GROUPS OF SCALINGS AND INVARIANT SETS FOR HIGHER-ORDER NONLINEAR EVOLUTION EQUATIONS

BY

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Abstract

We consider a nonlinear one-dimensional m-th order evolution equation of a general form

$$\text{(EE)} \qquad u_t = \mathbf{A}(u) \equiv A(x, u, u_1, ..., u_m), \quad u = u(x,t), \quad (x,t) \in Q = (0,1) \times [0,1],$$

where A is a C^∞-function, $u_t = \partial u/\partial t$ and u_i denote the i-th partial spatial derivatives $\partial^i u/\partial x^i$. It is known that if (EE) is invariant under a group of scalings with the infinitesimal generator $X = x\frac{\partial}{\partial x} + \mu t \frac{\partial}{\partial t}$ (μ is a constant "scaling order" of $\mathbf{A}$), then (EE) admits exact (self-similar) solutions depending on the single (invariant) variable $u(x,t) = \theta(\xi), \quad \xi = x/t^{1/\mu}$.

It is proved that when the operator in (EE) is composed from a sum of operators with different scaling orders and no group of scaling exists, the exact solutions can be constructed via the invariance of the set $S_0 = \{u: \ u_1 = F(u)/x\}$ with a smooth F to be determined. The time-evolution on S_0 is shown to be governed by a first-order ordinary differential equation. The results are illustrated with examples of nonlinear parabolic equations of the second and of the fourth-order.

AMS (MOS) Subject Classification: 35K55, 35K65.

Key Words. Nonlinear evolution equations, group of scalings, invariant sets and subspaces, exact solutions, scaling orders.

1. Introduction and main result

Consider a nonlinear one-dimensional m-th order evolution equation of the form

$$u_t = \mathbf{A}(u) \equiv A(x, u, u_1, ..., u_m), \quad u = u(x,t), \quad (x,t) \in Q = (0,1) \times [0,1], \tag{1.1}$$

where $u_t = \partial u/\partial t$ is the time derivative and u_i denote the i-th partial derivatives $\partial^i u/\partial x^i$ with respect to the spatial variable x. The function A is assumed to be smooth, $A \in C^\infty((0,1) \times \mathbf{R}^{m+1})$. The solutions of (1.1) are supposed to be smooth, $u \in C^\infty(Q)$. We will deal with invariant sets and exact, low-dimensional solutions of (1.1).

1.1. Group of scalings and scaling order for nonlinear operators. Equation (1.1) is invariant under the Lie group of scaling transformations

$$x^* = e^\varepsilon x, \quad t^* = e^{\mu\varepsilon} t, \tag{1.2}$$

where $\mu \in \mathbf{R}$ is a constant, with the infinitesimal generator

$$X = x\frac{\partial}{\partial x} + \mu t \frac{\partial}{\partial t}, \tag{1.3}$$

if the operator $\mathbf{A}$ satisfies the following homogenuity condition:

$$A(sx, u, \frac{1}{s}u_1, ..., \frac{1}{s^m}u_m) \equiv \left(\frac{1}{s}\right)^\mu A(x, u, u_1, ..., u_m) \quad \text{for all } s > 0. \tag{1.4}$$

See [BK] and [I]. From the invariance equation $Xu = 0$ we then conclude that (1.1) admits invariant (self-similar) solutions depending on a single variable:

$$u(x,t) = \theta(\xi), \quad \xi = x/t^{1/\mu}. \tag{1.5}$$

On the set of invariant solutions (1.5) the nonlinear partial differential equation (1.1) reduces to an ordinary differential equation (ODE) for the function θ

$$A(\xi, \theta, \theta', ..., \theta^{(m)}) + \frac{1}{\mu}\theta'\xi = 0. \tag{1.6}$$

We then say that $\mu = \mu(\mathbf{A})$ is the *scaling order* of the operator $\mathbf{A}$. For instance, for the following operator with power-like nonlinearities in the derivatives

$$\mathbf{A}(u) = \phi(u) x^\beta \Pi_{j=1}^m (u_j)^{\alpha_j}, \tag{1.7}$$

where ϕ is a smooth function, the scaling order is given by

$$\mu(\mathbf{A}) = \sum_{j=1}^m j\alpha_j - \beta. \tag{1.8}$$

1.2. Scaling index. We now consider a more general evolution equation of the form

$$u_t = \mathbf{B}(u) \equiv \sum_{i=1}^{M} \mathbf{A}_i(u), \tag{EE}$$

where the operators $\mathbf{A}_i$ are of the form (1.1) with the scaling orders μ_i in the sense of (1.4).

Definition. $I(\mathbf{B})$, the *scaling index* of operator $\mathbf{B}$, is the cardinal number of the set of the scaling orders of the components $\mathbf{A}_i$:

$$I(\mathbf{B}) = \#\{\mu_i\} \le M. \tag{1.9}$$

The following properties of the composition of scaling operators are straightforward.

Proposition 1.1. *The scaling orders $\mu_i(\mathbf{A})$ and the scaling index $I(\mathbf{B})$ do not depend on any invertible point transformation of the dependent variable $u = R(v)$, $R \in C^\infty$.*

Proposition 1.2. *The group of scalings (1.2) is admitted by the equation (EE) if and only if $I(\mathbf{B}) = 1$ (and $\mu(\mathbf{B}) = \mu$).*

1.3. Algebraic differentiation. If the index of the operator satisfies $I(\mathbf{B}) > 1$ then (EE) does not admit the Lie group (1.2). In this case we begin our construction of invariant sets and exact solutions by introdusing the set of functions

$$S_0 = \left\{ u \in C^\infty : \quad u_1 = \frac{1}{x} F(u) \right\}, \tag{1.10}$$

where F is a C^∞-function to be determined from the invariance condition

$$u(\cdot, 0) \in S_0 \Rightarrow u(\cdot, t) \in S_0 \ \text{ for } t \in (0, 1]. \tag{1.11}$$

The structure of the manifold (1.10) was used in [G2, Section 3] in the construction of first-order sign-invariants for nonlinear parabolic equations, i.e., the operators preserving sign on the evolution orbits.

The differential (contact) structure of S_0 makes it possible to introduce the following algebraic differentiation in the set.

Lemma 1.3. *Let $u \in S_0$. Then*

$$u_k = \frac{1}{x^k} \mathbf{G}_k(F), \quad k = 1, 2, ..., \tag{1.12}$$

where the ordinary differential operators $\mathbf{G}_k$ satisfy the recursion relation: for any $F \in C^\infty$

$$\mathbf{G}_{k+1}(F) = F \frac{d}{du} \mathbf{G}_k(F) - k \mathbf{G}_k(F), \quad k = 1, ...; \quad \mathbf{G}_1(F) = F. \tag{1.13}$$

Proof is straightforward: differentiating (1.12) we obtain

$$u_{k+1} = -\frac{k}{x^{k+1}} \mathbf{G}_k(F) + \frac{1}{x^k} \frac{d}{du} \mathbf{G}_k(F) u_1 \equiv \frac{1}{x^{k+1}} \mathbf{G}_{k+1}(F), \tag{1.14}$$

since $u_1 = F/x$ by definition (1.10). $\square$

Let us present some differential operators which are sufficient to deal with equations up to the fifth order:

$$
\begin{aligned}
&\mathbf{G}_2(F) = F(F' - 1), \\
&\mathbf{G}_3(F) = F\mathbf{G}_2' - 2\mathbf{G}_2 = F[(FF')' - 3F' + 2], \\
&\mathbf{G}_4(F) = F\mathbf{G}_3' - 3\mathbf{G}_3 = F[(F(FF')')' - 6(FF')' + 11F' - 6], \\
&\mathbf{G}_5(F) = F[(F(F(FF')')')' - 10(F(FF')')' + 35(FF')' - 50F' + 24], \;\; \text{etc.}
\end{aligned}
\tag{1.15}
$$

1.4. Main result: invariant sets for $I(\mathbf{B}) > 1$. Using the rule (1.12) and the scaling properties (1.4) with $\mu = \mu_i$ of operators $\mathbf{A}_i$, we conclude that on S_0

$$
\mathbf{B}(u) = \sum_{i=1}^{M} A_i(x, u, \frac{1}{x}\mathbf{G}_1, ..., \frac{1}{x^m}\mathbf{G}_m) \equiv \sum_{i=1}^{M} x^{-\mu_i} A_i(1, u, \mathbf{G}_1, ..., \mathbf{G}_m). \tag{1.16}
$$

Let $\lambda_1 < \lambda_2 < ... < \lambda_I$ be all the distinct scaling orders of the operators $\{\mathbf{A}_i\}$ so that (1.16) takes the form

$$
\mathbf{B}(u) = \sum_{j=1}^{I} x^{-\lambda_j} F\mathbf{C}_j(F), \tag{1.17}
$$

where the operators $\mathbf{C}_j$ are given by

$$
\mathbf{C}_j(F) = \frac{1}{F} \sum_{\{i:\mu_i = \lambda_j\}} \mathbf{A}_i(1, u, \mathbf{G}_1(F), ..., \mathbf{G}_m(F)), \quad j = 1, ..., I. \tag{1.18}
$$

We now state the main result.

Theorem 1.4. *The set S_0 is invariant if and only if the function $F(u)$ satisfies the system of $I(\mathbf{B})$ ODE's*

$$
F\frac{d}{du}\mathbf{C}_j(F) = \lambda_j \mathbf{C}_j(F), \quad j = 1, ..., I. \tag{1.19}
$$

Then the solutions $u(x,t)$ on S_0 take the form

$$
v \equiv \int_1^u \frac{dz}{F(z)} = \ln x + H(t), \tag{1.20}
$$

where $H(t)$ solves the ODE

$$
H' = \sum_{j=1}^{I} d_j e^{\lambda_j H}, \quad t \in (0, 1]; \quad d_j = \mathbf{C}_j(F)|_{u=1}, \tag{1.21}
$$

which is the evolution equation (EE) on S_0.

Proof. In view of (1.20) we have

$$u_t = F(u)H'(t). \tag{1.22}$$

Substituting (1.22) and (1.17) into equation (EE) restricted to S_0 we obtain the equation

$$H'(t) = \sum_{j=1}^{I} x^{-\lambda_j} \mathbf{C}_j(F(u)) \equiv D(x,t). \tag{1.23}$$

Therefore $D(x,t)$ does not depend on x whence

$$D_x = \sum_{j=1}^{I} \left[F\frac{d}{du}\mathbf{C}_j(F) - \lambda_j \mathbf{C}_j(F) \right] x^{-(\lambda_j+1)} \equiv 0. \tag{1.24}$$

Since the functions $\{x^{-(\lambda_j+1)}\}$ are linearly independent in $\mathbf{R}^I$ we then arrive at the system (1.19) which means that there holds

$$\frac{d}{dv}\mathbf{C}_j(F) \equiv F\frac{d}{du}\mathbf{C}_j(F) = \lambda_j \mathbf{C}_j(F). \tag{1.25}$$

Hence

$$\mathbf{C}_j(F) = d_j e^{\lambda_j v}. \tag{1.26}$$

Substituting (1.26) with $v = \ln x + H$ given by (1.20) into (1.23) we obtain the ODE (1.21). $\square$

Remark: group of scalings if $I = 1$. In the case $I = 1$, when the group of scalings exists, one can solve (1.21) explicitly. We set $\lambda_1 = \mu$, $d_1 = d$. Then from the equation with a single exponential function

$$H' = de^{\mu H},$$

one obtains

$$H(t) = -\frac{1}{\mu}\ln t + \mathrm{const}. \tag{1.27}$$

Substituting (1.27) into (1.20) we have that

$$\int_1^u \frac{dz}{F(z)} = \ln \frac{x}{t^{1/\mu}} + \mathrm{const}, \tag{1.28}$$

i.e., u depends on the single scaling group invariant (1.5).

In the subsequent sections we apply the above construction to a general parabolic equation of the second (Section 2) and of the fourth order (Section 3). Results on the third-order equations of the Korteveg-de Vries type

$$u_t = \mathbf{B}_3(u) \equiv \phi(u)u_{xxx} + \psi(u)u_x \tag{1.29}$$

(it is the KdV equation if $\phi = 1$ and $\psi(u) = u$) can be found in [G4].

In the last Section 4 we give further generalizations of the method and establish connections with the method of linear invariant subspaces for nonlinear operators.

2. Nonlinear parabolic equations

2.1. Quasilinear heat equation. Consider a general reaction-diffusion equation of parabolic type

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)u_{xx} + \psi(u)(u_x)^2 + f(u) \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) + \mathbf{A}_3(u), \tag{2.1}$$

which contains three smooth functions $\phi(u) \geq 0$, $\psi(u)$ and $f(u)$. By a point transformation $u = R(v)$, $R \in C^\infty$, it reduces to a quasilinear equation of the divergent form from the combustion or filtration theory $v_t = (\varphi(v))_{xx} + q(v)$ with two arbitrary smooth functions φ and q, $\varphi' \geq 0$.

The scaling orders of the three operators in (2.1) are $\mu_1 = \mu_2 = 2$ and $\mu_3 = 0$, so that the scaling index is $I(\mathbf{B}_2) = 2$. Substituting the differential rule (1.12) one can calculate that on S_0

$$\mathbf{B}_2(u) = F\left\{\frac{1}{x^2}[\phi(F'-1) + \psi F] + \frac{f}{F}\right\} \equiv F\left[\mathbf{C}_1(F) + \frac{1}{x^2}\mathbf{C}_2(F)\right], \tag{2.2}$$

so that $\lambda_1 = 0$ and $\lambda_2 = 2$. From Theorem 1.4 we have that the invariant conditions (1.19) take the form

$$F\left(\frac{f}{F}\right)' = 0, \quad F[\phi(F'-1) + \psi F]' = 2[\phi(F'-1) + \psi F], \tag{2.3}$$

which are equivalent to the system

$$f = \nu F, \quad FF'' - 2F' + 2 + \frac{\phi'}{\phi}F(F'-1) + \frac{\psi}{\phi}F(F'-2) + \frac{\psi'}{\phi}F^2 = 0, \tag{2.4}$$

where $\nu \in \mathbf{R}$ is an arbitrary constant. Equation (2.1) on S_0 reduces to the following ODE for the function $H(t)$ in the exact solution representation (1.20)

$$H' = d_1 + d_2 e^{2H}, \quad t \in (0, 1], \tag{2.5}$$

where

$$d_1 = \left(\frac{f}{F}\right)(1) = \nu, \quad d_2 = [\phi(F'-1) + \psi F](1). \tag{2.6}$$

It is worth mentioning that (2.3) is a system of two equations with four arbitrary unknown functions ϕ, ψ, f and F. Therefore the invariant sets and exact solutions are constructed for an infinite-dimensional family of the parabolic equations (2.1).

2.2. Radial N-dimensional equation. The quasilinear heat equation in $\mathbf{R}^N$

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)\Delta u + \psi(u)|\nabla u|^2 + f(u) \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) + \mathbf{A}_3(u)$$

(which reduces to the divergent form $v_t = \Delta\varphi(v) + q(v)$) in the radial case when

$$\Delta u = u_{xx} + \frac{N-1}{x}u_x, \quad |\nabla u| = u_x \geq 0 \ \ (x > 0),$$

exhibits the same scaling orders as (2.1). Then in (2.2) we have that

$$\mathbf{C}_1(F) = \phi(F' + N - 2) + \psi F.$$

Therefore we replace the second equation in (2.4) by the following one

$$FF'' - 2F' - 2(N-2) + \frac{\phi'}{\phi}F(F' + N - 2) + \frac{\psi}{\phi}F(F'-2) + \frac{\psi'}{\phi}F^2 = 0.$$

See also examples of exact solutions for different types of nonlinearities in [G2, Sect. 3].

2.3. Equation with gradient-dependent diffusivity. The quasilinear equation

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)(u_x)^\alpha u_{xx} + f(u)(u_x)^\beta \equiv \mathbf{A}_1(u) + \mathbf{A}_2(u) \tag{2.7}$$

is known as a model of curve shortening flows and in the mechanics of non-Newtonian liquids. We have $\mu_1 = \alpha + 2$ and $\mu_2 = \beta$. If $\alpha + 2 = \beta$ then (2.7) is invariant under the scaling group (1.2).

Let $\alpha + 2 \neq \beta$. Then on S_0

$$\mathbf{B}_2(u) = F\left[\frac{1}{x^{\alpha+2}}\phi F^\alpha (F' - 1) + \frac{1}{x^\beta} F^{\beta-1} f\right]. \tag{2.8}$$

Therefore the invariant conditions are

$$F[\phi F^\alpha (F'-1)]' = (\alpha+2)\phi F^\alpha (F'-1), \quad F[F^{\beta-1} f]' = \beta F^{\beta-1} f, \tag{2.9}$$

or, which is the same,

$$FF'' + \alpha(F')^2 - 2(\alpha+1)F' + (\alpha+2) + \frac{\phi'}{\phi} F(F'-1) = 0, \quad Ff' + [(\beta-1)F' - \beta]f = 0. \tag{2.10}$$

The ODE on the function H reads

$$H' = d_1 e^{(\alpha+2)H} + d_2 e^{\beta H}, \tag{2.11}$$

with

$$d_1 = [\phi F^\alpha (F'-1)](1), \quad d_2 = (F^{\beta-1} f)(1).$$

2.4. Fully nonlinear equation. We now consider a simple example of a fully nonlinear parabolic equation

$$u_t = \mathbf{B}_2(u) \equiv \phi(u)(u_x)^\alpha (u_{xx})^\gamma + f(u)(u_x)^\beta, \tag{2.12}$$

cf. (2.7) with $\gamma = 1$. Then $\mu_1 = \alpha + 2\gamma$, $\mu_2 = \beta$, and we assume that $\mu_1 \neq \mu_2$, so that $I = 2$. On S_0

$$\mathbf{B}_2(u) = F\left[\frac{1}{x^{\alpha+2}}\gamma\phi F^{\alpha+\gamma-1}(F'-1)^\gamma + \frac{1}{x^\beta} F^{\beta-1} f\right].$$

Therefore in the invariant conditions (2.9) we replace the first equation by

$$F[\phi F^{\alpha+\gamma-1}(F'-1)^\gamma]' = (\alpha+2\gamma)\phi F^{\alpha+\gamma-1}(F'-1)^\gamma,$$

and the ODE for $H(t)$ becomes

$$H' = d_1 e^{(\alpha+2\gamma)H} + d_2 e^{\beta H}, \quad d_1 = [\phi F^{\alpha+\gamma-1}(F'-1)^\gamma](1).$$

3. Fourth-order equations

Consider a fourth-order evolution equation of the form

$$u_t = \mathbf{B}_4(u) \equiv \mathbf{B}_2(u) + \rho(u)u_{xxxx}, \tag{3.1}$$

where $\mathbf{B}_2$ is the second-order operator given in (2.1). If $\phi = \rho \equiv -1$, $\psi \equiv 1$ and $f = 0$, (3.1) becomes the Kuramoto-Sivashinskii equation from the theory of flame propagation, which is a fourth-order semilinear parabolic equation.

The analysis of (3.1) is similar to the second-order parabolic equation (2.1). The fourth-order term in (3.1) exhibits the scaling order $\mu_4 = 4$, and $I(\mathbf{B}_4) = 3$, so that on S_0 we add to the right-hand side of (2.2) an extra term of the form

$$\mathbf{B}_4(u) = \mathbf{B}_2(u) + F\left[\frac{1}{x^4}\rho\mathbf{C}_3(F)\right], \quad \mathbf{C}_3(F) = \frac{1}{F}\mathbf{G}_4(F), \tag{3.2}$$

where $\mathbf{G}_4$ is as given in (1.15). Therefore we add to the system (2.4) the third equation

$$F(\rho\mathbf{C}_3)' = 4\rho\mathbf{C}_3, \tag{3.3}$$

and the ODE takes the form

$$H' = d_1 + d_2e^{2H} + d_3e^{4H}, \quad d_3 = [\rho\mathbf{C}_3(F)](1). \tag{3.4}$$

4. Generalizations, invariant subspaces

4.1. Invariant sets for operators of even orders. It was shown in [G2, Example 4.3] that under certain assumptions on the coefficients, the quasilinear parabolic equation (2.1) admits solutions on the invariant subspace $W = \mathrm{Span}\{\cos x\}$ (or $W = \mathrm{Span}\{\cosh x\}$). Let us prove that this corresponds to the invariant set of a different structure:

$$S_1 = \{u : \quad u_1 = \tan x\, F(u)\}. \tag{4.1}$$

The algebraic differentiation in S_1 is now different from that in S_0. Namely, setting for convenience

$$w(x) = \frac{1}{\cos^2 x},$$

we have

$$u_2 = wF(F' + 1) - FF' \equiv F(w\mathbf{P}_{21} + \mathbf{P}_{20}). \tag{4.2}$$

Denoting

$$\mathbf{Q}(F) = F[(FF')' + 3F' + 2],$$

we obtain

$$u_4 = F(w^2\mathbf{P}_{42} + w\mathbf{P}_{41} + \mathbf{P}_{40}), \tag{4.3}$$

where

$$\begin{aligned} \mathbf{P}_{42} &= 3\mathbf{Q} + (F\mathbf{Q})' \quad ((\cdot)' = d/du), \\ \mathbf{P}_{41} &= -2\mathbf{Q} - (F\mathbf{Q})' - (FF')' - (F(FF')')', \\ \mathbf{P}_{40} &= (F(FF')')'. \end{aligned} \tag{4.4}$$

We claim that the even derivatives u_{2k} belong to a $(k+1)$-dimensional linear subspace:

$$u_{2k} \in \mathrm{Span}\{1, w, ..., w^k\}. \tag{4.5}$$

Therefore the analysis on S_1 is more convenient for the equations of even orders (or for those which contain operators of even scaling orders).

Example 4.1. Consider the fourth order parabolic equation

$$u_t = \mathbf{B}_4(u) \equiv \phi(u)u_2 + \psi(u)(u_1)^2 + \rho(u)u_4. \tag{4.6}$$

Then since for $u \in S_1$

$$(u_1)^2 = wF^2 - F^2,$$

we have that in S_1

$$\begin{aligned} \mathbf{B}_4(u) &= F[w^2(\rho\mathbf{P}_{42}) + w(\rho\mathbf{P}_{41} + \phi\mathbf{P}_{21} + \psi F) + \rho\mathbf{P}_{40} + \phi\mathbf{P}_{20} - \psi F] \\ &\equiv F(w^2\Phi_2 + w\Phi_1 + \Phi_0) \in \mathrm{Span}\{1, w, w^2\}. \end{aligned} \tag{4.7}$$

The index of $\mathbf{B}_4$ (as usual, the number of linearly independent terms in (4.7)) is equal to 3. Substituting the general solution of $u_1 = \tan x\, F(u)$,

$$v \equiv \int_1^u \frac{dz}{F(z)} = \ln\cos\, x + H(t) \quad (\cos x > 0), \tag{4.8}$$

into (4.6) and using (4.7) we obtain

$$H'(t) = w^2(x)\Phi_2 + w(x)\Phi_1 + \Phi_0. \tag{4.9}$$

In view of the linear independence, we then conclude that the functions $\Phi_2(v)$, $\Phi_1(v)$ and $\Phi_0(v)$ must be exponential ones, d_2e^{4v}, d_1e^{2v} and d_0 respectively, whence the system of the three equations on F:

$$F\Phi_2' = 4\Phi_2, \quad F\Phi_1' = 2\Phi_1, \quad F\Phi_0' = 0. \tag{4.10}$$

Then (4.9) becomes

$$H' = d_2e^{4H} + d_1e^{2H} + d_0, \tag{4.11}$$

with the coefficients

$$d_2 = \Phi_2|_{u=1}, \quad d_1 = \Phi_1|_{u=1}, \quad d_0 = \Phi_0|_{u=1}. \tag{4.12}$$

In fact, (4.10) is a system of three differential equations with four arbitrary functions. Therefore, there exists in infinite-dimensional family of equations (4.6) which admit exact solutions. Observe that if we add to the right-hand side the operator $\kappa(u)u_1u_3$ with the odd derivatives, then it does not change the index of the operator since on S_1

$$u_1u_3 \in \mathrm{Span}\{1, w, w^2\}.$$

A similar analysis is performed for the set driven by the constraint $u_1 = \tanh x\, F(u)$.

Exact solutions on S_1 can be constructed for the third-order equations of the KdV-type (1.29), the corresponding solutions on S_0 can be found in [G4].

Remark on completeness. Consider the semilinear heat equation

$$u_t = u_{xx} + f(u), \tag{4.13}$$

and the set with the separate-variable constraint $u_x = g(x)F(u)$. It is proved in [GP, Theorem 3.2] that the only functions $g(x)$ which can generate invariant sets for nontrivial semilinear equations are functions $1/x$, $\tan x$ and $\tanh x$, which implies completeness of the analysis.

4.2. Operator representation. It can be shown that the set of solutions (1.20) or (4.8) can be treated as a one-dimensional subspace invariant under a certain nonlinear operator. The function $C(t) = \log H(t)$ then plays a role of the expansion coefficient in the subspace $W = \mathrm{Span}\{x\}$ (or $W = \mathrm{Span}\{\cos x\}$). The concept of linear subspaces admitted by nonlinear operator turned out to be useful in the study of the one-dimensional and multi-dimensional nonlinear evolution equations, see [G1] - [G3], [GPS1], [O] and references therein. It was shown that given a nonlinear operator, the problem of the determination of linear invariant subspaces reduces to an extremely difficult nonlinear eigenvalue problem [GPS2]. On the other hand, for an ODE operators of finite orders, given a linear subspace, the class of operators admitted it can be described in terms of the Lie-Bäcklund symmetries of the linear ODE giving the subspace, [S1] and [S2]. In the present paper we thus give a new representation of nonlinear operators admitting one-dimensional invariant subspaces of a special structure in the case $I(\mathbf{B}) > 1$. Thus in the limit case $I(\mathbf{B}) = 1$ of the minimally possible scaling index, the *nonlinear invariance* property (linear subspaces invariant under nonlinear operators) coincides with the classical concept of invariance under *groups of scalings*, which is essentially a linear one in terms of the construction of infinitesimal generators. We thus observe that there exists a certain extension (for a wider class of evolutions equations) of applications of the classical Lie group analysis, but in a nonlinear manner.

4.3. Higher-order evolution. The method of construction of the invariant set S_0 (or S_1) admits a similar extension to nonlinear equations with a higher-order evolution, for example to hyperbolic-like equations $u_{tt} = \mathbf{A}(u)$, $(x,t) \in Q$.

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