Minimal Blow-up Asymptotics of Quasilinear Heat Equations

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Abstract

We study the asymptotic properties of blow-up solutions $u = u(x,t) \geq 0$ of the quasilinear heat equation

$$u_t = (k(u)u_x)_x, \quad x > 0, \; t \in (0,1),$$

where $k(u)$ is a smooth nonnegative function, with a given blowing up regime on the boundary $u(0,t) = \psi(t) > 0$ for $t \in (0,1)$, where $\psi(t) \to \infty$ as $t \to 1^-$, and a bounded initial data $u(x,0) \geq 0$. We classify the asymptotic properties of the solutions near the blow-up time, $t \to 1^-$, in terms of the heat conductivity coefficient $k(u)$ and of boundary data $\psi(t)$, both are assumed to be monotone. We describe a domain, denoted by S_{11}^-, of minimal asymptotics corresponding to the data $\psi(t)$ with a slow growth as $t \to 1^-$ and a class of nonlinear coefficients $k(u)$.

We prove that for any problem in S_{11}^- such a blow-up singularity is asymptotically structurally equivalent to a singularity of the heat equation $u_t = u_{xx}$ described by its self-similar solution of the form $u_*(x,t) = -\ln(1-t) + g(\xi)$, $\xi = x/(1-t)^{1/2}$, where g solves a linear ODE. This particular self-similar solution is structurally stable upon perturbations of the boundary function and also upon nonlinear perturbations of the heat equation with the basin of attraction S_{11}^-.

AMS (MOS) Subject Classification: 35K55, 35K65.

Key Words. Quasilinear heat equation, asymptotic behaviour, approximate self-similar solutions, Maximum Principle, blow-up.

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1 Introduction

In this paper we classify the asymptotic behaviour of the solutions to the quasilinear heat equation in terms of the nonlinear coefficient of the equation and of the blow-up boundary function. We describe a new minimal domain of asymptotics on the plane of initial-boundary value problems. We show that the asymptotics in this minimal domain are given by a nonlinear transformation of a self-similar solution of the linear heat equation, which can be treated as a "stable low-dimensional attractor" of singularities in the minimal domain. We thus discuss an approach to a classification of the omega limits for a class of infinite-dimensional dynamical systems with variable coefficients.

1.1 Plane of initial-boundary value problems

We consider the initial-boundary value problem (IBVP) for the quasilinear heat equation of parabolic type

$$u_t = (k(u)u_x)_x, \quad x > 0, \;\; t \in (0, 1), \tag{1}$$

where $k(u) > 0$ for $u > 0$ is a given smooth function, with a blow-up boundary regime

$$u(0, t) = \psi(t) \to \infty, \quad t \to 1^-, \tag{2}$$

and bounded initial data

$$u(x, 0) = u_0(x) \geq 0, \quad x > 0. \tag{3}$$

The asymptotic behaviour of the solution as $t \to 1^-$ does not essentially depend on the initial data, so that the first term of the asymptotic expansion of the solution is expected to depend on two "coordinates": the behaviour of $\psi(t)$ for $t \approx 1^-$ and on the behaviour of $k(u)$ for $u \gg 1$. This makes it possible to classify the asymptotics on the so-called "plane of IBVPs", P_{IBVP}, introduced in [6], see details in Chapter 6 of [14] and in [2]. Any point of the plane P_{IBVP} can be associated with a class of IBVPs exhibiting the equivalent asymptotic behaviour. The classes of the coefficients $\{k\}$ and $\{\psi\}$ ordered with respect to their asymptotic growth form two "coordinates" of the plane. It turns out that the plane P_{IBVP} consists of a finite number of domains with essentially different types of asymptotic behaviour. See Figure in Section 12 of [6] where the plane P^+_{IBVP} is presented for the case of global solutions with

$$\psi(t) \to \infty, \quad t \to \infty.$$

In this case the plane consists of about 10 distinct domains.

In the case of blow-up functions on the boundary $x = 0$ the classification becomes more delicate and the structure of some domains in P^-_{IBVP} remained open. Namely, the *minimal* domain S_{11}^- which corresponds to both "minimal" behaviour of $k(u)$ as $u \to \infty$ and $\psi(t)$ as $t \to 1^-$.

1.2 Stable singular pattern for the heat equation

In a recent paper, [2], we studied minimal boundary blow-up singularities (boundary layers) for the *heat equation* (1) with $k(u) \equiv 1$:

$$u_t = u_{xx}, \quad x > 0, \;\; t \in (0,1), \tag{4}$$

with blow-up boundary function (2). We have shown that for a wide class of blow-up boundary data $\psi(t)$, the asymptotic properties of the rescaled blow-up flow are governed by the stable manifold of the linear self-adjoint operator in $L^2_{\rho_0}(\mathbf{R}_+)$:

$$\mathbf{A} = \frac{d^2}{d\xi^2} - \frac{1}{2}\xi\frac{d}{d\xi} \equiv \frac{1}{\rho}\frac{d}{d\xi}\left(\rho\frac{d}{d\xi}\right), \quad \rho(\xi) = e^{-\xi^2/4}. \tag{5}$$

The asymptotic behaviour as $t \to 1^-$ of the solutions to the heat equations was shown to be described by a nonlinear perturbation of the linear dynamical system

$$Y_s = \mathbf{A}Y, \quad s \to \infty. \tag{6}$$

We establish in [2] that such a structure of singularity is stable upon perturbations of the blowing up boundary data, and, moreover, we determined the domain of its stability. It can be formally treated as an "interval" I_1^- on the "straight line" (an "axis") $A^- = \{k \equiv 1\}$ on the plane P^-_{IBVP}, see Figure 1 in Section 5.

Let us give more precise description of such results. We showed that on the straight line $\{k = 1\}$ there exists a "dense" interval I_1^-, which can be considered as a basin of attraction of the self-similar solution of the heat equation (the logarithmic blow-up singularity),

$$u_1^-(t,x) = -\ln(1-t) + g(\xi), \quad \xi = x/(1-t)^{1/2}, \tag{7}$$

where the profile g is a unique minimal solution of the ODE

$$g'' - \frac{1}{2}g'\xi - 1 = 0, \quad \xi > 0, \;\; g(0) = 0, \tag{8}$$

satisfying $g(\xi) = -2\ln\xi + C_0 + o(1)$ for $\xi \gg 1$. We now describe the whole interval I_1^-. We prescribe the boundary function from the interval using the representation

$$\psi(t) = F(-\ln(1-t)), \tag{9}$$

where F is a smooth increasing function, $F(s) \to \infty$ as $s \to \infty$. We prove that $\psi \in I_1^-$ provided that

(10) $$\frac{F''(s)}{F'(s)} \to 0, \quad s \to \infty,$$

which is shown to be equivalent to the following standard condition similar to those in [4] and [6]:

(11) $$a_\psi^- = \lim_{t \to 1^-} \left(\frac{\psi(t)}{\psi'(t)} \right)' = -\infty.$$

For instance, $F(s) = \ln s$; $F(s) = s^\lambda$, $\lambda > 0$; $F(s) = \exp(s^\lambda)$, with $\lambda \in (0, 1)$, etc. In [2] we give a detailed description of the asymptotic behaviour of the solutions and derive an optimal rate of convergence to the transformed rescaled self-similar solution (7).

1.3 Main new results: the minimal domain

In the present paper we extend the stability results to quasilinear equations of the type (1). Namely, we prove that the self-similar solution (7), which represents a point S_* on the plane P_{IBVP}^-, is an attractor with a certain domain of attraction S_{11}^-. This domain is characterized as follows: (i) condition (10) (or (11)) for the boundary function $\psi(t)$, and (ii) a joint condition on both coefficients $\psi(t)$ and $k(u)$. Denoting $K(s) \equiv k(F(s))$, the condition is

(12) $$\frac{K'(s)}{K(s)} \equiv \frac{k'(F(s))F'(s)}{k(F(s))} \to 0, \quad s \to \infty,$$

where F is as in (9). Equation (12) defines two "monotone curves" $\gamma_\pm$ on the plane P_{IBVP}^- as the upper and lower boundaries of S_{11}^-. We emphasize a rather delicate non-rectangular structure of the minimal domain $S_{11}^- \subset P_{IBVP}^-$ (to be discussed in more detail in Section 5). We thus define the minimal domain as follows:

(13) $$S_{11}^- = \{\{\psi, k\} : \quad (11) \text{ and } (12) \text{ hold}\}.$$

We state the main result which establishes that for any IBVP from the minimal domain S_{11}^- the structure of blow-up singularity in the first approximation is governed by an asymptotically perturbed linear semigroup with the infinitesimal generator (5). This result exhibits some features of an approximate asymptotic version of Hartman-Grobman Theorem [13] of classification of hyperbolic equilibria for such infinite-dimensional dynamical systems. We thus establish a certain *structural equivalence* of the singularity generated by a class S_{11}^- of nonlinear IBVPs to a rescaled flow corresponding to the linear heat equation.

Theorem 1 *Let (11) and (12) be valid and $K(s)$ be monotone for $s \gg 1$. Then as $t \to 1^-$ uniformly on compact subsets,*

$$\theta(\xi,t) \equiv F^{-1}(u(\xi\phi(t),t)) + \ln(1-t) \to g(\xi), \tag{14}$$

where

$$\phi(t) = [(1-t)K(-\ln(1-t))]^{1/2} \to 0, \quad t \to 1^-. \tag{15}$$

In order to illustrate the theorem, we consider two examples of singularities for the special quasilinear heat equations with power nonlinearities which admit extra symmetries and are invariant under multi-parameter Lie groups of transformations, see [3]. Therefore these equations admit several exact invariant solutions. Nevertheless, the invariant solutions cannot describe the singularity we are going to present. In both cases, the singularity is governed by the flow generated by the linear operator $\mathbf{A}$ corresponding to the heat equation.

Example 1. Consider the IBVP

$$u_t = (u^\sigma u_x)_x, \quad x > 0, \ t \in (0,1), \tag{16}$$

with the blow-up boundary regime

$$u(0,t) = |\ln(1-t)|^\lambda, \quad \lambda > 0, \tag{17}$$

and bounded initial data. If $\sigma = 0$, then (16) is the heat equation and the asymptotic properties of the blow-up solutions can be found in [2]. On the other hand, (16) is the porous medium equation if $\sigma > 0$ and it is the fast diffusion equation if $\sigma < 0$. Let us apply Theorem 1. The boundary blow-up function (17) satisfies the first condition (11) for any exponent $\lambda > 0$. Consider the second condition (12). We have $F(s) = s^\lambda$, and $K(s) = k(F(s)) = s^{\lambda\sigma}$. Therefore, as $s \to \infty$

$$\frac{K'(s)}{K(s)} \sim \frac{1}{s} \to 0,$$

so that (12) holds. Finally, by the theorem we conclude that the corresponding blow-up solutions have the following structure as $t \to 1^-$:

$$u(x,t) \approx [|\ln(1-t)| + g(\xi)]^\lambda, \quad \xi = \frac{x}{\phi(t)}, \tag{18}$$

where

$$\phi(t) = [(1-t)|\ln(1-t)|^{\lambda\sigma}]^{1/2}.$$

Example 2. Consider now the IBVP for the equation with the exponential nonlinearity:

$$u_t = (e^u u_x)_x, \quad x > 0, \ t \in (0,1), \tag{19}$$

with

$$u(0,t) = \ln^{\lambda} |\ln(1-t)|, \quad \lambda > 0, \tag{20}$$

Then $F(s) = \ln^{\lambda} s$. Both conditions (11) and (12) hold, and therefore the asymptotic blow-up structure has the form

$$u(x,t) \approx \ln^{\lambda} [|\ln(1-t)| + g(\xi)], \quad \xi = \frac{x}{\phi(t)}, \tag{21}$$

where

$$\phi(t) = [(1-t)e^{\ln^{\lambda} |\ln(1-t)|}]^{1/2}.$$

Other examples are available in Section 5.

The plan of the paper is as follows. We prove Theorem 1 in Section 2 while in the next Section 3 we derive the exact rate of convergence. Section 4 is devoted to the optimal estimate of the final-time profile $u(x,1^-)$ of the solutions. In the last Section 5 we describe a formal geometric structure of the minimal domain S_{11}^- on the plane of IBVPs and consider further generalizations.

2 Convergence

2.1 Restriction to large solutions

We first prove a necessary preliminary result on the asymptotic behaviour of the solution $u(x,t)$. This shows that we can focus our attention on solutions $u(x,t)$ which are uniformly large close to the blow-up time. In fact, this implies that the asymptotic properties of the solutions depend on the behaviour of the nonlinear coefficient $k(u)$ for $u \gg 1$ and of the boundary function $\psi(t)$ for $t \approx 1^-$.

Proposition 2 *For any fixed $M \gg 1$, there exists a $\delta_0 > 0$ and $t_0 \approx 1^-$ such that*

$$u \geq M \quad \text{in } [0,\delta_0] \times [t_0, 1). \tag{22}$$

Proof. Consider a two-parameter family of the stationary solutions of equation (1) which are given by

$$U_{A,B}(x) = \varphi^{-1}(A - Bx), \text{ with } \varphi(s) = \int_0^s k(u)du, \tag{23}$$

with positive constants A and B. This is a set of decreasing functions which satisfy the following property: for any given $N \gg 1$ and positive $\rho \ll 1$, we can find positive values $A = \varphi(N)$ and $B = \varphi(N)/\rho$ such that $U_{A,B}(0) = N$ and $U_{A,B}(\rho) = 0$. This means that the stationary solution exhibits a big slope and is sufficiently steep near the origin. Now, let $M \gg 1$ be fixed and consider $N = 2M$. Condition of blow-up (9) implies that there exists $t_0 > 0$ such that $u(0, t_0) > 2M$. By continuity of the solution, there exists $\delta > 0$ such that $u(x, t_0) > 2M$ for every $x \in [0, \delta]$. Let A and B be the corresponding values associated with $N = 2M$ and $\rho = \delta$. It is clear by construction that $u(x, t_0) \geq U_{A,B}(x)$ on $[0, \delta]$ and $U_{A,B}(x) \geq M$ on $[0, \delta_0]$ with $\delta_0 = U_{A,B}^{-1}(M)$. By the standard comparison, the same inequality holds for every $t \in [t_0, 1)$ on $[0, \delta]$. This completes the proof. ■

2.2 Rescaled perturbed equation

We have denoted $K(s) = k(F(s))$. The rescaled function (14), $\theta(\xi, \tau)$, $\tau = -\ln(1-t) \to \infty$ as $t \to 1^-$, solves the following quasilinear parabolic equation:

$$\theta_\tau = \mathbf{L}(\theta) \equiv \mathbf{A}\theta - 1 + g_1(\tau, \theta)\theta_{\xi\xi} + g_2(\tau, \theta)(\theta_\xi)^2 + g_3(\tau)\xi\theta_\xi, \tag{24}$$

where $\mathbf{A}$ is a linear operator associated with the heat equation (see [2])

$$\mathbf{A}\theta = \theta_{\xi\xi} - \frac{1}{2}\,\xi\theta_\xi, \tag{25}$$

and the last three terms are non-autonomous perturbations

$$g_1 = \frac{K(\tau+\theta)}{K(\tau)} - 1, \quad g_2 = \frac{K(\tau+\theta)}{K(\tau)}\left[\frac{K'(\tau+\theta)}{K(\tau+\theta)} + \frac{F''(\tau+\theta)}{F'(\tau+\theta)}\right], \quad g_3 = \frac{1}{2}\frac{K'(\tau)}{K(\tau)}. \tag{26}$$

The boundary condition becomes

$$\theta(0, \tau) \equiv 0, \quad \tau > 0. \tag{27}$$

The corresponding initial data $\theta_0(\xi) = \theta(\xi, 0)$ are bounded.

We first prove that under the hypotheses of Theorem 1 all the perturbations in equation (24) are asymptotically small. Let us separate two kinds of perturbations. Denote

$$G_1(s) = F''(s)/F'(s), \quad G_2(s) = K'(s)/K(s).$$

In what follows we assume that $G_i(s) \neq 0$ for $s \gg 1$, $i = 1, 2$, and

$$G_i'(s)/G_i(s) \to 0 \ \text{ as } \ s \to \infty. \tag{28}$$

This condition implies "slow" decay or non-exponential perturbations (exponentially small perturbations and an oscillation-like behaviour can be considered as in [2]). We have the following result.

Proposition 3 *Under the hypotheses of Theorem 1, as* $\tau \to \infty$

$$g_3(\tau) \to 0, \tag{29}$$

and for any bounded function θ

$$g_1(\tau,\theta) \to 0, \quad g_2(\tau,\theta) \to 0. \tag{30}$$

Moreover, under hypothesis (28), we have that $|g_1(\tau,\theta)| = O(|G_2(\tau)|)$. *The same holds for* $g_2(\tau,\theta)$, *if* $G_1(\tau)/G_2(\tau) \to 0$ *as* $\tau \to \infty$. *If* $G_1(\tau)/G_2(\tau) \to \infty$ *then* $|g_2(\tau,\theta)| = O(|G_1(\tau)|)$.

Proof. (29) coincides with (12). The first limit in (30) follows from (12) by integrating the function $K'(s)/K(s)$ between τ and $\tau+\theta$ for $\tau \gg 1$. The second one is obtained using this limit together with the hypothesis on F and K.

On the other hand, from the conditions on G_i, (28), it follows that

$$\lim_{\tau\to\infty} \frac{K'(\tau+\theta)}{K'(\tau)} = 1 \text{ and } \lim_{\tau\to\infty} \frac{F''(\tau+\theta)}{F''(\tau)} = 1, \tag{31}$$

whence we obtain

$$\lim_{\tau\to\infty} \frac{g_1(\tau,\theta)}{G_2(\tau)} = \lim_{\tau\to\infty} \frac{K(\tau+\theta)-K(\tau)}{K'(\tau)} = \theta. \tag{32}$$

In a similar way one can see that

$$\lim_{\tau\to\infty} \frac{g_2(\tau,\theta)}{G_2(\tau)} = \lim_{\tau\to\infty} \frac{K'(\tau+\theta)}{K'(\tau)} = 1, \tag{33}$$

if $G_1(\tau)/G_2(\tau) \to 0$ as $\tau \to \infty$ and the same holds with $G_1(\tau)$ replaced by $G_2(\tau)$ if this limit is infinite. ■

Remark. We note that $G_1(\tau)/G_2(\tau) \to C$ as $\tau \to \infty$ with $C \neq 0$, or ∞, implies in general that $|g_2(\tau,\theta)|/|G_2(\tau)| = O(1)$. However, if the signs of $G_1(\tau)$ and $G_2(\tau)$ are different for $\tau \gg 1$, it is possible to have that

$$\frac{|g_2(\tau,A)|}{|G_2(\tau)|} = o(1)$$

which in the "limit case" $G_1(s) = -G_2(s)$ yields $g_2(\tau,\theta) = 0$ and this part of the perturbation is removed.

2.3 Compactness of rescaled orbits

Equation (24) is a nonlinear non-autonomous perturbation of the linear non-homogeneous equation

$$h_s = \mathbf{A}h - 1. \tag{34}$$

The linear ordinary differential operator $\mathbf{A}$ written in the standard Sturm-Liouville form

$$\mathbf{A} \equiv \frac{1}{\rho}\frac{d}{d\xi}\left(\rho\frac{d}{d\xi}\right), \quad \rho(\xi) = e^{-\xi^2/4},$$

is known to be self-adjoint in the weighted Hilbert space $L^2_{\rho_0}(\mathbf{R}_+)$ of functions vanishing at the origin. Its spectrum is composed from an increasing sequence of simple eigenvalues $\{\lambda_k = k/2,\ k = 0, 1, 2, ...\}$ and the corresponding eigenfunctions are Hermite polynomials $\{c_k H_k(\xi)\}$, which form an orthonormal basis in $L^2_{\rho_0}(\mathbf{R}_+)$, [1]. By $\|\cdot\|_\rho$ we denote the norm in $L^2_{\rho_0}(\mathbf{R}_+)$.

We begin with the following result based on construction of sub and supersolutions, cf. [2].

Proposition 4 *There exists a constant $D > 0$ such that*

$$-D + 2g(\xi) \leq \theta(\xi, \tau) \leq D \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+. \tag{35}$$

Proof. The upper bound is straightforward. In order to prove the lower one, we apply an argument of a special extension of the rescaled equation (24) in both τ and θ variables [2] which provides us with small bounds on the perturbation terms $g_2(\tau, \underline{\theta})$ and $g_3(\tau, \underline{\theta})$ with $\underline{\theta}(\xi, \tau) = 2g(\xi) - D$ and a constant D. However, this argument does not apply to the perturbation term $g_1(\tau, \theta)$ where an essential dependence on the constant D is available and another approach is necessary.

We first consider the case $K' < 0$. Note that in this case g_1 is positive and therefore the extension arguments gives a lower bound though g_2 can be unbounded. To begin with, assume that the functions $G_1(s)$ and $G_2(s)$ are small enough for every $s \in \mathbf{R}$. Then, the lower bound is given by $\underline{\theta}(\xi, \tau) = 2g(\xi) - D$ with D large enough. In fact if we denote $m_i = \max_{\{s\in\mathbf{R}\}} |G_i(s)|$ and take into account that g' is negative, it follows for $K(\tau + 2g(\xi) - D)/K(\tau) \leq 2$ that

$$\underline{\theta}_\tau - \mathbf{L}(\underline{\theta}) \leq -1 + 2(m_1 + m_2)4(g')^2 + m_2\xi|g'| \leq -1 + 2(m_1 + m_2)4\pi + 2m_2 \leq 0, \tag{36}$$

if $m_i \leq 1/32\pi$. On the other hand, if $K(\tau+2g(\xi)-D)/K(\tau) \geq 2$, under the same assumption on m_i and using that $2(g')^2(\xi) \leq 9g''(\xi)$ for every $\xi \geq 0$, we have that

$$g_1(\tau, \underline{\theta})\underline{\theta}_{\xi\xi} + g_2(\tau, \underline{\theta})(\underline{\theta}_\xi)^2 \geq 2\left\{\frac{K(\tau + 2g(\xi) - D)}{K(\tau)}[1 - 9(m_1 + m_2)] - 1\right\} \geq 0,$$

and therefore

$$\underline{\theta}_\tau - \mathbf{L}(\underline{\theta}) \leq -1 + m_2\xi|g'| \leq -1 + 2m_2 \leq 0.$$

In the general case, we proceed as follows. In order to derive (36) in $\mathbf{R}_+$, we begin with observing that $|G_i(s)| \leq 1/32\pi$ for $i = 1, 2$ if $s > s_0$ large enough by hypotheses (11) and (12). Now consider $t_0 \approx 1^-$ and $\delta \ll 1$ such that $-\ln(1-t_0) \geq s_0$ and $u > s_0$ in $[0,\delta] \times [t_0, 1)$.

We turn to the original problem for $x \in [0,\delta]$ and $t \in (t_0, 1)$. The solution $u(x,t)$ then can be considered as the solution of the initial-boundary value problem with the boundary conditions $u(0,t) = F(-\ln(1-t))$ and $u = u(\delta,t)$ at $x = \delta$ for every $t \geq t_0$. It is clear that in this domain our solution $u(x,t)$ depends only on the behaviour of the functions $F(s)$ and $k(s)$ on $[s_0, \infty)$. Therefore we can modify both functions for $s \leq s_0$ in a convenient way, constructing two new increasing functions $\hat{F}(s)$ and $\hat{k}(s)$, extensions of $F(s)$ and $k(s)$ for $s \leq s_0$, such that the corresponding $\hat{G}_i$ satisfy the required bound. Then, the rescaled function in Theorem 1 is defined using the extension as

$$\hat{\theta}(\xi,t) \equiv \hat{F}^{-1}(u(\xi\hat{\phi}(t),t)) + \ln(1-t), \tag{37}$$

where

$$\hat{\phi}(t) = [(1-t)\hat{k}(\hat{F}(-\ln(1-t)))]^{1/2}. \tag{38}$$

It is clear that $\hat{\theta}(\xi,\tau) = \theta(\xi,\tau)$ in the domain $[0,\delta] \times [t_0,1)$ and that $\underline{\hat{\theta}}(\xi,\tau) = 2g(\xi) - \hat{D}$, with $\hat{D} \gg 1$, is a lower bound for $\hat{\theta}(\xi,\tau)$.

In order to deal with the case $K' > 0$, we consider $\xi_0 \gg 1$ such that

$$-1 + 2g''(\xi) + 4(m_1+m_2)(g')^2(\xi) + 2m_2\xi g'(\xi) \leq 0, \tag{39}$$

for every $\xi \geq \xi_0$. We apply the extension ideas in the following way. On one hand, by Proposition 2, given $M = F(-2g(\xi_0))$, there exist positive values t_0^* and δ_0 such that $u(x,t) \geq M$ in $[0,\delta_0] \times [t_0^*, 1)$. On the other hand, by the hypotheses on F and K, given $\epsilon_0 = 1/32\pi$, there exists $s_0 \gg 1$ such that $|G_i(s)| \leq \epsilon_0$ for every $s \geq s_0$ and $i = 1, 2$. If we take $t_0 \approx 1^-$ such that $-\ln(1-t_0) \geq s_0$, $-\ln(1-t_0) > -2g(\xi_0)$ and $\xi_0(1-t_0)^{1/2} \leq \delta_0$, we can extend the functions k and F for every $s \leq s_0$ in a convenient way without modifying the solution of the initial-boundary value problem in $[0,\delta_0] \times [t_0, 1)$. In particular, if $-D$ is a bound for $u_0(x)$, we take non-decreasing extensions $\hat{k} > 0$ and $\hat{F}$ such that $m_i \leq \epsilon_0$ for $i = 1, 2$, $\hat{F}(0) = -D$ and $\hat{k}(-D) = 1$. In what follows, we use again F and k to denote the extended functions.

Now we define a new rescaled function $\hat{\theta}(\hat{\xi},\hat{\tau})$ as

$$\theta(\hat{\xi},\hat{\tau}) \equiv F^{-1}(u(x,t)) + \ln(1-t) - \ln(1-t_0) + 2g(\xi_0), \tag{40}$$

with new variables given by $\hat{\tau} = -\ln(1-t) + \ln(1-t_0)$ and $\hat{\xi} = x/\hat{\phi}(t)$, with $\hat{\phi}(t) = [(1-t)K(-\ln(1-t) + \ln(1-t_0))]^{1/2}$. The function $\hat{\theta}(\hat{\xi},\hat{\tau})$ satisfies equation (24) with perturbation terms

$$g_1 = \frac{K(\hat{\tau} + \hat{\theta} - 2g(\xi_0))}{K(\hat{\tau})} - 1, \quad g_3 = \frac{1}{2}\frac{K'(\hat{\tau})}{K(\hat{\tau})},$$

and

$$g_2 = \frac{K(\hat\tau+\hat\theta-2g(\xi_0))}{K(\hat\tau)}\left[\frac{K'(\hat\tau+\hat\theta-2g(\xi_0))}{K(\hat\tau+\hat\theta-2g(\xi_0))}+\frac{F''(\hat\tau+\hat\theta-2g(\xi_0))}{F'(\hat\tau+\hat\theta-2g(\xi_0))}\right].$$

Consider now $\underline{\theta}(\hat\xi,\hat\tau)=2g(\hat\xi)$. It follows from the previous construction, that the initial data verify $\hat\theta(\hat\xi,0)\geq 2g(\hat\xi)$ for every $\hat\xi\geq 0$ and $\hat\theta(0,\hat\tau)\geq 0$ for every $\hat\tau\geq 0$. On the other hand, (39) yields the inequality $\underline{\theta}_{\hat\tau}-\mathbf{L}(\underline{\theta})\leq 0$ for every $\hat\xi\geq\xi_0$. Finally, for $\hat\xi\leq\xi_0$ one can see that

$$g_1(\tau,\underline{\theta})\underline{\theta}_{\xi\xi}+g_2(\tau,\underline{\theta})\underline{\theta}_\xi^2\geq 2\left\{\frac{K(\hat\tau+2g(\hat\xi)-2g(\xi_0))}{K(\hat\tau)}\left[1-9(m_1+m_2)\right]-1\right\}\geq -36\epsilon_0,$$

since K is monotone increasing, whence

$$\underline{\theta}_{\hat\tau}-\mathbf{L}(\underline{\theta})\leq -1+36\epsilon_0+2m_2\hat\xi|g'|\leq -1+36\epsilon_0+4\epsilon_0\leq 0,$$

for every $\hat\tau\geq 0$. Therefore, applying the Maximum Principle, it follows that $\hat\theta(\hat\xi,\hat\tau)\geq 2g(\hat\xi)$ for every $\hat\tau\geq 0$ and $\hat\xi\geq 0$. Translating this result to the original rescaled function and variables $\theta(\xi,\tau)=F^{-1}(u(x,t))+\ln(1-t)$ and taking into account that

$$\hat\xi=\xi\left(\frac{K(-\ln(1-t))}{K(-\ln(1-t)+\ln(1-t_0))}\right)^{1/2}\leq 2\xi$$

for every $t\geq t_1\approx 1^-$, it follows that

$$\theta(\xi,\tau)\geq 2g(\hat\xi)-2g(\xi_0)+\ln(1-t_0)\geq -2g(\xi)-2g(\xi_0)+\ln(1-t_0)-\hat D$$

for every $\tau\geq\tau_1\gg 1$ with a certain positive constant $\hat D$. Taking an appropriate $D\geq\hat D$, we obtain the desired result for every $\tau\geq 0$. This completes the proof. ∎

As the next step, using the structure of the rescaled equation, we prove gradient bounds and hence the compactness of the orbit $\{\theta(\tau),\ \tau>0\}$ in $L^2_{\rho_0}(\mathbf{R}_+)$.

Proposition 5 *There holds*

$$|\theta_\xi|\leq C_1(1+\xi)\quad\text{in } \mathbf{R}_+\times\mathbf{R}_+, \tag{41}$$

and

$$\|\theta_\xi(\cdot,\tau)\|_\rho\leq C_2,\quad \tau>0, \tag{42}$$

where C_1 and C_2 are some positive constants.

Proof. The first estimate is the classical Bernstein estimate for the (extended) uniformly parabolic equation (24) (see e.g. [11]), and then (42) is a straightforward consequence. ∎

By $\omega(\theta_0)$ we denote the ω-limit set of the orbit with the uniform convergence on compact subsets and in $L^2_{\rho_0}(\mathbf{R}_+)$.

2.4 Uniform stability and passage to the limit

Proposition 6 *The ω-limit set $\Omega_0 = \{0\}$ of the linear homogeneous equation*

$$Y_s = \mathbf{A}Y, \quad s > 0, \tag{43}$$

is uniformly stable in the $L^2_{\rho_0}$-norm.

Proof. The operator $\mathbf{A}$ generates a semigroup of contractions in $L^2_{\rho_0}(\mathbf{R}_+)$. Multiplying equation (43) by $\rho(\xi)$ and $Y \in H^2_{\rho_0}(\mathbf{R}_+)$, and integrating by parts we obtain that

$$\frac{1}{2}\frac{d}{ds}\|Y(s)\|^2_\rho = -\frac{1}{2}(Y^2)_\xi(0) - \|Y_\xi(s)\|^2_{2,\rho} \le 0, \tag{44}$$

and the stability result follows. ∎

This implies that the ω-limit set $\Omega_1 = \{g\}$ of the non-homogeneous linear equation (34) is uniformly stable. Note that $g \in L^2_{\rho_0}(\mathbf{R}_+)$ is the unique stationary solution of equation (34), see Section 2 in [2].

We now prove the convergence result which is equivalent to Theorem 1.

Proposition 7 *There holds*

$$\omega(\theta_0) = \{g\}. \tag{45}$$

Proof. By the standard parabolic regularity theory [10], we conclude that given an increasing sequence $\{\tau_k\} \to \infty$ we have $\theta(\tau_k + s) \to h(s)$ in $L^\infty_{\text{loc}}(\mathbf{R}_+; C_{\text{loc}}(\mathbf{R}_+))$ where $h(s)$ is a weak solution of the uniformly parabolic equation (34). Hence $h(s)$ is a classical solution and moreover by (35)

$$\|h(s)\|_\rho \le C_3, \quad s > 0.$$

Using the uniform stability result in Proposition 6 and the estimates above, we then conclude that the Ω-limit set of equation (34) is also stable upon any asymptotically small nonlinear perturbations of the equation. See Theorem 3 in [7]. Thus, (45) holds completing the proof.

∎

Finally, let us show that the limit in (15) is zero. In fact, one can obtain that $\ln K(s)/s \to 0$ as $s \to \infty$ as a consequence of (12) which implies that $e^{-\epsilon s} \le K(s) \le e^{\epsilon s}$ for any $\epsilon > 0$, if $s \ge s_0(\epsilon) \gg 1$. Therefore, the fact that $\phi(\tau) \to 0$ as $t \to 1$ in (15) follows. This completes the proof of Theorem 1.

3 Rate of convergence

In this section we derive the exact rate of convergence to the stationary rescaled profile.

3.1 Extra rescaling

In order to get the exact rate of convergence we first define $V(\xi,\tau)=\theta(\xi,\tau)-g(\xi)$. This function satisfies the equation

$$V_\tau = \mathbf{A}V + g_1(\tau,g+V)(g''+V_{\xi\xi}) + g_2(\tau,g+V)(g'+V_\xi)^2 + g_3(\tau)\xi(g'+V_\xi). \tag{46}$$

The boundary condition becomes $V(0,\tau)\equiv 0$ and the initial data satisfies $|V(\xi,0)|\le -g(\xi)+D$. From the results of the previous sections, it follows that $|V(\xi,\tau)|\le -g(\xi)+D$ and that $V(\xi,\tau)\to 0$ as $\tau\to\infty$ in $L^2_{\rho_0}(\mathbf{R}_+)$ and uniformly on compact subsets.

We now define the scaling factor. We assume that the following limit, finite or infinite, exists (cf. Proposition 3):

$$\lim_{\tau\to\infty} G_2(\tau)/G_1(\tau) = C. \tag{47}$$

Then we set $G(\tau)=G_1(\tau)$ if C is finite and $G(\tau)=G_2(\tau)$ if $C=\infty$. We next introduce the new rescaled function

$$V(\xi,\tau)=G(\tau)H(\xi,\tau)$$

satisfying the rescaled equation

$$\begin{aligned} H_\tau = \mathbf{L}(H) \equiv \mathbf{A}H &+ \frac{g_1(\tau,g+G(\tau)H)}{G(\tau)}[g''+G(\tau)H_{\xi\xi}] \\ &+ \frac{g_2(\tau,g+G(\tau)H)}{G(\tau)}[g'+G(\tau)H_\xi]^2 + \frac{g_3(\tau)}{G(\tau)}\xi[g'+G(\tau)H_\xi] - \frac{G'(\tau)}{G(\tau)}H, \end{aligned} \tag{48}$$

with g_i defined in (26). The boundary function is given again by

$$H(0,\tau)\equiv 0, \quad \tau>0. \tag{49}$$

The corresponding initial data satisfy $|H_0(\xi)|\le[-g(\xi)+D]/|G(0)|$ with D as in (35). By Proposition 4 it follows that

$$|H(\xi,\tau)|\le[-g(\xi)+D]/|G(\tau)|. \tag{50}$$

We establish the following result.

Theorem 8 *Let the hypotheses (11) and (12) be valid. Then, under the given extra assumptions above, as $\tau\to\infty$ uniformly on compact subsets,*

$$H(\xi,\tau)\to H^*(\xi), \tag{51}$$

where $H^*\in L^2_{\rho_0}(\mathbf{R}_+)$ is a profile uniquely determined from the following linear inhomogeneous ODEs: if $C\in\mathbf{R}$ in (47) ($G=G_1$), then H^* solves

$$\mathbf{A}H^* + Cgg'' + (1+C)(g')^2 + \frac{1}{2}C\xi g' = 0, \tag{52}$$

and if $C = \pm\infty$ *(*$G = G_2$*), then*

$$\mathbf{A}H^* + (gg')' + \frac{1}{2}\xi g' = 0. \tag{53}$$

Therefore, under the above assumptions, for $\tau \gg 1$ uniformly on compact subsets and in $L^2_{\rho_0}(\mathbf{R}_+)$

$$\theta(\xi,\tau) = g(\xi) + G(\tau)H^*(\xi) + o(G(\tau)). \tag{54}$$

Proof. According to the general scheme of the analysis in Section 2, we first need to establish appropriate upper and lower bounds on the rescaled solution $H(\xi,\tau)$ for $\tau \geq \tau_0 \gg 1$. As in [2], Section 3, we fix a $\xi_0 \gg 1$ and denote $c_0 = -g(\xi_0) \ll 1$, where $c_0\xi_0 > 1$, and $c_1 = -g(\xi_0) - c_0\xi_0 \gg 1$. We next introduce the following supersolution

$$\bar{H}(\xi,\tau) = \{C_0[-g(\xi) + D], \quad \xi \leq \xi_0; \;\; C_0[c_0\xi + c_1 + D], \quad \xi > \xi_0\},$$

where the constant $C_0 > 0$ is such that $C_0|G(\tau)| \leq 1$ for $\tau \geq \tau_0$ and D is as in (35). Without loss of generality, we assume that $G(\tau) > 0$. Then for $\xi \leq \xi_0$ we obtain the inequality

$$\bar{H}_\tau - \mathbf{L}(\bar{H}) \geq \frac{1}{G(\tau)}\{C_0G(\tau) - g_1(\tau, g + G(\tau)\bar{H})(1 - C_0G(\tau))g''$$

$$- g_2(\tau, g + G(\tau)\bar{H})(1 - C_0G(\tau))^2(g')^2 - g_3(\tau)(1 - C_0G(\tau))\xi g' - C_0G'(\tau)[-g(\xi_0) + D]\}.$$

In the domain $\xi \geq \xi_0$ the corresponding parabolic differential inequality takes the form

$$\bar{H}_\tau - \mathbf{L}(\bar{H}) \geq \frac{1}{G(\tau)}\{C_0G(\tau)c_0\xi/2 - g_1(\tau, g + G(\tau)\bar{H})g'' - g_2(\tau, g + G(\tau)\bar{H})(g' + c_0C_0G(\tau))^2$$

$$-g_3(\tau)\xi[g' + c_0C_0G(\tau)] - C_0G'(\tau)[c_0\xi + c_1 + D]\}.$$

We now estimate the terms on the right-hand sides of both inequalities. In our analysis we use the following elementary properties of the profile $g(\xi)$ for $\xi \gg 1$: $g(\xi) \sim -2\ln\xi$, $g'(\xi) \sim -2/\xi$ and $g'' \sim 2/\xi^2$, so that $(g')^2 \sim g'' \sim e^g$. Using these properties and the assumptions on $G(s)$, one can see that there exist constants $M_0 > 0$ and $\tau_0 \gg 1$ such that for any $\tau \geq \tau_0$

$$|g_1(\tau, g + G(\tau)\bar{H})|g'' \leq M(\tau)|G(\tau)|,$$

where $M(\tau) = \{M_0, \;\xi \leq \xi_0; \quad M_0[1 + (C_0c_0)^2G(\tau)], \;\xi \geq \xi_0\}$. A similar estimate is true for the second term like $g_2(\tau, g + \bar{H})(g')^2$. The last term is estimated directly. Since τ_0 is arbitrarily large, we always assume that for every $\tau \geq \tau_0$

$$|G'(\tau)| \leq \max\left\{\frac{G(\tau)}{4[-g(\xi_0) + D]}, \; \frac{G(\tau)}{4(1 + c_1 + D)}\right\}.$$

We now can define the constant C_0 as follows: $C_0 = 1/G(\tau_0) \gg 1$.

We thus establish that for $\tau \geq \tau_0$ the first term on the right-hand sides is the leading one for the equation with suitable extended coefficients, and we arrive at the following inequalities. If $\xi \leq \xi_0$ then

$$\bar{H}_\tau - \mathbf{L}(\bar{H}) \geq \frac{1}{G(\tau)} \{(C_0/2)G(\tau) - 4M_0 G(\tau)\} \geq 0,$$

if $C_0 > 8M_0$. In the outer domain $\xi \geq \xi_0$ we have

$$\bar{H}_\tau - \mathbf{L}(\bar{H}) \geq \frac{1}{G(\tau)} \{(C_0/8)G(\tau) - 4M_0 G(\tau)\} \geq 0,$$

if $C_0 \geq 32M_0$. Note that in the estimate of the quadratic terms like $(c_0 C_0 G(\tau))^2 \leq c_0^2 C_0 G(\tau)$ this upper bound becomes negligible provided that $c_0 \ll 1$. It is not difficult to see that the comparison of the "initial" data at $\tau = \tau_0$ corresponding to H and $\bar{H}$ is guaranteed by (50) with the above choice of the large constant C_0. The construction of a similar subsolution is done in the same way.

We now apply the stability approach from Section 2. Namely, using the linearization argument and setting $Y(\xi, \tau) = H(\xi, \tau) - H^*(\xi)$ we arrive at a nonlinear asymptotically small perturbation of the linear equation (43). Therefore, using the uniform stability of the ω-limit set of (43) in $L^2_{\rho_0}(\mathbf{R}_+)$ and standard parabolic regularity estimates, we conclude that

$$\omega(\theta_0) = \{H^*\}.$$

The ODEs (52) and (53) follow from the properties (31) - (33). In particular, by (32) we deduce that, say, in the case $C \in \mathbf{R}$ $(G(\tau) \equiv G_1(\tau))$, as $\tau \to \infty$ there holds

$$\frac{g_1(\tau, g + o(1))}{G_1(\tau)} \equiv \frac{1}{G_1(\tau)} \left[\frac{K(\tau + g + o(1))}{K(\tau)} - 1\right]$$

$$= \frac{1}{G_1(\tau)}\, g G_2(\tau)(1 + o(1)) = Cg(1 + o(1)),$$

and this asymptotic expansion gives the second term Cgg'' in the ODE (52). ■

4 Final-time profile

Using an approach similar to that in [2] for the heat equation, we now derive the exact estimate of the final time profile $u(x, 1^-)$. In particular, we thus prove that in the minimal domain S_{11}^- all the blow-up solutions exhibit single-point blow-up.

Denote

$$\Phi(s) = e^{-s/2} K^{1/2}(s). \tag{55}$$

Then $\Phi(s) \to 0$ as $s \to \infty$ by (15) and

$$(\Phi^2)'(s) = e^{-s}[K'(s) - K(s)] < 0, \quad s \gg 1,$$

by the assumption (12). Therefore there exists the inverse function $\Phi^{-1}(x)$ for small $x > 0$.

Theorem 9 *Under the assumptions of Theorem 1, there holds*

$$u(x, 1^-) = F(\Phi^{-1}(x) + C_0 + o(1)), \quad x \to 0. \tag{56}$$

Proof. We introduce a one-parameter family of functions

$$v_\alpha(x, t) = \ln(1 - \alpha) + v(x\Phi(-\ln(1 - \alpha)), \alpha + (1 - \alpha)t),$$

where $\alpha \in [0, 1)$ is a parameter and $(x, t) \in S = [\delta, D] \times [0, 1]$, $\delta > 0$ and $D > \delta$ are arbitrary fixed constants. We list the following properties of the family:

(i) At $t = 0$ we have by Theorem 1 uniformly on any interval $I = [\delta, D]$

$$v_\alpha(x, t) \to g(x), \quad \alpha \to 1. \tag{57}$$

(ii) v_α solves a quasilinear parabolic equation of the form

$$v_t = h_1(\alpha, v)v_{xx} + h_2(\alpha, v)(v_x)^2, \tag{58}$$

where the coefficients are

$$h_1(\alpha, v) = \frac{K(-\ln(1 - \alpha) + v)}{K(-\ln(1 - \alpha))},$$

$$h_2(\alpha, v) = \frac{K'(-\ln(1 - \alpha) + v)}{K(-\ln(1 - \alpha))} + \frac{K(-\ln(1 - \alpha) + v)}{K(-\ln(1 - \alpha))} \frac{F''(-\ln(1 - \alpha) + v)}{F'(-\ln(1 - \alpha))}.$$

On the boundary we have

$$v_\alpha(0, t) = -\ln(1 - t), \quad t \in (0, 1).$$

For any bounded v, there holds

$$h_1(\alpha, v) \to 1, \quad h_2(\alpha, v) \to 0, \quad \alpha \to 1.$$

Therefore (58) is the heat equations with perturbations being uniformly small for the values of the parameter $\alpha \approx 1$.

(iii) By comparison with the solutions of the perturbed heat equation

$$V_t = (1 \pm \epsilon)V_{xx} \pm \epsilon(V_x)^2, \quad V(0, t) = -\ln(1 - t),$$

and unbounded initial data with at most logarithmic growth as $x \to \infty$, see (57), we then prove that $v_\alpha(x,t)$ is uniformly bounded on S. Consider for instance the corresponding estimate from above. We then compare for $\alpha \approx 1$ the solution v_α with the supersolution V satisfying the equation

$$V_t = (1+\epsilon)V_{xx} + \epsilon(V_x)^2, \quad V(0,t) = -\ln(1-t). \tag{59}$$

Moreover, choosing appropriate sufficiently large convex initial data we prove that $V_{xx} \geq 0$ by the Maximum Principle applied to the uniformly parabolic equation for $z = V_{xx}$:

$$z_t = (1+\epsilon)z_{xx} + 2\epsilon v_x z_x + 2\epsilon z^2.$$

Then $z \geq 0$ for $t \approx 1^-$ at the boundary $x = 0$ by a straightforward asymptotic convexity analysis of the corresponding stable self-similar solution of (59), see p. 168 in [14]. In general, this is a simple version of the semiconvexity estimate for the heat equation.

This makes it possible to compare the solutions of (58) and (59). Let us now prove that $V(x,t)$ is uniformly bounded above as $t \to 1^-$ on compact subsets. Indeed, by the transformation $V = E(W) \equiv (1+\epsilon)\ln W/\epsilon$ problem (59) reduces to the heat equation with a power-like boundary function

$$W_t = (1+\epsilon)W_{xx}, \quad W(0,t) = (1-t)^{-\epsilon/(1+\epsilon)}.$$

Using the convolution representation of the solution, we obtain that $W(x,t)$ is uniformly bounded on S, see e.g. p. 169 in [14], and by comparison we obtain that $V \leq W$ in S is also uniformly bounded above. A similar analysis applies to the corresponding estimate from below where the unbounded logarithmic initial data are taken into account.

(iv) By the standard regularity theory for the uniformly parabolic equations [10], we conclude that $\{v_\alpha\}$ is a compact subset in $C(S)$. Passing to the limit $\alpha \to 1$ in equation (58), we deduce that uniformly in S

$$v_\alpha \to \bar{v}$$

along a subsequence, where $\bar{v}$ solves the heat equation

$$v_t = v_{xx}, \quad v(0,t) = -\ln(1-t), \quad v(x,0) = g(x).$$

Therefore

$$\bar{v}(x,t) = -\ln(1-t) + g(x/(1-t)^{1/2}) \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+,$$

by uniqueness, and hence the whole sequence converges to $\bar{v}$.

(v) Finally, we conclude that as $\alpha \to 1$ uniformly on I

$$v_\alpha(x,1) \to \bar{v}(x,1) \equiv -2\ln x + C_0.$$

Setting here $x = 1 \in I$ we obtain that for all $\alpha \approx 1$ there holds

$$v(\Phi(-\ln(1-\alpha)), 1) = -\ln(1-\alpha) + C_0 + o(1).$$

Denoting $\Phi(-\ln(1-\alpha)) = x$, we complete the proof. ■

We now give a more detailed description of the final-time profile using the following result.

Proposition 10 *As $x \to 0$, there holds*

$$\Phi^{-1}(x) = -2\ln x + \ln K(-2\ln x)(1 + o(1)). \tag{60}$$

Proof. The expansion (60) is equivalent to

$$\frac{\Phi^{-1}(x) + 2\ln x}{\ln K(-2\ln x)} \to 1, \quad x \to 0.$$

Setting here $x = \Phi(s)$ and using (55), we arrive at the condition

$$\frac{\ln K(s - \ln K(s))}{\ln K(s)} \to 0, \quad s \to \infty.$$

Since by Lagrange's formula $\ln K(s - \ln K(s)) = \ln K(s) - (\ln K)'(\eta) \ln K(s)$ where $\eta(s) \in (s, s - \ln K(s))$, we obtain that

$$\frac{\Phi^{-1}(x) + 2\ln x}{\ln K(-2\ln x)} - 1 = -(\ln K)'(\eta) \to 0$$

by (12) since $\eta(s) \to \infty$ as $s \to \infty$. ■

Therefore, in general (56) reads

$$u(x, 1^-) = F(-2\ln x + \ln K(-2\ln x)(1 + o(1)) + C_0 + o(1)), \quad x \to 0. \tag{61}$$

5 On the geometric structure of the minimal domain

We now discuss a general structure of the minimal domain S_{11}^- on the plane of IBVPs. According to [6], Section 12 (see also [2]), the plane P_{IBVP}^- is composed from two axis, $\{\psi\}$-axis and $\{k\}$-axis, which represent classes of monotone boundary functions $\{\psi\}$ and monotone nonlinear heat conductivity coefficients $\{k\}$ ordered in terms of the rate of asymptotic growth calculated via special limits.

Due to Theorem 1, the self-similar solution (7) of the heat equation describes the asymptotic behaviour of the solutions of any IBVP from the domain S_{11}^-. Observe that this self-similar solution is not an exact solution but it is an approximate one.

We thus denote by $S_* = \{-\ln(1-t), 1\}$ (i.e., $\psi(t) = -\ln(1-t)$ and $k(u) = 1$) the corresponding point on the plane P^-_{IBVP}. It can be also identified with the singularity generated by the self-similar solutions of the heat equation. We say that S_* may be treated as the attractor with the basin of attraction which is exactly the minimal domain S^-_{11}.

5.1 On the curve of travelling waves

It is worth mentioning that, excluding the axis $A^- = \{k = 1\}$ of the linear heat equation [2], the domain S^-_{11} contains no other exact solutions. It is known [12], [3], that for rather arbitrary coefficients $k(u)$, the only nontrivial exact solutions of the quasilinear heat conduction equation (1), which are invariant under Lie or Lie-Bäcklund groups of transformations, are the travelling wave solutions of the form

$$u_k(x,t) = \phi_k^{-1}(1 + x - t), \text{ with } \phi_k(s) = \int_s^\infty \frac{k(u)}{u}\, du. \tag{62}$$

Since $k(u)$ is positive, $\phi_k(s)$ is a decreasing function and therefore $u_k(x,t)$ is well defined if $k(u)/u$ is integrable at $u = \infty$. Assuming that the above integral converges:

$$\int^\infty \frac{k(u)}{u}\, du < \infty, \tag{63}$$

we obtain a solution which blows up at $t = 1$. This solution is defined for a wide class of nonlinear coefficients $k(u)$ and represents the simplest (translational) structure of blow-up boundary layers for quasilinear heat equations. On the other hand, it plays an important role in the localization theory of blow-up solutions. It is not difficult to show (see [14], p. 386) that under assumption (63), an arbitrary blowing up boundary function $\psi(t)$ leads to the localization of the solutions of (1) in the sense that the blow-up set

$$B = \{x \geq 0 : \ u(x,t) \to \infty, \ t \to 1^-\},$$

is bounded and moreover, has zero measure: $B = \{0\}$. Roughly speaking, the set of travelling wave solutions determines a "continuous curve"

$$\Gamma_{TW} = \{\{\phi_k^{-1}(1-t), k(u)\} : \ (63) \text{ holds}\}$$

on the plane P^-_{IBVP}.

Let us describe some other properties of the curve of travelling wave solutions. By construction, Γ_{TW} may be treated as a "monotone decreasing" (in $\{k\}$) curve satisfying also the following property. One can find a point $\{\psi_k, k\} \in \Gamma_{TW}$ with an arbitrarily fast growing blow-up function $\psi_k(t)$ as $t \to 1^-$. For instance, the points $\{\exp\{(1-t)^{-1/(\alpha-1)}\}, (\alpha -$

$1)[\ln(u)]^{-\alpha},\}$ for any $\alpha > 1$ belong to the curve Γ_{TW}. This means that Γ_{TW} can be formally treated as an unbounded curve. Namely, it has a horizontal "asymptotic direction" described by (63): $\{\psi_k\} \to \infty$ if $\{k\}$ approaches from below the limit state corresponding to divergent integrals in (63).

We now prove that these exact travelling wave solutions do not belong to the minimal domain and therefore cannot play a role of an attractor there. Nevertheless, Γ_{TW} partially forms the lower boundary of S_{11}^-, see comments below. By Theorem 1, the singularities in S_{11}^- have another more delicate non-translational nonlinear scaling structure described by the heat equation.

Proposition 11 *We have*

$$\Gamma_{TW} \cap S_{11}^- = \emptyset,$$

which means that for any coefficient k satisfying (63) with the corresponding blow-up boundary function

$$\psi_k(t) = \phi_k^{-1}(1-t), \tag{64}$$

at least one of the asymptotic conditions (10) or (12) does not hold.

Proof. In fact we prove that the curve of travelling waves Γ_{TW} blowing up at $t = 1$ does not intersect the domain S_{11}^-. This domain is characterized by two limit conditions, condition (10) on $F(s)$ and condition (12) on $K(s)$. Note that in our context, $F_k(s) = \phi_k^{-1}(e^{-s})$ which implies, taking into account the definition of ϕ_k, that

$$F_k'(s) = \frac{e^{-s}\phi_k^{-1}(e^{-s})}{k(\phi_k^{-1}(e^{-s}))}.$$

Assume first that $F_k(s)$ is a superlinear function for $s \gg 1$. On one hand, condition (10) implies by the L'Hospital rule that

$$\lim_{s\to\infty} \frac{F_k'(s)}{F_k(s)} = \lim_{s\to\infty} \frac{e^{-s}}{k(\phi_k^{-1}(e^{-s}))} = 0, \tag{65}$$

and if the limits involved exist,

$$\lim_{s\to\infty} \frac{k(\phi_k^{-1}(e^{-s}))}{\phi_k^{-1}(e^{-s})k'(\phi_k^{-1}(e^{-s}))} = 0.$$

On the other hand, since in our class of nonlinearities $k(s) \to 0$ as $s \to \infty$, there holds $s/k(s) \to \infty$. Using the previous results, we see that condition (12) does not hold. In fact, we find that

$$\lim_{s\to\infty} \frac{k'(F_k(s))F_k'(s)}{k(F_k(s))} = \lim_{s\to\infty} \frac{\phi_k^{-1}(e^{-s})k'(\phi_k^{-1}(e^{-s}))}{k(\phi_k^{-1}(e^{-s}))} \; \frac{e^{-s}}{k(\phi_k^{-1}(e^{-s}))} = \infty.$$

Therefore, $\{\psi_k, k\} \notin S_{11}^-$. Finally, if $F_k(s)$ is a linear or a sublinear function for $s \gg 1$, it is clear that (65) also holds and we arrive at the same conclusion. ∎

5.2 On the general structure of the minimal domain

On Figure 1 we present a more detailed structure of the minimal domain S_{11}^- with the unique attracting point S_* (the heat equation with a special logarithmic blow-up boundary function). Observe that the domain is unbounded in the $\{k\}$-variable and is bounded in the $\{\psi\}$-variable which follows from condition (11). The condition (12) determines two monotone curves γ_- and γ_+ which bound the minimal domain from below and above respectively. We also indicate there the interval I_1^- for the heat equation, studied in [2], and the travelling wave curve Γ_{TW} which does not intersect the domain. Both curves (boundaries) $\gamma_\pm$ are defined on the interval I_1^-. In fact, we can show that $\Gamma_{TW} \subset \gamma_-$ on the interval $\{\psi\} \in I_1^-$ in the sense that the travelling wave solutions partially form the structure of this lower border curve. On the other hand, Γ_{TW} is not bounded in the $\{\psi\}$-direction.

We now consider some new examples of the IBVPs which belong and which does not belong to S_{11}^-, so that the "curves" (one-parameter families) of IBVPs intersect the corresponding boundaries $\gamma_\pm$.

Example 3. Consider a one parameter family $\{E_\lambda,\ \lambda \in (0,1)\}$ of the IBVPs for the following quasilinear parabolic equation with a supergrowing nonlinearity $k(u)$ for $u \gg 1$:

$$u_t = (e^{e^u} u_x)_x, \quad x > 0,\ t \in (0,1), \tag{66}$$

$$u(0,t) = \ln^\lambda |\ln(1-t)|, \quad \lambda \in (0,1). \tag{67}$$

Condition (11) holds. We have $F(s) = \ln^\lambda s$ and $K(s) \equiv k(F(s)) = \exp\{e^{\ln^\lambda s}\}$. From the second condition (12) as $s \to \infty$

$$\frac{K'(s)}{K(s)} \sim e^{\ln^\lambda s} \ln^{\lambda-1} s\, \frac{1}{s} \to 0$$

if and only if $\lambda < 1$. Therefore the curve E_λ intersects the upper boundary γ_+ at $\lambda = 1$, i.e., at $\psi(t) = \ln|\ln(1-t)|$. For any $\lambda \in (0,1)$, Theorem 1 gives the following structure of the blow-up singularity

$$u(x,t) \approx \ln^\lambda [|\ln(1-t)| + g(\xi)], \quad \xi = x/[(1-t)\exp\{e^{\ln^\lambda |\ln(1-t)|}\}]^{1/2}. \tag{68}$$

We cannot extend the set of singularities driven by the boundary function (67) for $\lambda \geq 1$. It is known that in the critical case $\lambda = 1$ the blow-up singularity can be described by the exact self-similar solution of the equation with the exponential nonlinearity $u_t = (e^u u_x)_x$, see [5]. Thus, the problem (point) E_1 does not belong to the minimal domain S_{11}^-, and in fact $E_1 \in \gamma_+$. The blow-up singularity has completely distinct asymptotic structure. This emphasizes the completeness of our analysis in the case of superlinear coefficients $k(u)$.

Example 4. We now consider the family $\{E_\lambda,\, \lambda \in (0,1/2)\}$, of the IBVPs with a superslow diffusion for $u \gg 1$

$$u_t = (e^{-u^2} u_x)_x, \quad x > 0,\ t \in (0,1), \tag{69}$$

$$u(0,t) = |\ln(1-t)|^\lambda. \tag{70}$$

Then (11) holds, $F(s) = s^\lambda$ and

$$K(s) = e^{-s^{2\lambda}}.$$

The second condition (12) holds provided that $\lambda < 1/2$ (which characterises a point where E_λ intersects the lower boundary γ_-). Thus, for $\lambda \in (0,1/2)$ as $t \to 1^-$

$$u(x,t) \approx [|\ln(1-t)| + g(\xi)]^\lambda, \quad \xi = x/\left[(1-t)e^{-|\ln(1-t)|^{2\lambda}}\right]^{1/2}. \tag{71}$$

We next show that $\lambda = 1/2$ corresponds to a distinct singular pattern. It turns out that in this critical case such singularities are expected to exhibit a simpler translational structure described by the travelling wave solution. It follows from (62) that in this case

$$\phi_k(s) = \frac{1}{2} s^{-2} e^{-s^2} + ..., \quad \phi_k^{-1}(z) = \sqrt{-\ln z} - \frac{1}{2}\frac{\ln|\ln z|}{\sqrt{-\ln z}} + ...\,,$$

so that the travelling wave boundary data have the form

$$u_k(0,t) = \phi_k^{-1}(1-t) = \sqrt{|\ln(1-t)|} - \frac{1}{2}\frac{\ln|\ln(1-t)|}{\sqrt{-\ln(1-t)}} + ...$$

This is an asymptotically perturbed boundary function (70) of the problem $E_{1/2}$ with the critical exponent $\lambda = 1/2$, so that this travelling wave singularity belongs to the border curve γ_-.

In order to characterize the rectangular-like part of the domain S_{11}^- on Figure 1, we present the last example.

Example 5. Consider the equation

$$u_t = (\ln^\sigma u\, u_x)_x \quad (u \gg 1),$$

where $\sigma \in \mathbf{R}$ is an arbitrary constant. Then as $s \to \infty$

$$\frac{K'(s)}{K(s)} = \frac{k'(F(s))}{k(F(s))} F'(s) \sim \frac{F'(s)}{F(s)\ln F(s)} = \frac{1}{\ln F(s)} \frac{F'(s)}{F(s)} \to 0$$

for any $F(s)$ satisfying (10), i.e., for any boundary function $\psi(t)$ satisfying (11). This means that for any fixed coefficient $k(u) = \ln^\sigma u$, Theorem 1 gives the asymptotic behaviour of the solutions for an arbitrary blow-up function $\psi(t)$ from the interval I_1^- (same as for the heat equation with $k \equiv 1$, i.e., $\sigma = 0$). The same is true for the family of IBVPs with $k(u) = u^\sigma$, $\sigma \in \mathbf{R}$. This class of power coefficients $k(u)$ defines the upper and lower horizontal segments of the rectangular-like domain on Figure 1.

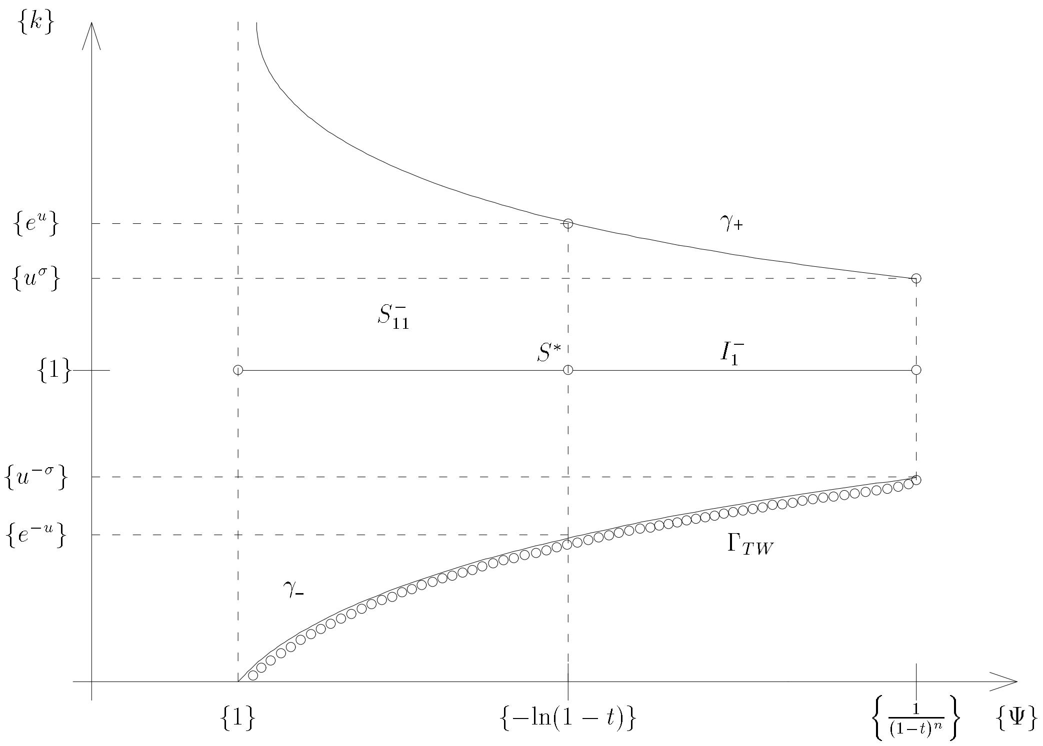

Figure 1.

5.3 On the extension to fully nonlinear equations

The minimal domain S_{11}^- of stability of the logarithmic singularity S_* for the heat equation admits a natural extension to more general nonlinear parabolic equations

$$u_t = \Phi(u, u_x, u_{xx}), \quad x > 0, \ t \in (0,1), \quad u(0,t) = \psi(t) \to \infty, \ t \to 1^-. \tag{72}$$

We impose the parabolicity condition on a sufficiently smooth function Φ:

$$\Phi_r(p, q, r) > 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R},$$

and assume that a unique classical solution exists on $\mathbf{R}_+ \times (0, 1-\epsilon]$, $\epsilon > 0$ small, for any bounded initial data (see e.g. [9]). We thus say that the heat equation singularity has a minimal domain of stability in the (infinite-dimensional) "space of IBVPs", where $A^- = \{\psi\}$ is one of the axes and the others (an infinite number, in general) depend on the operator Φ.

We now briefly discuss the main steps of the analysis.

(i) Extension problem. It can be solved by using the stationary solutions of (72) as in Subsection 2.1. Then we need to check some natural properties of sufficiently "steep" solutions of the corresponding ODE

$$\Phi(U, U', U'') = 0, \quad x > 0; \;\; U(0) = A, \;\; U'(0) = -B,$$

where A and B are large positive constants. Then we can restrict our attention to uniformly large solutions $u \gg 1$. We now impose an extra condition

$$\Phi(p, 0, 0) \le 0, \quad p \ge 0, \tag{73}$$

which guarantees that a new singularity cannot occur away from the origin $x = 0$. More precisely, one can see that in this case the derivative $z = u_x$ satisfying a quasilinear parabolic equation

$$z_t = \Phi_1(u, z, z_x)z + \Phi_2(u, z, z_x)z_x + \Phi_3(u, z, z_x)z_{xx} \tag{74}$$

cannot change its sign on the lateral boundary $x = 0$ by the Maximum Principle. Indeed, if $\psi'(t) \ge 0$ then $\bar u = \psi(t)$ is a supersolution of (72): $\bar u_t - \Phi(\bar u) = \psi'(t) - \Phi(\psi(t), 0, 0) \ge 0$ by (73). Therefore $z(0, t) \le 0$ and the Maximum Principle applies to equation (74). Then $u_x \le 0$ so that a boundary layer (not an internal layer) always occurs as $t \to 1^-$. By a slight modification of the previous approach, by intersection comparison with stationary solutions (see examples in [14], Chapter 4) we prove also that in the domain, where $u \gg 1$, there holds $u_x \ll -1$. This is an important estimate for the necessary extension of the equation.

(ii) Rescaled variables. Introducing as in (9) the nonlinear transformation by $\psi(t) = F(-\ln(1-t))$ and setting $u = F(v)$ we arrive at the equation

$$v_t = \frac{1}{F'(v)}\,\Phi(F(v), F'(v)v_x, F'(v)v_{xx} + F''(v)(v_x)^2), \quad v(0,t) = -\ln(1-t). \tag{75}$$

Finally, we introduce the rescaled variables corresponding to the heat equation singularity:

$$v(x,t) = -\ln(1-t) + \theta(\xi, \tau), \quad \tau = -\ln(1-t), \;\; \xi = x/\phi(t),$$

where $\phi(t) \to 0$ is a positive function to be determined. Then we arrive at the rescaled equation (cf. (24))

$$\theta_\tau = \frac{e^{-\tau}}{F'(\tau+\theta)}\Phi\left(F(\tau+\theta), F'(\tau+\theta)\frac{\theta_\xi}{\phi}, F'(\tau+\theta)\frac{\theta_{\xi\xi}}{\phi^2} + F''(\tau+\theta)\frac{(\theta_\xi)^2}{\phi^2}\right)$$

$$+e^{-\tau}\frac{\phi'}{\phi}\xi\theta_\xi - 1. \tag{76}$$

We now introduce the main assumptions. The first one plays a role of (10) (or (11)).

(H1) There exists a smooth positive function $\phi(t)$ such that as $\tau \to \infty$

$$\frac{1}{r}\frac{e^{-\tau}}{F'(\tau+p)}\Phi\left(F(\tau+p), q\frac{F'(\tau+p)}{\phi(\tau)}, r\frac{F'(\tau+p)}{\phi^2(\tau)} + q^2\frac{F''(\tau+p)}{\phi^2(\tau)}\right) \to 1 \tag{77}$$

uniformly on compact subsets from $(p, q, r) \in \mathbf{R}_+ \times \mathbf{R}_- \times \mathbf{R}_+$.

The second condition is equivalent to (12).

(H2) The function ϕ, defined from (**H1**), satisfies

$$e^{-\tau}\frac{\phi'(\tau)}{\phi(\tau)} \to -\frac{1}{2}, \quad \tau \to \infty. \tag{78}$$

(iii) Passage to the limit. The hypotheses above imply that the rescaled equation (76) has the form

$$\theta_\tau = \mathbf{A}\theta - 1 + \mathbf{P}(\tau, \theta), \quad \tau \gg 1, \tag{79}$$

where $\mathbf{A}$ is the linear operator (5) and $\mathbf{P}(\tau, \cdot)$ is an asymptotically small perturbation. If we prove by construction of a suitable super and subsolutions that the rescaled orbits are compact in $L^2_{\rho_0}(\mathbf{R}_+)$ (this can be done by an approach similar to that in the quasilinear case), then the uniform stability result (Proposition 6) for the unperturbed equation (43) will give the convergence result (45). We omit technical details. Consider more examples.

Example 6. The approach described above applies to the quasilinear equations with extra lower-order terms

$$u_t = (k(u)u_x)_x + f(u, u_x),$$

where $f(p, q)$ is a smooth function satisfying $f(p, 0) \le 0$ for any $p \ge 0$.

Example 7. As a curious example, we introduce a fully nonlinear parabolic equation with the same minimal domain as of the quasilinear one (1):

$$u_t = \ln\left[1 + e^{(k(u)u_x)_x}\right].$$

Since in a small neighbourhood of the logarithmic singularity we have that $u \to \infty$, $u_x \to -\infty$, $u_{xx} \to \infty$, and $(k(u)u_x)_x \to \infty$, as $t \to 1^-$ we actually deal with a fully nonlinear perturbation of the quasilinear heat equation

$$u_t = (k(u)u_x)_x + < \text{Fully nonlinear perturbation} > .$$

The same is true for the equation

$$u_t = (1 + u_x^2)\left[1 + \left(\frac{u_{xx}}{1 + u_x^2}\right)^3\right]^{1/3} + f(u, u_x),$$

or for more general ones related to mean curvature and curve shortening flows. In general, we expect that the minimal domain (basin) of attraction of the singularity S_* consists of IBVPs for the fully nonlinear equations which are asymptotically quasilinear ones on this singular structure.

Acknowledgements. This paper was written while the first author visited Department of Mathematical Sciences, the University of Bath, which has been supported by DGICYT Project PB94-0153, Spain. She is deeply grateful to both institutions for their hospitality. Both authors have been partially supported by TMR Grant EBR4061 PL97-0159.

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