ there holds*

$$u(x, t_0) \le V(x, t_0) \quad (u(x, t_0) \ge V(x, t_0)), \quad x \in \mathbf{R}, \tag{70}$$

where $V(x, t_0)$ is the tangent profile at the point (x_0, t_0).

Assume that the initial function $u_0(x)$ is chosen so that new intersections between the proper solution and the functions $V \in B$ cannot appear for $x \gg 1$. Then the property of B-concavity (convexity) is preserved in time.

Proposition 10. *Let B be complete. Then*

$$u_0(\cdot)\ \ is\ B-concave\ \Rightarrow\ u(\cdot, t)\ \ is\ B-concave\ for\ t \in (0, T). \tag{71}$$

Proof. It is established in [31], Section 6, that given a set of particular solutions B, the B-concavity (convexity) is preserved in time under assumptions of the completeness, continuity and the monotonicity of B relative to the parameters. The last two properties are obviously true for the set of TW solutions. Let us give a sketch of the proof. It is based on the fact that a B-concave (or B-convex) solution $u(x, t)$ satisfies for any $t \in [0, T)$

$$I(t; V) \le 2 \quad \text{for any}\ \ V \in B. \tag{72}$$

Therefore, given a B-concave (convex) initial data $u_0 \in C^1_+$, we conclude that if $u(x,t_0)$ with a $t_0 \in (0,T)$ is not B-concave (convex) in the sense of (70) then this evidently contradicts (72) (see the proof in Section 6 in [31]). $\square$

The B-concavity (convexity) for (61) with respect to general sets of the TW solutions is studied in [22], see some results below.

5.3. **Sign-invariant.** Thus, a proper set B of particular solutions defines the corresponding notion of the B-concavity (convexity) preserved in time, so that the appropriate concavity property essentially depends on the nonlinear parabolic equation under consideration. It follows from (70) that for the solutions $u(x,t) \in C^2_+$, the B-concavity is equivalent to the differential inequality

$$u_{xx} \le V_{xx} \quad \text{in } S_T \cap \{u > 0\}. \tag{73}$$

The B-convexity gives the opposite inequality. Calculating the derivative V_{xx} from the corresponding ODE for the TW profiles with the tangency conditions $u = V$, $u_x = V_x$

$$F(u, u_x, V_{xx}) + \lambda u_x = 0, \tag{74}$$

where $\lambda = \lambda(u, u_x)$ is uniquely determined from the tangential system (67), we obtain $V_{xx} = F^{-1}(u, u_x, -\lambda u_x)$ so that the inequality (73) implies that

$$u_{xx} \le F^{-1}(u, u_x, -\lambda(u, u_x)u_x) \quad \text{in } S_T \cap \{u > 0\}. \tag{75}$$

This means that the operator

$$H_B(u) \equiv u_{xx} - V_{xx} = u_{xx} - F^{-1}(u, u_x, -\lambda(u, u_x)u_x) \le 0 \tag{76}$$

preserves the negative sign on the proper parabolic orbits. It also preserves the positive sign due to the opposite property of the B-convexity. Therefore H is a *sign-invariant* [20] of the parabolic PDE which is determined from the complete set B of the TW solutions. One can see that by the construction of the sign-invariant, in the positivity domain

$$H_B(V) \equiv 0 \quad \text{for any } V \in B. \tag{77}$$

Observe that the backward problem: *given a sign-invariant, determine the corresponding set B of particular solutions*, is also of interest. Several new exact solutions of the quasilinear heat equations were found in [20] on the basis of this idea. The B-concavity/convexity properties exhibit a natural connection with the Maximum Principle.

Proposition 11. *$J(x,t) = H_B(u(x,t))$ solves a quasilinear homogeneous parabolic equation*

$$J_t = \mathbf{A}(J) \equiv aJ_{xx} + bJ_x + cJ \quad \text{in } S_T \cap \{u > 0\}, \tag{78}$$

where the coefficients $a > 0$, b and c depend on u, J and are sufficiently smooth.

Proof. Differentiating $J = H_B(u) \equiv \Phi(u, u_x, u_{xx})$ we deduce that

$$J_t = \Phi_p u_t + \Phi_q u_{xt} + \Phi_r u_{xxt}.$$

Calculating the derivatives u_t, u_{xt} and u_{xxt} from the PDE $u_t = F(u, u_x, u_{xx})$, and using a standard linearization-like procedure, we conclude that J solves a parabolic equation

$$J_t = \mathbf{A}(J) + W, \tag{79}$$

where $W(u, u_x)$ is the corresponding continuous non-homogeneous term. Using (77), we obtain that for any $V \in B$

$$W(V, V_x) \equiv 0.$$

Let us show that $W \equiv 0$ by completeness of B. The proof is based on a formal application of the Maximum Principle. Indeed, assuming that $W(\mu, \nu) > 0$ for some $\mu > 0$, $\nu > 0$, we choose a TW $V(x,t)$ such that $V(x_0, 0) = \mu$, $V_x(x_0, 0) = \nu$ and by the construction $J(\cdot, 0) \equiv 0$. Then $J(\cdot, t) \equiv 0$ for $t > 0$ by (77) and this contradicts equation (79) since by regularity $J \equiv H_B(V) > 0$ for all $t > 0$ small and $x \approx x_0$. $\square$

By the Strong Maximum Principle, any solution $J(x,t) \not\equiv 0$ (i.e., $u \not\equiv V \in B$ by (77)) of the quasilinear parabolic equation (78) has isolated zeros in the positivity domain for any $t > 0$. Therefore the results on the B-concavity/convexity follow from the Maximum Principle and say that $J(0) \geq 0$ (≤ 0) implies that $J(t) \geq 0$ (≤ 0) for $t > 0$. The analysis near the singular interfaces needs some extra observations obtained by the intersection comparison with the singular proper solutions $V \in B$ (see the results below).

Example 2. Consider the general filtration equation in the pressure representation

$$u_t = F(u)u_{xx} + (u_x)^2, \quad F \geq 0. \tag{80}$$

Under some natural assumptions on F ([31], Section 2) the complete TW-bundle is given by the liner profiles

$$f_\lambda(\xi) = (-\lambda)\xi, \quad \lambda \in (-\infty, 0) = \Delta_0$$

(hence $Q(u) = u$). Therefore the tangential system (67) has the form $(-\lambda)\xi = u$, $-\lambda = u_x$, so that $\lambda(u, u_x) = -u_x$. The equation (74) takes the form

$$F(u)V_{xx} + (u_x)^2 - u_x u_x = 0,$$

and then $V_{xx} = 0$. Therefore the sign-invariant is given by $H_B(u) = u_{xx}$. Hence the complete set B of piece-wise linear in x solutions generates the standard notions of concavity/convexity and both ones are preserved in time on the solutions of the parabolic PDE. See a complete analysis in [31] (moreover, for the compactly supported solutions, the B-concavity holds eventually in time; see also some results below).

The idea of intersections comparison with families of simple particular solutions with envelopes is a fruitful one in the study of essentially nonstationary combustion processes. The first results on the intersection comparison with a complete set of stationary solutions (the *method of stationary states* [24]) of the quasilinear heat equations are presented in [15], [17], see also Section 7 in [48]. The general monotonicity property of large solutions in [25], [26] can be treated as the eventual B-concavity result. Other examples are presented in [22].

5.4. B**-number.** We now consider more general proper solutions which are not B-concave or B-convex so that $H_B(u(x,t))$ changes sign in the positivity domain. The sign-invariant H_B makes it possible to introduce an important geometric characteristic of the proper solutions $u(x,t)$. We assume that $u(x,t_0) \in C_+^2$ for any $t_0 \in [0,T)$.

Definition 3. *Given the complete set B, the B-number $Z_B(t_0)$ of the proper profile $u(x,t_0)$ is the number of sign changes of $H_B(u(x,t_0))$ in the positivity domain.*

Given a point $P_0 = (x_0,t_0) \in \{u > 0\}$, $t_0 \in (0,T)$, where $H_B(u(x_0,t_0)) > 0$ (resp. < 0), the tangent solution $V \in B$ satisfies the strict inequality in (73)

$$u_{xx}(x_0,t_0) < V_{xx}(x_0,t_0) \quad (> V_{xx}(x_0,t_0)). \tag{81}$$

This follows by the Strong Maximum Principle applied to the linear parabolic equation (78) for the sign-invariant $J = H_B(u)$. On the other hand, if $H_B(u(x_0,t_0)) = 0$ and $H_B(u(x,t_0))$ changes sign at $x = x_0$ then this point is an *inflection* (intersection) point between the profile $u(x,t_0)$ and the tangent solution $V(x,t_0)$. Therefore, $Z_B(t)$ is equal to the number of the inflection points which the profile $u(x,t_0)$ can exhibit with all the tangent TW profiles $V(x,t_0)$.

We next characterize the main property of the B-number.

Theorem 12. *The finite B-number $Z_B(u(\cdot,t))$ of a proper solution u does not increase.*

Proof. It is not difficult to see that the appearance of a new zero of $H_B(u)$ at P_0 would mean that a new intersection between $u(x,t)$ and the tangent solution $V(x,t)$ at P_0 occurs for $t > t_0$ contradicting the Sturmian argument. The same is true at the singular interfaces and the proof is quite similar to that of Lemma 3.2 in [31]. □

For convenience, we state the last result separately.

Lemma 13. *If $Z_B(0) < \infty$ then no new sign changes of the sign-invariant $H_B(u(\cdot,t))$ are generated at the interfaces.*

The B-number gives a useful intersection bound.

Theorem 14. *Let B be complete and $Z_B(0) < \infty$. Then for any $t \in (0,T)$*

$$I(t;V) \le Z_B(t) + 2 \quad \text{for all } V \in B. \tag{82}$$

Proof. We apply the method of the evolution upon the parameters similar to those applied in [25], [26] and [31]. We argue by contradiction. Assume that there exists a TW solution $V = f(\xi + a, \lambda) \in B$ such that $I(t_0;V) \ge Z_B(t_0) + 3$. Let $x_1 < x_2 < x_3$ be any three neighbour points of intersections. Checking the continuous evolution upon the parameters, we conclude that there exists at least one inflection point $\tilde{x} \in (x_1,x_3)$ between the profile $u(x,t_0)$ and the corresponding tangent profile $\tilde{V}(x,t_0)$. Therefore, there exist more than $Z_B(t_0)$ inflection points on the profile $u(x,t_0)$ which are sign changes of $H_B(u(x,t_0))$. This contradicts the assumption. □

5.5. **Eventual B-concavity.** As a new example, we prove a general result on the property of the *eventual* B-concavity of the compactly supported solutions. We assume that $u_0(x)$ is compactly supported and has a bell-shaped form with a single maximum. Such bell-shaped form is preserved in time which follows from the Maximum Principle for the linear parabolic equation for the derivative $z = u_x$ (see Section 3.1), and $u(x,t)$ has two interfaces. Since u_x takes now both the positive, $u_x > 0$ and the negative sign, $u_x < 0$, we introduce the tangent TW solutions defined by (67) with $\nu < 0$. For simplicity, we impose the condition

$$F(p, -q, r) \equiv F(p, q, r), \tag{83}$$

so that the parabolic PDE is invariant under the reflection transformation $x \mapsto -x$. This is always true for the general quasilinear heat operators (or the filtration ones)

$$F(u, u_x, u_{xx}) = (\varphi(u))_{xx} + \psi(u) \equiv \varphi'(u)u_{xx} + \varphi''(u)(u_x)^2 + \psi(u). \tag{84}$$

Therefore, the tangent solutions with $\mu > 0$ and $\nu < 0$ are given by the reflection $f_\lambda(\xi) \mapsto f_{-\lambda}(\xi)$. We also impose the condition

$$F(p, 0, 0) \equiv 0. \tag{85}$$

Then the flat solution given by the ODE (65) is a constant one: $V_*(t) \equiv \mu$ for any $t \geq 0$. (One can consider the more general case where $F(p, 0, 0) \leq 0$, so that $\psi(p) \leq 0$ in (84), then $V_*(t)$ is decreasing.) In order to follow the lines of the proof of the eventual concavity for the filtration equation (see [31]), we assume that $u(x,t)$ exists in $(0, \infty) \times \mathbf{R}$ and

$$\sup_x u(x,t) \to 0, \quad t \to \infty.$$

We assume that the set B with $\Delta_0 = (-\infty, 0)$ is complete and the profiles $f_\lambda(\xi)$ are strictly monotone and are defined for all $\xi > 0$.

Proposition 15. *Assume that the initial B-number is finite: $Z_B(0) < \infty$. Then there exists a finite $t_* \geq 0$ such that $u(\cdot, t)$ is B-concave and $Z_B(t) = 0$ for any $t \geq t_*$.*

Proof. It is quite similar to that in Section 3 of [31] (see also [22]). The key idea of the proof is the compactness argument (see p. 752-753 in [31]) establishing that all the inflection curves on which $H_B(u(x,t)) = 0$ disappear at the interfaces in finite time. We also should take into account that the interfaces of $u(x,t)$ propagate for $t \gg 1$ slower than any travelling wave. This follows by comparison with the TWs $f_{\pm\lambda}(x \pm \lambda t \pm a)$, $a \gg 1$. Indeed, $u_0(x) \leq f_{\pm\lambda}(x \pm a)$ in $\mathbf{R}$ provided that $a \gg 1$ whence the necessary bounds on the interfaces. $\square$

5.6. **Strong Maximum Principle for interfaces.**

Proposition 16. *Let $B = \{f_\lambda,\ \lambda \in D_0\}$ be complete and sufficiently smooth in λ. Assume that for any $\lambda \in \Delta_0$*

$$\left(\frac{\partial f}{\partial \lambda}\right) = o\left(\frac{\partial f}{\partial \xi}\right), \quad \xi \to +0. \tag{86}$$

Fix a TW solution $V(x,t) = f_c(x-ct-a) \in B$. Assume that for some $t_0 \in (0,T)$ the interfaces of $u(x,t_0)$ and $V(x,t_0)$ coincide: $s(t_0) = s_c(t_0) = ct_0 + a$, and the B-number is finite, $Z_B(t_0) < \infty$. If $s(t) \not\equiv s_c(t)$ on $[0,t_0]$, then

$$s(t) \neq s_c(t) \quad \text{on any small interval } (t_0, t_0+\varepsilon),\ \varepsilon > 0. \tag{87}$$

Proof. Since $Z_B(t_0) < \infty$, assuming that $u(\cdot, t_0)$ is, say, B-convex near the interface, we consider the tangent TW profiles. Let $s(t_0) = 0$. Namely, for any small $x_0 > 0$ we consider the tangential system

$$u(x_0,t_0) = f_\lambda(\xi_0), \quad u_x(x_0,t_0) = f'_\lambda(\xi_0), \tag{88}$$

where $\xi_0 > 0$ and $\lambda < c$ are uniquely determined and depend on $x_0 \approx +0$. Then the tangent TW profiles at the points (x_0,t_0) are given by $V(x,t_0) = f_\lambda(x-(x_0-\xi_0))$. By the B-convexity of $u(x,t_0)$ near the interface, we have that $u(x,t_0) \geq f_c(x)$ for small $x \geq 0$ and $u(x,t_0) \geq f_\lambda(x-(x_0-\xi_0))$ for any $x_0 \approx +0$. By the standard local comparison of the solutions $u(x,t)$ and any tangent solution $V(x,t)$ we conclude that these inequalities are true for small $x \approx 0$ and $t - t_0 > 0$. Therefore, the interface $s_\lambda(t) = \lambda(t-t_0) + (x_0 - \xi_0)$ of the tangent solution V overtakes the interface $s_c(t) = c(t-t_0)$ of the fixed TW after the time

$$\Delta t(x_0) = \left(\frac{x_0 - \xi_0}{c - \lambda}\right)(x_0).$$

We now pass to the limit $x_0 \to +0$. Then by continuity, $\lambda \to c-0$ and $\xi_0 \to 0$. Using the L'Hospital rule, we conclude that

$$\lim_{x_0 \to +0} \Delta t(x_0) = \lim_{x_0 \to +0} \left(1 - \frac{d\xi_0}{dx_0}\right)\left(-\frac{d\lambda}{dx_0}\right)^{-1}. \tag{89}$$

Differentiating the first equation in (88), we deduce that

$$\frac{\partial u}{\partial x}(x_0,t_0) = \frac{\partial f_\lambda}{\partial \xi}\frac{d\xi_0}{dx_0} + \frac{\partial f_\lambda}{\partial \lambda}\frac{d\lambda}{dx_0}.$$

Using the second equation, we compute that

$$\frac{d\xi_0}{dx_0} = 1 - \frac{d\lambda}{dx_0}\left(\frac{\partial f_\lambda}{\partial \lambda}\right)\left(\frac{\partial f_\lambda}{\partial \xi}\right)^{-1}.$$

Substituting it into (89), we obtain

$$\lim_{x_0 \to +0} \Delta t(x_0) = \lim_{x_0 \to +0} \left(\frac{\partial f_\lambda}{\partial \lambda}\right)\left(\frac{\partial f_\lambda}{\partial \xi}\right)^{-1}(\xi_0) = 0$$

by the assumption (86) and the smoothness of $f_\lambda(\xi)$ in λ.

Finally, $s(t) < s_c(t) = c(t-t_0)$ for any small $t - t_0 > 0$. $\square$

6. Regularity analysis in B-classes

6.1. **B-classes and gradient bounds.** Given the set $B = \{f_\lambda,\ \lambda \in \Delta_0\}$, we introduce two functional classes $B^{\pm}(c)$, where $c \in \Delta_0$ is a parameter. We first consider the *local* B-classes which describe some differential properties of the proper solutions near the singular interface, in a small ν-neighbourhood of $x = s(t)$, $\nu > 0$. Since the analysis is essentially local in space, without loss of generality we assume that for all $\lambda \in \Delta_0$ the proper profiles $f_\lambda(\xi)$ are monotone increasing in $\xi \geq 0$. Fix a $t_0 \in [0, T)$ and, as usual, assume that the solution profile $u(x, t_0)$ is continuous and monotone with the interface at $x = s(t_0) = 0$.

By $I_\nu(t_0; V)$ we denote the number of intersections between $u(x, t_0)$ and $V(x, t_0)$ on the right-hand ν-neighbourhood of the interface $x = 0$. The definition is local in space and it applies to rather arbitrary solutions, e.g., to bounded, compactly supported and bell-shaped profiles $u(x, t_0)$ with the left-hand interface at $s(t_0) = 0$.

Definition 4. *Let $c \in \Delta_0$. (i) We say that $u(x, t_0)$ belongs to the local class $B^+_{\rm loc}(c)$ if there exists a small $\nu > 0$ such that the TW solutions $V(x, t; c, a) \equiv f_c(x - c(t - t_0) - a) \in B$ satisfy*

$$I_\nu(t_0; V) = 1 \quad for\ all\ a \approx +0\ and\ I_\nu(t_0; V) = 0\ \ if\ \ a = 0. \tag{90}$$

(ii) $u(x, t_0)$ belongs to the local class $B^-_{\rm loc}(c)$ if

$$I_\nu(t_0; V) = 1 \quad for\ all\ a \approx -0\ and\ I_\nu(t_0; V) = 0\ \ if\ \ a = 0. \tag{91}$$

It follows from the definition that in the local class $B^+_{\rm loc}(c)$ at $t = t_0$ the TW profiles $\{f_c(x - a)\}$ with the shifting parameter to the right, $a \approx +0$, are sufficiently "steep" to intersect the solution profile $u(x, t_0)$ exactly once in a right ν-neighbourhood of the interface. On the contrary, in the second class $B^-_{\rm loc}(c)$ the TW profiles are flat enough to intersect $u(x, t_0)$ once if they are shifted to the left with $a \approx -0$. In both cases, by continuity, no intersection exists in the ν-neighbourhood with the TW without shifting, $a = 0$. Observe that the character of intersections of $u(x, t_0)$ and $f_c(x - a)$ are different for both B-classes. Namely, it is from below for the first class and from above for the second one.

From the monotonicity of the G-function upon λ we obtain the following gradient bounds. Recall the definition of the pressure variable $v = Q(u)$.

Proposition 17. *(i) If $u(x, t_0) \in B^+_{\rm loc}(c)$ then for small $x - s(t_0) \geq 0$*

$$\overline{D}_x v(x, t_0) \equiv \limsup_{x \to x_0} \frac{v(x, t_0) - v(x_0, t_0)}{x - x_0} \leq G(c, v_0). \tag{92}$$

(ii) If $u(x, t_0) \in B^-_{\rm loc}(c)$ then for small $x - s(t_0) \geq 0$

$$\underline{D}_x v_0 \equiv \liminf_{x \to x_0} \frac{v(x, t_0) - v(x_0, t_0)}{x - x_0} \geq G(c, v_0). \tag{93}$$

If $v(x, t_0) \in C^1_+$ (by the assumptions, this is true by the interior regularity if $t_0 > 0$) then these estimates mean that for $x \approx s(t_0) + 0$

$$(i)\ \ v_x(x, t_0) \leq G(c, v_0) \quad \text{and} \quad (ii)\ \ v_x(x, t_0) \geq G(c, v_0). \tag{94}$$

Proof. (i) Let $t_0 = 0$. It follows from the definition and the monotonicity property of the G-function (Lemma 3) that the steep enough profiles $Q(V(x,0)) \equiv Q(f_c(x-a))$ intersects $v_0 = Q(u_0(x))$ from below, i.e., for $a \approx +0$ at the intersection point the differences $Q(V(x,0)) - v_0(x)$ changes sign from minus to plus when x crosses the intersection point. This implies (92). (ii) Similarly, if $u_0 \in B^-_{\rm loc}(c)$ then for $a \approx -0$ near the interface the intersection is from above and (93) follows. □

Fix a $t_0 \in (0,T)$, so that the spatial profile $u(x,t_0)$ is sufficiently smooth in the positivity domain. Then the inclusions $u(x,t_0) \in B^+_{\rm loc}(c_1)$ and $u(x,t_0) \in B^-_{\rm loc}(c_2)$, $c_{1,2} \in \Delta_0$, imply that the solution profile $v = v(x,t_0)$ satisfies the following gradient bound near the singular interface where $v_x > 0$:

$$\frac{v_x}{G(c_1,v)} \le 1, \quad \frac{v_x}{G(c_2,v)} \ge 1. \tag{95}$$

Both bounds give a Bernstein-type estimate on the first derivative near the singularity. Observe that this estimate is expected to be optimal since the corresponding equalities are true for the set of the TW solutions.

The above definition of the local classes is extended to the global one when we count all the intersections in $\mathbf{R}$ of the proper solution with the TW solutions.

Definition 5. *Let $c \in \Delta_0$ and $t_0 \in [0,T)$. (i) $u(x,t_0)$ belongs to the global class $B^+(c)$ if*

$$I(t_0;V) \le 1 \quad \textit{for all } a \in \mathbf{R} \textit{ and } u(x,t_0) \le f_c(x) \text{ in } \mathbf{R}. \tag{96}$$

(ii) $u(x,t_0)$ belongs to the global class $B^-(c)$ if

$$I(t_0;V) \le 1 \quad \textit{for all } a \in \mathbf{R} \textit{ and } u(x,t_0) \ge f_c(x) \text{ in } \mathbf{R}. \tag{97}$$

By this definition, global B-classes control the type of intersections in the whole space which gives gradient bounds for all $x \in \mathbf{R}$. Evidently, (96) and (97) impose certain restrictions on the behaviour of the solutions $u(x,t_0)$ as $x \to \infty$. This gives the following result. In the proof we always assume that due to the natural properties of the proper solutions in the positivity domain, new intersections between $u(x,t)$ and the TW solutions $f_\lambda(x - \lambda t - a)$ cannot appear at $x = \infty$.

Proposition 18. *If $u_0 \in B^\pm(c)$ then $u(\cdot,t) \in B^\pm(c)$ for all $t \in (0,T)$.*

6.2. **Instantaneous smoothing in the local B-classes.** By the intersection comparison, using the local interior regularity for positive solutions, we state the *smoothing effect* as follows. Denote by $\xi_*(\lambda,\varepsilon) > 0$ the minimal intersection point of the profile $Q(f_\lambda(\xi))$ with the positive ε-level:

$$Q(f_\lambda(\xi_*)) = \varepsilon.$$

Theorem 19. *(i) Let $(-\infty,c] \subseteq \Delta_0$ and for any fixed small $\varepsilon > 0$*

$$G(\lambda,\varepsilon) \to \infty, \quad \lambda \to -\infty. \tag{98}$$

Given an arbitrary u_0, for any $\delta > 0$ there exists $c = c(\delta) \in \Delta_0$ such that $u(\cdot,t) \in B^+_{\rm loc}(c)$ for small $t - \delta > 0$.

(ii) Let $[c, \infty) \subseteq \Delta_0$ and for any fixed small $\varepsilon > 0$

$$G(\lambda, \varepsilon) \to 0, \quad \lambda \to \infty. \tag{99}$$

Let

$$\lim_{\lambda \to \infty} \frac{\xi_*(\lambda, \varepsilon)}{\lambda} \to 0, \quad \varepsilon \to 0. \tag{100}$$

Then for any $\delta > 0$ there exists a $c(\delta) \in \Delta_0$ such that $u(\cdot, t) \in B^-_{\rm loc}(c)$ for small $t - \delta > 0$.

(iii) If $u_0 \in B^{\pm}_{\rm loc}(c)$, then $u(\cdot, t) \in B^{\pm}_{\rm loc}(c)$ for all small $t > 0$.

Proof. (i) Let $s(\delta/2) = 0$. The proof is based on the intersection comparison with a subset $\{f_\lambda\} \subset B$ where $\lambda \ll -1$. The profile $u(x, \delta/2)$ is continuous and hence there exists a $\lambda \in \Delta_0$, $\lambda \le -2$, such that $u(x, \delta/2) \le f_\lambda(x + \delta/2)$ in a neighbourhood of the origin, say, for $x \in (-1, 1)$. Since the solution $u(x, t)$ is uniformly smooth in the positivity domain where $u \ge \mu = u(\delta, \delta/2) > 0$, there exists a constant $c \ll 2\lambda$ such that the steep profiles $f_c(x - a)$ intersect $u(x, \delta/2)$ exactly once, say, for any $a \in [\delta/2, 1/2]$. Denote such subset of the TW solutions $V(x, t) = f_c(x - c(t - \delta/2) - a)$ as $B_{c,\delta} \subset B$.

Since the speeds of these chosen TW's, f_λ and f_c, differ strongly, $c \ll -2|\lambda|$, it follows that after the *time of smoothing*

$$\Delta t = -\frac{\delta}{c - \lambda} \le \frac{\delta}{|\lambda|} \le \frac{\delta}{2},$$

the solution profile $u(x, \delta)$ is completely "covered" near the interface by the profiles $V(x, \delta) \in B_{c,\delta}$ in the sense that for a small $\nu > 0$ and $t = \delta$

$$I_\nu(t; V) \le 1 \ \text{ for any } V \in B_{c,\delta}, \tag{101}$$

and moreover for any $x_* > s(\delta)$ small enough there exists a unique $V_* \in B_{c,\delta}$ such that $V_*(x, \delta)$ intersects $u(x, \delta)$ exactly once at $x = x_*$, $I_\nu(\delta; V_*) = 1$. Since by construction the steep profile $V_*(x, \delta)$ intersects $u(x, \delta)$ at $x = x_*$ from below, i.e., the difference $V_*(x, \delta) - u(x, \delta)$ changes sign from minus to plus when x crosses x_*, we conclude that $u(x, \delta) \in B^+_{\rm loc}(c)$. By the same intersection comparison argument this remains true for $t - \delta > 0$ sufficiently small.

(ii) The proof is similar. Let us perform necessary estimates. We fix small $\mu = u(\delta, \delta/2) > 0$ and we first find values of the TW speed $c \gg 1$ and the translational parameter $a = a_0 < 0$ of the TW profile $f_c(x - a_0)$ satisfying the conditions

$$Q(f_c)(a_0) = Q(\mu) = \varepsilon, \quad (Q(f_c))'(a_0) \ll 1. \tag{102}$$

These equations mean that on the ε-level the profile $Q(f_c(x - a_0))$ is very flat to intersect the pressure profile $v(x, \delta/2)$ exactly once in a neighbourhood of $x = 0$, excluding the points $x \in [0, \delta]$ near the singular interface at $x = s(\delta/2) = 0$ with unknown regularity. Remind that in the positivity set $\{v \ge \varepsilon\}$ the monotone profile $v(x, \delta/2)$ is sufficiently smooth and by the Strong Maximum Principle applied to the linear parabolic equation for the derivative $z = v_x$, it is assumed to be strictly positive and bounded away from zero, say, $v_x \ge \nu_\delta > 0$ (see Section 2.1).

It follows from the properties of the TW profiles that such constants $c \gg 1$ and a_0 exist and moreover for a fixed $c \gg 1$ one can chose $a_0 = -\xi_*(c, \varepsilon) < 0$. We then form the corresponding subset $B_{c,\delta} \subset B$ of the TW solutions $V(x,t) = f_c(x - c(t - \delta/2) - a)$ with $a \le a_0$. With such a choice of the parameters, the time of the regularization is estimated as follows

$$\Delta t \sim \frac{1}{c}\, \xi_*(c, \varepsilon), \tag{103}$$

so that we need the extra assumption (100) which says that it can be arbitrarily small for $\varepsilon \ll 1$. This means that after the time $\Delta t \le \delta/2$, in a right-hand neighbourhood of the interface $x = s(\delta)$ the profile $u(x, \delta)$ will be completely covered by the TW profiles $f_c(x - c\delta/2 - a)$ with a single intersection from above. This means that $u(x, \delta) \in B^-_{\rm loc}(c)$. By the intersection comparison with the subset $B_{c,\delta}$ we obtain the same result for small $t - \delta > 0$. In the case (iii) initial data u_0 are already smooth enough and the proof is straightforward. □

For convenience, we state the following general result on the gradient bounds near the interface and on the interface for singular proper solutions.

Theorem 20. *Let $(-\infty, c] \cup [c, \infty) \subseteq \Delta_0$. Then for any small $\delta > 0$ there exists a $c = c(\delta) \gg 1$ such that for all $t \in [\delta, T - \delta]$ near the interface*

$$G(c, u) \le (Q(u))_x \le G(-c, u) \tag{104}$$

and

$$f_c(x - s(t)) \le u(x,t) \le f_{-c}(x - s(t)). \tag{105}$$

In the case where either $(-\infty, c] \notin \Delta_0$ or $[c, \infty) \notin \Delta_0$ we obtain the corresponding one-sided estimates in (104) and (105).

Proof. (104) is the consequence of (95). The estimates on the interface (105) follows from the intersection comparison with the steep ($-c \ll -1$) and the flat ($c \gg 1$) TW solutions as in the proof of the regularization effect. See also the definition of the local B-classes. □

6.3. **Eventual smoothing and waiting time effect.** The absence of the instantaneous smoothing can lead to singular effects like "waiting time" for the PME. It is possible if at least one class $B^+_{\rm loc}(-c)$ or $B^-_{\rm loc}(c)$ is not well defined for $c \gg 1$ due to the nonexistence of the corresponding proper TW solutions or in view of their "bad" properties. In particular, for the PME it is the class $B^-_{\rm loc}(c)$ since the TW profiles f_λ does not exist for $\lambda \ge 0$.

We now describe the *eventual smoothing* phenomenon. The statement below is motivated by the waiting time effect for the PME and for convenience we choose the same range Δ_0 of the TWs.

Proposition 21. *Let $\Delta_0 = (-\infty, 0)$ and the proper set $B = \{f_\lambda\}$ is complete. Assume (98) holds. Then for any $t_0 \in (0, T)$:*

(i) $u(\cdot, t_0) \in B^+_{\rm loc}(c(t_0))$.

(ii) $\bar{D}^+ s(t_0) \equiv \limsup_{t\to t_0+0}\,[s(t)-s(t_0)]/(t-t_0) \le 0$.
(iii) Let u_0 *be* B*-convex. If* $s(t_0) \ne s(0)$ *then* $u(\cdot,t_0) \in B^-(c(t_0))$.

The last property (iii) means that once the interface moves, the solution attains the lower gradient bound (the upper one is always available for any $t>0$ by (i)).

Proof. (i) Follows from Theorem 19, (i). (ii) This is proved by local comparison. By completeness of B, $f_\lambda \to 0$ as $\lambda \to 0$, so that for any $a \approx s(t_0)+0$ there exists a $\lambda_a \approx -0$ such that $u(x,t_0) \ge f(x-a;\lambda_a)$, $\quad x \approx s(t_0)+0$. Hence $u(x,t) \ge f(x-\lambda_a(t-t_0)-a;\lambda_a)$ by comparison so that $s(t)-s(t_0) \le \lambda_a(t-t_0)+a-s(t_0)$ whence the result by passing to the limit $a \to s(t_0)+0$.

(iii) Let $s(0)=0$ and let $u_0(x)$ touches 0 at $x=0$ so smoothly that $u_0 \notin B^-(c)$ for any $c<0$ (otherwise, if such $c<0$ exists, the result is true for any $t_0>0$). Assume $s(t_0)<0$. Set $a_0 = s(t_0)/2 < 0$ and choose $c = s(t_0)/2t_0 < 0$, $c \in \Delta_0$. The TW solution $V(x,t) = f_c(x-ct-a_0)$, with the interface at $a_0 = s(t_0)/2<0$ for $t=0$ and at $s(t_0)$ for $t=t_0$, has a unique intersection with the B-convex initial function u_0: $I(0,V)=1$. This follows from the property (72) of the B-convexity. Indeed, if $I(0,V)\ge 2$ then using the mutual location of the interfaces of both the solutions, by slightly varying the parameters c and a_0 one can find a new solution $V' \in B$ such that $I(0,V') \ge 3$ contradicting (72). This is the main feature of the B-convexity/concavity, see [31] and [22]. This unique intersection between $u(x,t)$ and $V(x,t)$ for $t \in (0,t_0)$ must disappear at $t=t_0$ where the interfaces coincide (see [31], p. 1097). Indeed, if the intersection stays uniformly away from the interface $x=s(t)$ for $t \approx t_0-0$, then by a slight shifting $V(x,t)$ in x to the right-hand side, by the standard comparison we get that the interface $s(t)$ cannot reach $s(t_0)$ at $t=t_0$.

Thus, at $t=t_0$ we have $I(t_0,V)=0$, so that $u(x,t_0) \ge V(x,t_0) = f_c(x-s(t_0))$. Since the profile $u(x,t_0)$ is B-convex, from (72) we conclude that it belongs to $B^-(c)$ (see Definition 5, (ii)). Moreover, by Proposition 18, $u(\cdot,t)\in B^-(c)$ for all $t\in[t_0,T)$. $\square$

We now present a simple condition where the above eventual smoothing effect becomes the instantaneous one since no waiting time is available. We again impose a convenient assumption of the B-convexity, which is not that restrictive and in the local B-classes can be dispensed with (see the comment below).

Proposition 22. *Let* u_0 *be* B*-convex,* $s(0)=0$*, and let* $u_0(x)$ *be the envelope to the family* $\{f(x-a,\lambda)\}$ *of the proper TW profiles so that for any* $x_0>0$ $f(x-a(x_0);\lambda(x_0))$ *is the tangent profile to* $u_0(x)$ *at* $x=x_0$*. Assume*

$$\limsup_{x_0\to+0} \frac{a(x_0)}{\lambda(x_0)} \ge 0. \tag{106}$$

Then $s(t)<0$ *and* $u(\cdot,t)\in B^-(c(t))$ *for any small* $t>0$*.*

Proof. Fix a small $t>0$. By the assumption, there exists a sequence $\{x_{0k}\}\to+0$ such that for $k \gg 1$

$$\frac{a(x_{0k})}{\lambda(x_{0k})} \ge -\frac{t}{2}.$$

By the B-convexity, $u_0(x) \le f(a - a(x_{0k}); \lambda(x_{0k}))$, so that

$$s(t) \le \lambda(x_{0k})t + a(x_{0k}) \equiv \lambda(x_{0k})[t + a(x_{0k})/\lambda(x_{0k})].$$

Choosing $k \gg 1$, we deduce that $s(t) < 0$. Then Proposition 21, (iii) applies to guarantee that $u(\cdot,t) \in B^-(c)$. $\square$

7. Interface equation and regularity

Let us remind the definition of the interface operators from Section 2.2. For the proper solutions $u(x,t)$ these are defined in the same way with the natural translation $x \mapsto s(t) + \xi$.

7.1. The case $\Delta_0 = \mathbf{R}$.

Theorem 23. *Let the set B with $\Delta_0 = \mathbf{R}$ be complete and continuous upon λ. Assume that (98) - (100) hold. Then:*

(i) The corresponding interface operators are finite on the proper solution $u(\cdot,t)$ for any $t \in (0,T)$. If the TW interfaces are of the first-order, then the interface equation is

$$D^+s(t) = \mathbf{N}_1(u(\cdot,t)) \equiv A_0^{-1}((Q(u))_x(s(t),t)), \quad t \in (0,T). \tag{107}$$

If the TW interfaces are of the second-order then the interface equation is the system

$$\mathbf{M}_1(u(\cdot,t)) = A_0, \quad D^+s(t) = \mathbf{N}_2(u(\cdot,t)), \quad t \in (0,T). \tag{108}$$

(ii) The one-sided derivatives $D^{\pm}s(t)$ are uniformly bounded on $[\delta, T-\delta]$ for arbitrarily small $\delta > 0$ and hence the interface is uniformly Lipschitz continuous there.

Proof. (i) We first prove that the interface operators are well-defined on the proper solutions. Fix an arbitrary $t_0 \in (0,T)$ and consider the solution profile $u(x,t_0)$ on the complete TW-bundle. Let $s(t_0) = 0$. Denote

$$\Lambda(t_0) = \{\lambda \in \Delta_0 : \ u(x,t_0) \text{ intersects } f_\lambda(x)\} \ne \emptyset. \tag{109}$$

In view of completeness of B, $\Lambda(t_0) = \emptyset$ would mean $u(x,t) \equiv f_\lambda(\xi)$ for some $\lambda \in \Delta_0$ by uniqueness of the proper solution. We exclude this trivial case from the consideration. Moreover, by the Strong Maximum Principle for uniformly parabolic equations we have that $u(x,t_0) \not\equiv f_\lambda(x)$ on any compact subset in x from the positivity domain of both solutions. The set $\Lambda(t_0)$ is open by continuity. It follows from Theorem 17 that

$$\Lambda(t_0) \text{ is bounded.} \tag{110}$$

Given a small $X > 0$, we introduce the following two functions:

$$\lambda_-(X) = \inf\{\lambda \in \Lambda(t_0) : \ u(x,t_0) \text{ intersects } f_\lambda(x) \text{ on } (0,X)\},$$

$$\lambda_+(X) = \sup\{\lambda \in \Lambda(t_0) : \ u(x,t_0) \text{ intersects } f_\lambda(x) \text{ on } (0,X)\}.$$

Then $\lambda_-(X) < \lambda_+(X)$, $\lambda_-(X)$ is increasing and $\lambda_+(X)$ is decreasing by construction. It follows that

$$f(x;\lambda_+(X)) \le u(x,t_0) \le f(x;\lambda_-(X)), \quad x \in (0,X). \tag{111}$$

By monotonicity, there exist finite limits

$$\lim_{X \to +0} \lambda_\pm(X) = \lambda^\pm, \quad \lambda^+ \ge \lambda^-. \tag{112}$$

If $\lambda^- < \lambda^+$ then for any $\lambda \in (\lambda^-, \lambda^+)$ the profile $u(x,t_0)$ intersects the TW profiles $f_\lambda(x)$ infinitely many times which contradicts the following observation.

Proposition 24. *For $t_0 \in (0,T)$ the proper solution profile $u(x,t_0)$ cannot have infinitely many intersections with two different TW profiles $f(x;\lambda_1)$ and $f(x;\lambda_2)$ with $\lambda_1 \ne \lambda_2$.*

We thus do not need a result on the disappearance for any $t > 0$ of an infinite number of intersections between a fixed pair of solutions $u(x,t)$ and $V(x,t)$ with coinciding interfaces. This is known for uniformly parabolic equations [1], [43]. For the fully nonlinear equation (1), this procedure gives a linearized parabolic equation with a strong moving degeneracy of the coinciding interfaces and such a result on finite number of intersections, in general, is not straightforward.

Proof. We argue by contradiction. Consider the time-evolutions of the intersections of the solutions $u(x,t)$ and $V_1(x,t) = f(x - \lambda_1(t-t_0); \lambda_1)$, $V_2(x,t) = f(x - \lambda_2(t-t_0); \lambda_2)$. By the Sturmian argument, at $t = 0$ the continuous initial function $u_0(x)$ must have an infinite number of intersections with both $V_1(x,0)$ and $V_2(x,0)$ which now have the disjoint interfaces, $s_1(0) = -\lambda_1 t_0 \ne s_2(0) = -\lambda_2 t_0$. Therefore, the intersections with either $V_1(x,0)$ or $V_2(x,0)$ are concentrated in a domain of the uniform strict positivity of the corresponding solutions. Therefore, at $t = t_0/2$ such an infinite number becomes finite due to the general results for uniformly parabolic equations [1], [43]. Hence, for $t > t_0/2$ one of these numbers is finite contradicting the assumption. $\square$

We continue the proof of (i). Thus

$$\lambda^- = \lambda^+ = \lambda_0(t_0). \tag{113}$$

Consider the complete, continuous and ordered TW-bundle $B = \{f_\lambda(\xi),\ \lambda \in \Delta_0\}$ (Section 5). It follows from (109) - (111) that the first interface operator $\mathbf{M}_1$ is well-defined on the profile $u(x,t_0)$ and

$$\mathbf{M}_1(u(\cdot,t_0)) = A_0. \tag{114}$$

In the case of the first-order TW interface equation this means that there exists

$$\mathbf{N}_1(u(\cdot,t_0)) = \lambda_0. \tag{115}$$

Then $D^+ s(t_0) = \lambda_0$ by the standard comparison with the TW solutions. In the case of the second-order TW interface equation, in addition, we then obtain from (111) as $X \to +0$ that

$$\mathbf{N}_2(u(\cdot,t_0)) = \lambda_0 \in \mathbf{R}, \tag{116}$$

where $\lambda_0 = D^+ s(t_0)$ by the usual comparison.

(ii) We have already proved that $D^+s(t)$ exists and is finite on $(0,T)$. Moreover, we have shown that on $[\delta, T-\delta]$

$$|D^+s(t)| \le c, \tag{117}$$

where $c(\delta)$ depends on small $\delta > 0$. Consider the left-hand derivative $D^-s(t_0)$. It follows from Proposition 24 that the interface function $s(t)$ cannot oscillate infinitely many times around any fixed level $\lambda_0 \in \mathbf{R}$ with non-decaying oscillations as $t \to t_0 - 0$. This would mean that $u(x,t)$ has infinite number of intersections with a subset of the TW solution $f_\lambda(x - \lambda(t-t_0))$ for all $\lambda \approx \lambda_0$. Cf. [30], p. 1097. Therefore, $D^-s(t_0) \in \bar{\mathbf{R}}$ exists. From (117) it follows that $D^-s(t_0)$ is finite and satisfies the same bound. This implies the uniform Lipschitz continuity of the interface on $[\delta, T-\delta]$. This completes the proof of the theorem. ∎

We give extra regularity properties of the interfaces. The proof, performed in [29] for the quasilinear heat equation with strong absorption, applies to general nonlinear equations.

Theorem 25. *Under the hypotheses of Theorem 21, for any $t_0 \in (0,T)$ there hold*

$$\lim_{t \to t_0 - 0} D^+\eta(t) = \lim_{t \to t_0 - 0} D^-\eta(t) = D^-\eta(t_0), \tag{118}$$

$$\lim_{t \to t_0 + 0} D^+\eta(t) = \lim_{t \to t_0 + 0} D^-\eta(t) = D^+\eta(t_0). \tag{119}$$

7.2. **Interfaces in the case $\Delta_0 \ne \mathbf{R}$.** We now discuss the properties of interfaces in the case where $\Delta_0 \ne \mathbf{R}$. The first result is motivated by the paper [34] where the blow-up equation (36) is considered and the proper TW subset B is given by (37) with $\Delta_0 \ne \mathbf{R}$ and the improper set $\bar{B}$, (38), is non-empty. Observe that this implies that the blow-up interface of a proper solution in not a C^2-function.

Proposition 26. *Let $\Delta = \Delta_0 = [b, \infty)$, $b \in \mathbf{R}$, and any $\lambda \approx b - 0$ is in the steep limit. Then for $t_0 \in [0,T)$*

$$\underline{D}^+ s(t) \equiv \liminf_{t \to t_0 + 0} \frac{s(t) - s(t_0)}{t - t_0} \ge b. \tag{120}$$

Proof. Let $\{u_n \ge 1/n\}$ be a monotone sequence of the regularized solutions and $u = \lim u_n$. Fix an arbitrarily small $\varepsilon > 0$ and consider the monotone increasing sequence of the regular TW profiles $\{f^n(\xi; b-\varepsilon)\}$ satisfying the corresponding ODE with the conditions (26). Since $b - \varepsilon \notin \Delta$, by assumption we have that as $n \to \infty$ $P_n(f) \to \infty$ for any $f > 0$. Profile $f^n(\xi)$ becomes arbitrarily steep on any fixed level for $n \gg 1$. Therefore, for any $n \gg 1$ there exists a small $\delta = \delta(\varepsilon) > 0$ such that $u_{0n}(x) \le f^n(x + \delta; b - \varepsilon)$ for small $x > 0$. Hence, $u_n(x,t) \le f^n(x - (b-\varepsilon)t + \delta; b - \varepsilon)$ for small $t > 0$ and $x \approx 0$ by comparison. Passing to the limit $n \to \infty$ we obtain $u(x,t) \le H_\infty(x - (b-\varepsilon)t + \delta)$, where H_∞ is the infinite-step Heaviside function $H_\infty(z) = \{\infty,\ z > 0;\ 0,\ z \le 0\}$. Therefore $s(t) \ge (b-\varepsilon)t - \delta$, and we get (120) by passing to the limit $\varepsilon \to 0$ (then $\delta(\varepsilon) \to 0$). We also obtain that

$$s(t) \ge bt. \tag{121}$$

$\square$

As a typical property of the proper solutions in the case $\Delta_0 \neq \mathbf{R}$ we present the following result on the uniform interface propagation of the improper TWs, cf. [34].

Proposition 27. *Let $\Delta = \Delta_0 = [b, \infty)$, any $\lambda \approx b - 0$ is in the steep limit and $\bar{\Delta}_0 \neq \emptyset$. Fix an arbitrary $c \in \bar{\Delta}_0$ and the corresponding improper TW profile $\bar{f}_c \in \bar{B}$. Let $u(x,t)$ be the proper solution with the initial data $u_0(x) = \bar{f}_c(x)$. Then*

$$s(t) = bt, \quad t > 0. \tag{122}$$

Proof. Since $c \in \bar{\Delta}_0 \subseteq \Delta_0$, by the order property of B we have that $\bar{f}_c(x) \geq f_b(x) \geq f_c(x)$. Therefore, $u(x,t) \geq f_b(x - bt)$ and hence $s(t) \leq bt$. Combining with (121), we get (122). $\square$

Thus, if the TW-diargam consists of continuous proper and improper branches, we observe interfaces moving with constant speed for all $t > 0$, representing moving singular internal layers. These can occur eventually in time and the interface looses C^2-regularity at the moment of straightening. See an example for equation (36) in [34], Sections 5,6, where a slow logarithmic convergence to the self-similar internal layer is studied.

The case of nonexistence of the proper TW of flat type, motivated by the PME, is simpler.

Proposition 28. *Let $\Delta = \Delta_0 = (-\infty, b)$, $b \in \mathbf{R}$, and $\lambda = b$ is in the flat limit. Then*

$$\underline{D}^+ s(t) \leq b. \tag{123}$$

Proof. We compare $u_n(x,t)$ with arbitrarily small solutions $f_b^n(x - bt - a_n)$ and pass to the limit $n \to \infty$. $\square$

8. Applications to absorption and blow-up singularities

In the examples below the construction of the regularized sequence of non-singular solutions is done by replacing the strong absorption term, say $-u^p$, or similar ones, by a smooth uniformly Lipschitz continuous function $-\psi_n(u) \equiv -u^p$ for $u \geq 2/n$, $\psi_n(u) = 0$ for $u \leq 1/n$ and $\{-\psi_n(u)\}$ is strictly monotone decreasing. By the Maximum Principle, the solutions from the monotone decreasing sequence $\{u_n\}$ does not reach the singular level, and standard known results from the theory of quasilinear and fully nonlinear equations are supposed to guarantee the existence of the regularized solutions. By monotonicity, the proper solution $u = \lim u_n$, trivial (entirely singular) or nontrivial, is defined, and next we study its existence and regularity properties using the theorems given above. Then the interfaces are Lipschitz continuous and the derivatives $D^{\pm}s(t)$ exist and uniformly bounded on the corresponding time-intervals away from 0, T and, if necessary, from the waiting time T_w.

In several examples below, equations can admit other types of singularities or degeneracy beyond the level $\{u = 0\}$ depending on the diffusion-like terms. We then consider a class of solutions where those stay away from the main singular zero-level. Indeed, the TWs with straight characteristic propagation only cannot explain the interaction between different types of singularities. Such new singular patterns can be studied by using other sets $\tilde{B}$ of exact solutions (like self-similar ones) which contain the desired focusing-type collision of singularities. These are typical questions of blow-up, see e.g. [3], [15] - [19], [24] - [26], [30], [33], [34] and Chapter 4 in [48].

8.1. **Quasilinear heat equations with absorption.** We begin our study with the quasilinear equation having power-type nonlinearities

$$u_t = [|(u^m)_x|^\sigma (u^m)_x]_x - u^p, \quad m > 0, \;\; \sigma > -1, \tag{124}$$

where monotone smooth initial data $u_0(x)$ satisfy all the necessary hypotheses. The case $\sigma = 0$ has been studied in [28] and [29]. If $\sigma > 0$ we assume that $u_x \neq 0$ in an open right-hand neighbourhood of the interface so that $u = 0$ is the only possible singular point. As we mentioned above, the TWs with constant level propagation do not describe the focusing effect of different singular levels $\{u = 0\}$ and $\{u_x = 0\}$.

The TW solutions $V(x,t) = f(\xi)$, $\xi = x - \lambda t$, are determined from the second-order ODE

$$[|(f^m)'|^\sigma (f^m)']' - f^p + \lambda f' = 0. \tag{125}$$

Setting

$$P = \frac{df^m}{d\xi} > 0, \quad f > 0,$$

we obtain the first-order ODE

$$(\sigma+1) P^{\sigma+1} \frac{dP}{df} = G(f) - \lambda P, \quad G(f) = m f^{p+m-1}. \tag{126}$$

Consider first the problem of existence/nonexistence of a nontrivial proper solution of the Cauchy problem for (124).

Theorem 29. *There holds*

(i) If $p \leq -m$ then $\Delta = \emptyset$ and the proper solution $u = \lim u_n \equiv 0$ for any small $t > 0$.

(ii) If $p > -m$ then $\Delta \neq \emptyset$ and there exists $u(x,t) \not\equiv 0$ for small $t > 0$.

Observe that in the case (ii) the proper solutions can have a finite interface (if $\Delta_0 \neq \emptyset$) or can be strictly positive (if $\Delta_0 = \emptyset$). See the classification below. The theorem follows from the following criterion of the existence of the minimal orbit $P_{\min}$ (cf. [32], Section 6).

Lemma 30. *The convergence of the integral*

$$\int_0^1 G(s)ds < \infty \Longleftrightarrow p > -m \tag{127}$$

is the necessary and sufficient condition for the existence of the minimal orbit $P_{\min} = \lim P_n < \infty$.

Proof. (i) *Nonexistence.* Since $G(f) \ge 0$ we have

$$(\sigma+1)P^\sigma \frac{dP}{df} \ge -\lambda.$$

Integrating this inequality over $(0,\delta)$, we conclude that $P(0)$ exists and finite. Therefore in the domain $\{P \le c\}$

$$(\sigma+1)P^\sigma \frac{dP}{df} = \frac{G(f)}{P} - \lambda \ge \frac{G(f)}{c} - \lambda.$$

Thus, if the integral diverges,

$$\int_0^1 G(s)ds = \infty, \tag{128}$$

then for any $\lambda \in \mathbf{R}$ the minimal orbit does not exist and $P_n(f) \to \infty$ as $n \to \infty$ for any small $f > 0$ (the steep limit).

(ii) *Existence of* $P_{\min}$. Obviously, under convergence (127) $P_{\min}$ exists for $\lambda = 0$. Let us prove that it then also exists for any $\lambda > 0$. Define the function

$$Y(f) = A + \frac{1}{\sigma+1} A^{-\sigma-1} \int_0^f G(s)ds, \quad A > 0. \tag{129}$$

Then

$$\frac{dP}{df}|_{P=Y(f)} = \frac{G(f)}{(\sigma+1)Y^{\sigma+1}(f)} - \frac{\lambda}{(\sigma+1)Y^\sigma(f)} < \frac{G(f)}{(\sigma+1)A^{\sigma+1}} \equiv \frac{dY}{df}.$$

This means that the orbits intersecting the curve $P = Y(f)$ stay in the domain $\{P > Y(f),\ f > 0\}$ and are well-defined for all $f > 0$ small. Then the minimal orbit $P = P_{\min}(f)$ is the separatrix which is constructed as the limit $P_{\min} = \lim P_n$. $\square$

We now consider the case $p > -m$ and perform a detailed analysis of the TW set B in the case of finite propagation where $\Delta_0 \ne \emptyset$. We begin with the most interesting case of the second-order interface equation where the proper solutions exhibit a special behaviour near the interfaces.

1. Second-order interfaces, $\Delta_0 = \mathbf{R}$. We impose the conditions

$$-m < p < p_c = \frac{\sigma+2}{\sigma+1} - m, \quad p < m(\sigma+1), \tag{130}$$

where p_c is a critical exponent separating the parameter ranges with the interfaces of the second and of the first order (see below). The first inequality says that the first term $G(f)$ on the right-hand side of (126) is the leading one for small $f > 0$ and, as $f \to 0$, the minimal orbit has the form

$$P_{\min}(f) = f^\gamma[c_0 - \lambda c_1 f^{\gamma+1-p-m}(1+o(1))], \quad \gamma = \frac{p+m}{\sigma+2}, \tag{131}$$

where $c, c_*, c_0, c_1, c_2, ...$ denote different positive constants depending on the parameters m, σ and p. One can calculate that

$$c_0 = \left[\frac{m(\sigma+1)}{(\sigma+1)(p+m)} \right]^{1/(\sigma+2)}.$$

Integrating the equation $df^m/d\xi = P_{\min}(f)$ and taking into account the second inequality in (130), we obtain that for any $\lambda \in \mathbf{R}$ there exists a unique TW profile with the following behaviour near the interface as $\xi \to +0$

$$f^\alpha(\xi) = c_*\xi - \lambda c_2 \xi^\beta (1 + o(1)), \tag{132}$$

where

$$\alpha = \frac{m(\sigma+1)-p}{\sigma+2} > 0, \;\; \beta = \frac{(\sigma+2)(1-p)}{m(\sigma+1)-p} > 1, \;\; c_* = \frac{m(\sigma+1)-p}{m(\sigma+2)} c_0.$$

We thus prove

Proposition 31. *If (130) holds then $\Delta_0 = \mathbf{R}$ and the TW-bundle B is complete, sufficiently smooth and strictly monotone decreasing in λ.*

It follows from (132) that the pressure variable is $v = Q(u) = u^\alpha$ and the following two interface operators are well defined on the complete TW-bundle: for any $f \in B$ there exist the first-order operator

$$\mathbf{M}_1(f) \equiv \lim_{\xi \to +0} \frac{1}{\xi} f^\alpha(\xi) = c_*, \tag{133}$$

and the second-order one

$$\mathbf{N}_2(f) \equiv \frac{1}{c_2} \left\{ \lim_{\xi \to +0} \frac{[c_*\xi - f^\alpha(\xi)]^{1/\beta}}{\xi} \right\}^\beta = \lambda. \tag{134}$$

Let us determine the G-function. It follows from (132) that for small $\varepsilon > 0$

$$G(\lambda, \varepsilon) \equiv \frac{d}{d\xi} Q(f)|_{Q(f)=\varepsilon} = c_* - \lambda c_3 \varepsilon^{\beta-1}(1 + o(1)). \tag{135}$$

We now perform the regularity analysis in the B-classes. One can get from (135) and equation (126) that (98) and (99) hold. In order to check the condition (100) (which is necessary for the instantaneous smoothing in $B^-_{\rm loc}$) we observe that given small $\varepsilon > 0$ we have for $\lambda \gg 1$ that $P_{\min}(f) \approx P_0(f) - 0$ for $f \gg 1$, with the zero cline $P_0(f) = (m/\lambda) f^{m+p-1}$. Therefore $df/d\xi \approx mf^p/\lambda$ for $\xi \gg 1$. Integrating, we conclude that $f^\alpha(\xi) \approx h(\xi/\lambda)$ for $\xi \gg 1$, where h is a smooth monotone function satisfying $h(0) = 0$. Solving the algebraic equation for the intersection point $\xi_*(\lambda, \varepsilon)$: $h(\xi_*/\lambda) \approx \varepsilon$, we get $\xi_*/\lambda \approx h^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$. We summarize the results on the properties of solutions to the PDE (124).

Theorem 32. *In the range (130), on* $[\delta, T-\delta]$ *with any small* $\delta > 0$ *there hold:*

(i) The gradient Bernstein-type estimate near the interface has the form

$$|(u^\alpha)_x - c_*| \le c(\delta) u^\mu, \quad \mu = \alpha(\beta - 1) = \frac{\sigma + 2 - (\sigma+1)(p+m)}{\sigma+2} > 0. \tag{136}$$

(ii) The interface equation is the system

$$\mathbf{M}_1(u(\cdot,t)) = c_*, \quad D^+ s(t) = \mathbf{N}_2(u(\cdot,t)). \tag{137}$$

2. First-order interfaces for $p = p_c$**,** $\Delta_0 = \mathbf{R}$**.** In the critical case

$$p = p_c \quad (m(\sigma+1) > 1) \tag{138}$$

for any $\lambda \in \Delta = \mathbf{R}$, the ODE (126) admits the exact solution

$$P_{\min}(f) = A_0(\lambda) f^{1/(\sigma+1)}, \tag{139}$$

where $A_0(\lambda) > 0$ is a unique positive root of the algebraic equation

$$A_0^{\sigma+2} + \lambda A_0 - m = 0. \tag{140}$$

The function $A_0(\lambda)$ is well-defined, strictly monotone and continuous for all $\lambda \in \mathbf{R}$. There exists the inverse one A_0^{-1} and the TW-diagram consists of a single analytic monotone decreasing branch

$$\lambda = A_0^{-1}(S) = -S^{\sigma+1} + \frac{m}{S}, \quad S > 0.$$

Integrating the equation

$$(f^m)' = A_0(\lambda) f^{1/(\sigma+1)},$$

we obtain the proper TW-bundle $\{f_\lambda(\xi),\ \lambda \in \mathbf{R}\}$ with the profiles

$$f^\alpha(\xi) = cA_0(\lambda)\xi, \quad \alpha = \frac{m(\sigma+1)-1}{\sigma+1} > 0, \ \ c = \frac{\alpha}{m}. \tag{141}$$

Thus the pressure variable stays the same, $v = u^\alpha$, but due to (141) the interface is of the first order (see Section 2). We introduce the first-order interface operator

$$\mathbf{M}_1(f) = \lim_{\xi \to +0} \frac{1}{\xi} f^\alpha(\xi) \equiv (f^\alpha)'(0) = cA_0(\lambda), \tag{142}$$

and set

$$\mathbf{N}_1(f) \equiv A_0^{-1}(\mathbf{M}_1(f)/c). \tag{143}$$

The G-function does not depend on the level, $f^\alpha = \varepsilon$:

$$G(\lambda, \varepsilon) \equiv \frac{d}{d\xi} f^\alpha = cA_0(\lambda). \tag{144}$$

Using the completeness, continuity and monotonicity of B, we introduce the local B-classes. The conditions (98) and (99) are obviously valid. Consider (100). Computing the intersection point ξ_* with the ε-level, we get for the solutions (141) that $\xi_*(\lambda,\varepsilon)/\lambda \sim \varepsilon/\lambda A_0(\lambda)$ for $\lambda \gg 1$. Since by (140) $A_0(\lambda) \approx m/\lambda$ for $\lambda \gg 1$, we have that $\lambda A_0(\lambda) \to m$ as $\lambda \to \infty$ so that (100) is true. We thus obtain

Theorem 33. *In the critical case (138) on any interval* $[\delta, T-\delta]$ *there hold:*

(i) The gradient estimate near the interface has the form $0 < c_2(\delta) \le (u^\alpha)_x \le c_1(\delta)$.

(ii) The interface equation is $D^+ s(t) = \mathbf{N}_1(u(\cdot,t))$.

3. First-order interfaces for $p \in (p_c, 1)$**,** $\Delta_0 = \mathbf{R}$**.** Consider the supercritical case

$$p_c < p < 1 \quad (m(\sigma+1) > 1). \tag{145}$$

The minimal orbit is $P_{\min} = ((-\lambda)f)^{1/(\sigma+1)}(1+o(1))$, $\lambda < 0$, $P_{\min} = cf^{(p+m)/(\sigma+2)}$ if $\lambda = 0$ and $P_{\min} = \lambda^{-1} m f^{p+m-1}(1+o(1))$, $\lambda > 0$, so that the TW-bundle is composed from the profiles

$$f^\alpha(\xi) = \frac{1}{\alpha m}(-\lambda)^{1/(\sigma+1)}\xi(1+o(1)), \;\; \lambda < 0, \;\; \alpha = \frac{m(\sigma+1)-1}{\sigma+1}; \tag{146}$$

$$f^\alpha(\xi) = c_*\xi, \;\; \lambda = 0, \;\; \alpha = \frac{m(\sigma+1)-p}{\sigma+2}; \quad f^{1-p}(\xi) = \frac{1}{\lambda}(1-p)\xi(1+o(1)), \;\; \lambda > 0.$$

Thus, $\Delta_0 = \mathbf{R}$, the pressure is $Q(u) = u^\alpha$ for $\lambda < 0$, $Q(u) = u^\gamma$ for $\lambda = 0$ and $Q(u) = u^{1-p}$ for $\lambda > 0$. Calculating the G-function and checking the necessary conditions for the instantaneous smoothing in the B-classes, we obtain

Theorem 34. *In the parameter range (145) on* $[\delta, T-\delta]$ *there hold:*

(i) The gradient bound near the interface is

$$(u^\alpha)_x \le c_1(\delta), \quad (u^{1-p})_x \ge c_2(\delta).$$

(ii) The interface equation is $D^+ s(t) = \mathbf{N}_1(u(\cdot,t))$, *where*

$$\mathbf{N}_1(u(\cdot,t)) = -[\alpha m (u^\alpha)_x(s(t),t)]^{\sigma+1} < 0 \quad if \;\; (u^\alpha)_x > 0,$$

$$\mathbf{N}_1(u(\cdot,t)) = 0 \quad if \;\; (u^\gamma)_x = 0,$$

$$\mathbf{N}_1(u(\cdot,t)) = (1-p)[(u^{1-p})_x(s(t),t)]^{-1} > 0 \quad if \;\; (u^{1-p})_x > 0.$$

4. First-order interfaces for $p \ge 1$**,** $\Delta_0 = (-\infty, 0]$**.** In the case

$$1 \le p < (\sigma+1)m \quad (m(\sigma+1) > 1), \tag{147}$$

for any $\lambda < 0$ the first term on the right-hand side of (126) is negligible and we obtain solution (146). If $\lambda = 0$ then we arrive at the profile (132). The instantaneous smoothing effect in the local B^+-class is no more available and the waiting time phenomenon is possible.

Theorem 35. *In the parameter range (147) on* $[T_w + \delta, T-\delta]$, *where* $T_w \in [0, T)$ *is the waiting time there hold:*

(i) The gradient bound near the interface is $c_2(\delta) \le (u^\alpha)_x \le c_1(\delta)$.

(ii) The interface equation is

$$D^+ s(t) = \mathbf{N}_1(u(\cdot,t)) \equiv -[\alpha m (u^\alpha)_x(s(t),t)]^{\sigma+1} < 0.$$

5. First-order interfaces for $p \geq m(\sigma+1)$, $\Delta_0 = (-\infty, 0)$. In this range the complete TW-bundle is composed from the functions (146), and the previous theorem holds.

6. Diffusion-absorption equation with strong convection. As a new example, we add to the quasilinear equation an extra convection term

$$u_t = [|(u^m)_x|^\sigma (u^m)_x]_x - u^p - (u^\nu)_x. \tag{148}$$

Then the first-order ODE takes the form

$$(\sigma+1)P^{\sigma+1}\frac{dP}{df} = G(f) - \lambda P + \nu f^{\nu-1}P, \quad G(f) = mf^{p+m-1}. \tag{149}$$

The analysis of (149) is quite similar. In particular, one can check that a similar second-order interface with $\Delta_0 = \mathbf{R}$ occurs if the convection term is not that strong and in addition to (130) there holds

$$\nu > \nu_c = (p+m)\frac{\sigma+1}{\sigma+2}.$$

In the critical case (138) with $\nu = \nu_c$ the exact solution (139) with $A_0(\lambda) > 0$ satisfying

$$A_0^{\sigma+2} + (\lambda + \nu_c)A_0 - m = 0,$$

generates a complete TW-bundle with the first-order interface equation. The rest of the cases are straightforward.

On the other hand, if $\nu < \nu_c$, $\nu < 1$ $(\nu < m(\sigma+1))$, then the last term in (149) is the leading one, and we arrive at the second-order interface equation governed by the convection term. In [49] the second-order interface equation and further regularity results were established for the divergent PME with convection $u_t = (u^m)_{xx} - (u^\nu)_x$, $m > 1$, $\nu \in (0,1)$, where the method of Lagrangian coordinates applies. See also the last section in [29].

7. A fully nonlinear generalization. Consider now the fully nonlinear equation

$$|u_t|^{\gamma-1}u_t = [|(u^m)_x|^\sigma (u^m)_x]_x - u^p, \quad m > 0, \ \sigma > -1, \ \gamma > 0. \tag{150}$$

Then the ODE for the TW profiles $V = f(\xi)$ takes the form

$$[|(f^m)'|^\sigma (f^m)']' - u^p + B_0|f'|^{\gamma-1}f' = 0, \quad B_0(\lambda) = |\lambda|^{\gamma-1}\lambda.$$

Setting $P = (f^m)'$ and using that $P > 0$ and $dP/df > 0$ near the interface, we obtain the equation similar to (126):

$$(\sigma+1)P^{\sigma+1}\frac{dP}{df} = G(f) - B_0 m^{1-\gamma} f^{(1-m)(\gamma-1)}P^\gamma, \quad G(f) = mf^{p+m-1}. \tag{151}$$

Then we prove

Lemma 36. *There holds*

$$\Delta \neq \emptyset \Longleftrightarrow \int_0^1 G(s)ds < \infty \Longleftrightarrow p > -m. \tag{152}$$

Let us perform necessary computations for the second-order interfaces in the subcritical range:

$$(153) \qquad -m < p < p_c = \gamma \frac{\sigma + 2 - m(\sigma+1)}{\sigma + 2 - \gamma}, \quad p < m(\sigma+1) \quad (\gamma < \sigma + 2).$$

Integrating (151) two times, we obtain the complete proper TW-bundle composed from

$$(154) \qquad f^\alpha(\xi) = c_* \xi - B_0(\lambda) c_2 \xi^\beta (1 + o(1)),$$

$$\alpha = \frac{m(\sigma+1) - p}{\sigma + 2} > 0, \quad \beta = \frac{(\sigma+2)(1-p) + (\gamma - 1)[p + m - (\sigma+2)(m-1)]}{m(\sigma+1) - p} > 1.$$

Next using operators (133) and $\tilde{\mathbf{N}}_2$ given by (134), we define the main second-order interface operator as follows

$$(155) \qquad \mathbf{N}_2(f) = B_0^{-1}(\tilde{\mathbf{N}}_2(f)).$$

Differentiating (154), we get the G-function as $\varepsilon \to 0$

$$G(\lambda, \varepsilon) = c_* - B_0(\lambda) c_3 \varepsilon^{\beta - 1}(1 + o(1)).$$

Checking the conditions of the smoothing in the B-classes, finally we arrive at Theorem 29. The exponent in the Bernstein estimate is $\mu = \alpha(\beta - 1) > 0$. In the critical range $p = p_c$, where equation (151) admits an exact solution, and in the supercritical one $p > p_c$ the interfaces are of the first-order. The analysis is similar.

8.2. **Blow-up interfaces for the quasilinear equation with source.** We now consider blow-up solutions of the quasilinear equation

$$(156) \qquad v_t = [|(v^m)_x|^\sigma (v^m)_x]_x + v^p, \quad m > 0, \ \sigma > -1, \ p > 1.$$

The transformation $u = 1/v$ reduces the blow-up level $\{v = +\infty\}$ to the propagation on the zero level $\{u = 0\}$ for the non-divergent equation with strong absorption

$$(157) \qquad u_t = -u^2 [|(u^{-m})_x|^\sigma (u^{-m})_x]_x - u^q, \quad q = 2 - p < 1.$$

The corresponding ODE for the TW solutions $V = f(\xi)$ is

$$-f^2[(-(f^{-m})')^\sigma (f^{-m})']' - f^{2-p} + \lambda f' = 0.$$

Setting $P = -(f^{-m})' = m f^{-m-1} f' > 0$, we obtain the first-order ODE

$$(158) \qquad (\sigma + 1) P^{\sigma+1} \frac{dP}{df} = G(f) - \lambda \frac{P}{f^2}, \quad G(f) = m f^{-p-m-1}.$$

Since $\int_0 G(s)\,ds$ diverges, unlike the absorption case, we always have that

$$(159) \qquad \Delta \cap (-\infty, 0] = \emptyset.$$

We also conclude that the blow-up interfaces are always of the first order.

1. $p = p_c$**.** In the critical case $p = p_c$ (see (8)) there exists the exact minimal orbit $P_{\min} = A_0 f^{-1/(\sigma+1)}$, where $A_0(\lambda) > 0$ solves the algebraic equation $A_0^{\sigma+2} - \lambda A_0 + m = 0$. As we have shown, the proper profiles are the minimal ones so that

$$\Delta = \Delta_0 = [\lambda_0, \infty), \quad \lambda_0 = (\sigma+2)\left(\frac{m}{\sigma+1}\right)^{(\sigma+1)/(\sigma+2)},$$

and the slope $S = A_0(\lambda)$ is given by the minimal branch. This means that always $D^+ s(t) \geq \lambda_0$ by Proposition 26. The TW-bundle is composed from the profiles

$$f^{\alpha}(\xi) = cA_0(\lambda)\xi, \;\; \lambda \in \Delta_0, \quad \alpha = \frac{1 - m(\sigma+1)}{\sigma+1} > 0, \;\; c = \frac{\alpha}{m}.$$

The TW-diagram consists of a monotone decreasing branch

$$\lambda = S^{\sigma+1} + \frac{m}{S}, \quad 0 < S \leq S_0 = \left(\frac{m}{\sigma+1}\right)^{1/(\sigma+2)}.$$

This gives the corresponding first-order interface operator, the G-function and we arrive at a standard first-order interface equation. The waiting time phenomenon is not possible (since $\{\lambda \gg 1\} \subset \Delta_0$) and the interface is moving for all $t \in (0, T)$.

2. $p < p_c$**.** In the subcritical case $p \in (1, p_c)$ the minimal orbit

$$P_{\min} = \frac{m}{\lambda} f^{-p-m+1}(1 + o(1)), \quad \lambda \in \Delta_0 = \mathbf{R}_+,$$

generates the TW-bundle $f^{p-1}(\xi) = [(p-1)/\lambda]\xi(1+o(1))$, and the corresponding interface operator and the G-function. The interface is moving for arbitrarily small $t \in (0, T)$.

8.3. **Dual PME with absorption.** Consider the dual PME with absorption

$$u_t = |u_{xx}|^{m-1} u_{xx} - u^p, \quad m > 0. \tag{160}$$

For $m > 1$, this equation degenerates on the set $\{u_{xx} = 0\}$. Therefore, we consider the class of solutions with no spatial inflections for small $u > 0$. On the other hand we can consider a modified ("regularized") equation $u_t = [u^2 + (u_{xx})^2]^{(m-1)/2} u_{xx} - u^p$ (or any similar one) with the only singular/degenerate level at $u = 0$. The interaction between the singular sets $\{u = 0\}$ and $\{u_{xx} = 0\}$ for (160) (focusing or collision-like patterns) may be studied by using the self-similar solutions.

Consider the TW solutions of (160). Setting $f' = P$ in the ODE for the TW profiles $|f''|^{m-1} f'' - f^p + \lambda f' = 0$, we obtain $|PP'_f|^{m-1} PP'_f = f^p - \lambda P$. The proper solutions satisfy $f'' > 0$ near the interface so that for small $f > 0$ the first-order ODE takes the form

$$P\frac{dP}{df} = (f^p - \lambda P)^{1/m}. \tag{161}$$

Lemma 37. *$\Delta \neq \emptyset$ if and only if $p > -m$.*

The proof is the same as above. We assume that $p > -m$, hence $\Delta \neq \emptyset$.

1. Second-order interfaces, $p < p_c$**.** Consider first the subcritical range

$$-m < p < p_c = \frac{m}{2m-1} > 0, \quad p < m, \;\; m > \frac{1}{2}. \tag{162}$$

The minimal orbit is given by

$$P_{\min} = f^{\gamma}[c_0 - \lambda c_1 f^{\gamma - p}(1 + o(1))], \quad \gamma = \frac{p+m}{2m}, \;\; c_0 = \sqrt{\frac{2m}{p+m}}.$$

The TW-bundle, $\Delta_0 = \mathbf{R}$, is given by (132) where

$$\alpha = \frac{m-p}{2m}, \quad \beta = \frac{2m(1-p)}{m-p} > 1, \quad c_* = \frac{m-p}{\sqrt{2m(p+m)}}.$$

The pressure is $v = u^{\alpha}$ and we introduce the interface operators (133), (134) and the G-function (135). Finally, Theorem 29 is true with the exponent μ in the Bernstein-type estimate replaced by

$$\mu = \alpha(\beta - 1) = \frac{m - (2m-1)p}{2m} > 0.$$

2. $p = p_c$**.** In the critical case $p = p_c > 0$ ($m > 1$) the ODE (161) admits the exact minimal orbit $P_{\min}(f) = A_0(\lambda) f^p$, where $A_0(\lambda) > 0$ solves the algebraic equation $p^m A_0^{2m} + \lambda A_0 - 1 = 0$. The TW-bundle is given by

$$f^{\alpha}(\xi) = c A_0(\lambda)\xi, \quad \lambda \in \Delta_0 = \mathbf{R}, \quad \alpha = c = \frac{m-1}{2m-1} > 0.$$

We introduce the first-order operator (142), (143), the G-function (144) and arrive at Theorem 30.

3. $p \in (p_c, 1)$**.** In this supercritical range the TW-bundle is ($m > 1$)

$$f^{\alpha}(\xi) = c(-\lambda)^{1/(2m-1)} \xi (1 + o(1)), \quad \lambda < 0, \;\; \alpha = \frac{m-1}{2m-1};$$

$$f^{\alpha}(\xi) = c_* \xi, \;\; \lambda = 0, \;\; \alpha = \frac{m-p}{2m}; \quad f^{1-p}(\xi) = \frac{1}{\lambda}(1-p)\,\xi(1 + o(1)), \quad \lambda > 0.$$

We then determine the corresponding pressures, the G-functions and the first-order interface operators. Finally we arrive at the result similar to Theorem 31.

4. $p \geq 1$**.** The analysis is quite similar. We have that $\Delta_0 = (-\infty, 0]$, $p < m$ and $\Delta_0 = (-\infty, 0)$, $p \geq m$.

5. A doubly fully nonlinear generalization. Consider a generalization of equation (160):

$$|u_t|^{\gamma-1} u_t = |u_{xx}|^{m-1} u_{xx} - u^p, \quad \gamma > 0. \tag{163}$$

We then obtain the ODE for $P = f' > 0$, $f > 0$ small:

$$P \frac{dP}{df} = (f^p - B_0 P^{\gamma})^{1/m}, \quad B_0(\lambda) = |\lambda|^{\gamma-1}\lambda.$$

The computations are similar. For instance, $\Delta \neq \emptyset$ if $p > -m$. The second-order interface equation takes place if

$$p < p_c = \frac{\gamma m}{2m - \gamma}, \quad m > \frac{\gamma}{2}.$$

If $p = p_c > 0$, then the explicit complete TW-bundle is given by $f^\alpha(\xi) = cA_0(\lambda)\xi$, $\lambda \in \mathbf{R}$, $\alpha = (\gamma - p)/\gamma > 0$, where $A_0(\lambda) > 0$ solves the algebraic equation $(p/\gamma)^m A_0^{2m} + |\lambda|^{\gamma-1}\lambda A_0^\gamma - 1 = 0$. This TW-bundle determines the first-order interface operators, the G-function and the interface equation. The analysis for $p > p_c$ is straightforward.

8.4. **Blow-up for the dual PME with source.** Consider the blow-up propagation for the dual PME with source

$$v_t = |v_{xx}|^{m-1} v_{xx} + v^p, \quad m \in (0,1), \;\; p > 1. \tag{164}$$

The function $u = 1/v$ solves the fully nonlinear equation with strong absorption

$$u_t = -u^2 |(u^{-1})_{xx}|^{m-1} (u^{-1})_{xx} - u^q, \quad q = 2 - p < 1. \tag{165}$$

Setting $P = f^{-2} f' > 0$ in the ODE for the TW solutions $V = f(\xi)$, we obtain

$$P \frac{dP}{df} = -\frac{1}{f^2} \left(\lambda P - f^{-p} \right)^{1/m}.$$

As usual for the blow-up interfaces, we get (159). In the critical case $p = p_c$, $m \in (1/2, 1)$, see (162), we obtain the exact minimal orbit $P_{\min} = A_0 f^{-p}$, where $A_0(\lambda) > 0$ is the minimal root of the algebraic equation $p^m A_0^{2m} - \lambda A_0 + 1 = 0$. Therefore,

$$\Delta = \Delta_0 = [\lambda_0, \infty), \quad \lambda_0 = 2m\sqrt{p}(2m-1)^{-(2m-1)/2m}.$$

The TW-bundle is $f^{p+1}(\xi) = (p+1)A_0(\lambda)\xi$, $\lambda > 0$, and the proper TW-diagram $\lambda = A_0^{-1}(S)$ consists of the monotone decreasing branch

$$\lambda = p^m S^{2m-1} + \frac{1}{S}, \quad 0 < S \leq S_0 = \frac{1}{\sqrt{p}} (2m-1)^{-1/2m}.$$

In the subcritical case $p \in (1, p_c)$, we have got the TW-bundle $f^{p+1}(\xi) = [(1+p)/\lambda]\xi(1 + o(1))$, $\lambda > 0$. The interface equation is of the first order and interfaces are moving for $t > 0$. One can study a more fully nonlinear equations like

$$|v_t|^{\gamma-1} v_t = |v_{xx}|^{m-1} v_{xx} + v^p, \quad \gamma > 0.$$

8.5. **General quasilinear heat equation with absorption.** We consider the general quasilinear parabolic equation with arbitrary nonlinearities in both the diffusion and the absorption operators

$$u_t = (\varphi(u))_{xx} - \psi(u), \tag{166}$$

where $\varphi(u)$ and $\psi(u)$ are given sufficiently smooth functions satisfying natural conditions including $\varphi'(u) > 0$ (parabolicity), $\psi(u) > 0$ for $u > 0$, $\varphi(0) = 0$. The TWs for equations like (166) with arbitrary nonlinearities were considered in a number of papers, [14], [32],

[35], see also a survey in [36]). We need to derive some extra properties of the TW profiles. Setting $P = d\varphi(f)/d\xi > 0$, we obtain the ODE

$$P\frac{dP}{df} = G(f) - \lambda P, \quad G(f) = \varphi'(f)\psi(f). \tag{167}$$

As in [32], Section 6, we obtain

Lemma 38. $\Delta \neq \emptyset \Longleftrightarrow \int_0^1 G(s)ds < \infty$.

1. Subcritical range: second-order interface equation. Let us derive a criterion of this the most interesting case.

Lemma 39. *Assume that*

$$G'(s) \to \infty, \quad s \to 0. \tag{168}$$

Then $\Delta = \mathbf{R}$ *and as* $f \to 0$

$$P_{\min} = Y(f) - \frac{\lambda}{Y(f)} \int_0^f Y(\eta)d\eta(1 + o(1)), \quad Y(f) = \sqrt{2\int_0^f G(s)ds}. \tag{169}$$

Proof is straightforward and is based on the fact that the leading term in the right-hand side of (167) is the first one. Integrating $\varphi'(f)f'_\xi = P_{\min}(f)$ we conclude that $\Delta_0 = \mathbf{R}$ provided that

$$\int_0^1 \frac{\varphi'(s)}{Y(s)}ds < \infty$$

(a well known condition, [40]). Let us perform some extra necessary computations. Denote by $f_0(\xi)$ the TW profile with $\lambda = 0$ given by the pressure equality

$$Q(f_0) \equiv \int_0^f \frac{\varphi'(s)}{Y(s)}ds = \xi,$$

i.e., $f_0 = Q^{-1}(\xi)$. Using a standard iteration procedure, under necessary hypotheses, we conclude that for any $\lambda \in \mathbf{R}$ there holds

$$Q(f(\xi)) = \xi - \lambda R(\xi)(1 + o(1)), \quad R(\xi) = \int_0^\xi \left(\frac{1}{Y^2(f_0(\eta))} \int_0^{f_0(\eta)} Y(\zeta)d\zeta \right) d\eta, \tag{170}$$

which gives the complete TW-bundle. Thus, $v = Q(u)$ is the pressure and as $\varepsilon \to +0$, the G-function is given by

$$G(\lambda, \varepsilon) = 1 - \lambda R'(\varepsilon)(1 + o(1)). \tag{171}$$

Therefore, the optimal gradient Bernstein bound on the proper solutions near the singular interface takes the form

$$|(Q(u))_x - 1| \leq cR'(Q(u)). \tag{172}$$

Using this bound, we define the first-order operator

$$\mathbf{M}_1(f) = (Q(f))_x(0) \equiv 1. \tag{173}$$

The expansion (170) makes it possible to define the second-order operator as follows

$$\mathbf{N}_2(f) \equiv \lim_{\xi \to +0} \frac{\xi - Q(f(\xi))}{R(\xi)} = \lambda. \tag{174}$$

After checking the conditions of the instantaneous smoothing in B-classes, we arrive at the interface system

$$\mathbf{M}_1(u(\cdot,t)) \equiv (Q(u))_x(s(t),t) = 1,$$
$$D^+s(t) = \mathbf{N}_2(u(\cdot,t)) \equiv \lim_{\xi \to +0} \frac{\xi - Q(u(s(t)+\xi,t))}{R(\xi)}.$$

2. Critical case. We define the critical case as follows: equation (167) admits an exact solution of the form $P_{\min}(f) = A_0(\lambda)G(f)$. Obviously, substituting into the equation, we conclude that the critical case occurs if the coefficients φ and ψ satisfy the condition

$$G'(s) = (\varphi'\psi)'(s) \equiv a_0 > 0. \tag{175}$$

Then $A_0(\lambda) > 0$ for any $\lambda \in \mathbf{R} = \Delta$ is uniquely determined from the quadratic equation $a_0 A_0^2 + \lambda A_0 - 1 = 0$. The TW-diagram is presented by the monotone analytic branch

$$\lambda = A_0^{-1}(S) = -a_0 S + \frac{1}{S}, \quad S > 0.$$

In the case of finite propagation we obtain the TW-bundle

$$Q(f(\xi)) = A_0(\lambda)\xi, \quad \lambda \in \mathbf{R} = \Delta_0, \;\; Q(f) = \int_0^f \frac{ds}{\psi(s)},$$

and hence the Bernstein estimate is

$$c_2(\delta) \le (Q(u))_x \le c_1(\delta).$$

The first-order interface equation takes the form

$$D^+s(t) = A_0^{-1}((Q(u))_x(s(t),t)).$$

3. Supercritical range. We now assume that unlike (168) and (175) there holds

$$G'(s) \to 0, \quad s \to 0. \tag{176}$$

Then the minimal proper orbit is given by

$$P_{\min} = (-\lambda)f(1+o(1)), \;\; \lambda < 0; \quad P_{\min} = \frac{1}{\lambda} G(f)(1+o(1)), \;\; \lambda > 0.$$

This gives the TW-bundle

$$Q_1(f) \equiv \int_0^f \frac{\varphi'(s)}{s}\, ds = (-\lambda)\xi(1+o(1)), \;\; \lambda < 0; \quad Q_0(f) \equiv \int_0^f \frac{\varphi'(s)}{Y(s)}\, ds = \xi, \;\; \lambda = 0;$$

$$Q_2(f) \equiv \int_0^f \frac{ds}{\psi(s)} = \frac{1}{\lambda}\xi(1+o(1)), \;\; \lambda > 0.$$

Therefore, $\Delta_0 = \mathbf{R}$, provided that all the pressure functions are well defined, and $\Delta_0 = (-\infty, 0]$ or $\Delta_0 = (-\infty, 0)$ if the corresponding integrals diverge. We then conclude that, under, if necessary, a waiting time, on $[T_w+\delta, T-\delta]$, $T_w \in [0,T)$, the gradient bound has the form

$$(Q_1(u))_x \le c_1(\delta), \quad (Q_2(u))_x \ge c_2(\delta),$$

where the pressure $Q_{1,2}$ depends on the slope (or, which is the same, on the interface direction). In the first-order interface equation

$$D^+ s(t) = \mathbf{N}_1(u(\cdot,t))$$

the operator $\mathbf{N}_1$ is determined by the above TW-bundle.

4. Generalizations. The analysis can be easily performed for the doubly nonlinear equations

$$u_t = [\Phi((\varphi(u))_x)]_x - \psi(u), \quad \Phi' \ge 0,$$

or for fully nonlinear equations like

$$Z(u_t) = [|(\varphi(u))_x|^\sigma (\varphi(u))_x]_x - \psi(u), \quad \sigma > -1, \;\; Z' \ge 0.$$

Different types of the interface equations and optimal regularity bounds can be established for the equations with extra convection terms like

$$u_t = (\varphi(u))_{xx} - (\rho(u))_x - \psi(u).$$

8.6. **Fully nonlinear equation from detonation theory.** We consider the nonlinear parabolic equation

$$u_t = g(uu_{xx}) - (u_x)^2 - \psi(u); \quad g' > 0, \;\; \psi(u) > 0, \;\; u > 0. \tag{177}$$

This equation with the coefficients

$$g(s) = 2\ln\left(\frac{e^s - 1}{s}\right) \quad \text{and} \quad \psi(u) = -2\ln u \tag{178}$$

describes the evolution of the perturbations of Zel'dovich-von Neumann-Doering's (ZDN) square wave occurring during a detonation in a duct [5]. The equation is known to create finite-time singularity when initially positive solutions touch the singular level $\{u=0\}$. A mathematical theory of such singular extinction behaviour is presented in [23].

We now consider equation (177) with rather arbitrary nonlinearities. As for the ZDN-model, for convenience, we impose the condition of strong absorption

$$\psi(f) \to +\infty, \quad f \to 0.$$

The ODE for the minimal orbit $P = f'$ takes the form

$$P\frac{dP}{df} = \frac{1}{f}\, g^{-1}(\psi(f) - \lambda P + P^2). \tag{179}$$

We then conclude that

$$\Delta \ne \emptyset \quad \text{if} \quad \int_0^1 G(s)ds < \infty, \quad G(s) = \frac{1}{s}\, g^{-1}(\psi(s)). \tag{180}$$

For the coefficients of the ZDN-wave equation

$$g(s) = 2(s - \ln s + ...), \quad s \to +\infty, \tag{181}$$

so that the integral diverges. Therefore, the proper (maximal) solutions cannot be continued in a nontrivial way beyond singularity, cf. [23], Section 9.

We now assume finite propagation, $\Delta_0 \neq \emptyset$, which is guaranteed by the convergence

$$\int_0^1 \frac{ds}{Y(s)} < \infty, \quad Y(s) = \sqrt{\int_0^s G(\eta)d\eta}.$$

We need to study second-order interfaces. This means that in the right-hand side of (179) the function $\psi(f)$ forms the leading term. Comparing it with the other terms, we derive the following sufficient condition of the second-order interface propagation

$$\left(\frac{g^{-1}(\psi(s))}{\psi(s)\psi'(s)}\right)' \to 0, \quad s \to 0. \tag{182}$$

Then $f' = P_{\min} = \sqrt{2}Y(f)(1+o(1))$. Hence, the pressure function is given by

$$v = Q(u) = \frac{1}{\sqrt{2}} \int_0^u \frac{ds}{Y(s)}.$$

Consider, for instance, equation (177) with the function g given in (178), so that (181) holds. The condition (182) reads

$$(1/\psi'(s))' \to 0, \quad s \to 0.$$

The ODE for small $f > 0$ takes the asymptotic form

$$P\frac{dP}{df} = \frac{1}{2f}\,\psi(f) - \frac{\lambda}{2f}\,P + \dots, \quad \lambda \neq 0. \tag{183}$$

We thus arrive at a perturbed two-term ODE similar to (167) which is studied in the same way. In particular, we obtain a similar minimal orbit

$$P_{\min} = X(f) - \frac{\lambda}{2X(f)} \int_0^f \frac{X(s)}{s}\,ds + \dots, \;\; X(f) = \sqrt{\int_0^f \frac{\psi(s)}{s}\,ds}.$$

The pressure variable is

$$v = Q(u) = \int_0^u \frac{ds}{X(s)}.$$

Integrating the equation $f' = P_{\min}(f)$, we derive the complete TW-bundle:

$$Q(f) = \xi - \frac{1}{2}\lambda R(\xi)(1+o(1)), \quad R(\xi) = \int_0^{\xi} \left(\frac{1}{X^2(f_0(\eta))} \int_0^{f_0(\eta)} \frac{X(s)}{s}\,ds\right) d\eta, \tag{184}$$

where $f_0(\xi) = Q^{-1}(\xi)$. If $\lambda = 0$ then the quadratic term P^2 in (179) has to be taken into account. The analysis of the TW-bundle (184) is standard and under necessary conditions we arrive at the G-function, derive the optimal gradient estimate similar to (172) and the interface operators like (171) - (174). Finally we obtain the interface system.

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