NONLINEAR DIFFERENCE OPERATORS ADMITTING INVARIANT SUBSPACES

by

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Dedicated to Professor A.A. Samarskii on the occasion of his 80th birthday

Abstract

Finite-dimensional reductions of nonlinear difference equations are studied. A general representation of p-th order nonlinear difference operators admitting a given finite-dimensional invariant subspace is established. We prove a criterion of invariance of a linear subspace with respect to nonlinear difference operators. This solves the forward problem of invariant subspaces. Examples of nonlinear discrete operators are presented and the corresponding reductions and exact invariant solutions are discussed. We consider some aspects of the backward problem. A description of invariant subspaces for a special quadratic first-order operator is given.

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1. Main result: forward problem of invariant subspaces

Let $\mathbf{Z}$ be the set of integers, and let V be a linear space of real-valued functions defined on $\mathbf{Z}$. For $u \in V$, $i \in \mathbf{Z}$, denote $u_i = u(i)$. Fix a natural p. Consider a nonlinear difference operator $\Phi : V \to V$ of the form

$$\Phi[u]_i \equiv \phi(u_i, u_{i+1}, ..., u_{i+p}, i), \;\; i \in \mathbf{Z}, \tag{1.1}$$

where $\phi : \mathbf{R}^{p+1} \times \mathbf{Z} \to \mathbf{R}$ is a real-valued function. Operator (1.1) is a p-th order difference operator. Given a function $g \in V$, we study a difference equation of the form

$$\Phi[u] = g \quad \text{on } \mathbf{Z}, \tag{1.2}$$

which is a system of nonlinear algebraic equations.

System (1.2) consists of an infinite number of equations with the solutions $u \in V$. Finite-difference approximations of nonlinear ordinary and partial differential equations are known to lead to systems like (1.2). The value p is then known to be the order of the difference equation and depends on the order of the approximation of the differential operators. For instance, $p = 2$ corresponds to a three-point approximation of the second derivative $u'' \sim (u_i - 2u_{i+1} + u_{i+2})/h^2$, where $h > 0$ is a constant step of the discretization and u_i denotes the value of u at the i-th point of the grid. On the other hand, it might be a three-point symmetric approximation of the first derivative u' of the form $(u_{i+2} - u_i)/2h$. See examples in Sections 3 and 4.

We give a description of equations (1.2) which reduce to a finite system of algebraic equations. Namely, we are looking for solutions which belong to a linear n-dimensional subspace $W_n \subset V$ given by the linear span

$$W_n = \mathcal{L}\{f_1, f_2, ..., f_n\} \tag{1.3}$$

of the functions $f_j \in V$, $j = 1, ..., n$, which are assumed to be linearly independent. We assume that this set of functions represents a *fundamental system* of solutions of a linear difference equation of the n-th order with the operator $L : V \to V$,

$$L[u]_i \equiv \sum_{k=0}^{n} a_k(i) u_{i+k} = 0, \quad i \in \mathbf{Z}, \tag{1.4}$$

where $a_k \in V$, $k = 0, 1, ..., n$; $a_n(i) \neq 0$, $a_0(i) \neq 0$ on $\mathbf{Z}$.

We say that a function $F(u_i, u_{i+1}, ..., u_{i+n-1}, i) : V \times \mathbf{Z} \to \mathbf{R}$ is a *first integral* of the linear equation (1.4) if for any solution u of (1.4) it takes a constant value independent on i: $F(u_i, u_{i+1}, ..., u_{i+n-1}, i) \equiv const$ on $\mathbf{Z}$. We use also the short notations: $F(u)_i$ and $F(u)$. Linear first integrals $I(u)$ of the n-th order are defined by linear functions:

$$I(u)_i \equiv I(u_i, u_{i+1}, ..., u_{i+n-1}, i) = \sum_{k=0}^{n-1} l_k(i) u_{i+k}, \quad i \in \mathbf{Z}, \tag{1.6}$$

where $l_k \in V$, $k = 0, 1, ..., n$; $l_{n-1}(i) \neq 0$ on $\mathbf{Z}$. We suppose that the linear n-th order equation (1.4) admits n linearly independent first integrals $I_1(u), ..., I_n(u)$.

Subspace (1.3) is said to be *invariant* under the nonlinear operator Φ if $\Phi[W_n] \subseteq W_n$. We now state the main result on the *forward problem*: given a subspace, describe the set of nonlinear difference operators admitting the subspace.

Theorem 1.1. *Operator Φ in (1.1) with $p \le n-1$ admits the invariant subspace (1.3) if and only if Φ has the form*

$$\Phi[u] = \sum_{j=1}^{n} A_j(I_1(u), ..., I_n(u)) f_j, \tag{1.7}$$

where $A_j : \mathbf{R}^n \to \mathbf{R}$, $j = 1, ..., n$, are arbitrary functions.

Note that we can always assume that $p \le n-1$. Indeed, for any operator (1.1) restricted to W_n, all the higher-order variables u_{i+k} with $k \ge n$ are excluded by means of the subspace equation (1.4). Theorem 1.1 establishes the general structure of operators (1.1) admitting a given invariant subspace. The theorem is proved in Section 2 with examples in Section 3. Note also that the set of operators admitting a given invariant subspace is a linear space. In a similar way as in the continuous case [6], these operators can be considered as the generators of the higher-order symmetries of linear descrete equations. Theorem 1.1 then gives a complete description of such symmetries.

The first application of the theorem concerns a class of systems (1.2) with Φ possessing an invariant subspace W_n and $g \in W_n$. The following corollary shows that every such system on subspace W_n, i.e., on the set of functions

$$u = \sum_{m=1}^{n} c_m f_m \tag{1.8}$$

with constants c_m, reduces to a finite-dimensional system of algebraic equations (for coefficients c_m).

Corollary 1.2. *Let*

$$g = \sum_{k=1}^{n} \alpha_k f_k \in W_n, \tag{1.9}$$

where α_k, $k = 1, ..., n$, are constants. Given an operator Φ of the form (1.7), equation (1.2), restricted to the invariant subspace W_n of functions (1.8), is equivalent to the system of n algebraic equations for coefficients $c_1, ..., c_n$:

$$A_j(C_1, ..., C_n) = \alpha_j, \quad j = 1, ..., n, \tag{1.10}$$

where

$$C_i(c_1, ..., c_n) = \sum_{m=1}^{n} c_m I_i(f_m), \quad i = 1, ..., n. \tag{1.11}$$

Indeed, substituting (1.8) into equation (1.2), (1.7) and using the linear properties of the functionals $\{I_i(u)\}$ we derive the system (1.10), (1.11).

As the second application, we consider an infinite-dimensional dynamical system

$$\frac{du}{dt} = \Phi[u], \tag{1.12}$$

where $u(t,x) : \mathbf{R}^{p+1} \times \mathbf{Z} \to \mathbf{R}$. If Φ admits the invariant subspace (1.3) and hence is represented in the form (1.7), then on W_n the system (1.12) reduces to a finite-dimensional dynamical system on the coefficients of the expansion (1.8):

$$\frac{d}{dt}c_j = A_j(C_1, ..., C_n), \;\; C_i(c_1, ..., c_n) = \sum_{m=1}^{n} c_m I_i(f_m), \;\; i, j = 1, ..., n,$$

where c_j are functions of t. A similar reduction exists in the completely discrete case where the continuous derivative d/dt is replaced by the corresponding discrete operator. The corresponding finite-dimensional reduction is constructed for higher-order equations

$$P\left(\frac{d}{dt}\right) u = \Phi[u], \tag{1.13}$$

where P is an arbitrary linear polynomial-type operator (e.g. $P(d/dt) = d^2/dt^2$). Equations (1.13) can be considered as a discrete model of a nonlinear PDE

$$P\left(\frac{\partial}{\partial t}\right) u = \mathbf{A}[u],$$

so that Φ is a discrete approximation of a nonlinear differential operator $\mathbf{A}$. See several examples of such operators admitting invariant subspaces in the parabolic, hyperbolic and higher-order evolution equations in [1] and [5]. Such discrete models are typical in the theory of finite-difference schemes and moving-mesh methods applied to nonlinear PDEs. Observe that the idea of the so-called *invariant sets on linear subspaces* developed in [5] for continuous operators can be applied to some discrete ones [3]. In general, an evolution equation on such an invariant set reduces to an overdetermined dynamical system which sometimes admits a solution, see examples in [2] and [5]. As in the continuous case [5], the problem of invariant sets reduces to a nonlinear eigenvalue problem. In the last Section 4 we discuss some aspects on the backward problem of the invariant subspaces.

2. Proof of Theorem 1.1

Proof. Sufficiency is obvious. Let us prove the necessity. The invariance of (1.3) means that image $\Phi[u]$ of $u \in W_n$ has the form

$$\Phi[u] = \sum_{j=1}^{n} \tilde{A}_j(u) f_j$$

where coefficients $\tilde{A}_j$, $j = 1, ..., n$ are constant for given u. Note that every $u \in W_n$, being a solution of equation (1.4), can be uniquely represented by the values of the linear independent first integrals (1.6) $\{I_k(u),\ k = 1, ..., n\}$. So one can write $\tilde{A}_j(u) \equiv A_j(I_1, ..., I_n)$, which completes the proof.

3. Examples of first integrals and nonlinear operators for $\mathbf{n = 2}$

We consider the case $n = 2$, i.e., the second-order linear difference equation (1.4) of the form

$$L[u]_i \equiv a_i u_{i+2} + b_i u_{i+1} + c_i u_i = 0. \tag{3.1}$$

The adjoint equation with the adjoint operator $M[v]$ satisfying the identity

$$(L[u], v) = (u, M[v]), \tag{3.2}$$

where we introduce a formal scalar product $(u, v) = \sum_{k \in \mathbf{Z}} u_k v_k$, takes the form

$$M[v]_i \equiv c_i v_i + b_{i-1} v_{i-1} + a_{i-2} v_{i-2} = 0. \tag{3.3}$$

Proposition 3.1. *Any function of the form*

$$I(u)_i = a_{i-1} v_{i-1} u_{i+1} - c_i v_i u_i, \tag{3.4}$$

where v is a solution of the adjoint equation (3.3), is the first integral of equation (3.1).

Proof. One can see that the condition

$$I(u)_i = I(u)_{i+1} \quad \text{in } \{u \in V : \ L[u] = 0\} \tag{3.5}$$

reduces to

$$u_{i+1} M[v]_{i+1} = 0, \tag{3.6}$$

whence the conclusion. $\square$

Example 3.2. Consider the linear equation

$$L[u]_i \equiv u_{i+2} - \frac{1}{4} u_i = 0 \tag{3.7}$$

($a_i = 1,\ b_i = 0,\ c_i = -1/4$). The characteristic equation is $\lambda^2 - 1/4 = 0$ whence the subspace W_2 of solutions to (3.7):

$$W_2 = \mathcal{L}\{2^{-i}, (-2)^{-i}\}. \tag{3.8}$$

The adjoint equation takes the form

$$M[v]_i \equiv -\frac{1}{4} v_i + v_{i-2} = 0 \tag{3.9}$$

and admits two linearly independent solutions

$$v_{1,i} = 2^i, \quad v_{2,i} = (-2)^i. \tag{3.10}$$

¿From Proposition 3.1, we then conclude that there exist two linearly independent first integrals of equation (3.7)

$$I_1(u)_i = 2^i(u_{i+1} + \frac{1}{2}u_i), \quad I_2(u)_i = (-2)^i(u_{i+1} - \frac{1}{2}u_i). \tag{3.11}$$

By Theorem 1.1 we prove that the family of all the first order nonlinear operators admitting the two-dimensional subspace (3.8) is given by

$$\Phi[u]_i = A_1(I_1(u)_i, I_2(u)_i)2^{-i} + A_2(I_1(u)_i, I_2(u)_i)(-2)^{-i}, \tag{3.12}$$

where $A_1, A_2 : \mathbf{R}^2 \to \mathbf{R}$ are arbitrary functions. Due to Corollary 1.2, for any $g \in W_2$ the equation $\Phi(u) = g$ is equivalent to a system of two algebraic equations.

For applications, it is important to derive all the *autonomous* operators admitting W_2, i.e., those which do not depend on the variable $i \in \mathbf{Z}$. Denoting

$$2^i(u_{i+1} + \frac{1}{2}u_i) = a, \quad (-2)^i(u_{i+1} - \frac{1}{2}u_i) = b,$$

one can conclude that such operators invariant under translation $i \to i+1$ satisfy

$$[A_1(2a, (-2)b) - 2A_1(a, b)]2^{-i-1} + [A_2(2a, (-2)b) + 2A_2(a, b)](-2)^{-i-1} = 0$$

for all $i \in \mathbf{Z}$ and $a, b \in \mathbf{R}$. In view of the linear independence of the functions, we than deduce that

$$A_1(2a, (-2)b) = 2A_1(a, b), \quad A_2(2a, (-2)b) = -2A_2(a, b), \ \ a, b \in \mathbf{R}. \tag{3.13}$$

Thus the homogeneous operators have the form

$$\Phi[u]_i = A_1(u_{i+1} + \frac{1}{2}u_i, u_{i+1} - \frac{1}{2}u_i) + A_2(u_{i+1} + \frac{1}{2}u_i, u_{i+1} - \frac{1}{2}u_i), \tag{3.14}$$

where A_1 and A_2 are arbitrary functions satisfying (3.13). For instance, the operators

$$\Phi_1[u]_i = \frac{(u_{i+1} + \frac{1}{2}u_i)^3}{(u_{i+1} - \frac{1}{2}u_i)^2}, \quad \Phi_2[u]_i = \frac{(u_{i+1} + \frac{1}{2}u_i)^2}{(u_{i+1} - \frac{1}{2}u_i)}$$

admit the subspace W_2. A more general operator, possessing W_2, is given by

$$\Phi[u]_i = C_1\Phi_1[u]_i + C_2\Phi_2[u]_i + \tilde{L}[u]_i$$

where $C_1, \ C_2 \in \mathbf{R}$ are constants and $\tilde{L}$ is an arbitrary linear operator admitting W_2.

4. On the backward problem

Several examples of subspaces and sets invariant under nonlinear second-order difference operators of reaction-diffusion type are presented in [3]. On the other hand, several invariant subspaces for different types of ordinary and partial differential operators can be found in [1], [2], [6], see also references therein. We note that the general backward problem: given a nonlinear finite-order difference operator determine all the subspaces which it admits, remains open. In the case of nonlinear differential operators this problem is shown to be reduced to a difficult *nonlinear eigenvalue problem* [4].

In this section we give an example of a particular solution of the backward problem for the following first-order operator

$$\Phi[u]_i = (u_{i+1} - u_i)^2. \tag{4.1}$$

Observe that the backward problem for the corresponding differential operator of the Hamilton-Jacobi type

$$\Phi[u] = (u_x)^2 \tag{4.1$'$}$$

((4.1) is then the difference approximation $((u_{i+1} - u_i)/h))^2$ with $h = 1$) can be solved directly [6]. Namely, up to translations in x, it admits the one-dimensional subspaces

$$W_1 = \mathcal{L}\{x^2\}, \quad W_1 = \mathcal{L}\{1\},$$

the two two-dimensional subspaces $W_2 = \{1, x\}$ and $W_2 = \mathcal{L}\{1, x^2\}$ given by the equations

$$u_{xx} = 0, \quad \text{and} \quad u_{xx} = \frac{u_x}{x},$$

and a single three-dimensional subspace

$$W_3 = \mathcal{L}\{1, x, x^2\} \quad (u_{xxx} = 0).$$

For convenience, we give, step by step, a classification of invariant subspaces W_n for (4.1) with $n \le 2$, $n = 3$, and $n \ge 4$ respectively.

4.1. $\mathbf{n \le 2}$.

Theorem 4.1. *Operator (4.1) admits the invariant subspace W_n, with $n \le 2$, given by the linear equation with constant coefficients*

$$L[u]_i \equiv u_{i+1} - Bu_i - Cu_{i-1} = 0 \quad (B, C \in \mathbf{R}), \tag{4.2}$$

only in the following four cases:

(i) $B = C = 0$ (equation $u_{i+1} = 0$), $W_0 = \{0\}$;
(ii) $B = 1,\ C = 0$ ($u_{i+1} - u_i = 0$), $W_1 = \mathcal{L}\{1\}$;
(iii) $B = 0,\ C = 1$ ($u_{i+1} - u_{i-1} = 0$), $W_2 = \mathcal{L}\{1, (-1)^i\}$;

(iv) $B = 2,\ \ C = -1$ *(*$u_{i+1} - 2u_i + u_{i-1} = 0$*)*, $W_2 = \mathcal{L}\{1, i\}$.

It is worth mentioning that the case (iii) is not available in the classification of the differential operators given above. The subspace in (iii) contains 2-periodic functions.

Proof. The invariance criterion

$$L[\Phi[u]] = 0 \quad \text{if} \quad L[u] = 0 \tag{4.3}$$

reduces to

$$(u_{i+2} - u_{i+1})^2 - B(u_{i+1} - u_i)^2 - C(u_i - u_{i-1})^2 = 0$$

on the solutions of (4.2). Using (4.2) to eliminate u_{i+1} and u_{i+2} and taking into account that the functions u_i^2, $u_i u_{i-1}$ and u_{i-1}^2 are linearly independent, we arrive at the system

$$\begin{aligned}
&(B^2 + C - B)^2 = B(B-1)^2 + C,\\
&C(B^2 + C - B)(B - 1) = C(B(B-1) - 1),\\
&C^2(B-1)^2 = C(BC + 1).
\end{aligned} \tag{4.4}$$

Consider two cases.

1. Case $C = 0$. Then from the first equation we get $B(B-1)^3 = 0$ so that

$$\text{either } B = 0 \text{ or } B = 1. \tag{4.5}$$

2. Case $C \neq 0$. Then substituting C from the second equation (4.4),

$$C = -B(B-2) - \frac{1}{B-1} \quad (B \neq 1), \tag{4.6}$$

we obtain the following two equations on B:

$$\begin{aligned}
&\left(B - \frac{1}{B-1}\right)^2 - B(B-1)^2 = -B(B-2) - \frac{1}{B-1},\\
&-[(B-1)^2 - B]\left[B(B-2) + \frac{1}{B-1}\right] = 1,
\end{aligned} \tag{4.7}$$

which reduce to a single algebraic equation

$$B^2(B-2)^3 = 0.$$

Thus, either $B = 2$, $C = -1$ or $B = 0$, $C = 1$.

□

4.2. n = 3. In this case we have the following result.

Theorem 4.2. *Operator (4.1) admits three-dimensional invariant subspace W_3 given by the equation*

$$L[u]_i \equiv u_{i+3} - Bu_{i+2} - Cu_{i+1} - Du_i = 0 \quad (B, C, D \in \mathbf{R}), \tag{4.8}$$

if and only if
(i) $B = 3,\ C = -3,\ D = 1,\ W_3 = \mathcal{L}\{1, i, i^2\}$;
(ii) $B = C = 0,\ D = 1,\ W_3 = \mathcal{L}\{1, \cos(2\pi i/3), \sin(2\pi i/3)\}$;
(iii) $B = C = 1,\ D = -1,\ W_3 = \mathcal{L}\{1, i, (-1)^i\}$.

The proof is a slight modification of the proof of the previous theorem.

4.3. n $\geq$ 4. Finally we consider multi-dimensional invariant subspaces when the linear operator takes the form

$$L[u]_i \equiv u_{i+n} - \sum_{k=0}^{n-1} a_k u_{i+k} = 0, \quad n \geq 4 \quad (a_k \in \mathbf{R}). \tag{4.9}$$

Theorem 4.3. *The only subspaces W_n with $n \geq 4$ invariant under operator (4.1) are*
(i) $u_{i+n} = u_{i+n-1} + u_{i+1} - u_i$,

$$W_n = \mathcal{L}\{1, i, e^{2\pi i I/(n-1)}, ..., e^{2\pi i I(n-2)/(n-1)}\} \quad (I^2 = -1);$$

(ii) $u_{i+n} = u_i$,

$$W_n = \mathcal{L}\{1, e^{2\pi i I/n}, ..., e^{2\pi i I(n-1)/n}\}.$$

A real-valued representation of these subspaces is straightforward from the corresponding characteristic equations.

Proof. Consider the invariance criterion

$$\begin{aligned}\Phi(u)_{i+n} \equiv (u_{i+n+1} - u_{i+n})^2 &= \sum_{k=0}^{n-1} a_k (u_{i+k+1} - u_{i+k})^2 \\ &= a_{n-1}(u_{i+n} - u_{i+n-1})^2 + \sum_{k=0}^{n-2} a_k (u_{i+k+1} - u_{i+k})^2\end{aligned} \tag{4.10}$$

on the solutions of (4.9). We next exclude the variables

$$\begin{aligned}u_{i+n+1} &= \sum_{k=0}^{n-1} a_k u_{i+k+1} = \sum_{k=1}^{n} a_{k+1} u_{i+k} = a_{n-1} u_{i+n} + \sum_{k=1}^{n-1} a_{k-1} u_{i+k} \\ &= a_{n-1} \sum_{k=0}^{n-1} a_k u_{i+k} + \sum_{k=1}^{n-1} a_{k-1} u_{i+k} = \sum_{k=1}^{n-1} (a_{n-1} a_k + a_{k-1}) u_{i+k} + a_{n-1} a_0 u_i,\end{aligned}$$

and

$$u_{i+n} = \sum_{k=0}^{n-1} a_k u_{i+k},$$

so that

$$u_{i+n+1} - u_{i+n} = \sum_{k=1}^{n-1} [(a_{n-1} - 1)a_k + a_{k-1}]u_{i+k} + (a_{n-1} - 1)a_0 u_i,$$

$$u_{i+n} - u_{i+n-1} = (a_{n-1} - 1)u_{i+n-1} + \sum_{k=1}^{n-2} a_k u_{i+k} + a_0 u_i.$$

Setting for convenience

$$\beta = a_{n-1} - 1 \tag{4.11}$$

and substituting the representations given above into (4.10), we obtain

$$\begin{aligned}
&\left(\sum_{k=1}^{n-1} (\beta a_k + a_{k+1})u_{i+k}\right)^2 + 2\beta a_0 u_i \sum_{k=1}^{n-1} (\beta a_k + a_{k-1})u_{i+k} + \beta^2 a_0^2 u_i^2 \\
&= a_{n-1}\{(\beta u_{i+n-1} + \sum_{k=1}^{n-2} a_k u_{i+k})^2 + 2a_0 u_i(\beta u_{i+n-1} + \sum_{k=1}^{n-2} a_k u_{i+k}) + a_0^2 u_i^2\} \\
&\quad + a_{n-2}(u_{i+n-1} - u_{i+n-2})^2 + a_{n-3}(u_{i+n-2} - u_{i+n-3})^2 + ... \\
&\quad + a_1(u_{i+2} - u_{i+1})^2 + a_0(u_{i+1} - u_i)^2.
\end{aligned} \tag{4.12}$$

Equating the coefficients of the factors u_i^m, $m = 0, 1, 2$, respectively, we arrive at the following equations. The equation corresponding to $u_i^0 = 1$ is given by

$$\begin{aligned}
&\left(\sum_{k=1}^{n-1} (\beta a_k + a_{k+1})u_{i+k}\right)^2 = a_{n-1}\left(\beta u_{i+n-1} + \sum_{k=1}^{n-2} a_k u_{i+k}\right)^2 \\
&+ a_{n-2}(u_{i+n-1} - u_{i+n-2})^2 + ... + a_1(u_{i+2} - u_{i+1})^2 + a_0 u_{i+1}^2.
\end{aligned} \tag{4.13}$$

The coefficients of u_i are as follows

$$a_0\beta \sum_{k=1}^{n-1} (\beta a_k + a_{k-1})u_{i+k} = a_0 \left[a_{n-1}\left(\beta u_{i+n-1} + \sum_{k=1}^{n-2} a_k u_{i+k}\right) - u_{i+1}\right], \tag{4.14}$$

whence since $a_0 \neq 0$ we obtain: the equation of u_{i+n-1}, $\beta(\beta a_{n-1} + a_{n-2}) = a_{n-1}\beta$, leads to

$$\beta[(\beta - 1)a_{n-1} + a_{n-2}] = 0, \tag{4.15}$$

the equation of u_{i+k} for all $2 \leq k \leq n-2$ are

$$\beta(\beta a_k + a_{k-1}) = a_{n-1}a_k, \tag{4.16}$$

and of u_{i+1} ($k=1$) is given by

$$\beta(\beta a_1 + a_0) = a_{n-1}a_1 - 1. \tag{4.17}$$

Finally, the last equation of u_i^2 reads $\beta^2 a_0^2 = a_{n-1}a_0^2 + a_0$ whence ($a_0 \neq 0$)

$$\beta^2 a_0 = a_{n-1}a_0 + 1. \tag{4.18}$$

I. If $\beta = 0$, i.e., $a_{n-1} = 1$, then the system (4.15) - (4.18) takes the form

$$a_k = 0, \quad 2 \le k \le n-2; \quad a_1 = 1, \;\; a_0 = -1, \tag{4.19}$$

and we arrive at the case (i) of the theorem. The invariant subspace is determined from the characteristic equation

$$\lambda^n = \lambda^{n-1} + \lambda - 1,$$

so that $\lambda = 1$ and $\lambda^{n-1} = 1$, i.e.,

$$\lambda_k = e^{2\pi k I/(n-1)}, \quad k = 1, 2, ..., n-2; \quad \lambda_{n-1} = \lambda_n = 1,$$

where $I^2 = -1$.

II. Let $\beta \neq 0$, i.e., $a_{n-1} \neq 1$. Then the system (4.15)-(4.18) reads

$$\begin{aligned}
&(\beta - 1)a_{n-1} + a_{n-2} = 0, \\
&(\beta^2 - a_{n-1})a_k + \beta a_{k-1} = 0, \quad k = 2, ..., n-2, \\
&(\beta^2 - a_{n-1})a_1 + \beta a_0 = -1, \\
&(\beta^2 - a_{n-1})a_0 = 1.
\end{aligned}$$

Denoting here

$$\alpha = \beta^2 - a_{n-1} \equiv \beta^2 - \beta - 1, \tag{4.20}$$

we obtain the solution

$$\begin{aligned}
&a_0 = \frac{1}{\alpha} \;\; (\alpha \neq 0), \quad a_1 = -\frac{\alpha + \beta}{\alpha^2}, \\
&a_k = \left(-\frac{\beta}{\alpha}\right)^{k-1} a_1, \quad k = 2, ..., n-2, \\
&a_{n-2} = 1 - \beta^2 \quad (a_{n-1} = 1 + \beta \;\text{ from } (4.11)).
\end{aligned} \tag{4.21}$$

Consider again (4.13). The coefficient of u_{i+1}^2 is

$$(\beta a_1 + a_0)^2 = a_{n-1}a_1^2 + a_1 + a_0,$$

whence by (4.11) and (4.21)

$$\alpha^2 - 2\alpha\beta(\alpha + \beta) + \beta^2(\alpha + \beta)^2 = (1 + \beta)(\alpha + \beta)^2 - \beta\alpha^2.$$

Substituting α from (4.20) we obtain $\beta^2(\beta+1)(\beta-2) = 0$. If $\beta = -1$ then $\alpha = 1$ by (4.20) and we arrive at the subspace (ii). If $\beta = 2$ then $\alpha = 1$ and we get $a_{n-1} = 1 + \beta = 3$, $a_0 = 1$, $a_1 = -3$ and $a_k = -3(-2)^{k-1}$, $k = 2, ..., n-2$. Then the last equation in (4.21) means that $-3(-2)^{n-3} = -3$ whence $n = 3$ contradicting the assumption.

This completes the proof. $\square$

One can see that the equation in (ii),

$$u_{i+n} = u_i, \tag{4.22}$$

implies that

$$\Phi[u]_{i+n} = \Phi[u]_i$$

for any operator Φ, which does not depend explicitly on i. Therefore, the linear subspace W_n given by equation (4.22) is invariant under an arbitrary autonomous operator.

Final remark on the maximal dimension of invariant subspaces. In the differential case, the order p of a *nonlinear* ordinary differential operator F is known to satisfy [6]

$$n \le 2p+1, \tag{4.23}$$

where n is the maximal dimension of an invariant subspace. If $n > 2p+1$ then F is a linear operator. It follows from Theorem 4.3 that this is not the case for the discrete operators where the maximal dimension n can be arbitrarily large even for the first-order quadratic operator (4.1). Obviously, the set of difference operators is wider than that of the differential ones. The differential operator

$$\Phi_0[u] = F(u, u', u'', ..., u^{(p)})$$

can be obtained from a one-parameter family of the difference operators

$$\Phi_h[u] = F(u, u_x, u_{xx}, ..., u\underbrace{x...x}_{p}),$$

where the arguments denote the discrete approximation of the derivatives, say,

$$u_x = (u_{i+1} - u_{i-1})/2h, \quad u_{xx} = (u_{i-1} - 2u_i + u_{i+1})/h^2, \ \ ..., \tag{4.24}$$

in the sense of a formal limit $\Phi_0 = \lim \Phi_h$ as $h \to 0$. Therefore, under extra restrictions estimate (4.23) may be useful for the subset of such difference operators $\{\Phi_h\}$.

Example 4.4. The discrete second-order quadratic operator

$$\Phi_h[u] = uu_{xx} - \frac{3}{4}(u_x)^2,$$

with the derivatives approximations (4.24), is generated by a differential second-order one

$$\Phi_0[u] = uu'' - \frac{3}{4}(u')^2.$$

Both discrete and differential operators admit the five-dimensional invariant subspace

$$W_5 = \mathcal{L}\{1, x, x^2, x^3, x^4\},$$

so that the equality in (4.23) holds. Therefore, both continuous and discrete parabolic-like equations $u_t = \Phi_0[u]$ and $u_t = \Phi_h[u]$ on W_5 reduce to the corresponding five-dimensional dynamical systems on the coefficients $\{C_k(t)\}$ of the expansion

$$u(x,t) = \sum_{k=0}^{4} C_k(t) x^k \in W_5.$$

The hyperbolic-like second-order equations $u_{tt} = \Phi_0[u]$ and $u_{tt} = \Phi_h[u]$ on W_5 reduce to ten-dimensional dynamical systems. Other examples can be found in [3].

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