Bifurcation from the essential spectrum for almost-periodic perturbations of Hill's equation

E.C.M. Crooks
Department of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, U.K.
email: ecmc@maths.bath.ac.uk

Abstract

We show that the infimum of the essential spectrum of the linearisation of the equation

$$-\ddot{u}(x) + V(x)u(x) - r(x)|u(x)|^{p-1}u(x) = \lambda u(x) \tag{1}$$

is a bifurcation point for (1). Here the potential V is periodic and the function r almost-periodic. Thus (1) is an almost-periodic perturbation of Hill's equation. Bifurcation results of Küpper and Stuart are combined with existence theory for almost-periodic problems developed by Serra, Tarallo and Terracini.

1 Introduction

Many results have been obtained on bifurcation from points in the essential spectra of linearisations of nonlinear equations - for a survey, see [11]. In particular, a series of works initiated by T. Küpper and C.A. Stuart has explored such phenomena for equations such as

$$\begin{cases} -\ddot{u}(x) + V(x)u(x) - r(x)|u(x)|^{p-1}u(x) = \lambda u(x) \\ u \in H^2(\mathbb{R}) \setminus \{0\}, \lambda \in \mathbb{R}, \end{cases} \tag{2}$$

under the assumptions that $2 < p < 6$, $V, r \in L^\infty(\mathbb{R}), r \geq 0$ and V is periodic (see [10] and the references therein). Their approach is variational. Non-zero solutions for a given $\lambda \in \mathbb{R}$ are obtained as critical points of a functional, using either constrained minimisation or an application of the Mountain Pass Lemma. For these results, it is necessary to suppose either that $r(x) \to 0$ as $|x| \to \infty$ or that r is periodic with the same period as V.

Besicovitch [1] defines a uniformly almost-periodic (UAP) function to be a function $q : \mathbb{R} \to \mathbb{R}$, continuous on $\mathbb{R}$, such that for each $\epsilon > 0$, there exists $L(\epsilon) > 0$ so that each interval of $\mathbb{R}$ of length $L(\epsilon)$ contains a number τ, satisfying

$$|q(x+\tau) - q(x)| \leq \epsilon \quad \text{for each} \quad x \in \mathbb{R}. \tag{3}$$

Clearly, a continuous periodic function is UAP. If r is merely assumed to be a UAP-function, the study of (2) becomes more complicated because the associated functional has neither compactness of the nonlinear term (as in the case $r(x) \to 0$) nor invariance under the action of a group of symmetries (as in the case where r is periodic with the same period as V). The difficulties arising from almost-periodicity are addressed in an interesting paper of E. Serra, M. Tarallo and S. Terracini [8]. There, the existence of non-zero solutions of almost-periodic equations is obtained using concentration compactness, a version of the Mountain Pass Lemma due to E. Séré and a study of the asymptotic form of certain functions.

In this paper, the almost-periodic existence theory of [8] is employed to extend a bifurcation result of [10] to the case in which r is a UAP-function. Our aim is to show that under the hypotheses

(L1) $V \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$; without loss of generality, suppose that $V \geq 1$;

(L2) $V(x+1) = V(x)$ for each $x \in \mathbb{R}$ (V is periodic),

and

(N0) $2 < p < 6$;

(N1) $r \geq 0, r \not\equiv 0$;

(N2) r is a UAP-function,

the infimum of the essential spectrum of the linearisation of the right-hand side of (2) about $u \equiv 0$ is a bifurcation point for (2). Note that (N2) implies that $r \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ - see [1, p 2]. Suppose henceforth that (L1), (L2) and (N0), (N1), (N2) hold. Define $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{S}u = -\ddot{u} + Vu \quad \text{for each} \quad u \in \mathcal{D}(\mathcal{S}) = H^2(\mathbb{R}). \tag{4}$$

The equation $\mathcal{S}u = 0$ is the reknowned Hill's equation. Under hypothesis (L1), it is well-known that $\mathcal{S}$ is self-adjoint and bounded below. Let

$$\Lambda = \inf_{u \in H^2(\mathbb{R}) \setminus \{0\}} \frac{(\mathcal{S}u, u)}{(u, u)}. \tag{5}$$

(Here $(\cdot, \cdot)$ denotes the usual L^2-inner product.) Then $\Lambda \geq 1$ is the infimum of the spectrum of $\mathcal{S}$, $\sigma(\mathcal{S})$ [6, p 278]. Moreover, the periodicity assumption on V, (L2), implies that $\sigma(\mathcal{S})$ is purely continuous and consists of a countable number of closed intervals [4]. So $\lambda \in \sigma_{ess}(\mathcal{S})$, the essential spectrum of $\mathcal{S}$. It will be shown that Λ is a bifurcation point for the nonlinear equation (2), in the sense that there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset (-\infty, \Lambda) \times H^2(\mathbb{R})$ of solutions of (2) such that $u_n \not\equiv 0$ for each $n \in \mathbb{N}$, and

$$\|u_n\|_{H^1(\mathbb{R})} \to 0 \quad \text{and} \quad \lambda_n \to \Lambda \quad \text{as} \quad n \to \infty.$$

To adopt a variational approach, we formulate the problem in $H^1(\mathbb{R})$. Note first that since $H^2(\mathbb{R})$ is dense in $H^1(\mathbb{R})$,

$$\Lambda = \inf_{u \in H^2(\mathbb{R}) \setminus \{0\}} \frac{(\mathcal{S}u, u)}{(u, u)} = \inf_{u \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \dot{u}^2 + Vu^2 \, dx}{(u, u)}.$$

Also, for $0 < \lambda < \Lambda$,

$$\int_{\mathbb{R}} \dot{u}^2 + \{V - \lambda\} u^2 \, dx \geq \left(1 - \frac{\lambda}{\Lambda}\right) \|u\|^2_{H^1(\mathbb{R})} \quad \text{for each} \quad u \in H^1(\mathbb{R}). \tag{6}$$

Now define a functional $J : \mathbb{R} \times H^1(\mathbb{R}) \to \mathbb{R}$ by

$$J(\lambda, u) = \frac{1}{2} \int_{\mathbb{R}} \dot{u}^2 + \{V - \lambda\} u^2 \; dx - \frac{1}{p} \int_{\mathbb{R}} r|u|^p \; dx, \quad (\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}).$$

Since $p > 2$ (by (N0)), [9, Lemma 5.1] yields that $J \in C^1(\mathbb{R} \times H^1(\mathbb{R}), \mathbb{R})$ and

$$\nabla_u J(\lambda, u)v = \int_{\mathbb{R}} \dot{u}\dot{v} + \{V - \lambda\} uv \; dx - \int_{\mathbb{R}} r|u|^{p-2} uv \; dx, \quad u, v \in H^1(\mathbb{R}), \lambda \in \mathbb{R}.$$

Given $\lambda \in \mathbb{R}$, we will seek solutions of

$$\nabla_u J(\lambda, u) = 0, \quad u \in H^1(\mathbb{R}) \backslash \{0\}. \tag{7}$$

Since u is defined on $\mathbb{R}$, it is straightforward to show that if u satisfies (7), then $u \in H^2(\mathbb{R}) \cap C^2(\mathbb{R})$ is a classical solution of (2).

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2 Existence for fixed λ via the Mountain Pass Lemma

Throughout this section, fix $\lambda \in (0, \Lambda)$ and suppose that there exists $v \in H^1(\mathbb{R})$ such that $J(\lambda, v) < 0$. Let $c(\lambda, v) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\lambda, \gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R})) : \gamma(0) = 0, \gamma(1) = v\}$.

Lemma 2.1 **(a)** *$J(\lambda, 0) = 0$ is a strict local minimum for J; there exists $\rho(\lambda) > 0$ such that for each $\rho \in (0, \rho(\lambda))$, there exists $\alpha_\rho > 0$ with*

$$J(\lambda, u) \geq \alpha_\rho$$

for each $u \in H^1(\mathbb{R})$ with $\|u\|_{H^1(\mathbb{R})} = \rho$;

(b) *$c(\lambda, v) > 0$;*

(c) *given $\epsilon \in (0, c(\lambda, v))$ sufficiently small, there exists $\tilde{c} \in (c(\lambda, v) - \epsilon, c(\lambda, v) + \epsilon)$ and a sequence $\{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R})$ such that as $n \to \infty$,*

$$\begin{aligned} &J(\lambda, u_n) \to \tilde{c}, \\ &\|\nabla_u J(\lambda, u_n)\|_{(H^1(\mathbb{R}))^*} \to 0, \\ &\|u_n - u_{n-1}\|_{H^1(\mathbb{R})} \to 0. \end{aligned} \tag{8}$$

Proof. **(a)** is clear from estimate (6) and the definition of J. **(b)** is immediate from **(a)**. **(a)** and the fact that there exists $v \in H^1(\mathbb{R})$ with $J(\lambda, v) < 0$ form the geometric hypotheses of the Mountain Pass Lemma (see [2]), which yields the existence of a sequence $\{u_n\}_{n=1}^{\infty} \in H^1(\mathbb{R})$ with $J(\lambda, u_n) \to c(\lambda, v)$ and $\|\nabla_u J(\lambda, u_n)\|_{(H^1(\mathbb{R}))^*} \to 0$ as $n \to \infty$. That there is a sequence $\{u_n\}_{n=1}^{\infty}$ with the additional property $\|u_n - u_{n-1}\|_{H^1(\mathbb{R})} \to 0$ and the functional converging to a modified level $J(\lambda, u_n) \to \tilde{c}$, follows from work of Séré [3, 7]. A proof of a form of Séré's result from which (8) follows is given in [8, Theorem 1.2]. □

In the rest of this section, let $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R})$ be a *fixed* sequence satisfying (8) for some $\tilde{c} \in (\frac{1}{2}c(\lambda, v), \frac{3}{2}c(\lambda, v))$. We will construct a nonzero solution u of the equation $\nabla_u J(\lambda, u) = 0$ as a weak limit of translates of the sequence $\{u_n\}_{n=1}^{\infty}$. First some standard technical observations.

Lemma 2.2 **(a)** $\{\|u_n\|_{H^1(\mathbb{R})}\}_{n=1}^{\infty}$ *is a bounded sequence;*

(b) $\liminf_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R})} > 0$*;*

(c) $\nabla_u J(\lambda, \cdot) : H^1(\mathbb{R}) \to (H^1(\mathbb{R}))^*$ *is weakly sequentially continuous.*

Proof. **(a), (b)** and **(c)** are proved in [8, Proposition 2.8, Proposition 2.11 and Propostion 2.2] respectively. □

For $\tau \in \mathbb{R}$, $u \in H^1(\mathbb{R})$, define

$$(T_\tau u)(x) = u(x - \tau), \ x \in \mathbb{R}. \tag{9}$$

Now Lemma 2.2 **(a), (b)** immediately imply the existence of $\alpha > 0$, a sequence $\{\tau_k\}$, a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and $u \in H^1(\mathbb{R})$ such that $T_{\tau_k} u_{n_k} \rightharpoonup u$ as $k \to \infty$ and $(T_{\tau_k} u_{n_k})(0) \geq \alpha$ for each k. Since $(T_{\tau_k} u_{n_k})(0) \not\to 0$, $u \neq 0$. But u may not be a solution of $\nabla_u J(\lambda, u) = 0$. A more careful choice of sequence $\{\tau_k\}_{k=1}^{\infty}$ is required. We seek a sequence $\{\tau_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that

(T1) no subsequence of $T_{\tau_k} u_{n_k}$ converges weakly to zero;

(T2) $\|\nabla_u J(\lambda, T_{\tau_k} u_{n_k})\|_{(H^1(\mathbb{R}))^*} \to 0$ as $k \to \infty$;

(T3) $J(\lambda, T_{\tau_k} u_{n_k}) \to \tilde{c}$ as $k \to \infty$.

Before proving that such sequences exist, we show that (T1), (T2), (T3) imply the existence of a non-zero solution of $\nabla_u J(\lambda, u) = 0$.

Lemma 2.3 *Suppose that* $\{\tau_k\}_{k=1}^{\infty} \subset \mathbb{R}, \{u_{n_k}\}_{k=1}^{\infty}$ *satisfy (T1), (T2), (T3). Then there exists* $u_\lambda \in H^1(\mathbb{R})\backslash\{0\}$ *such that* $\nabla_u J(\lambda, u_\lambda) = 0$ *and*

$$\left(1 - \frac{\lambda}{\Lambda}\right) \|u_\lambda\|^2_{H^1(\mathbb{R})} \leq \left(\frac{3p}{p-2}\right) c(\lambda, v). \tag{10}$$

Proof. By Lemma 2.2 **(a)**, $\|T_{\tau_k} u_{n_k}\|_{H^1(\mathbb{R})}$ is bounded independently of $k \in \mathbb{N}$. So there exists $u_\lambda \in H^1(\mathbb{R})$ and subsequences of $\{\tau_k\}_{k=1}^{\infty}, \{u_{n_k}\}_{k=1}^{\infty}$ (still labelled τ_k, u_{n_k}) such that $T_{\tau_k} u_{n_k} \rightharpoonup u_\lambda$

in $H^1(\mathbb{R})$. By (T1), $u_\lambda \neq 0$. Moreover, Lemma 2.2 **(c)** shows that $\nabla_u J(\lambda, T_{\tau_k} u_{n_k}) \rightharpoonup \nabla_u J(\lambda, u_\lambda)$ in $(H^1(\mathbb{R}))^*$ as $k \to \infty$. So by (T2) and the uniqueness of weak limits, $\nabla_u J(\lambda, u_\lambda) = 0$.
Now for each $u \in H^1(\mathbb{R})$,

$$2J(\lambda, u) - \nabla_u J(\lambda, u)u = \left(\frac{p-2}{p}\right) \int_{\mathbb{R}} r|u|^p \, dx.$$

So $\mathcal{C} : H^1(\mathbb{R}) \to \mathbb{R}$, $\mathcal{C}(u) = 2J(\lambda, u) - \nabla_u J(\lambda, u)u$ is a convex function of u, and thus $\mathcal{C}$ is weakly sequentially lower semicontinuous. It follows that

$$\begin{aligned} \mathcal{C}(u_\lambda) &\leq \liminf_{k\to\infty} \mathcal{C}(T_{\tau_k} u_{n_k}) \\ &= \liminf_{k\to\infty} \{2J(\lambda, T_{\tau_k} u_{n_k}) - \nabla_u J(\lambda, T_{\tau_k} u_{n_k}) T_{\tau_k} u_{n_k}\} \\ &= 2\tilde{c} \;<\; 3c(\lambda, v), \end{aligned} \tag{11}$$

since (T2), (T3) hold and $\tilde{c} \in (\frac{1}{2}c(\lambda, v), \frac{3}{2}c(\lambda, v))$. To obtain estimate (10), it remains to observe that by estimate (6) and the fact that $\nabla_u J(\lambda, u_\lambda) = 0$,

$$\begin{aligned} \left(1 - \frac{\lambda}{\Lambda}\right) \|u_\lambda\|^2_{H^1(\mathbb{R})} &\leq \int_{\mathbb{R}} \dot{u_\lambda}^2 + \{V - \lambda\} u_\lambda^2 \, dx \\ &= \int_{\mathbb{R}} r|u_\lambda|^p \, dx \;=\; \left(\frac{p}{p-2}\right) \mathcal{C}(u_\lambda). \end{aligned} \tag{12}$$

(10) follows from (11) and (12). □

To obtain the existence of sequences $\{\tau_k\}_{k=1}^\infty, \{u_{n_k}\}_{k=1}^\infty$ satisfying (T1), (T2), (T3), we invoke a key result of [8] that relies on the property $\|u_n - u_{n-1}\|_{H^1(\mathbb{R})} \to 0$ of the sequence $\{u_n\}_{n=1}^\infty$.

Lemma 2.4 *There exist a sequence $\{\tau_k\}_{k=1}^\infty \subset \mathbb{R}$ and a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ which satisfy (T1), (T2), (T3).*

Proof. By Lemma 2.2 **(a)**, $\{u_n\}_{n=1}^\infty$ has a weakly convergent subsequence, say $u_{n_k} \rightharpoonup u_0 \in H^1(\mathbb{R})$ as $k \to \infty$. If $u_0 \neq 0$, then it follows from the properties (8) of $\{u_n\}_{n=1}^\infty$ that (T1), (T2), (T3) are satisfied using this sequence $\{u_{n_k}\}_{k=1}^\infty$ if we set $\tau_k = 0$ for each $k \in \mathbb{N}$.

So suppose that $u_0 = 0$. Now since the full sequence $\{u_n\}_{n=1}^\infty$ has the properties (8), [8, Proposition 3.13] yields the existence of a sequence $\{\tau_n\}_{n=1}^\infty \subset \mathbb{R}$ such that

$$\begin{aligned} &(i) \quad \liminf_{n\to\infty} |T_{\tau_n} u_n(0)| > 0, \quad \text{and} \\ &(ii) \quad \lim_{n\to\infty} |\tau_n - \tau_{n-1}| = 0. \end{aligned}$$

(We emphasize that the *full* sequence $\{u_n\}_{n=1}^\infty$ is used because of the importance of the property $\|u_n - u_{n-1}\|_{H^1(\mathbb{R})} \to 0$, which is not inherited by subsequences - see [8] for details.)

It will be shown that there is a subsequence $\{(\tau_{n_k}, u_{n_k})\}_{k=1}^\infty$ of $\{(\tau_n, u_n)\}_{n=1}^\infty$ such that if $\tau_k = \tau_{n_k}$ for each $k \in \mathbb{N}$, then $\{\tau_k\}_{k=1}^\infty, \{u_{n_k}\}_{k=1}^\infty$ satisfy (T1), (T2), (T3). Note first that it follows from (i) that any subsequence of $\{(\tau_n, u_n)\}_{n=1}^\infty$ satisfies (T1). Also, if any subsequence of $\{\tau_n\}_{n=1}^\infty$ were bounded, then (i) would imply that $u_0 \neq 0$. So since we are supposing here that $u_0 = 0$, it follows that $|\tau_n| \to \infty$ as $n \to \infty$. Suppose in the sequel that $\tau_n \to \infty$; a similar argument applies if $\tau_n \to -\infty$. We will use this, together with (ii), to construct a subsequence of $\{(\tau_n, u_n)\}_{n=1}^{\infty}$ for which (T2), (T3) also hold.

It is shown in [1, p 5, Theorem 11] that given $\epsilon > 0$ and two UAP-functions q_1 and q_2, there exist $L(\epsilon)$ and τ such that (3) holds for both q_1 and q_2. Since V and r are both UAP-functions, it thus follows that there exists a sequence $\{\sigma_k\}_{k=1}^{\infty} \subset \mathbb{R}$, with $\sigma_k \to \infty$ as $k \to \infty$, such that

$$\|r(\cdot + \sigma_k) - r(\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{and} \quad \|V(\cdot + \sigma_k) - V(\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad k \to \infty.$$

Now $|\tau_n - \tau_{n-1}| \to 0$ and $\tau_n \to \infty$ as $n \to \infty$, so there is a subsequence $\{\tau_{n_k}\}_{k=1}^{\infty}$ of $\{\tau_n\}_{n=1}^{\infty}$ such that $|\sigma_k - \tau_{n_k}| \to 0$ as $k \to \infty$. Thus as UAP-functions are uniformly continuous ([1, p 2]),

$$\|r(\cdot + \tau_{n_k}) - r(\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{and} \quad \|V(\cdot + \tau_{n_k}) - V(\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad k \to \infty.$$

Let $u \in H^1(\mathbb{R}), \|u\|_{H^1(\mathbb{R})} = 1$ be arbitrary. Then

$$\begin{aligned}
\nabla_u J(\lambda, T_{\tau_{n_k}} u_{n_k})u &= \nabla_u J(\lambda, u_{n_k})u(\cdot - \tau_{n_k}) + \int_{\mathbb{R}} \{V(x) - V(x - \tau_{n_k})\} u_{n_k}(x - \tau_{n_k}) u(x)\ dx \\
&\quad - \int_{\mathbb{R}} \{r(x) - r(x - \tau_{n_k})\} |u_{n_k}(x - \tau_{n_k})|^{p-2} u_{n_k}(x - \tau_{n_k}) u(x)\ dx \\
&\leq \|\nabla_u J(\lambda, u_{n_k})\|_{(H^1(\mathbb{R}))^*} + K_1 \|V(\cdot) - V(\cdot - \tau_{n_k})\|_{L^\infty(\mathbb{R})} \\
&\qquad + K_2 \|r(\cdot) - r(\cdot - \tau_{n_k})\|_{L^\infty(\mathbb{R})}
\end{aligned}$$

for some $K_1,\ K_2 > 0$ independent of $k \in \mathbb{N}$. So $\|\nabla_u J(\lambda, T_{\tau_{n_k}} u_{n_k})\|_{(H^1(\mathbb{R}))^*} \to 0$ as $k \to \infty$. A similar argument shows that $J(\lambda, T_{\tau_{n_k}} u_{n_k}) \to \tilde{c}$ as $k \to \infty$. Thus $\{\tau_{n_k}\}_{k=1}^{\infty}$, $\{u_{n_k}\}_{k=1}^{\infty}$ satisfy (T2), (T3). The result follows. $\square$

3 Bifurcation

Exploiting estimate (10), we marry the existence results of the previous section to an approach to proving bifurcation discussed in [10]. First a lemma on the existence of the key ingredients of the Mountain Pass method when λ is close to Λ. Note that it is for this result that the constraint $p < 6$ in condition (N0) is essential.

Lemma 3.1 *There exist a sequence $\lambda_n \to \Lambda$ and a sequence $\{v_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R})\backslash\{0\}$ such that $J(\lambda_n, v_n) < 0$ for each $n \in \mathbb{N}$ and*

$$\frac{c(\lambda_n, v_n)}{\Lambda - \lambda_n} \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. This follows from an amalgamation of [10, Lemma 7.1, (1) and (3)] and [10, Lemma 9.5 (b)]. Note that although [10, Lemma 9.5 (b)] is obtained under the assumption that r is periodic, the Riemann-Lebesgue lemma for UAP-functions (see [5, Proposition 3.1]) enables the same conclusion to be drawn when r is UAP. $\square$

Our main result is the following.

Theorem 3.2 *There exists a sequence* $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset (-\infty, \Lambda) \times H^1(\mathbb{R}) \backslash \{0\}$ *such that* $\nabla_u J(\lambda_n, u_n) = 0$ *for each* n*, and as* $n \to \infty$*,*

$$\lambda_n \to \Lambda \quad \text{and} \quad \|u_n\|_{H^1(\mathbb{R})} \to 0.$$

Proof. Let $\{(\lambda_n, v_n)\}_{n=1}^{\infty}$ be as given by Lemma 3.1. For each $n \in \mathbb{N}$, $J(\lambda_n, v_n) < 0$, so Lemmas 2.3 and 2.4 together yield the existence of $u_n \in H^1(\mathbb{R}) \backslash \{0\}$ such that $\nabla_u J(\lambda_n, u_n) = 0$ and

$$\left(1 - \frac{\lambda_n}{\Lambda}\right) \|u_n\|^2_{H^1(\mathbb{R})} \leq \left(\frac{3p}{p-2}\right) c(\lambda_n, v_n).$$

Since $\lambda_n \to \Lambda$ and $\dfrac{c(\lambda_n, v_n)}{\Lambda - \lambda_n} \to 0$ as $n \to \infty$, the result follows. $\square$

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