Nash-Moser Theory for Standing Water Waves

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Abstract

Unlike progressive (or steady) Stokes waves, standing waves are a truly time-dependent phenomenon in the sense that they can not be regarded as stationary relative to a moving reference frame and we believe that, until now, there has been no existence theory, even for the case of small amplitude waves.

Our purpose here is to show how the Nash-Moser iteration method can be used to give a rigorous proof of the existence of certain small-amplitude standing water waves on an incompressible irrotational flow of finite depth over a horizontal bottom under gravity, when surface tension is neglected. The waves in question are solutions of the Euler equations, formulated in Lagrangian coordinates, for an ideal incompressible fluid which satisfy exactly the constant-pressure and kinematic boundary conditions on the free surface. But in addition, the solutions for which existence is established satisfy further constraints which are expressed in terms of the normal component of pressure gradient on the free surface. These constraints are designed to facilitate the mathematical development here, with particular regard to the method for obtaining *a priori* bounds necessary for the Nash-Moser method. They have no obvious analogue in the theory of Stokes waves.

The presentation here, which is more-or-less self contained, includes a proof of an elementary version of the Nash-Moser theorem sufficient for the purpose in hand and a detailed account from first principles of the *a priori* bounds required to implement the method.

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1 Introduction

We consider an infinite two-dimensional layer of ideal fluid at rest under gravity between a free surface at atmospheric pressure and a rigid horizontal bottom. The small-amplitude standing water-wave problem is to find a nearby irrotational, incompressible free-surface flow in which the pressure at the free surface is constant and which is non-trivially periodic in both space and time, but which is not steady with respect to any moving reference frame. This problem, which is genuinely time-dependent, has been open for some time.

Here we seek irrotational solutions of the Euler equations which satisfy the constant-pressure free-boundary condition, which are even in time and have vertical lines of spatial symmetry at intervals of half the spatial wavelength. To fix ideas, we seek solutions that have fixed time-period 2π and fixed depth h_0, and regard the spatial period $2l$ and g, the acceleration due to gravity, as parameters in the problem.

When $l = \pi$ the Euler equation linearised about the trivial solution has non-zero 2π-periodic solutions if and only if $g = g_0$, $h = h_0$ are such that the dispersion equation $n^2 = gm\tanh(mh)$ has a solution $(m, n) \in \mathbb{N}^2$ ($\mathbb{N}$ denotes the positive integers, excluding 0). In that case there is a linearised standing wave with minimal spatial and temporal periods $2\pi/m$ and $2\pi/n$, respectively.

We can explain the present approach schematically as follows. Suppose that $g = g_0$, $h = h_0$ are chosen so that the dispersion equation has a unique positive integer solution. (In Section 2.3 it is shown that that, in the set of (g, h) for which the dispersion equation has a positive integer solution, the set where it has a unique solution has full measure.) Now for parameter values (g, l) with $\|(g-g_0, l-\pi)\|$ sufficiently small, we seek a symmetric solution of the Euler equations as a perturbation of the unique solution of the linearised problem. More precisely , if the Euler equation is written as $E(z, g, l) = 0$, where z is a 2π-periodic symmetric function, $\emptyset$ denotes the rest solution and the solution of the linearised problem is θ, we seek solutions of the form $z = \emptyset + \varepsilon^2\theta + \varepsilon^3 X$ for $\varepsilon > 0$ sufficiently small. Note that, for small $\varepsilon > 0$ and (g, l) sufficiently close to (g_0, π), $\emptyset + \varepsilon^2\theta$ is an approximate solution of the Euler equations.

Since $n^2 - g_0 m\tanh(mh_0)$ can be arbitrarily small for arbitrarily large values of m, n (the so-called 'small divisors' problem'), neither Newton iteration nor the Nash-Moser (accelerated convergence) method [5, 6, 10, 11, 22, 23] give a non-trivial solution of the full problem. If they had done so, the result would be the existence of standing waves for all ε sufficiently small. *We claim no such result.*

Here is what we do. It is convenient to reformulate the problem in terms of new parameters ω_1, ω_2, μ and λ close to 0 by putting

$$g = (1+\varepsilon\lambda)(g_0 + \varepsilon\omega_1),\ \ l = \pi(1+\varepsilon\lambda),\ \ \varepsilon^2 = \kappa\mu(1+\omega_2)^{1/2},$$

where κ is a constant determined by (m, n), the solution of the dispersion equation (see Section 3). Suppose the Euler equations now have the form

$$E(\omega_1, \omega_2, \mu, \lambda, X) = 0. \tag{1.1}$$

The strategy for overcoming the small divisors problem is to seek only solutions of (1.1) which satisfy two additional constraints in the form

$$\Psi_1(a) = g_0, \quad \Psi_2(a) = -\mu^2. \tag{1.2}$$

Here Ψ_i, $i = 1, 2$, are real-valued functionals of $a = a(\omega_1, \omega_2, \mu, \lambda, X)$, the *normal component of pressure gradient at the free surface* of flows determined by $(\omega_1, \omega_2, \mu, \lambda, X)$. In Section 3.1 equations (1.2) are solved for ω_i, $i = 1, 2$, locally as functions of (μ, λ, X) and solutions of the Euler equation (1.1) which satisfy the normal-pressure constraints (1.2) therefore satisfy a reduced equation in the form

$$\Phi(\mu, \lambda, X) = 0. \tag{1.3}$$

The essential observation in this paper is that when $-\mu^2$ sufficiently small is a fixed element of a certain set $\mathcal{M}$, equation (1.3) *can be solved* for (λ, X) using the Nash-Moser method. The set $\mathcal{M}$, which is defined in Section 2.3, has zero as a Lebesgue point and we can conclude that there are many small amplitude standing waves (but the existence of a continuum is not a consequence of the method).

Crucial to the Nash-Moser method is the problem of obtaining *a priori* bounds on solutions v of the equation $\Lambda v = f$, where $\Lambda(\mu, \lambda, X)$ is an approximate right-inverse of the partial derivative $\partial_{(\lambda,X)}\Phi(\mu, \lambda, X)$ for fixed small μ. The bulk of this paper, from Section 7 onwards, is committed to the task of showing that when $-\mu^2 \in \mathcal{M}$, the hypotheses (a), (b) and (c) of the Nash-Moser method in Section 3.2 can be verified. In fact the constraints (1.2) are suggested by the method for obtaining these priori bounds. At this stage their motivation is mathematical; any physical significance they might have awaits explanation.

Standing-wave background. The standing-wave problem goes back to the beginning of modern hydrodynamics theory; for example Stokes [16] discussed linear theory based on the dispersion equation. A review of nonlinear perturbation theory, including the Russian literature and the important work of Sekerzh-Zen'kovich (for example [15]) up to 1965, is to be found in [20, 21]. During this period an important set of papers by G. I. Taylor [17] and by Penney & Price [13] appeared. While concerned with *large-amplitude* physical phenomena, [13, page 268] is very pessimistic about the likelihood of there ever being a proof of existence of small-amplitude waves because of mathematical complexity of the problem. More recent theory which ensures the existence of formal power series satisfying the Euler equations and the full nonlinear boundary conditions in the case of infinite depth is to be found in [1, 7, 14]. Because these power series are not known to converge, this does not lead to a rigorous existence theory. Indeed, since their convergence would lead to the existence of a continuum of standing waves, it is now unclear (in the light of present findings) whether or not such convergence result is to be expected in generality.

Mathematical background. There are three aspects of our method which deserve brief mention. First we choose to work in Lagrangian coordinates (Section 2.1). While this seems to complicate the initial formulation, it has distinct advantages in the linear PDE theory which occupies much of our work. The second is the Nash-Moser method (Section 3.2) which goes back to Nash's famous theorem in geometry [12] (see [8]) but is now an integral part of nonlinear functional analysis [3]. For our purposes the treatment by Hörmander [5] (see also [6]) is an essential reference. Finally, the needed *a priori* bounds are obtained here from first principles (Section 7 onwards). Using changes of variables to obtain an equivalent 'canonical' operator equation which has constant coefficients in the main term a descent method is developed which yields the required bounds when (1.2) hold. Because this is an unsteady (moving free-boundary) problem, these changes of variables demand careful analysis. We are particularly concerned with how they effect the

Hilbert transform and the Neumann operator in Section 7. To facilitate our estimation of these effects in the descent method, a notion of 'quasi-one-dimensional operators' is introduced in Section 7.2.

The growing literature on small-divisor theory for infinite-dimensional periodic systems, due to Craig, Wayne and Kuksin [2, 18, 9] for example, is nicely surveyed in [19].

Notation. According to the circumstances, we use an assortment of standard notation, such as $\partial_t u$, $(\partial/\partial t)u$ and u_t, to represent the partial derivative of u with respect to a variable t. If $u = (u_1, u_2)$ is a vector-function, we often write $u_{1,t}$ or $u_{1,tt}$ to denote partial derivatives of its component u_1. (Similar standard notation is used for other independent variables.) If $\xi \mapsto u(t, \xi)$ is a time-dependent displacement vector field we sometimes denote its Jacobian matrix by $u'_\xi(t, \xi)$.

Hilbert Transforms. Theory of the standard Hilbert transform K is to be found in Zygmund [24] (where the Hilbert transform of u is referred to as the function conjugate to u). We also use a finite-depth Hilbert transform $K(h)$, which is defined for sufficiently smooth, 2π-periodic functions u on $\mathbb{R}$ as follows: suppose $h > 0$ and

$$\begin{aligned} \Delta\phi = 0 \text{ on } \mathbb{R} \times (-h, 0), \qquad & \phi(x, y) = \phi(x + 2\pi, y) = 0, \\ \phi_y(x, -h) = 0,\ x \in \mathbb{R} \qquad & \phi(x, 0) = u(x),\ x \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Then $K(h)u(x) \equiv -\psi(x, 0)$, $x \in \mathbb{R}$, when ψ is the harmonic conjugate of ϕ normalised so that $\psi(x, -h) = 0$. Integral representations for K and $K(h)$ are given in Section 7.

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2 Formulation and main results

2.1 Euler's equations for standing waves

The standing water-wave problem refers to an incompressible irrotational flow, which is periodic in space and time, of an ideal fluid with a free surface contained in a channel with a horizontal bottom under gravity. We consider the two-dimensional flow of an incompressible fluid which at time t occupies the region $\mathcal{D}_t = \{-h_0 < x_2 < \eta(x_1, t)\}$ in the plane of points $x = (x_1, x_2)$ bounded by a fixed horizontal bottom $x_2 = -h_0$ and a free surface $\mathcal{S}_t = \{(x_1, x_2) : x_2 = \eta(x_1, t)\}$. In $\mathcal{D}_t$ the liquid motion is modelled by the Euler equations for a velocity field $\vec{v}(t, x) = (v_1(t, x), v_2(t, x))$ and the pressure $p(t, x)$:

$$\text{in } \mathcal{D}_t : \quad \text{div } \vec{v} = 0, \quad \vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} + g\vec{e_2} + \nabla p = 0.$$

Here $\vec{e_i}$, $1 = 1, 2$, are the standard basis vectors in $\mathbb{R}^2$ and gravity g acts vertically in the $-\vec{e}_2$ direction. It is assumed that the free boundary $\mathcal{S}_t$ evolves through the motion of its points with velocity $\vec{v}$ subject to pressure on $\mathcal{S}_t$ remaining constant (atmospheric). Therefore the following kinematic and dynamic boundary conditions hold

$$\text{on } \mathcal{S}_t : \quad \eta_t + v_1\eta_{x_1} - v_2 = 0, \quad (\vec{v_t} + (\vec{v} \cdot \nabla)\vec{v} + g\vec{e_2}) \cdot \vec{s} = 0, \tag{2.1}$$

where $\vec{s}$ is tangential to the free surface. When the flow is supposed irrotational, the governing equations become

$$\text{on } \mathcal{D}_t : \text{ div } \vec{v} = 0, \quad \text{rot } \vec{v} = 0, \tag{2.2}$$

and the impermeability condition at the bottom is

$$\text{on } x_2 = -h_0 : \quad \vec{v} \cdot \vec{e_2} = 0. \tag{2.3}$$

Equations (2.1)-(2.3) are supplemented with periodicity and symmetry conditions. The spatial conditions are given by the requirement that

$$\eta(t, x_1 + 2l) = \eta(t, x_1),\ (-1)^i v_i(t, -x_1, x_2) = v_i(t, x) = v_i(t, x + 2l\vec{e_1}). \tag{2.4}$$

It follows that there is no flux through the lines $x_1 = nl$, $n \in \mathbb{Z}$ and that any solution of (2.1)-(2.4) gives the motion of liquid partially filling a rectangular container. Clearly this problem has a trivial solution $\vec{v} = 0$, $\eta = \text{const.}$ for any positive h_0 and l.

The standing water-wave problem then is to find non-trivial solutions of equations (2.1)-(2.4) with *fixed* temporal period and temporal symmetry as follows:

$$\vec{v}(t + 2\pi, x) = \vec{v}(t, x) \quad \eta(t + 2\pi, x_1) = \eta(t, x_1), \quad \vec{v}(-t, x) = -\vec{v}(x, t). \tag{2.5}$$

Note that the kinetic energy of such a flow vanishes instantaneously when $t = n\pi$, $n \in \mathbb{Z}$.

Standing water waves of small amplitude are solutions near to an equilibrium of the infinite-dimensional Hamiltonian dynamical system given by (2.1)-(2.3), a so-called centre problem. In this setting we consider l and g as unknown parameters whose values form part of the solution to the problem. We suppose that $l - \pi$ and $g - g_0$ are small, where g_0 is fixed below. Note also that problem (2.1)-(2.5) is a non-stationary free boundary problem for the Cauchy-Riemann equations. It is convenient to reduce it to a boundary-value problem in a fixed domain using Lagrangian coordinates.

2.2 Lagrangian coordinates

Let $\Omega \subset \mathbb{R}^3$ be the strip $\mathbb{R}^2 \times (-h_0, 0)$ in the space of points (t, ξ), where $\xi = (\xi_1, \xi_2)$, bounded by the planes $\Gamma_{h_0} = \{\xi_2 = -h_0\}$ and $\Gamma_0 = \{\xi_2 = 0\}$. We denote by Ω_0 the set $\mathbb{R} \times (-h_0, 0)$ which, in the plane of the Lagrangian variable ξ, is the image of Ω under the projection $(t, \xi) \mapsto \xi$. We deal with mappings (displacement vector fields) $\xi \mapsto x(t, \xi)$ of $\bar{\Omega}_0$ into $\mathbb{R}^2$, where x is of class $C^2(\bar{\Omega})$. Denote by $M_0(t, \xi)$ the Jacobi matrix $x'_\xi(t, \xi)$, of $\partial_\xi x(t, \xi)$, M_0^* its transpose, set $|M_0| = \det M_0$, and consider the following equations

$$\Omega : \quad \mathcal{L}_1(x) \equiv \partial_t |M_0| = 0, \quad \mathcal{L}_2(x) \equiv \operatorname{rot}_\xi (M_0^* x_t) = 0, \tag{2.6}$$

$$\Gamma_0 : \quad \mathcal{L}_3(x) \equiv M_0^* (x_{tt} + g\vec{e_2}) \cdot \vec{e_1} = 0, \quad \Gamma_{h_0} : x_2 = -h_0. \tag{2.7}$$

Lemma 2.1. *Suppose that $x \in C^2(\bar{\Omega})$ is such that (2.6)-(2.7) hold and that, for $t \in \mathbb{R}$, the mapping $\xi \mapsto x(t, \xi)$ is a diffeomorphism of Ω_0 onto a curvilinear strip $\mathcal{D}_t$ bounded by the straight line $x_2 = -h_0$ and a curve $\mathcal{S}_t \equiv \{x = x(t, \xi_1, 0) : \xi_1 \in \mathbb{R}\}$. Then the vector field $\vec{v}(t, x) = x_t(t, \xi(t, x))$ satisfies equations and boundary conditions (2.1)-(2.3).*

Proof. Fix an arbitrary smooth function $\psi(\xi)$ compactly supported in Ω_0. Using the identity

$$\partial_t |M_0| = \operatorname{div}_\xi (|M_0| M_0^{-1} x_t), \tag{2.8}$$

we obtain that

$$0 = \int_{\Omega_0} \psi \operatorname{div}_\xi (|M_0| M_0^{-1} x_t) d\xi = -\int_{\Omega_0} \nabla_\xi \psi \cdot M_0^{-1} x_t |M_0| d\xi$$

$$= -\int_{\Omega_0} M_0^{-1*}\nabla_\xi\psi\cdot x_t\,|M_0|d\xi = -\int_{\mathcal{D}_t}\nabla_x\psi\cdot x_t\,dx = \int_{\mathcal{D}_t}\psi\,\mathrm{div}_x x_t\,dx.$$

Hence x_t is solenoidal in x. Denoting the standard symplectic matrix by S recall that

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S^* = -S, \quad S^2 = -I, \quad SB^*S^{-1} = |B|B^{-1}, \tag{2.9}$$

for any 2×2 matrix B, and that $\mathrm{rot}\,\vec{a} = \mathrm{div}(S\vec{a})$ for any smooth vector field $\vec{a}(\xi)$. This implies that

$$\begin{aligned} 0 &= \int_{\Omega_0}\psi\,\mathrm{rot}_\xi(M_0^* x_t)d\xi = -\int_{\Omega_0}\nabla_\xi\psi\cdot SM_0^* x_t d\xi \\ &= -\int_{\mathcal{D}_t} M_0^*\nabla_x\psi\cdot(SM_0^*S^{-1})Sx_t|M_0|^{-1}dx = -\int_{\mathcal{D}_t}\nabla_x\psi\cdot Sx_t dx \\ &= \int_{\mathcal{D}_t}\psi\,\mathrm{rot}_x\,x_t dx. \end{aligned}$$

Hence x_t is x-irrotational when (2.6) is satisfied, as required.

The kinematic boundary condition is satisfied automatically. It is clear that $x_{tt}(t,\xi) = \big(x_t(t,\xi(t,x))\big)_t + (x_t\cdot\nabla_x)x_t(t,\xi(t,x))$. Hence $x_{tt} = \vec{v}_t + (\vec{v}\cdot\nabla_x)\vec{v}$ and boundary condition (2.7) becomes $(\vec{v}_t + (\vec{v}\cdot\nabla_x)\vec{v} + g\vec{e}_2)\cdot(M_0\vec{e}_1) = 0$. It remains to note that the vector $M_0\vec{e}_1$ is tangential to $\mathcal{S}_t$. □

To guarantee the periodicity and symmetry conditions for $\vec{v}$ we need additional restrictions on solutions of equations (2.6), (2.7). The following assertion is an obvious consequence of Lemma 2.1. Let

$$x = J_l(\xi + Z(t,\xi)), \quad J_l = \mathrm{diag}\{l/\pi, 1\}, \tag{2.10}$$

Corollary 2.2. *Suppose, under the assumptions of Lemma 2.1, that the diffeomorphism $\xi \mapsto x(t,\xi)$ satisfies periodicity conditions given by*

$$Z(t,\xi + 2\pi\vec{e}_1) = Z(t+2\pi,\xi) = Z(t,\xi), \tag{2.11}$$

that the component Z_1 of the vector field Z is even in t and odd in ξ_1, while the component Z_2 is even in both t and ξ_1. Then the vector field $\vec{v}(t,x) = x_t(t,\xi(t,x))$ satisfies (2.4)-(2.5). □

Note that, because any solution of the Lagrangian formulation can be composed with a diffeomorphism of Ω_0 onto itself to yield another solution of the Lagrangian formulation, there is no question of a one-to-one correspondence between solutions of (2.6)-(2.7) and standing waves. However, any non-trivially time-dependent solution of (2.6)-(2.7) yields a non-trivial standing wave.

The presence of periodic solutions of nonlinear Hamiltonian systems near an equilibrium is strongly influenced by the existence and multiplicity of periodic solutions of the linearised problem. Therefore a preliminary step in the standing-wave problem is to understand the periodic solutions of the equations linearised at a trivial solution. Our further considerations of the *nonlinear* problem will use a Nash-Moser-type iteration method which necessitates a detailed examination of the equations linearised at points other than the equilibrium. To address both these issues we begin with a lemma which gives the Gateaux derivatives $\mathcal{L}_i'(x)$, defined as usual by

$$\mathcal{L}_i'(x)y = \lim_{\delta \to 0}\{\mathcal{L}_i(x+\delta y) - \mathcal{L}_i(x)\}/\delta.$$

Henceforth partial differential operators, rot, div, ∇ *and* Δ, *refer to differentiation with respect to* ξ *only, and not* t.

Lemma 2.3. *If* $x(t,\xi)$, $y(t,\xi)$ *are sufficiently smooth, time-dependent vector fields and* $w = (|M_0|M_0^{-1}y)_t$, *then*

$$\mathcal{L}_1'(x)y = \operatorname{div} w, \quad \mathcal{L}_2'(x)y = \operatorname{rot}\,(A^{-1}w + R_2 y), \tag{2.12}$$
$$\mathcal{L}_3'(x)y = \partial_t(A^{-1}w)\cdot \vec{e_1} + \partial_{\xi_1}(a|M_0|M_0^{-1}y)\cdot \vec{e}_2 + R_3 y. \tag{2.13}$$

Here the positive-definite, symmetric matrix A, *the coefficient* a *and the linear operators* R_i *are given by the formulae*

$$A = |M_0|(M_0^* M_0)^{-1}, \quad a = |M_0|^{-1}M_0^*(x_{tt} + g\vec{e_2})\cdot \vec{e}_2, \tag{2.14}$$
$$R_2 y = -\{\mathcal{L}_1(x)A^{-1} + \mathcal{L}_2(x)S\}M_0^{-1}y, \tag{2.15}$$
$$R_3 y = \partial_t(R_2 y)\cdot \vec{e_1} + \partial_{\xi_1}(\mathcal{L}_3(x)M_0^{-1}y)\cdot \vec{e_1}. \tag{2.16}$$

Remark. The coefficient a, which plays an important role from Section 7 onwards, is the *normal component of the pressure gradient* on the free surface, and the special form of the remainders R_i, which vanish when $x(t,\xi)$ satisfies governing equations (2.6)-(2.7), is important in verifying hypothesis (b) of the Nash-Moser theorem in Section 3.2.

Proof. We begin by showing that, for any smooth displacement vector fields $x(t,\xi)$, $y(t,\xi)$, the vector field $\vec{q} = M_0^* y_t - M_{0,t}^* y$ satisfies the identity

$$\vec{q} = A^{-1}w - \{\mathcal{L}_1(x)A^{-1} + \mathcal{L}_2(x)S\}M_0^{-1}y. \tag{2.17}$$

In order to prove this note that since $(M_0M_0^{-1})_t = 0$

$$\vec{q} = M_0^* M_0(M_0^{-1}y)_t + (M_0^* M_{0,t} - M_{0,t}^* M_0)M_0^{-1}y.$$

The identity $M_0^* M_{0,t} - M_{0,t}^* M_0 = -\mathcal{L}_2(x)S$ then implies that

$$\vec{q} = M_0^* M_0(M_0^{-1}y)_t - \mathcal{L}_2(x)SM_0^{-1}y,$$

and $w = (|M_0|M_0^{-1}y)_t$ and $\mathcal{L}_1(x) = |M_0|_t$ imply that

$$(M_0^{-1}y)_t = |M_0|^{-1}w - \mathcal{L}_1(x)|M_0|^{-1}M_0^{-1}y.$$

Substituting these relations in the previous identity gives (2.17)

Let us turn to proving equalities (2.12). Note first from (2.8) and (2.9) that $\mathcal{L}_1(x) = \operatorname{rot}((SM_0)^* x_t)$ and hence that $\mathcal{L}_1' y = \operatorname{rot}((SN)^* x_t + (SM_0)^* y_t)$ where N denotes the Jacobi matrix of $\partial_\xi y(t,\xi)$. Since $\nabla(Sx_t \cdot y) = (SM_{0,t})^* y - (SN)^* x_t$, we conclude that

$$\mathcal{L}_1'(x)y = \operatorname{rot}((SM_{0,t})^* y + (SM_0^*)y_t) = \operatorname{rot}((SM_0)^* y)_t = \operatorname{div} w.$$

Similarly,

$$\begin{aligned}\mathcal{L}_2'(x)y &= \operatorname{rot}(M_0^* y_t + N^* x_t) = \operatorname{rot}(M_0^* y_t + N^* x_t - \nabla(x_t \cdot y)) \\ &= \operatorname{rot}(M_0^* y_t - M_{0,t}^* y) = \operatorname{rot} \vec{q}\end{aligned}$$

which, along with (2.17), implies (2.12). Next consider the operator $\mathcal{L}_3'$. Note that

$$\mathcal{L}_3'(x)y = (N^* x_{tt} + M_0^* y_{tt}) \cdot \vec{e_1} + g N^* \vec{e_2} \cdot \vec{e_1}, \tag{2.18}$$

and that $N^* x_{tt} + M_0^* y_{tt} = (N^* x_t + M_0^* y_t)_t - \nabla(y_t \cdot x_t)$. Since

$$\nabla(y_t \cdot x_t) = \nabla\big((y \cdot x_t)_t - y \cdot x_{tt}\big) = (M_{0,t}^* y + N^* x_t)_t - \nabla(y \cdot x_{tt}),$$

this gives that $N^* x_{tt} + M_0^* y_{tt} = \vec{q_t} + \nabla(y \cdot x_{tt})$. Therefore, from (2.18) and the identity $N^* \vec{e_2} \cdot \vec{e_1} = y_{2,\xi_1}$, we conclude that $\mathcal{L}_3'(x)y = \vec{q_t} \cdot \vec{e_1} + ((x_{tt} + g\vec{e_2}) \cdot y)_{\xi_1}$ and hence that

$$\mathcal{L}_3'(x)y = \vec{q_t} \cdot \vec{e_1} + \big(M_0^*(x_{tt} + g\vec{e_2}) \cdot M_0^{-1} y\big)_{\xi_1}. \tag{2.19}$$

Since, on Γ_0, $M_0^*(x_{tt} + g\vec{e_2}) \cdot \vec{e_1} = \mathcal{L}_3(x)$ and $M_0^*(x_{tt} + g\vec{e_2}) \cdot \vec{e_2} = |M_0| a$, it now follows follows from (2.5) that

$$\mathcal{L}_3'(x)y = \partial_t \vec{q} \cdot \vec{e_1} + \partial_{\xi_1}\big(a|M_0| M_0^{-1} y \cdot \vec{e_2} + \mathcal{L}_3(x) M_0^{-1} y \cdot \vec{e_1}\big).$$

Substituting formula (2.17) for $\vec{q}$ gives (2.13) and the lemma is proven. □

2.3 Resonance and the dispersion equation

Now we turn to the linearised equations and the choice of g_0. Equations (2.6),(2.7), (2.10), (2.11) with $l = \pi$ and $g = g_0 \in (0, \infty)$ have the trivial solution $x(t, \xi) = \xi$, with $M_0 = I$, $a = g_0$ and $Z \equiv 0$ in (2.10). From Lemma 2.3, the linearisation of the governing equations at this solution leads to the classical boundary-value problem of linear water-wave theory

$$\begin{aligned}&\Omega: \quad \operatorname{div} Y_t = 0, \quad \operatorname{rot} Y_t = 0, \\ &\Gamma_0: \quad Y_{1,tt} + g_0 Y_{2,\xi_1} = 0, \quad \Gamma_{h_0}: \quad Y_2 = 0.\end{aligned}$$

Every 2π-periodic solution of this problem with Y_1 even in t and odd in ξ_1, and Y_2 even in both t and ξ_1 has the form $Y = \text{const.}\theta(t, \xi)$ where

$$\theta = \cos nt\big(\sin m\xi_1 \cosh m(\xi_2 + h_0), -\cos m\xi_1 \sinh m(\xi_2 + h_0)\big), \tag{2.20}$$

and the positive integers m, n satisfy the dispersion equation

$$n^2 - g_0 m \tanh(h_0 m) = 0. \tag{2.21}$$

For the nonlinear standing-wave problem we will be interested in solutions with g in (2.6)-(2.7) close to g_0 when (2.21) has a unique positive integer solution (m, n). Resonance is said to occur when (2.21) has more than one positive integer solution. The following proposition shows that the value of g_0 alone determines whether the system is resonant and that there is no resonance for typical g_0.

Denote by $\mathcal{N}(\nu)$, $\nu > 0$, the set of all $g > 0$ for which there exists $c(g, \nu)$ with

$$|n^2 - g|m||^{-1} \leq c(\nu, g)(1 + |n|)^\nu \text{ for all } (m, n) \in \mathbb{Z}^2 \setminus \{0\},$$

and, for fixed g, denote by $\Sigma(g)$ the spectral set all h for which (2.21) has at least one positive integer solution with $(g_0, h_0) = (g, h)$. Recall $0 \notin \mathbb{N}$.

Proposition 2.4. *Let $\nu \in (1,2)$ be an arbitrary real number. Then the set $\mathcal{N}(\nu)$ has full measure and there exists a set $\mathcal{N}_0(\nu) \subset \mathcal{N}(\nu)$ such that the complement of $\mathcal{N}_0(\nu)$ in $(0,\infty)$ is a set of zero measure, and the dispersion equation has at most one positive integer solution for any $g_0 \in \mathcal{N}_0(\nu)$ and any h_0. The spectral set $\Sigma(g_0)$, for $g_0 \in \mathcal{N}(\nu)$, is countable and has only one limit point, namely 0.*

Moreover, for $\nu \in (1,2)$, $g_0 \in \mathcal{N}(\nu)$ and $\tau > 1 - \nu^{-1}$, there exists a set $\mathcal{M}(g_0,\tau) \subset (-1,1)$ such that 0 is a Lebesgue point of $\mathcal{M}(g_0,\tau)$ and

$$|d(\zeta)-\mu|^{-1} \le c|\mu|^{-\tau}(1+|\zeta_1|^2) \text{ and } |d(\zeta)-\mu|^{-1} \le c|\mu|^{-\tau}(1+|\zeta_2|)$$

for all $\zeta \in \mathbb{Z}^2 \setminus \{0\}$, where $d(\zeta) = g_0|\zeta_2| - \zeta_1^2$. Here $c = \min\{1, g_0/4\}$. □

Proof. The proof depends on the following considerations. Let $\nu \in (1,2)$ be fixed and, for $k \in \mathbb{N}$, let $E_k \equiv \{g > 0 : |\zeta_1^2 - g\zeta_2| \ge (k(1+\zeta_1))^{-\nu}$ for all $\zeta = (\zeta_1,\zeta_2) \in \mathbb{N}^2\}$. Then for any $g \in \mathcal{N}(\nu) \equiv \cup_{k\in\mathbb{N}} E_k$ there exists $c(g,\nu) > 0$ such that

$$|d(\zeta)^{-1}| \le c(g,\nu)(1+|\zeta_1|)^{\nu} \quad\text{and}\quad |d(\zeta)^{-1}| \le c(g,\nu)(1+|\zeta_2|)^{\nu/2}. \tag{2.22}$$

(The second inequality follows from the first and the definition of $\mathcal{N}(\nu)$.)

Lemma 2.5. *The complement of $\mathcal{N}(\nu)$ in $(0,\infty)$ is a set of zero measure.*

Proof. For each fixed $N \in \mathbb{N}$, let

$$E_k^N = \{g^{-1} \in (0,N] : \left|\frac{1}{g} - \frac{\zeta_2}{\zeta_1^2}\right| < \frac{1}{k\zeta_1^{2+\nu}} \text{ for some } \zeta \in \mathbb{N}^2\}.$$

It is easy to see that this set is covered by a countable union of intervals centred at the points ζ_2/ζ_1^2 with lengths $2/k\zeta_1^{2+\nu}$, and that the number of such intervals corresponding to any given ζ_1 does not exceed $c(N)\zeta_1^2$, where $c(N)$ is a constant. Therefore

$$\text{meas}\, E_k^N \le \frac{c(N)}{k}\sum_{\zeta_1\in\mathbb{N}} \zeta_1^{-\nu} \to 0 \text{ as } k \to \infty.$$

Hence $\cap_{k\in\mathbb{N}} E_k^N$ has zero measure. This shows that $\text{meas}\,([N^{-1},\infty)\setminus\mathcal{N}(\nu)) = 0$ for all N, and the result follows. □

We are interested in the unique solvability of the dispersion equation

$$\zeta_1^2 - g\zeta_2 \tanh(h\zeta_2) = 0, \quad \zeta \in \mathbb{Z}^2 \setminus \{0\} \tag{2.23}$$

when $g \in \mathcal{N}_0(\nu)$, which we now define. Let H denote the set of all $h > 0$ such that $z = e^h$ is a solution of a polynomial equation in the form

$$m_2 n_1^2 (z^{2m_2} - 1)(z^{2m_1} + 1) = m_1 n_2^2 (z^{2m_1} - 1)(z^{2m_2} + 1), \tag{2.24}$$

where (m_1,n_1), (m_2,n_2) are distinct elements of $\mathbb{N}^2$. Then H is countable. When $h \in H$ and e^h satisfies (2.24), let $g(h) = n_i^2 m_i^{-1} \coth(m_i h)$, $i = 1, 2$. Then (2.23) has at least two positive integer solutions when $g = g(h)$, $h \in H$. Let $G \equiv \{n^2 m^{-1}\coth(mh) : (m,n) \in \mathbb{N}^2,\ h \notin H\}$. Then $(0,\infty)\backslash G$ is countable and for each $g \in G$ (i) there exists $h > 0$ for which (2.23) has exactly one positive integer solution and (ii) there is no $h > 0$ for which (2.23) has more than one positive integer solution. Let $\mathcal{N}_0(\nu) = \mathcal{N}(\nu) \cap G$ and note that $\Sigma(g) = \{m^{-1}\tanh^{-1}(n^2/gm) : (n,m) \in \mathbb{N}\}$.

Lemma 2.6. *For $g_0 \in \mathcal{N}(\nu)$, $\nu \in (1,2)$, 0 is the only limit point of $\Sigma(g_0)$.*

Proof. For each $h \in \Sigma(g_0)$ there exists $(m,n) \in \mathbb{N}^2$ such that $n^2 < g_0 m$ and

$$h = \frac{1}{m} \tanh^{-1}\left(\frac{n^2}{g_0 m}\right) = \frac{1}{2m} \log\left(\frac{g_0 m + n^2}{g_0 m - n^2}\right).$$

Therefore if $g_0 \in \mathcal{N}(\nu)$, $\nu \in (1,2)$,

$$h \le (2m)^{-1} \log\left(2 g_0 m\, c(g_0,\nu)(1+\sqrt{g_0 m})^{\nu}\right).$$

This proves that 0 is the only limit point of $\Sigma(g_0)$ in this case. □

It remains to define $\mathcal{M}(g_0,\tau)$.

Lemma 2.7. *Let $\nu \in (1,2)$, $g_0 \in \mathcal{N}(\nu)$ and $\tau > 1-\nu^{-1}$. Then there exists a set $\mathcal{M}(g_0,\tau) \subset (-1,1)$ such that 0 is a Lebesgue point of $\mathcal{M}(g_0,\tau)$ and for $\mu \in \mathcal{M}(g_0,\tau)$*

$$|d(\zeta)-\mu|^{-1} \le c|\mu|^{-\tau}(1+|\zeta_1|^2) \quad \textit{and} \quad |d(\zeta)-\mu|^{-1} \le c|\mu|^{-\tau}(1+|\zeta_2|) \tag{2.25}$$

for all $\zeta \in \mathbb{Z}^2 \setminus \{0\}$, where $c = \min\{1, g_0/4\}$.

Proof. Let $c = \min\{1,\, g_0/4\}$ and, for each $\zeta_1 \in \mathbb{Z}$, observe that there is at most one $\zeta_2 \in \mathbb{N}$ such that

$$|d(\zeta_1,\zeta_2)-\mu| < c|\mu|^{\tau}/(1+|\zeta_1|^2) \text{ for some } \mu \in (-c,c).$$

Let

$$\mathcal{M}(g_0,\tau) \equiv \{\mu \in (-c,c) : |d(\zeta_1,\zeta_2)-\mu| \ge c\mu^{\tau}/(1+|\zeta_1|^2) \text{ for all } \zeta \in \mathbb{Z}^2 \setminus \{0\}\}.$$

We have just seen that for $\lambda \in (0,c)$ the set $(-\lambda,\lambda) \setminus \mathcal{M}(g_0,\tau)$ is a subset of the union of the intervals of length $2c\lambda^{\tau}/(1+|\zeta_1|^2)$ with centres at $d(\zeta_1,\zeta_2)$, where $|d(\zeta_1,\zeta_2)-\mu| < c\mu^{\tau}/(1+|\zeta_1|^2)$ for some $\mu \in (-\lambda,\lambda)$. Since $g_0 \in \mathcal{N}(\nu)$ these centres $d(\zeta_1,\zeta_2)$ satisfy

$$\frac{1}{c(g_0,\nu)(1+|\zeta_1|)^{\nu}} \le |d(\zeta_1,\zeta_2)| \le \lambda + \frac{c\lambda^{\tau}}{1+|\zeta_1|^2}, \quad \nu \in (1,2).$$

It follows that for $\lambda > 0$ sufficiently small there exists $\hat{c}(g_0,\nu)$ such that the centres satisfy $(1+|\zeta_1|)^{-1} \le \hat{c}(g_0,\nu)\lambda^{1/\nu}$.

Together these observations give that

$$\frac{\text{meas}\left((-\lambda,\lambda)\setminus\mathcal{M}\right)}{\lambda} \le \sum_{\{|\zeta_1| : (1+|\zeta_1|)^{-1} \le \hat{c}(g_0,\nu)\lambda^{1/\nu}\}} \frac{2c\lambda^{\tau-1}}{1+|\zeta_1|^2} \le \text{const.}\lambda^{\tau-1+1/\nu} \to 0$$

as $\lambda \to 0$. Therefore 0 is a Lebesgue point of $\mathcal{M}$. The required inequalities are immediate from the definition of $\mathcal{M}(g_0,\tau)$. □

2.4 The main results

The main result of this article is the following theorem on existence of non-trivial solutions to the standing water-wave problem.

Theorem 2.8. *Suppose that* $\nu \in (1, 24/23)$, $g_0 \in \mathcal{N}_0(\nu)$, $h_0 \in \Sigma(g_0)$, $r \geq 25$ *and* (m, n) *is the unique solution of the corresponding dispersion equation. Then there exists an infinite set* $\mathcal{E} \subset \mathbb{R}^+$ *with* 0 *as a limit point such that, for any* $\varepsilon \in \mathcal{E} \cap (0, \varepsilon_0]$, *the boundary-value problem* (2.6)-(2.7) *has a solution of a class* C^r *in the form*

$$x(t, \xi) = J_l\big(\xi + \varepsilon^2\theta(t, \xi) + \varepsilon^3 X_\varepsilon(t, \xi)\big).$$

The functions $X_\varepsilon = (X_{1\varepsilon}, X_{2\varepsilon})$ *are* 2π*-periodic with respect to* t *and* ξ_1, $X_{1\varepsilon}$ *is even in* t *and odd in* ξ_1 *while* $X_{2\varepsilon}$ *is even both in* t *and* ξ_1. *Moreover*

$$\|X_\varepsilon\|_{C^r(\Omega)} \leq c, \quad (|g - g_0| + |l - \pi|) \leq c\varepsilon,$$

where c *depends only on* ν, g_0, h_0 *and* r. □

It is obvious that, under the assumptions of Theorem 2.8, the mapping $\xi \mapsto x(t, \xi)$ is diffeomorphism for sufficiently small ε and meets the requirements of Lemma 2.1 and Corollary 2.2. This leads to the following result.

Theorem 2.9. *Under the assumptions of Theorem 2.8, for any* $\varepsilon \in \mathcal{E} \cap (0, \varepsilon_0]$ *problem* (2.2)-(2.3) *has a solution of class* C^{r-1} *which can be represented as*

$$\eta(t, x_1) = -\varepsilon^2 \sinh(mh_0) \cos nt \cos(\pi m x_1/l) + \varepsilon^3 \eta_\varepsilon(t, x_1),$$

$$\vec{v}(t, x) = \varepsilon^2 \partial_t \theta(t, J_l^{-1} x) + \varepsilon^3 \vec{v_\varepsilon}(t, x)$$

such that

$$\|\eta_\varepsilon\|_{C^r(\mathbb{R}^2)} + \|\vec{v_\varepsilon}\|_{C^{r-1}(\mathcal{D}_t)} \leq c(g_0, \nu, h_0, r).$$

□

3 Scaling: reduction to an operator equation

Let $g_0 \in \mathcal{N}_0(\nu)$, $\nu \in (1, 2)$ and $h_0 \in \Sigma(g_0)$ be fixed throughout. Then let g, l, and $\varepsilon > 0$ be given in terms of new parameters λ, μ, ω_j, $j = 1, 2$, by

$$g = (1 + \varepsilon\lambda)(g_0 + \varepsilon\omega_1), \quad l = \pi(1 + \varepsilon\lambda), \quad \varepsilon^2 = \kappa\mu(1 + \omega_2)^{1/2}, \tag{3.1}$$

where $\kappa = 4(mn)^{-1}\sinh(2h_0 m)$. In what follows, the new small parameter $\mu > 0$, as well as g_0 and h_0, will be specified *a priori*. On the other hand, the parameters λ, ω_j, $j = 1, 2$, will be regarded as unknowns the values of which form part of the solution to the problem. Therefore, with $\varepsilon > 0$ regarded as a function of the fixed $\mu > 0$ and the variable ω_2, we seek a solution to the standing-wave problem in the form

$$x(t, \xi) = J_\lambda\big(\xi + \varepsilon^2\theta(t, \xi) + \varepsilon^3 X(t, \xi)\big), \quad J_\lambda = \mathrm{diag}\,\{1 + \varepsilon\lambda, 1\}. \tag{3.2}$$

Here $X(t,\xi)$ is a new unknown vector-function, ε is expressed in terms μ and ω_2 by (3.1), and θ is given by formula (2.20) in which (m,n) is the unique positive integer solution to the dispersion equation. In this notation $M_0 = J_\lambda M$ where $M = I + \varepsilon^2 M_\theta + \varepsilon^3 M_X$,

$$
\begin{aligned}
&M_\theta = \theta'_\xi, \;\; M_X = X'_\xi, \;\; A = (1+\varepsilon\lambda)|M|(M^* J_\lambda^2 M)^{-1}, \\
&a = |M|^{-1} M^* \left(\frac{J_\lambda}{1+\varepsilon\lambda} x_{tt} + (g_0 + \varepsilon\omega_1)\vec{e}_2 \right) \cdot \vec{e}_2.
\end{aligned}
\tag{3.3}
$$

We now define nonlinear differential operators $\Phi_i(X,\lambda,\omega_1,\omega_2)$ by putting

$$
\Phi_i(X,\lambda,\omega_1,\omega_2)(t,\xi) = \varepsilon^{-3} \mathcal{L}_i \left(J_\lambda(\xi + \varepsilon^2\theta(t,\xi) + \varepsilon^3 X(t,\xi)) \right). \tag{3.4}
$$

From this, (2.6), (2.7) and the identity (see (2.8) and (2.9))

$$
\partial_t |M_0| = (1+\varepsilon\lambda)\mathrm{div}\,\{S M^* S^{-1} J_\lambda^{-1} x_t\},
$$

the standing-wave problem is reduced to the following equations and boundary conditions:

$$
\Omega: \;\; \Phi_1 \equiv (1+\varepsilon\lambda)(\mathrm{div} X_t + \varepsilon\Phi_{10}) = 0, \quad \Phi_2 \equiv \mathrm{rot} X_t - 2\lambda\theta_{1,t\xi_2} + \varepsilon\Phi_{20} = 0, \tag{3.5}
$$

$$
\Gamma_0: \;\; \Phi_3 \equiv X_{1,tt} + g_0 X_{2,\xi_1} + \omega_1\theta_{2,\xi_1} + \lambda\theta_{1,tt} + \varepsilon\Phi_{30} = 0, \quad \Gamma_{h_0}: X_2 = 0. \tag{3.6}
$$

Here Φ_{j0} are nonlinear differential operators depending on the parameters λ, ω_j, μ:

$$
\Phi_{10} = \mathrm{div}\left\{ S(M_\theta + \varepsilon M_X)^* S^{-1}(\theta_t + \varepsilon X_t) \right\}, \tag{3.7}
$$

$$
\Phi_{20} = \mathrm{rot}\left\{ (M_\theta + \varepsilon M_X)^* J_\lambda^2 (\theta_t + \varepsilon X_t) \right\} - \lambda^2\theta_{1,t\xi_2} - (2\lambda + \varepsilon\lambda^2) X_{1,t\xi_2}, \tag{3.8}
$$

$$
\begin{aligned}
\Phi_{30} = {} & (M_\theta + \varepsilon M_X)^* J_\lambda^2 (\theta + \varepsilon X)_{tt} \cdot \vec{e_1} + \lambda^2\theta_{1,tt} + \lambda\omega_1\theta_{2,\xi_1} \\
& + (2\lambda + \varepsilon\lambda^2) X_{1,tt} + (\omega_1 + \lambda g_0 + \varepsilon\lambda\omega_1) X_{2,\xi_1}.
\end{aligned}
\tag{3.9}
$$

For fixed ε, the boundary-value problem (3.5)-(3.6) involves three unknown parameters λ, ω_i, $i = 1, 2$.

The next step is strategically important. We impose two additional constraints on the solutions and solve the constraint equations for ω_i, $i = 1, 2$ as functions of (X, λ, μ). These constraints have no obvious analogy in the theory of steady waves; here they are introduced to overcome the famous 'small divisor' problem in the mathematics of standing waves.

With a defined in terms of $(X, \lambda, \mu, \omega_1, \omega_2)$ by (3.1) and (3.3), let $\Psi_i(a)$ be defined by

$$
\Psi_1(a) \equiv \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta(\xi_1)^{-2} d\xi_1 \right)^{-1}, \quad \Theta(\xi_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{a(t,\xi_1,0)}\,dt, \tag{3.10}
$$

$$
\Psi_2(a) \equiv \frac{g_0}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(a_1(t,\xi_1)^2 - a_2(t,\xi_1)^2 \right) \Theta^{-3}(\xi_1) \sqrt{a(t,\xi_1,0)}\,dt\,d\xi_1, \tag{3.11}
$$

where

$$a_1(t,\xi_1) = \Theta(\xi_1)\partial_t \frac{1}{\sqrt{a(t,\xi_1,0)}}, \quad a_2(t,\xi_1) = \Theta^2(\xi_1)\partial_{\xi_1}\Big\{\frac{1}{\Theta(\xi_1)}\int_0^t \sqrt{a(\tau,\xi_1,0)}d\tau\Big\}.$$

Fix an arbitrary number τ, $1-\nu^{-1} < \tau < 1/24$ and let $\mathcal{M}(g_0,\tau)$ be the set in Proposition 2.4.

We now seek standing-wave solutions of (2.6)-(2.7) such that the corresponding function a (the normal component of pressure gradient at the free surface) satisfies the following two constraints:

$$\Psi_1(a) = g_0, \quad \Psi_2(a) = -\mu^2 \in \mathcal{M}(g_0,\tau). \tag{3.12}$$

Once it is proved that for sufficiently small μ with $-\mu^2 \in \mathcal{M}(g_0,\tau)$ equations (3.2)-(3.12) have solutions $X(t,\xi_1), \lambda, \omega_1, \omega_2$, with bounds that are independent of μ, then the existence of small amplitude standing water waves will have been established.

3.1 The operator equation

First we will show how to solve the constraint equations (3.12) for the parameters ω_i as functions of X, μ and λ.

Lemma 3.1. *There exist functionals $\omega_i(X,\lambda,\mu)$, defined when $\|X\|_{C^3(\Omega)^2} + |\lambda| \le 1$ and $\mu \in [0,\mu_1]$, such that*

$$\Psi_1(a(X,\lambda,\omega_i(X,\lambda,\mu))) \equiv g_0, \quad \Psi_2(a(X,\lambda,\omega_i(X,\lambda,\mu))) \equiv -\mu^2.$$

They are infinitely differentiable with respect to (X,λ) and continuous with respect to μ with $|\omega_i| \le c|\mu|^{1/2}$. Moreover, as a bounded linear functional on $C^3(\Omega)^2\times\mathbb{R}$, for fixed μ, the derivative $\|\omega'_{i,(X,\lambda)}\|_{L(C^3(\Omega)^2\times\mathbb{R},\mathbb{R})} \le c|\mu|^{1/2}$. The positive constants $|\mu_1|$, c depend on g_0 and h_0 only.

Proof. Note that the functions a, a_i have the following expansions

$$a = g_0 + \varepsilon\omega_1 + \varepsilon^2\phi + \varepsilon^3 A_0,\ a_1 = -\frac{\varepsilon^2}{2g_0}\phi_t + \varepsilon^3 A_1,\ a_2 = \frac{\varepsilon^2}{2}\frac{\partial}{\partial\xi_1}\int_0^t \phi dt + \varepsilon^3 A_2,$$

where $\phi = \theta_{2,tt} + g_0\theta_{2,\xi_2}$, and the A_j's are functions of X, λ, ω_j which are real-analytic in a neighbourhood of 0. It follows that $\Psi_1 = g_0 + \varepsilon\omega_1 + O(\varepsilon^2)$ and $\Psi_2 = -(\varepsilon^2/\kappa)^2 + O(\varepsilon^5)$. Substituting this expansions in (3.12) and replacing ε^2 by $\kappa\mu(1+\omega_2)^{1/2}$ we can rewrite these equations in the form

$$\Psi_1 - g_0 = \varepsilon(\omega_1 + \mu^{1/2}Q_1(X,\lambda,\omega_i,\mu)) = 0,$$

$$\Psi_2 + \mu^2 = -\mu^2(\omega_2 - \mu^{1/2}Q_2(X,\lambda,\omega_i,\mu)) = 0.$$

Thus we get the following equations for ω_i

$$\omega_1 + \mu^{1/2}Q_1 = 0,\ \omega_2 - \mu^{1/2}Q_2 = 0.$$

By construction the functionals Q_j are analytical with respect to X, λ and continuous with respect to μ on the set $\|X\|_{C^3(\Omega)} + |\lambda| \le 1$, $\mu \in [0,\mu_1]$, for suitable $\mu_1 > 0$, depending only on g_0, h_0. Hence the existence and the smoothness properties of the functionals ω_i follow from the implicit function theorem. $\square$

Now we are in a position to reduce the standing-wave problem to a nonlinear operator equation. Set $\vec{U} = (X, \lambda)$ and consider the mapping $\vec{U} \mapsto \Phi(\vec{U}, \mu)$ which, for fixed μ, assigns to $\vec{U}$ the collection of functions

$$\Phi(\vec{U}, \mu) = \{\Phi_1, \Phi_2, \Xi\Phi_3\}, \quad \Phi_i(\vec{U}, \mu) = \Phi_i(X, \lambda, \omega_j(X, \lambda, \mu)),$$

where $\Xi\Phi_3$ denotes the restriction of Φ_3 to Γ_0 and the Φ_i are defined in (3.5)-(3.9).

For any integer $r \geq 0$ let E_r denote the closed subspace of $\vec{U} = (X, \lambda) \in C^r(\Omega)^2 \times \mathbb{R}$ such that X is 2π-periodic with respect t and ξ_1 and even in t, X_1 is odd in ξ_1 and X_2, which is even in ξ_1, vanishes on Γ_{h_0}. Let $\|\vec{U}\|_{E_r} = \|X\|_{C^r(\Omega)^2} + |\lambda|$.

For any integer $\rho \geq 0$ let F_ρ denote the closed subspace of $C^\rho(\Omega)^2 \times C^\rho(\Gamma_0)$, which consists of all vector-valued functions $\vec{f} = (f_1, f_2, f_3)$ satisfying the following periodicity and symmetry conditions. The functions $f_1, f_2 : \Omega \to \mathbb{R}$ and $f_3 : \Gamma_0 \to \mathbb{R}$ are 2π-periodic with respect to t and ξ_1. The functions f_1, f_2 are odd in t, f_1 is even in ξ_1, f_2 is odd in ξ_1 and f_3 is even in t and odd in ξ_1.

From the definitions of E_r and Φ, it follows that any sufficiently smooth solution of

$$\Phi(\vec{U}, \mu) = 0 \tag{3.13}$$

is also a solution to a problem (3.2)-(3.6). The solvability of this nonlinear operator equation (3.13) is based on a version of Nash-Moser iteration.

3.2 Some Nash-Moser theory

The proof of the following abstract implicit function theorem on the solvability of a nonlinear operator equation in scales of Banach spaces will be given in an Appendix.

Denote by $(E_t, \|\cdot\|_t)$, $(F_s, |\cdot|_s)$, t, $s \geq 0$, two scales of Banach spaces. It is supposed that for t, $s \geq 0$ there exists $c(t,s)$ such that for any $t \leq s$ and $\lambda \in [0,1]$

(i) $\|u\|_t \leq c(t,s)\|u\|_s$, $\quad |f|_t \leq c(t,s)|f|_s$.

(ii) $\|u\|_{\lambda t+(1-\lambda)s} \leq c(t,s)\|u\|_t^\lambda \|u\|_s^{(1-\lambda)}$, $\quad |f|_{\lambda t+(1-\lambda)s} \leq c(t,s)|f|_t^\lambda |f|_s^{(1-\lambda)}$.

(iii) There exists a family of smoothing operators S_ϑ defined over the first scale such that for $\vartheta > 0$

$$\|S_\vartheta u\|_t \leq c(t,s)\|u\|_s, \quad \|S_\vartheta u\|_s \leq c(t,s)\vartheta^{t-s}\|u\|_t,$$
$$\|S_\vartheta u - u\|_t \leq c(t,s)\vartheta^{s-t}\|u\|_s.$$

Let ρ and r be fixed non-negative numbers and consider an operator Φ defined in a neighbourhood of 0 of the space E_r taking its values in F_ρ and depending on a small parameter μ with the following properties. There exist numbers

$$0 \leq k \leq \rho \leq r-1, \ R > 0, \ \mu_0 > 0, \quad 0 \leq 3\varrho \leq \gamma$$

and, for all $l \geq 0$, $c(l) > 0$ and $\mu(l) \in [0, \mu_0]$ such that

(a) For any $\mu \in [0, \mu_0]$ the operator $\Phi(\cdot, \mu)$ is twice continuously differentiable in the ball $B = \{u : \|u\|_r \leq R\}$. If $u \in B \cap E_{r+l}$ for some $l \geq 0$, then

$$|\Phi(u,\mu)|_{\rho+l} \leq c(l)(1 + \|u\|_{r+l}), \quad |\Phi(0,\mu)|_{\rho+l} \leq c(l)|\mu|^\gamma.$$

If $v,\, w \in E_{r+l} \cap B$, $l \geq 0$, then

$$|\Phi(u,\mu) - \Phi(v,\mu) - \Phi'_u(u,\mu)(u-v)|_{\rho+l} \leq c(l, \|u\|_{r+l}, \|v\|_{r+l})\|u-v\|^2_{r+l},$$

where c is bounded on bounded sets.

(b) There exists a family of bounded linear operators $\Lambda(u,\mu) : E_r \to F_\rho$ depending on $(u,\mu) \in B \times [0,\mu_0]$ which approximate Φ' as follows:

$$|\Lambda(u,\mu)v|_\rho \leq c(0)\|v\|_r.$$

If $u,\, v \in E_{r+l} \cap B$, $l \geq 0$, then

$$|\Lambda(u,\mu)v - \Phi'_u(u,\mu)v|_{\rho+l} \leq c(l)|\Phi(u,\mu)|_{\rho+l+1}\|v\|_{r+l}(1 + \|u\|_{r+l}).$$

(c) There is a set $\mathcal{M} \subset (0,\infty)$ with the property that 0 is a limit point of $\mathcal{M}$ and, for $l \geq 0$, $\mu \in (0,\mu(l)] \cap \mathcal{M}$, $u \in B \cap E_{r+l}$ and $f \in F_{\rho+l}$, the linear operator equation $\Lambda(u,\mu)v = f$ has at least one solution v satisfying the inequality

$$\|v\|_{\rho-k+l} \leq c(l)\mu^{-\varrho}\left(|f|_{\rho+l} + \|u\|_{r+l}|f|_\rho\right).$$

Theorem 3.2. A Nash-Moser Theorem. *Under the above assumptions there exists $\mu_1 > 0$ such that for any $\mu \in [0,\mu_1] \cap \mathcal{M}$ the equation $\Phi(u,\mu) = 0$ has a solution $u \in E_r$ satisfying the inequality $\|u\|_r \leq R$.* □

Thus solvability of (3.13) is reduced to showing that Φ satisfies the hypotheses of this theorem when $-\mu^2 \in \mathcal{M}(g_0,\tau)$ for some $g_0 \in \mathcal{N}_0(\nu)$, $\nu \in (1, 24/23)$ and $1 - \nu^{-1} < \tau < 1/24$.

4 Φ and its approximate derivative

In this section we define an operator Λ which approximates the Gateaux derivative of Φ and show that Φ and Λ satisfy conditions (a), (b) of the implicit function theorem. We start by constructing Λ. The partial derivative of Φ with respect to $\vec{U}$ at $\vec{U} = (X,\lambda)$ is given by

$$\Phi'_{\vec{U}}(\vec{U},\mu)\vec{V} = \Phi'_X(X,\lambda,\omega_i,\mu)Y + \Phi'_\lambda(X,\lambda,\omega_i,\mu)\sigma + \sum_{i=1,2}\Phi'_{\omega_i}(X,\lambda,\omega_i,\mu)\omega'_{i,\vec{U}}(X,\lambda,\mu)\vec{V},$$

for any vector $\vec{V} = (Y,\sigma)$, $Y \in C^2(\Omega)$, where the functionals ω_i are given by Lemma 3.1. A direct calculation shows that

$$\Phi'_\lambda(X,\lambda,\omega_i,\mu)\sigma = \sigma\{0, -2\theta_{1,t\xi_2}, \Xi\theta_{1,tt}\} + \varepsilon\sigma\mathcal{Q}_\lambda, \tag{4.1}$$

$$\Phi'_{\omega_i}(X,\lambda,\omega_i,\mu)\omega'_{i,\vec{U}}(X,\lambda,\mu)\vec{V} = (\omega'_{i,\vec{U}}(X,\lambda,\mu)\vec{V})\mathcal{Q}_{\omega_i}, \tag{4.2}$$

where

$$\mathcal{Q}_\lambda = \left\{\frac{\Phi_1}{1+\varepsilon\lambda}, \frac{\partial\Phi_{20}}{\partial\lambda}, \Xi\frac{\partial\Phi_{30}}{\partial\lambda}\right\}, \quad \mathcal{Q}_{\omega_i} = \left\{\frac{\partial\Phi_1}{\partial\omega_i}, \frac{\partial\Phi_2}{\partial\omega_i}, \Xi\frac{\partial\Phi_3}{\partial\omega_i}\right\}. \tag{4.3}$$

Now (recall Lemma 2.3) define linear differential operators $\Lambda_i = \Lambda_i(\vec{U}, \mu)$ by

$$\Lambda_1(\vec{U}, \mu)\vec{V} = \operatorname{div} w_\lambda, \quad \Lambda_2(\vec{U}, \mu)\vec{V} = \operatorname{rot}(A^{-1}w_\lambda) - 2\sigma\theta_{1,t\xi_2}, \tag{4.4}$$

$$\Lambda_3(\vec{U}, \mu)\vec{V} = \partial_t(A^{-1}w_\lambda \cdot \vec{e}_1) + \partial_{\xi_1}(a|M_0|M_0^{-1}J_\lambda Y) \cdot \vec{e}_2 + \sigma\theta_{1,tt}, \tag{4.5}$$

where a is given by (3.3) and now

$$w_\lambda = (|M_0|M_0^{-1}J_\lambda Y)_t = (SM_0^* S^{-1}J_\lambda Y)_t, \quad M_0 = J_\lambda M. \tag{4.6}$$

Here A, M, a and ε are expressed in terms X, λ, μ by (3.1)-(3.3) and Lemma 3.1.

With the $(\vec{U}, \mu)$-dependence understood let

$$\Lambda\vec{V} = \Lambda_0\vec{V} + \varepsilon\sigma\mathcal{Q}_\lambda + \sum_{i=1}^{2}(\omega'_{i,\vec{U}}(\vec{U}, \mu)\vec{V})\mathcal{Q}_{\omega_i}, \quad \Lambda_0 = \{\Lambda_1, \Lambda_2, \Xi\Lambda_3\}. \tag{4.7}$$

The proof that the operators Φ, Λ satisfy conditions (a), (b) of the implicit function theorem is based on two auxiliary lemmas. The first is an estimate on derivatives of composite functions from [5, Appendix].

Lemma 4.1. *For $k > 0$ let $D_k = \{p \in \mathbb{R}^d : \|p\| \le k\}$ and suppose $\vec{\varphi} : \Omega \to D_k$, $f : D_k \to \mathbb{R}$. Then for any $s \ge 0$ there exists a constant c depending only on s, k such that*

$$\|f \circ \vec{\varphi}\|_{C^s(\Omega)} \le c\left(\|f\|_{C^s(D)}\|\vec{\varphi}\|^s_{C^1(\Omega)} + \|f\|_{C^1(D)}\|\vec{\varphi}\|_{C^s(\Omega)} + \|f\|_{C(D)}\right).$$

In particular, for any functions $v_i \in C^s(\Omega)$, $i = 1, 2$,

$$\|v_1 v_2\|_{C^s(\Omega)} \le c(\|v_1\|_{C^s(\Omega)}\|v_2\|_{C(\Omega)} + \|v_2\|_{C^s(\Omega)}\|v_1\|_{C(\Omega)}).$$

□

Next we give estimates of the coefficients of Λ_i. Let B denote the unit ball in E_r.

Lemma 4.2. *Under the assumptions of Lemma 3.1 there exists $\mu_2 > 0$ such that whenever $|\mu| \le \mu_2$, $(X, \lambda) = \vec{U} \in B \cap E_{r+l}$, $l \ge 0$, $r \ge 3$,*

$$\|a^{\pm 1} - g_0^{\pm 1}\|_{C^{r+l-2}(\Omega)} + \|I - M^{\pm 1}\|_{C^{r+l-1}(\Omega)} \le |\mu|c(1 + \|\vec{U}\|_{E_{r+l}}), \tag{4.8}$$

$$\|I - A\|_{C^{r+l-1}(\Omega)} + \|I - A^{-1}\|_{C^{r+l-1}(\Omega)} \le |\mu|^{1/2}c(1 + \|\vec{U}\|_{E_{r+l}}), \tag{4.9}$$

where the constant c depends only on g_0,h_0, and r, l.

Proof. Denote by $\vec{\varrho} : \Omega \to \mathbb{R}^{40}$ the vector-function with components the partial derivatives of the components of X and θ with respect to (t, ξ_1, ξ_2) of order up to 2. Under the assumptions of the lemma, there exists k (depending, via θ, only on g_0 and h_0) such that $\vec{\varphi} \in D_k$ and $\|\vec{\varphi}\|_{C^{s-2}(\Omega)} \le c(s)(1 + \|\vec{U}\|_{E_s})$.

Using (3.1) and Lemma 3.1, choose μ_2 so small that $(2\sqrt{\kappa})^{-1}\varepsilon \le \mu^{1/2} \le 2\varepsilon/\sqrt{\kappa}$ and $|\omega_i| \le 2^{-1}$ for any $\mu \le \mu_2$. Since $M = I + \varepsilon^2 M_\theta + \varepsilon^3 M_X$ and since $\operatorname{Tr} M_\theta = 0$, $|M| \equiv f_M(\vec{\varphi}) = 1 + \mu^{3/2}P_M(\vec{\varphi})$, where P_M is a polynomial with uniformly bounded coefficients. Therefore there exists $\mu_2 > 0$ such that for any $\mu \le \mu_2$ the functions $f_M^{\pm 1}$ are real-analytic in the ball D_k and their derivatives have the bounds depending only on g_0

and h_0. Since f_M does not depends on the second derivatives of X, we conclude from this and Lemma 4.1 that

$$\||M|^{\pm 1}\|_{C^{r+l-1}(\Omega)} \le c(1 + \|\vec{U}\|_{E_{r+l}}). \tag{4.10}$$

Now note that inequality (4.8) for $I - M$ is a simple consequence of the formula for M, and that estimate (4.8) for $M^{-1} - I$ follows from that for $M - I$, (4.10) and the identity (see (2.9)) $M^{-1} - I = |M|^{-1}SM^*S^{-1}(I - M)$.

Estimate (4.9) on the matrix $I - A$ then follows from inequalities (4.8), (4.10) for M and the identity

$$A - I = S(M_0^* - I)S^{-1}M_0^{*-1} + (M_0^{-1} - I)^*,$$

since $\|M_0 - M\|_{C^{r+l-1}} \le c|\mu|^{1/2}(1 + \|\vec{U}\|_{E_{r+l}})$.

Inequality (4.9) for $A^{-1} - I$ is a consequence of that for the matrix $A - I$ and the identity $A^{-1} - I = (I - A)|M_0|^{-1}M_0^*M_0$.

Finally recall from Lemma 3.1 that, under the assumptions of the lemma, $\varepsilon\omega_1 \le c\mu$. From this and formula (3.3) for the function a we conclude that

$$a = f_a(\vec{\varphi}) = g_0 + \mu P_a(\vec{\varphi})(1 + \mu^{3/2}P_M(\vec{\varphi}))^{-1},$$

where P_a is a polynomial with uniformly bounded coefficients. Therefore we can choose $\mu_2 > 0$ such that for any $\mu \le \mu_2$ the functions $f_a^{\pm 1}$ are real-analytic in the ball D_k and their derivatives have bounds depending only on g_0 and h_0. An appeal to Lemma 4.1 completes the proof. $\square$

Proposition 4.3. *Let $2 \le \rho \le r - 3$, $l \ge 0$ and $\mu \in (0, \mu_2]$. Then the operator $\Phi : B \cap E_{r+l} \to F_{\rho+l}$ is infinitely differentiable and*

$$\|\Phi(0, \mu)\|_{\rho+l} \le c\mu^\gamma \quad \textit{with } \gamma = 1/2. \tag{4.11}$$

Whenever $\vec{U} \in B \cap E_{r+l}$,

$$\|\Phi\|_{F_{\rho+l}} \le c(1 + \|\vec{U}\|_{E_{r+l}}), \quad \|\Lambda\vec{V}\|_{F_{\rho+l}} \le c\|\vec{V}\|_{E_{r+l}}(1 + \|\vec{U}\|_{E_{r+l}}), \tag{4.12}$$

$$\begin{aligned}\|\Phi(\vec{U}, \mu) - \Phi(\vec{V}, \mu) - \Phi'_{\vec{U}}(\vec{U}, \mu)(\vec{U} - \vec{V})\|_{E_{\rho+l}}& \\ \le c\left(l, \|\vec{U}\|_{E_{r+l}}, \|\vec{V}\|_{E_{r+l}}\right)\|\vec{U} - \vec{V}\|^2_{E_{r+l}}.&\end{aligned} \tag{4.13}$$

$$\|\Lambda\vec{V} - \Lambda_0\vec{V}\|_{F_{\rho+l}} \le c\mu^{1/2}\|\vec{V}\|_{E_3}(1 + \|\vec{U}\|_{E_{r+l}}), \tag{4.14}$$

$$\|\Lambda\vec{V} - \Phi'_{\vec{U}}(\vec{U}, \mu)\vec{V}\|_{F_{\rho+l}} \le c\|\Phi(\vec{U}, \mu)\|_{F_{\rho+l+1}}\|\vec{V}\|_{E_{r+l}}(1 + \|\vec{U}\|_{E_{r+l}}). \tag{4.15}$$

Here the constant c depends only on r, l and does not depend on $\vec{U}$, and the function c is bounded on bounded sets.

Proof. The smoothness properties of the operator Φ is a consequence of its definition and Lemma 3.1. This gives (4.13). Formulae (3.7)-(3.9) show that $\|\Phi_i(0,0,\omega_j(0,0,\mu))\|_{C^{\rho+l}} \leq c\varepsilon \leq c\mu^\gamma$. This gives (4.11) Note from (3.5)-(3.9) that Φ is a polynomial of the vector-function $\vec{\varphi}$ defined in the proof of Lemma 4.2. Therefore estimates (4.12) on Φ is a consequence of Lemma 4.2. Formula (4.6) and standard estimates on a product of smooth functions imply that

$$\|w_\lambda\|_{C^{\rho+l-2}(\Omega)} \leq c(\|M\|_{C^{\rho+l-1}(\Omega)}\|Y\|_{C(\Omega)} + \|Y\|_{C^{\rho+l-1}(\Omega)}\|M\|_{C(\Omega)})$$

which, since $\|\vec{U}\|_{E_r} \leq 1$, gives

$$\|w_\lambda\|_{C^{\rho+l-2}(\Omega)} \leq c\|\vec{V}\|_{E_0}\|\vec{U}\|_{E_{\rho+l}} + c\|\vec{V}\|_{E_{\rho+l-1}}. \tag{4.16}$$

Therefore formulae (4.4) and (4.5) for the Λ_i and estimates on the product of smooth functions from Lemma 4.1 yield

$$\|\Lambda_i\vec{V}\|_{C^{\rho+l-3}(\Omega)} \leq c(\|w_\lambda\|_{C^{\rho+l-2}(\Omega)} + \|Y\|_{C^{\rho+l-2}(\Omega)} + |\sigma|)$$

$$+c(\|w_\lambda\|_{C(\Omega)} + \|Y\|_{C(\Omega)})(\||M_0|M_0^{-1}\|_{C^{\rho+l-2}} + \|A^{-1}\|_{C^{\rho+l-2}(\Omega)} + \|a\|_{C^{\rho+l-2}(\Omega)}).$$

From this (4.8), (4.9) and inequality (4.16) we finally obtain the estimate on $\Lambda_0 = \{\Lambda_1, \Lambda_2, \Xi\Lambda_3\}$,

$$\|\Lambda_0\vec{V}\|_{F_{\rho+l}} \leq c(\|\vec{V}\|_{E_{r+l}} + \|\vec{V}\|_{E_1}\|\vec{U}\|_{E_{r+l}}). \tag{4.17}$$

Next we estimate $\Lambda - \Lambda_0$ in (4.7). Note from (3.5), (3.6) and (4.3) that the vector-functions $\mathcal{Q}_\alpha$, where α represents ω_i or λ, are polynomials in $\vec{\varphi}$ with uniformly bounded coefficients. Hence $\|\mathcal{Q}_\alpha\|_{E_{r+l-2}} \leq c(1+\|\vec{U}\|_{E_{r+l}})$. From this Lemma 3.1 and formula (4.7) we conclude that

$$\|(\Lambda-\Lambda_0)\vec{V}\|_{F_{\rho+l}} \leq c\{\varepsilon|\sigma| + \sum_{i=1}^{2}|\omega'_{i,\vec{U}}\vec{V}|\}(\max_\alpha \|\mathcal{Q}_\alpha\|_{E_{r+l-2}}) \leq c\mu^{1/2}\|\vec{V}\|_{E_3}(1+\|\vec{U}\|_{E_{r+l}}),$$

which proves (4.14). Note that estimate (4.12) follows from (4.14) and inequality (4.17)

It remains to estimate the discrepancy $(\Lambda - \Phi')(\vec{U},\mu)$. Definition (3.4) and (3.1) imply that $\Phi'_{i,X}Y = \mathcal{L}'_{i,x}J_\lambda Y$, which along with Lemma 2.3 gives

$$(\Phi'_{\vec{U}} - \Lambda)(\vec{U},\mu)\vec{V} = \varepsilon^3\{0, \mathrm{rot}\,(R_2Y), \Xi R_3Y\}$$

where

$$R_2Y = -(\Phi_2S + \Phi_1A^{-1})M^{-1}Y, \quad R_3Y = \partial_t(R_2Y)\cdot\vec{e_1} + \partial_{\xi_1}(\Phi_3M^{-1}Y)\cdot\vec{e_1}.$$

Now (4.8), (4.9) and Lemma 4.1 gives (4.15) and the proposition follows. $\square$

5 Approximate right inverse of linearisation

Proposition 4.3 shows that the operators Φ and Λ satisfy conditions (a), (b) of the implicit function theorem. Thus we have reduced the proof of solvability of (3.13) to that for the linear equation $\Lambda\vec{V} = \vec{f}$. In this section we consider the truncated equation $\Lambda_0\vec{V} = \vec{f}$, $\vec{f} = (f_1, f_2, f_3) \in F_{\rho+l}$, which is equivalent to the boundary-value problem for $\vec{V} = (Y, \nu)$.

$$\Omega: \quad \operatorname{div} w = f_1, \quad \operatorname{rot}(A^{-1}w) - 2\sigma\theta_{1,t\xi_2} = f_2, \tag{5.1}$$

$$\Gamma_0: \quad \partial_t(A^{-1}w)\cdot\vec{e}_1 + \partial_{\xi_1}(a|M_0|M_0^{-1}J_\lambda Y)\cdot\vec{e}_2 + \sigma\theta_{1,tt} = f_3, \tag{5.2}$$

$$\Gamma_{h_0}: \quad w_2 = 0, \tag{5.3}$$

where the vector-valued function w is defined in terms Y by relation (4.6) (with $w_\lambda = w$), when $\vec{U} \in E_{r+l}$. We show that this problem is equivalent to an extended system which consists of an auxiliary elliptic problem and a scalar integro-differential equation. In order to formulate this equivalence we need some definitions, the most important of which is that of a Neumann operator. For $\vec{U} \in E_{r+l}$, let a and A be defined by (2.14). The well-posedness of problems described in the following discussion is established in Lemma 6.2 and Corollary 6.3.

Definition 5.1. *By the Neumann operator we mean the linear operator N defined on the space of smooth functions $u(t,\xi_1)$, which are 2π-periodic in both variables, by the formula*

$$Nu = A\nabla\varphi\cdot\vec{e}_2, \quad (t,\xi) \in \Gamma_0, \tag{5.4}$$

where φ is the solution to the boundary-value problem

$$\Omega: \quad \operatorname{div}(A\nabla\varphi) = 0, \tag{5.5}$$

$$\Gamma_0: \quad \varphi = u, \quad \Gamma_{h_0}: \quad A\nabla\varphi\cdot\vec{e}_2 = 0. \tag{5.6}$$

We introduce some auxiliary functions. First define a divergence-free vector field as $P(t,\xi) = n\sin nt\, S\nabla p(\xi)$ where

$$p(\xi) = \sin m\xi_1\left(\xi_2\cosh m(\xi_2+h_0) - \frac{\sinh m(\xi_2+h_0)}{m} + h_0\frac{\cosh m\xi_2}{\cosh mh_0}\right).$$

Thus $\operatorname{rot} P(t,\xi) = 2\theta_{1,t\xi_2}$, $P_2 \equiv 0$ on Γ_{h_0} and $P_1 \equiv 0$ on Γ_0. Let

$$\chi(t,\xi_1) = -n^2m^{-1}\cos nt\cos m\xi_1\ \cosh m(h_0+\xi_2) \text{ so that } \chi_{\xi_1} = -\theta_{1,tt}.$$

Let ψ_0 be the solution to the elliptic boundary-value problem

$$\Omega: \quad \operatorname{div}(SA^{-1}S^{-1}\nabla\psi_0) = \operatorname{rot}(P - A^{-1}P), \tag{5.7}$$

$$\Gamma_0: \quad SA^{-1}S^{-1}\nabla\psi_0\cdot\vec{e}_2 = (A^{-1}-I)P\cdot\vec{e}_1, \quad \Gamma_{h_0}: \quad \psi_0 = 0, \tag{5.8}$$

and introduce the function $q: \Gamma_0 \to \mathbb{R}$ defined in terms of ψ_0 by

$$\Gamma_0: \quad q = -P_2 - \psi_{0,\xi_1} + (a^{-1}\chi)_t = c_q\sin nt\cos m\xi_1 + q_0 \tag{5.9}$$

where $q_0 = -\psi_{0,\xi_1} + ((a^{-1} - g_0^{-1})\chi)_t$, and $c_q = \{nmh_0/\cosh mh_0\} \neq 0$.

Now we are in a position to formulate the extended system. Its first part is an elliptic boundary-value problem. (See [4] for an account of the general theory.)

Problem A. Find functions $\psi_1(t,\xi)$ and $\varphi_1(t,\xi)$ satisfying the following equation and boundary conditions

$$\Omega: \quad \mathrm{div}(A\nabla\varphi_1) = f_1, \tag{5.10}$$
$$\Gamma_0: \quad \varphi_1 = 0, \quad \Gamma_{h_0}: \quad A\nabla\varphi_1\cdot\vec{e}_2 = 0, \tag{5.11}$$

and

$$\Omega: \quad \mathrm{div}(SA^{-1}S^{-1}\nabla\psi_1) = f_2, \tag{5.12}$$
$$\Gamma_0: \quad SA^{-1}S^{-1}\nabla\psi_1\cdot\vec{e}_2 = 0, \quad \Gamma_{h_0}: \quad \psi_1 = 0. \tag{5.13}$$

Now define a 2π-periodic function $f:\Gamma_0\to\mathbb{R}$ in terms of solutions of Problem A by

$$\Gamma_0: \quad f = -\partial_{\xi_1}\psi_1 - A\nabla\varphi_1\cdot\vec{e}_2 + \frac{\partial}{\partial t}\left(\frac{1}{a}\int_0^{\xi_1} f_3 d\xi_1\right). \tag{5.14}$$

Recall that f_3 is odd in ξ_1.

Problem B. For q and f given by (5.9) and (5.14), Problem B is to find a function $u(t,\xi_1)$ (and, *a fortiori*, the function φ in the definition of the Neumann operator), and a parameter σ, satisfying the integro-differential equation

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial u}{\partial t}\right) + Nu = \sigma q + f. \tag{5.15}$$

Problems A and B, along with equations (5.5), (5.6) and formulae (5.9) and (5.14), form a system of equations for the functions φ_1, ψ_1, u, φ and the parameter σ. The next proposition shows that any solution to this system generates a solution to the boundary-value problem (5.1)-(5.3). Let $\Pi_t v = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(s,\xi)ds$.

Proposition 5.2. *Suppose that 2π-periodic-in-time C^3-functions φ_1, ψ_1 are solutions of Problem A, and that 2π-periodic-in-time C^3-functions u, φ, and a parameter $\sigma\in\mathbb{R}$, satisfy Problem B with $(f_1,f_2,f_3)\in F_1$. Let*

$$w = A(\nabla\varphi + \nabla\varphi_1) + S^{-1}(\nabla\psi_1 + \sigma\nabla\psi_0) + \sigma P, \tag{5.16}$$
$$W(t,\xi) = a^{-1}\{\varphi_t(t,\xi) - \sigma\chi(t,\xi) - \int_0^{\xi_1} f_3(t,s)ds\}. \tag{5.17}$$

Then the parameter σ and the vector field Y defined by the formula

$$J_\lambda Y = |M_0|^{-1}M_0(\partial_t^{-1}w - \Pi_t W\vec{e}_2), \tag{5.18}$$

satisfy equations and boundary conditions (5.1)-(5.3).

Proof. Since P is divergence-free and $\mathrm{div}\,(S\nabla\cdot)\equiv 0$,

$$\mathrm{div}\, w = \mathrm{div}\,(A\nabla\varphi + A\nabla\varphi_1) = f_1.$$

Hence w satisfies the first equation (5.1). Next

$$\mathrm{rot}(A^{-1}w) = \mathrm{div}\,\big(SA^{-1}S^{-1}(\nabla\psi_1 + \sigma\nabla\psi_0)\big) + \sigma\,\mathrm{rot}(A^{-1}P)$$

$$= f_2 + \sigma\,\mathrm{rot}(P - A^{-1}P) + \sigma\,\mathrm{rot}(A^{-1}P) = f_2 + \sigma\,\mathrm{rot}\,P = f_2 + 2\sigma\theta_{1,t\xi_2},$$

which yields the second equation (5.1). Let us check that w, Y satisfy the boundary condition (5.2). Using formulae (5.14) and (5.17), rewrite the integro-differential equation (5.15) in the form $W_t + w \cdot \vec{e}_2 = 0$. An application of ∂_t^{-1} to both sides and the identity $\partial_t^{-1}\partial_t W = W - \Pi_t W$ yields

$$\Gamma_0: \quad W + \partial_t^{-1} w \cdot \vec{e}_2 - \Pi_t W = 0.$$

Multiplying both sides by a and using the identity $\partial_t^{-1} w \cdot \vec{e}_2 - \Pi_t W = |M_0| M_0^{-1} J_\lambda Y \cdot \vec{e}_2$, which is a consequence of (5.18), we obtain

$$\Gamma_0: \quad u_t + a|M_0|M_0^{-1}J_\lambda Y \cdot \vec{e}_2 = \sigma\chi + \int_0^{\xi_1} f_3(t,s)ds.$$

Differentiating this equality and using identity $\chi_{\xi_1} = -\theta_{1,tt}$ yields

$$\partial_t(u_{\xi_1}) + \partial_{\xi_1}(a|M_0|M_0^{-1}J_\lambda Y).\vec{e}_2 + \sigma\theta_{1,tt} = f_3. \tag{5.19}$$

Note that formula (5.16) gives

$$A^{-1}w \cdot \vec{e}_1 = \varphi_{\xi_1} + \varphi_{1,\xi_1} - SA^{-1}S^{-1}(\nabla\psi_1 + \sigma\nabla\psi_0) \cdot \vec{e}_2 + \sigma A^{-1}P \cdot \vec{e}_1.$$

Since P_1 vanishes on Γ_0, we have $A^{-1}P \cdot \vec{e}_1 = (A^{-1} - I)P \cdot \vec{e}_1$ on Γ_0. From this, the previous equality and the boundary conditions (5.8) (5.11) and (5.13). we conclude that $A^{-1}w \cdot \vec{e}_1 = \varphi_{\xi_1} = u_{\xi_1}$ at Γ_0, which along with (5.19) gives the needed equation (5.2).

It remains to note that boundary condition (5.3) on Γ_{h_0} is the consequence of formula (5.16) and boundary conditions (5.6), (5.8), (5.11), (5.13). $\square$

6 Proof of the main result (Theorem 2.8)

In this section we show that the operator Λ defined by the formula (4.7) has a right inverse which satisfies condition (c) of the Nash-Moser theorem. This will complete the proof of Theorem 2.8.

We start with a proof of solvability of Problem A based on the following lemma. Let $Q(t,\xi)$ be a positive-definite, symmetric 2×2 matrix of class $C^{s+6}(\Omega)$ with elements Q_{ij} such that

$$\|Q\|_{C^6(\Omega)} + \|Q^{-1}\|_{C(\Omega)} \le c_Q,$$

Q_{ij} are even in t, 2π-periodic with respect to t, ξ_1, even in ξ_1 for even $i+j$, and odd in ξ_1 for odd $i+j$. Consider the following boundary-value problem

$$\Omega: \quad \mathrm{div}(Q\nabla v) = G, \tag{6.1}$$

$$\Gamma_0: \quad v = G_0, \quad \Gamma_{h_0}: \quad Q\nabla v \cdot \vec{e}_2 = G_{h_0}. \tag{6.2}$$

Lemma 6.1. *Under the above assumptions, for any $G \in C^{s+3}(\Omega)$, $G_i \in C^{s+3}(\Gamma_i)$, $s \ge 0$, $i \in \{0, h_0\}$, problem (6.1)-(6.2) has a unique solution satisfying the inequality*

$$\|v\|_{C^s(\Omega)} \le c(c_Q,s)(\|G\|_{C^{s+3}(\Omega)} + \|G_0\|_{C^{s+3}(\Gamma_0)} + \|G_{h_0}\|_{C^{s+3}(\Gamma_{h_0})})$$
$$+ c(c_Q,s)\|Q\|_{C^{s+6}(\Omega)}(\|G\|_{C^3(\Omega)} + \|G_0\|_{C^3(\Gamma_0)} + \|G_{h_0}\|_{C^3(\Gamma_{h_0})}).$$

This solution is 2π-periodic in t, ξ_1 and has the same oddness or evenness properties as G, G_0, G_{h_0}.

Proof. The proof is given in the Appendix. $\square$

Note that under the assumptions of Proposition 4.3 the matrices A and $SA^{-1}S^{-1}$ for any $\mu \in [0, \mu_2]$ and $\vec{U} \in B \cap E_{r+l}$ meet all requirements on Q in Lemma 6.1 with $r \geq 7$. From this and estimates on $A^{\pm 1}$ from Lemma 4.2 we get the following result on the existence and uniqueness of solutions to problems (5.5)-(5.6), (5.7)-(5.8), (5.10)-(5.11) and (5.12)-(5.13).

Lemma 6.2. *Let $\mu \leq \mu_2$, $10 \leq \rho \leq r-7$ and $\vec{U} \in B \cap E_{r+l}$. Then*

(i) Problem A has a unique solution which satisfies the inequalities

$$\|\varphi_1\|_{C^{\rho+l-3}(\Omega)} + \|\psi_1\|_{C^{\rho+l-3}(\Omega)} \leq c\|\vec{f}\|_{F_\rho}\|\vec{U}\|_{E_{r+l}} + c\|\vec{f}\|_{F_{\rho+l}}.$$

These functions are 2π-periodic with respect to t, ξ_1. The function φ_1 is odd in t and even in ξ_1 and the function ψ_1 is odd in t, ξ_1 .

(ii) Problem (5.5)-(5.6) *has a unique solution which is 2π-periodic with respect to t, ξ_1, has the same evenness and oddness properties as u, and satisfies the inequality*

$$\|\varphi\|_{C^{\rho+l-10}(\Omega)} \leq c\|u\|_{C^{\rho-7}(\Gamma_0)}\|\vec{U}\|_{E_{r+l}} + c\|u\|_{C^{\rho+l-7}(\Gamma_0)}.$$

(iii) Problem (5.7)-(5.8) *has a unique solution ψ_0 which satisfies*

$$\|\psi_0\|_{C^{\rho+l+1}(\Omega)} \leq c|\mu|^{1/2}(1 + \|\vec{U}\|_{E_{r+l}})$$

and is 2π-periodic with respect to t, ξ_1 and odd in t, ξ_1.

The constant c depends only on l, r. $\square$

Formula (5.9) for the function q_0 and (5.14) for f, along with estimates on the function a_{-1} from Lemma 4.2 and estimates on ψ_i from Lemma 6.2, imply the following.

Corollary 6.3. *Under the assumptions of Lemma 6.2*

$$\|q_0\|_{C^{\rho+l}(\Gamma_0)} \leq c|\mu|^{1/2}(1 + \|\vec{U}\|_{E_{r+l}}),$$
$$\|f\|_{C^{\rho+l-4}(\Gamma_0)} \leq c\|\vec{f}\|_{F_\rho}(1 + \|\vec{U}\|_{E_{r+l}}) + c\|\vec{f}\|_{F_{\rho+l}}. \quad \square$$

The proof of the existence of solutions to Problem B is based on the following theorem on solvability of the integro-differential equation (5.15). This result is one of the main results of this article.

Assume that the matrix A in the definition of the Neumann operator and functions a, q satisfy the following conditions.

(α) The matrix A has representation

$$A = (1 + \varepsilon\lambda)|M|(M^* J_\lambda^2 M)^{-1}, \quad J_\lambda = \operatorname{diag}\{1 + \varepsilon\lambda, 1\}, \tag{6.3}$$

where M is the Jacobi matrix of the mapping $\xi \mapsto \xi + \mu Z(\xi, t)$. We will suppose, for some $r \geq 25$ and $l \geq 0$, that $(Z, \lambda) \in E_{r+l}$, $\|(Z, \lambda)\|_{E_r} \leq c_0$ and $\varepsilon \leq c_0\mu^{1/2}$.

(β) The coefficients a and q are 2π-periodic with respect to both ξ_1 and t, a is even in both ξ_1 and t, and q is even in t and odd in ξ_1. Also, for some $c_0 > 0$,

$$\mu^{-1}\|g_0 - a\|_{C^{r-2}(\mathbb{R}^2)} + \|q\|_{C^s(\mathbb{R}^2)} \leq c_0, \quad \langle q, \vartheta\rangle \geq c_0^{-1} > 0.$$

Here $\vartheta = \sin nt \cos m\xi_1$, (m, n) is the solution to dispersion equation (2.21) with $g_0 \in \mathcal{N}_0(\nu)$ for some $\nu \in (1, 2)$, $h_0 \in \Sigma(g_0)$. (Here $\langle ., . \rangle$ denotes the scalar product in the Hilbert space $L_2([-\pi, \pi]^2)$.)

(γ) The coefficient a satisfies the relations $\Psi_1(a) = g_0$, $\Psi_2(a) = -\mu^2$, where $\Psi_j(a)$ is defined by (3.10)-(3.11).

Theorem 6.4. *Let hypotheses $(\alpha)-(\gamma)$ be satisfied with $\nu \in (1,4/3)$ where $5 \le s \le r-18$, $l \ge 0$, and suppose that $f \in C^{s+l}(\mathbb{R}^2)$ is a 2π-periodic function which is odd in t and even in ξ_1. Then there exists $\delta > 0$ such that, whenever $-\mu^2 \in \mathcal{M}(g_0,\tau) \cap (-\delta^2, 0)$ for $\tau \in (1-\nu^{-1}, 4^{-1})$, $\nu \in (1,4/3)$, the integro-differential equation (5.15) has a unique 2π-periodic solution such that u is odd in t, even in ξ_1, and $\langle u, \vartheta\rangle = 0$. Moreover*

$$|\sigma| + \|u\|_{C^{s-3+l}(\Gamma_0)} \le c\mu^{-4\tau}\left\{\|f\|_{C^{s+l}(\Gamma_0)} + \|f\|_{C^s(\Gamma_0)}\left(\|(Z,\lambda)\|_{E_{r+l}} + \|a\|_{C^{r+l-2}(\Omega)} + \|q\|_{C^{s+l}(\Gamma_0)}\right)\right\},$$

where the constant c depends only on c_0, r, l, g_0, h_0 and τ.

Proof. The proof begins in Section 7 and is finally completed at the end of Section 10.3. □

We are now in a position to prove the solvability of the equation $\Lambda\vec{V} = \vec{f}$

Proposition 6.5. *Suppose that $11 \le \rho \le r-14$, $\nu \in (1,2)$ and $\tau \in (1-\nu^{-1}, 8^{-1})$. Then there exists $\delta_1 > 0$ and $c > 0$, depending only on r, $l \ge 0$ and the parameters $g_0 \in \mathcal{N}_0(\nu)$ and $h_0 \in \Sigma(g_0)$, such that whenever $-\mu^2 \in \mathcal{M}(g_0,\tau) \cap (-\delta_1^2, 0)$ and $\vec{U} \in B \cap E_{r+l}$, the equation $\Lambda\vec{V} = \vec{f}$, for any $\vec{f} \in F_{\rho+l}$, has a solution $\vec{V} \in E_{\rho+l-11}$ satisfying the inequality*

$$\|\vec{V}\|_{E_{\rho+l-11}} \le c\mu^{-4\tau}(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}) \tag{6.4}$$

Proof. First we establish the existence of a solution to problem (5.1)-(5.3) which is equivalent to the truncated operator equation $\Lambda_0\vec{V} = \vec{f}$. By Proposition 5.2, the solution to (5.1)-(5.3) is given by formulae (5.16)-(5.18) and it is therefore sufficient to prove that the terms in these formulae are all well defined. Note that they are divided into two groups. The first consists of the functions φ_1, ψ_1 which solve Problem A. The second involves a solution u to Problem B and the function φ satisfying equations (5.5)-(5.6).

We begin by proving solvability of Problem B. It is sufficient to show that the integro-differential equation in Problem B meets all requirements of Theorem 6.4. To start observe that, by the formula for M preceding (3.3), the matrix A has the needed representation (6.3), in which M is the Jacobi matrix of the mapping $\xi \mapsto \xi + \varepsilon^2 Z$ in which $Z = \theta + \varepsilon X$. Since, by hypothesis, $\|X\|_{E_r} \le 1$ for $r \ge 25$ and, by (3.1) and Lemma 3.1, $|\varepsilon| \le 2\mu^{1/2}$ for any $\mu \le \mu_2$, we conclude that the matrix A satisfies condition (α) of Theorem 6.4.

Lemma 4.2 yields that for any $\mu \le \mu_2$ the function a satisfies condition (β) of this theorem. It follows from formula (5.9) for the coefficient q and estimate on the remainder q_0 from Corollary 6.3 that

$$\langle q, \vartheta\rangle = c_q\langle\vartheta,\vartheta\rangle - \langle q_0, \vartheta\rangle \ge \pi^2 c_q - c\mu^{1/2},$$

where the positive constants c_q and c depend only on r and (m,n). Hence for any small μ the coefficient q also satisfies condition (β) of Theorem 6.4 with $s = \rho - 4$.

Condition (γ) is satisfied automatically by the construction of the operator Φ. Therefore the integro-differential equation (5.15) satisfies all the condition of Theorem 6.4 with the constant c_0 depending only on r and the parameters g_0 , h_0. Hence there exists $\delta_2 > 0$ such that for any $\mu \in (0,\delta_2)$, $-\mu^2 \in \mathcal{M}(g_0,\tau)$ equation (5.15) has a unique 2π-periodic solution satisfying the orthogonality condition. Setting $s = \rho - 4$, the estimates on u from Theorem 6.4 and on f and q from Corollary 6.3 give that

$$|\sigma| + \|u\|_{C^{\rho+l-7}(\Gamma_0)} \le c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}\right). \tag{6.5}$$

In particular, when $l = 0$,

$$|\sigma| \le c|\mu|^{-4\tau}\|\vec{f}\|_{F_\rho} \tag{6.6}$$

From (6.5) and assertion (ii) of Lemma 6.2 we conclude that the boundary-value problem (5.5)-(5.6) has a unique solution which is 2π-periodic with respect to t, ξ_1 , odd in t, even in ξ_1 and satisfies the inequality

$$\|\varphi\|_{C^{\rho+l-10}(\Omega)} \le c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}\right). \tag{6.7}$$

Hence by Lemma 6.2 (i), (iii), we conclude that all the terms in formulae (5.17)-(5.18) for the vector field Y are well defined and, by Proposition 5.2, the vector $\vec{V} = (Y, \sigma)$ solves (5.1)-(5.3)

To estimate $\vec{V}$, Lemmas 4.1 and 4.2 with formula (5.18) imply that

$$\begin{aligned}\|Y\|_{C^{\rho+l-11}(\Omega)} \le c(\|w\|_{C^{\rho+l-11}(\Omega)} + \|W\|_{C^{\rho+l-11}(\Omega)})\\ + c\|\vec{U}\|_{E_{r+l}}(\|w\|_{C^{\rho-11}(\Omega)} + \|W\|_{C^{\rho-11}(\Omega)}).\end{aligned} \tag{6.8}$$

Also Lemma 4.2 gives an estimate for w in (5.16)

$$\begin{aligned}\|w\|_{C^{\rho+l-11}(\Omega)} \le c\|\varphi\|_{C^{\rho+l-10}(\Omega)} + c\|\varphi_1\|_{C^{\rho+l-10}(\Omega)} + c\|\psi_1\|_{C^{\rho+l-10}(\Omega)} + c|\sigma|\\ + c|\sigma|\|\psi_0\|_{C^{\rho+l-10}(\Omega)} + c\|A\|_{C^{\rho+l-11}(\Omega)}(\|\varphi\|_{C^1(\Omega)} + \|\varphi_1\|_{C^1(\Omega)}).\end{aligned}$$

Estimates on φ_1, ψ_i from Lemma 6.2, inequalities (6.7) for φ and (6.6) for σ now give

$$\|w\|_{C^{\rho+l-11}(\Omega)} \le c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|f\|_{E_\rho}\right). \tag{6.9}$$

Next the formula (5.17) for W yields the inequality

$$\begin{aligned}\|W\|_{C^{\rho+l-11}(\Omega)} \le c\|\varphi\|_{C^{\rho+l-10}(\Omega)} + c|\sigma| + c\|f_3\|_{C^{\rho+l-11}(\Omega)}\\ +c\|a^{-1}\|_{C^{\rho+l-11}(\Omega)}(\|\varphi\|_{C^1(\Omega)} + |\sigma| + \|f_3\|_{C(\Omega)}).\end{aligned}$$

From inequalities (6.7) for φ, (6.6) for σ and estimate on a^{-1} from Lemma 4.2, we conclude that

$$\|W\|_{C^{\rho+l-11}(\Omega)} \le c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}\right). \tag{6.10}$$

Substituting (6.9), (6.10) into (6.8) gives the following estimate on the solution $\vec{V} = (Y, \sigma)$ to boundary-value problem (5.1)-(5.3)

$$\|\vec{V}\|_{E_{\rho+l-11}} \le c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}\right). \tag{6.11}$$

Note that this problem has an infinite set of solutions. But formulae (5.16)-(5.18) select one, and only one, element of this set. Hence they define a linear operator $\mathcal{R}_0 : \vec{f} \to \vec{V}$ which assign to any vector $\vec{f} \in F_\rho$ the solution to problem (5.1)-(5.3) given by (5.16)-(5.18). It is clear that $\mathcal{R}_0$ is a right inverse of the operator Λ_0 and is well defined on the Banach space $F_{\rho+l}$ and that $\vec{V} = \mathcal{R}_0\vec{f}$ satisfies (6.11).

Let us now prove the solvability of the equation $\Lambda\vec{V} = \vec{f}$. First rewrite it in the form $\Lambda_0\vec{V} = (\Lambda_0 - \Lambda)\vec{V} + \vec{f}$. It is therefore sufficient to prove the solvability of

$$\vec{V} = \mathcal{R}_0(\Lambda_0 - \Lambda)\vec{V} + \mathcal{R}_0\vec{f}. \tag{6.12}$$

Estimate (4.14) on the norm $\Lambda_0 - \Lambda$ along with inequality (6.11) for $\vec{V} = \mathcal{R}_0\vec{f}$ give

$$\|\mathcal{R}_0(\Lambda_0 - \Lambda)\vec{V}\|_{E_{\rho-11}} \leq c\mu^{-4\tau}\|(\Lambda_0 - \Lambda)\vec{V}\|_{F_\rho} \leq c\mu^{1/2-4\tau}\|\vec{V}\|_{E_3}.$$

Since $\tau < 1/8$ and $\rho - 11 \geq 3$, the operator $\mathcal{R}_0(\Lambda_0 - \Lambda)$ is a contraction in the space $F_{\rho-11}$ for sufficiently small μ. Hence the existence of a solution to the equation $\Lambda\vec{V} = \vec{f}$ and estimate (6.4) for $l = 0$ follows from the contraction mapping principle.

If, in addition, $\vec{f} \in F_{\rho+l}$, then by estimate (6.11),

$$\begin{aligned}\|\mathcal{R}_0(\Lambda_0 - \Lambda)\vec{V} + \mathcal{R}_0\vec{f}\|_{E_{\rho+l-11}} \leq{}& c\mu^{-4\tau}\|\vec{U}\|_{E_{r+l}}(\|(\Lambda_0 - \Lambda)\vec{V}\|_{F_\rho} + \|\vec{f}\|_{F_\rho}) \\ &+ c\mu^{-4\tau}(\|(\Lambda_0 - \Lambda)\vec{V}\|_{F_{\rho+l}} + \|\vec{f}\|_{F_\rho+l}).\end{aligned}$$

Inequalities (4.14) and (6.4) with $l = 0$ show that the solution to equation (6.12) satisfy the inequalities

$$\|(\Lambda_0 - \Lambda)\vec{V}\|_{F_{\rho+l}} \leq c\mu^{1/2}(1 + \|\vec{U}\|_{E_{r+l}})\|\vec{V}\|_{E_{\rho-11}}, \quad \|\vec{V}\|_{E_{\rho-11}} \leq c\mu^{-4\tau}\|\vec{f}\|_{F_\rho}.$$

Since $\tau < 1/8$, substituting these relations into the previous estimate gives

$$\|\mathcal{R}_0(\Lambda_0 - \Lambda)\vec{V} + \mathcal{R}_0\vec{f}\|_{E_{\rho+l-11}} \leq c\mu^{-4\tau}\left(\|\vec{f}\|_{F_{\rho+l}} + \|\vec{U}\|_{E_{r+l}}\|\vec{f}\|_{F_\rho}\right),$$

which implies (6.4) for any $l \geq 0$ and the proposition follows. □

Therefore the operator $\Phi(\vec{U}, \mu)$ satisfies the hypotheses of Theorem 3.2

Proof of Theorem 2.8. Note that equations (3.5), (3.6) and (3.12) along with formulae (3.1), (3.2) form a boundary-value problem for a vector-valued function $X(t, \xi)$ and parameters l, g, ε. It follows from Lemma 3.1 that for any $\mu \in [0, \mu_1]$ this problem in the scales of Banach spaces E_r, F_ρ is equivalent to the operator equation (3.13). Hence it is sufficient to prove the existence of non-trivial solutions of this equation. First note that these scales of Banach spaces have been proved to satisfy the hypotheses (i)-(iii) in [5, Appendix].

Now fix integers $r \geq 25$, $\rho \in [14, r - 11]$ and the exponents $\gamma = 1/2$, $\varrho = 4\tau$ and $1 - \nu^{-1} < \tau < 1/24$. Set $\mu_0 = \min\{\mu_2, \delta_1\}$, where $\mu_2 \leq \mu_1$ and δ_1 are the numbers from Propositions 4.3 and 6.5.

Now inequalities (4.11)-(4.13) in Proposition 4.3 give that the operator $\Phi(\vec{U}, \mu)$ satisfies condition (a) of the implicit function theorem. Also, estimates (4.14) and (4.15) from Proposition 4.3 shows that the operator $\Lambda(\vec{U}, \mu)$ satisfies condition (b) of the same theorem. Next note that, under the assumptions of Theorem 2.8, the operator Λ meets all requirements of Proposition 6.5. It follows that, for $\mu \leq \mu_0 \leq \delta_1$, $-\mu^2 \in \mathcal{M}(g_0, \tau)$ and $\vec{U} \in B \cap E_{r+l}$, the operator $\Lambda(\vec{U}, \mu)$ has a right inverse satisfying condition (c) of the implicit function theorem with the exponent $k = 11$.

Applying the theorem now shows that for all μ sufficiently small with $-\mu^2 \in \mathcal{M}(g_0, \tau)$, the operator equation (3.13) has a solution (X, λ) with $|X|_{C^r(\Omega)} + |\lambda| \leq 1$. It remains only to note that for any twice continuously differentiable solution to (3.13) the vector-valued function $x(t, \xi)$ satisfies equations (2.6), (2.7) and $\varepsilon \to 0$ as $\mu \to 0$. □

7 The Neumann operator

The rest of the paper is devoted to a proof of Theorem 6.4, which we try to make as self-contained as possible. Recall from Definition 5.1 that the Neumann operator is defined for any doubly 2π-periodic function $u(t,\xi_1)$ by

$$Nu = A\nabla\varphi\cdot\vec{e}_2, \quad (t,\xi)\in\Gamma_0,$$

where φ is a solution to the elliptic boundary-value problem

$$\Omega: \quad \mathrm{div}(A\nabla\varphi)=0, \quad \Gamma_0: \quad \varphi=u, \quad \Gamma_{h_0}: \quad A\nabla\varphi\cdot\vec{e}_2=0. \tag{7.1}$$

We intend to show how to extract the main part, estimate a remainder term and examine the problem of changing independent variables in the Neumann operator N. (It is known that changing the independent variables in pseudo-differential operators yields operators which consist of a main term given by an explicit formula and a remainder term of lower order. Here however the Neumann operator is not pseudo-differential, and the problem of changing independent variables is more delicate. Remainders may be unbounded even when the initial operator is bounded, and the notion of "lower order" is not so simple.) We use a notion of quasi-one-dimensionality to explain our results.

Henceforth the following notation will be used. For any non-negative integer s we denote by H_s the Hilbert space of 2π-periodic functions $u:\mathbb{R}^2\to\mathbb{R}$ with the norm

$$\|u\|_s = \Big\{\sum_{\zeta\in\mathbb{Z}^2}(1+|\zeta|)^{2s}|\mathbf{F}u(\zeta)|^2\Big\}^{1/2}, \quad \mathbf{F}u(\zeta)=\frac{1}{4\pi^2}\int\limits_{[-\pi,\pi]^2}\exp(-i\zeta\cdot y)u(y)dy,$$

and by H_s^- the subspace of H_s which consists of all functions $u(y)$, $y=(y_1,y_2)$, which are odd in y_1 and even in y_2.

Let C^s be the subspace of $C^s(\mathbb{R}^2)$ which consists of all 2π-periodic functions and write $|u|_s$ for the usual 'sup-norm' $\|u\|_{C^s(\mathbb{R}^2)}$ of functions $u\in C^s$.

The Hilbert transform K and what we have called (at the end of Section 1) the finite-depth Hilbert transform $K(h)$, with respect to the second variable y_2, are defined in terms of the Fourier transform $\mathbf{F}$ in the space H_0 by the relations

$$\mathbf{F}Ku(\zeta) = -i\,(\mathrm{sgn}\zeta_2)\mathbf{F}u(\zeta), \quad \mathbf{F}K(h)u(\zeta) = -i\,(\mathrm{sgn}\zeta_2)\tanh(h|\zeta_2|)\mathbf{F}u(\zeta).$$

They have the integral representations

$$Ku(y_1,y_2)=\int_{-\pi}^{\pi}k(y_2-s)u(y_1,s)ds, \quad K(h)u(y_1,y_2)=\int_{-\pi}^{\pi}k(y_2-s,h)u(y_1,s)ds,$$

where $k(s)=(2\pi)^{-1}\cot(s/2)$ and $k(s,h)=k(s)+k_1(s,h)$, where $k_1(\cdot,h)$ is a real-analytic, odd, 2π-periodic function. The operators D_j^{-1} and $|D_j|^s$, $j=1$, 2, are are defined in terms of the Fourier transform by

$$\mathbf{F}(D_j^{-1}u)(\zeta)=(i\zeta_j)^{-1}\mathbf{F}u(\zeta) \ \text{ for } \ \zeta_j\neq 0, \quad \mathbf{F}(D_j^{-1}u)(\zeta)=0 \ \text{ for } \ \zeta_j=0,$$

$$\mathbf{F}(|D_j|^s u)(\zeta)=|\zeta_j|^s\mathbf{F}u(\zeta) \ \text{ for } \ \zeta_j\neq 0, \quad \mathbf{F}(|D_j|^s u)(\zeta)=0 \ \text{ for } \ \zeta_j=0,$$

and operators such as $(1+|D_j|)^\alpha$ are defined analogously in terms of Fourier transforms. For periodic functions $v(y_1,y_2)$ and $u(t,\xi_1)$ let

$$\Pi_i v(y_j)=(2\pi)^{-1}\int_{-\pi}^{\pi}v(y_1,y_2)dy_i,\ i\neq j,\quad \Pi_t u(\xi_1)=(2\pi)^{-1}\int_{-\pi}^{\pi}u(t,\xi_1)dt.$$

Thus $\Pi_t\partial_t^{-1}u(\xi_1)\equiv 0\equiv \Pi_j D_j^{-1}v(y_i)$. The following result is both suggestive and useful.

Lemma 7.1. *In the notation of Section 3 suppose, for each t, that $A=|M_0|(M_0^*M_0)^{-1}$ where $M_0(t,\xi)$ is the Jacobian of the transformation $\xi\mapsto x(t,\xi)$. Suppose also that, for each t, $\operatorname{div}(A\nabla u)=0$ in Ω_0. Then u is a harmonic function of x on $x(t,\Omega_0)$ for each t.*

Proof. Let v be infinitely differentiable with compact support in Ω. Then

$$\begin{aligned}0&=\int_{\Omega_0}v\operatorname{div}(A\nabla u)d\xi=-\int_{\Omega_0}\nabla v\cdot A\nabla u d\xi\\&=-\int_{\Omega_0}|M_0|(M_0^*)^{-1}\nabla v\cdot(M_0^*)^{-1}\nabla u d\xi=-\int_{x(t,\Omega_0)}\nabla_x v\cdot\nabla_x u\,dx,\end{aligned}$$

and the result follows. □

Eventually we will require $r\geq 25$, so there is no loss of generality in assuming that from the outset.

7.1 Decomposition of the Neumann operator N

Theorem 7.2. *In the notation of Section 3, suppose that*

$$A(t,\xi)=(1+\varepsilon\lambda)|M|(M^*J_\lambda^2M)^{-1},$$

where M is the Jacobi matrix of the mapping $\xi\mapsto\xi+\varepsilon^2Z(t,\xi)$, $Z=\theta+\varepsilon X$ and X is the first component of $\vec U=(X,\lambda)$, $\|\vec U\|_{E_r}\leq 1$. Then there exists $\delta_1>0$ such that whenever $\varepsilon^2\leq\delta_1$ the operator N is given by $N=\partial_{\xi_1}K(h_\lambda)+R_0$, where $h_\lambda=(1+\varepsilon\lambda)^{-1}h_0$, $K(h_\lambda)$ is a finite-depth Hilbert transform and R_0 is the Fredholm operator

$$R_0u=\int_{-\pi}^{\pi}r_0(t,\xi_1,s)u(t,s)ds.$$

The kernel r_0 is even in all its variables separately, 2π-periodic, symmetric in the sense that $r_0(t,\xi_1,s)=r_0(t,s,\xi_1)$ and, if $\vec U\in E_{l+r}$, $l\geq 0$, then $\|r_0\|_{C^{r+l-4}(\mathbb{R}^3)}\leq c(r,l)\varepsilon^2(1+\|\vec U\|_{E_{r+l}})$.

Proof. Note from (7.1) that $A\nabla\varphi$ is a time-dependent solenoidal vector field on Ω_0. Therefore a 'stream function' $\psi(t,\xi)$, which vanishes at Γ_{h_0}, is defined in terms of the 'potential' φ by the relation $A\nabla\varphi=S\nabla\psi$. Green's formula applied to problem (7.1) shows that ψ is 2π-periodic with respect to ξ_1, t. Thus $Nu=-\partial_{\xi_1}\psi(t,\xi_1,0)$. Now consider ξ as a complex variable $\xi=\xi_1+i\xi_2$ and define a complex function on Ω_0 in the complex ξ-plane by $\chi(\xi)=i(\varphi+i\psi)(\xi)$.

Let $x(t,\xi)=J_\lambda(\xi+\varepsilon^2Z(t,\xi))$ and note that $x(t,\xi+2\pi\vec e_1)=x(t,\xi)+2\pi(1+\varepsilon\lambda)\vec e_1$. Then, for ε sufficiently small $(1+\varepsilon\lambda)^{-1}x(t,\cdot)$ can be regarded as a diffeomorphism from

Ω_0 in the complex ξ-plane onto a curvilinear strip D_t bounded in the complex ζ-plane by the line $Im\,\zeta = -h_\lambda$ and a periodic curve defined in parametric form by the equation

$$\zeta = \xi_1 + \varepsilon^2 Z_1(t,\xi_1) + i(1+\varepsilon\lambda)^{-1}\varepsilon^2 Z_2(t,\xi_1) \equiv \gamma(t,\xi_1).$$

It follows from Lemma 7.1 and equation (7.1) that $\chi(t,\zeta) \equiv \chi(t,\xi(t,\zeta))$ is a holomorphic function of the variable ζ in the strip D_t. At the upper curve this function equals $-\psi + iu$. Since the real part of χ vanishes at the bottom of D_t, the function χ has a holomorphic extension across the line $Im\,\zeta = -h_\lambda$ onto the curvilinear strip bounded by the curves $\zeta = \gamma(t,\xi_1)$ and $\zeta = -2ih_\lambda + \bar{\gamma}(t,\xi_1)$. Note that, in the extended domain, χ has the property that $\chi(t,\zeta) = -\overline{\chi}(t, -2ih_\lambda + \bar{\zeta})$.

The conformal mapping $z = e^{i\zeta}$ takes this strip onto a ring-like region in the z-plane bounded by $l_0 \equiv \{z = e^{i\gamma(t,\xi_1)} : \xi_1 \in [-\pi,\pi]\}$ and $l_{h_\lambda} \equiv \{z = e^{2h_\lambda + i\bar{\gamma}(t,\xi_1)} : \xi_1 \in [-\pi,\pi]\}$. Note that increasing ξ_1 corresponds to l_i in the positive (anti-clockwise) direction. Cauchy's integral formula for a point z inside the ring gives

$$\chi(t,\zeta(z))) = -\frac{1}{2\pi i}\int_{l_0} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z} + \frac{1}{2\pi i}\int_{l_{h_\lambda}} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z}.$$

If $z \to z_0 = e^{i\gamma(t,\xi_1)} \in l_0$, then

$$\frac{1}{2\pi i}\int_{l_0} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z} \to -\frac{1}{2}\chi(t,\zeta(z_0)) + \frac{1}{2\pi i}\int_{l_0} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z_0}.$$

From this, and the previous equality we conclude that

$$-\chi(t,\zeta(z_0)) = \frac{1}{\pi i}\int_{l_0} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z_0} - \frac{1}{\pi i}\int_{l_{h_\lambda}} \frac{\chi(t,\zeta(\eta))d\eta}{\eta - z_0}.$$

The definition of γ and the symmetry of χ now give that $-\chi(t,\xi_1)$ equals

$$\frac{1}{\pi i}\int_{-\pi}^{\pi} \frac{[e^{i\gamma(t,s)}]'_s}{e^{i\gamma(t,s)} - e^{i\gamma(t,\xi_1)}}\chi(t,s)ds + \frac{1}{\pi i}\int_{-\pi}^{\pi} \frac{e^{2h_\lambda}[e^{i\bar{\gamma}(t,s)}]'_s}{e^{2h_\lambda + i\bar{\gamma}(t,s)} - e^{i\gamma(t,\xi_1)}}\bar{\chi}(t,s)ds.$$

Hence

$$-\chi(t,\xi_1) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{\chi(t,s)}{1 - e^{i(\xi_1 - s)}} + \frac{\bar{\chi}(t,s)}{1 - e^{-2h_\lambda + i(\xi_1 - s)}}\right\}ds + \int_{-\pi}^{\pi}(\rho_1\chi + \rho_2\bar{\chi})ds. \tag{7.2}$$

Here the kernels ρ_j are defined by the formulae

$$\pi\rho_1(t,\xi_1,s) = \frac{\gamma'_s(t,s)}{1 - e^{i\gamma(t,\xi_1) - i\gamma(t,s)}} - \frac{1}{1 - e^{i(\xi_1 - s)}}, \tag{7.3}$$

$$-\pi\rho_2(t,\xi_1,s) = \frac{\bar{\gamma}'_s(t,s)}{1 - e^{-2h_\lambda + i\gamma(t,\xi_1) - i\bar{\gamma}(t,s)}} - \frac{1}{1 - e^{-2h_\lambda + i(\xi_1 - s)}}.$$

Note that

$$\begin{aligned} Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\chi(t,s)ds}{1 - e^{i(\xi_1 - s)}}\right\} &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t,s)ds - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\left(\frac{\xi_1 - s}{2}\right)u(t,s)ds \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t,s)ds - Ku(t,\xi_1). \end{aligned}$$

Equating real parts in (7.2) gives a Fredholm equation for ψ

$$(1+P_0)\psi + \Gamma_3\psi = (P_1 - \Gamma_4 - K)u, \tag{7.4}$$

where the Poisson operators P_j and Fredholm operators Γ_j are given by

$$P_0\psi = \frac{1}{\pi}\int_{-\pi}^{\pi} Re\left\{\frac{1}{2} + \frac{1}{1-e^{-2h_\lambda + i(\xi_1 - s)}}\right\}\psi(t,s)ds, \quad \Gamma_j u = \int_{-\pi}^{\pi} \rho_j(\xi_1, s, t)u ds,$$
$$P_1\psi = \frac{1}{\pi}\int_{-\pi}^{\pi} Im\left\{\frac{1}{1-e^{-2h_\lambda + i(\xi_1 - s)}}\right\}u(t,s)ds,$$

with $\rho_3 = Re(\rho_1 + \rho_2)$ and $\rho_4 = Im(\rho_1 - \rho_2)$. It follows from (7.3) and the estimates of compositions of smooth functions that there exists $\delta > 0$ (independent of l) such that, whenever $\varepsilon^2 \le \delta$ and $X \in C^{r+l}$, $l \ge 0$,

$$\|\rho_j\|_{C^{r+l-2}(\mathbb{R}^3)} \le \varepsilon^2 c(r,l)(|X|_{r+l} + 1). \tag{7.5}$$

For $\varepsilon = 0$ the operators $\Gamma_j = 0$ and D_t coincides with the strip $-h_\lambda < Im\,\zeta < 0$. In this case $u, -\psi$ are related by the finite-depth Hilbert transform. Therefore the function $\psi = -K(h_\lambda)u$ is the explicit solution (see (1.4)) to equation (7.4) with $\Gamma_j = 0$. It follows from (7.4) that

$$\psi = -K(h_\lambda)u + v, \quad (1+P_0)v + \Gamma_3 v = [\Gamma_3 K(h_\lambda) - \Gamma_4]u. \tag{7.6}$$

The last equality can be regarded as an integral equation for the function v. The proof of its solvability is based on the following two lemmas. The first is proved in the Appendix.

Lemma 7.3. *The operator $I + P_0$ is invertible in the space of 2π-periodic functions and*

$$((1+P_0)^{-1} - 1)u \equiv P_2 u = \int_{-\pi}^{\pi} p_2(\xi_1 - s, h_\lambda)u(t,s)ds.$$

Here the kernel $p_2(s, h_\lambda)$ is real-analytic and 2π-periodic. □

Now rewrite equation (7.6) as

$$v + A_1 v = A_2 u, \tag{7.7}$$

where $A_1 = (I + P_2)\Gamma_3$ and $A_2 = A_1 K(h_\lambda) - (I + P_2)\Gamma_4$. Now we estimate the kernel of A_i.

Lemma 7.4. *Under the assumptions of Theorem 7.2, δ can be chosen so that*

$$A_j v = \int_{-\pi}^{\pi} a_j(\xi_1, s, t)v(t,s)ds \quad \textit{where} \quad \|a_j\|_{C^{r+l-3}(\mathbb{R}^3)} \le c(r,l)\varepsilon^2(|X|_{r+l} + 1). \tag{7.8}$$

Proof. By (7.5), the kernel of the operator Γ_3 satisfies inequality (7.8) with $r + l - 3$ replaced by $r + l - 2$. The kernel of the operator $P_2\Gamma_3$ is given by the formula

$$\int_{-\pi}^{\pi} p_2(\xi_1 - s_1, h_\lambda)\rho_3(s_1, s, t)ds_1.$$

The required estimates are then a consequence of (7.3), (7.5) and estimates on convolutions of smooth functions. Hence the kernel a_1 satisfies inequality (7.8) with $r+l-3$ replaced by $r+l-2$. Replacing ρ_3 by ρ_4 gives a similar estimate for the kernel of operator $(I+P_2)\Gamma_4$. It remains to show that $A_1K(h_\lambda)$ is a Fredholm operator and the C^{r+l-3}-norm of its kernel does not exceed the right side of (7.8). Recall that $K(h_\lambda) = K + K_1(h_\lambda)$ where $K_1(h_\lambda)$ is a smoothing operator with real-analytic kernel. The estimate of the kernel of A_1K_1 is obvious. Now the kernel of A_1K is $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cot((s-s_1)/2)\, a_1(\xi_1, s_1, t)ds_1$. Since for non-integer $r+l$ the Hilbert transform is a bounded operator in the space C^{r+l} of 2π-periodic functions, the needed C^{r+l-3} estimate of the kernel of A_1K follows from the C^{r+l-2} estimate for a_1. □

The rest of the proof of Theorem 7.2 follows from the next assertion, the proof of which is in the Appendix.

Lemma 7.5. *Suppose that the kernel $a_1 \in C^{r+l-3}(\mathbb{R}^3)$ satisfies the inequality $|a_1| \le \varepsilon^2 c_0$. Then there exists $\delta_1 > 0$ depending only on r, l, and c_0 such the equation $v + A_1 v = f$ has a unique solution*

$$v(t,\xi_1) = f(t,\xi_1) + \int_{-\pi}^{\pi} r_1(t,\xi_1,s) f(t,s)ds$$

for any $f \in C^0$ and $\varepsilon^2 \in [0,\delta_1]$. The kernel r_1 is 2π-periodic, even and $\|r_1\|_{C^{r+l-3}(\mathbb{R}^3)} \le c(r,l,c_0)\|a_1\|_{C^{r+l-3}(\mathbb{R}^3)}$. □

Now by Lemma 7.4, choose $\delta_1 > 0$ such that a_1 meets all requirements to Lemma 7.5 for $\varepsilon^2 \le \delta_1$ and the corresponding kernel r_1 satisfies the inequality $\|r_1\|_{C^{r+l-3}(\mathbb{R}^3)} \le c(r,l)\varepsilon^2(|X|_{r+l-3}+1)$. Hence

$$v = \int_{-\pi}^{\pi} r_2(t,\xi_1,s)u(t,s)ds \quad \text{and} \quad r_2(t,\xi_1,s) = a_2(t,\xi_1,s) + \int_{-\pi}^{\pi} r_1(t,\xi_1,s_1)a_2(t,s_1,s)ds_1.$$

Since $\psi = -K(h_\lambda)u + v$ and $Nu = -\partial_{\xi_1}\psi$, we conclude that the operator N has the needed representation in which the Fredholm operator R_0 has kernel $r_0 = -\partial_{\xi_1} r_2$. The estimates on r_0 follow from the estimates on r_1 and a_2. Since N and $\partial_{\xi_1}K(h)$ are symmetric operators in H_0, the kernel of R_0 is also symmetric and the theorem follows.

□

7.2 Quasi-one-dimensional operators

This section introduces the notion of quasi-one-dimensional operators which will be used to treat changes of variables in N. For any real α, β denote by $G_{\alpha,\beta}$ the operator on H_s defined by $G_{\alpha,\beta} = (1+|D_1|)^{-\alpha}(1+|D_2|)^{\beta}$.

Definition 7.6. *For any non-negative exponents α, β we say that R is a quasi-one-dimensional (Q1D) operator of order (α,β) in H_s if*

$$|||R|||_{\alpha,\beta,s} \equiv \sup_{\|u\|_{H_s}=1} \|G_{\alpha,\beta}Ru\|_s < \infty. \quad \square$$

It follows from this definition that if R is a Q1D operator of order (α, β) in H_s then, for any $\alpha' \geq \alpha$ and $\beta' \leq \beta$, the operator R has order (α', β') and $|||R|||_{\alpha',\beta',s} \leq |||R|||_{\alpha,\beta,s}$. Note also that KR, $\Pi_j R$ and $D_i^{-1}R$ are Q1D operator with the same $|||\cdot|||_{\alpha,\beta,s}$-norm as R and that $|||D_1^p D_2^q R|||_{\alpha,\beta,s} \leq |||R|||_{\alpha-p,\beta+q,s}$, for any p, $q \geq 0$. The next result is more complicated.

Proposition 7.7. *If R is a Q1D operator of order (α, β) in H_s, then*

$$|||aR|||_{\alpha,\beta,s} \leq c(\alpha, \beta, s)|a|_l|||R|||_{\alpha,\beta,s},$$

for any 2π-periodic function $a(y_1, y_2)$ of class C^l and for any exponent $\alpha+\beta+s < l-2$.

Proof. Note that

$$\mathbf{F}(G_{\alpha,\beta}(aRu))(\zeta) = \sum_{\eta\in\mathbb{Z}^2}(1+|\zeta_1|)^{-\alpha}(1+|\zeta_2|)^{\beta}\mathbf{F}a(\zeta-\eta)\mathbf{F}(Ru)(\eta).$$

We have the standard inequality for any $\alpha, \beta, s \in \mathbb{R}$:

$$(1+|\zeta|)^s(1+|\zeta_1|)^{-\alpha}(1+|\zeta_2|)^{\beta} \leq c(1+|\zeta-\eta|)^{|\alpha|+|\beta|+|s|})(1+|\eta_1|)^{-\alpha}(1+|\eta_2|)^{\beta}(1+|\eta|)^s.$$

With $M(\eta) = (1+|\eta_1|)^{-\alpha}(1+|\eta_2|)^{\beta}(1+|\eta|)^s|\mathbf{F}(Ru)(\eta)|$ we obtain that

$$\|G_{\alpha,\beta}(aRu)\|_s^2 = \sum_{\zeta\in\mathbb{Z}^2}(1+|\zeta|)^{2s}|\mathbf{F}(G_{\alpha,\beta}(aRu))(\zeta)|^2$$

$$\leq c|a|_l^2\{\sum_{\zeta,\eta\in\mathbb{Z}^2}(1+|\zeta-\eta|)^{\alpha+\beta+s-l}M(\eta)\}^2.$$

Since $\alpha+\beta+s-l < -2$, classical convolution estimates yield that

$$\|G_{\alpha,\beta}(aRu)\|_s^2 \leq c|a|_l^2\sum_{\eta\in\mathbb{Z}^2}M(\eta)^2 = c|a|_l^2\|G_{\alpha,\beta}Ru\|_s^2,$$

which completes the proof. □

Examples of Q1D operators are the identity I and the Hilbert transform K, which are Q1D of order $(\alpha, 0)$ for any $\alpha \geq 0$, and the operator $K_1(h) = K(h) - K$ which is Q1D of any order. Now we discuss Q1D properties of integral operators in the form

$$Au(y) = \int_{-\pi}^{\pi}\int_0^1 a(y,s,\tau)u(y_1+\tau\omega(y,s),s)dsd\tau,$$
$$Bu(y) = \int_{-\pi}^{\pi} b(y,s)u(y_1+\omega(y,s),s)ds,$$

in which the kernels $a(y,s,\tau)$, $b(y,s)$ and the function $\omega(y,s)$ are 2π-periodic with respect to y_i, s. Denote by Σ the strip $\mathbb{R}^3 \times [0,1]$ and identify D_i with $(\partial/\partial y_i)$, $i = 1, 2$.

Proposition 7.8. *Suppose that $a \in C^l(\Sigma)$, $b \in C^l(\mathbb{R}^3)$ for some $l \leq r-3$, $\|\omega\|_{C^{r-2}(\mathbb{R}^2)} \leq c_1$ and $1 + D_1\omega(y,s) > c_1^{-1}$ for some $c_1 > 0$. Then*

$$\|Au\|_s \leq c\|a\|_{C^s(\Sigma)}\|u\|_s, \quad \|Bu\|_s \leq c\|b\|_{C^s(\mathbb{R}^3)}\|u\|_s, \quad \textit{for } \; s \leq l, \tag{7.9}$$
$$|||A|||_{\beta,\beta,s} \leq c\|a\|_{C^l(\Sigma)}, \quad |||B|||_{\beta,\beta,s} \leq c\|b\|_{C^l(\mathbb{R}^3)}, \quad \textit{for } \; 0 \leq \beta \leq l-s. \tag{7.10}$$

Here the constant c depends only on r and c_1.

Proof. We begin by proving (7.9) for A. For $s=0$, Cauchy's inequality implies

$$|Au(y)|^2 \le c\|a\|^2_{C(\Sigma)} \int_{-\pi}^{\pi}\int_0^1 |u(y_1+\tau\omega(y,s),s)|^2 ds d\tau.$$

Under the assumptions of the lemma, the mapping $z_1 = y_1+\tau\omega(y,s)$, $z_2=y_2$, $z_3=s$, $z_4=\tau$ is a diffeomorphism of the parallelepiped $\Sigma_\pi = [-\pi,\pi]^3\times[0,1]$ onto itself with Jacobian $1+\tau D_1\omega \in [c_1^{-1}, 1+c_1] \subset (0,\infty)$. Therefore

$$\|Au\|_0^2 \le c\|a\|^2_{C(\Sigma)}\int_{\Sigma_\pi} |u(z_1,z_3)|^2 dz \le c\|a\|^2_{C(\Sigma)}\|u\|_0^2.$$

Assuming (7.9) holds for some exponent $s\le l-1$, we will prove it for $s+1$. Fix $i=1,2$. It is easy to check that $D_iAu = A'D_1u + A''u$ where

$$A'u(y) = \int_{-\pi}^{\pi}\int_0^1 a(y,s,\tau)(\delta_{1i}+\tau D_i\omega)D_1u(y_1+\tau\omega,s)dsd\tau,$$

$$A''u(y) = \int_{-\pi}^{\pi}\int_0^1 D_ia(y,s,\tau)u(y_1+\tau\omega,s)dsd\tau.$$

By the induction hypothesis, since $s\le l\le r-3$,

$$\|A'D_1u\|_s \le c\|a\|_{C^s(\Sigma)}\|D_1u\|_s \le c\|a\|_{C^s(\Sigma)}\|u\|_{s+1} \quad\text{and}\quad \|A''u\|_s \le c\|a\|_{C^{s+1}(\Sigma)}\|u\|_s,$$

which gives $\|D_iAu\|_s \le c\|a\|_{C^{s+1}(\Sigma)}\|u\|_{s+1}$ and the assertion follows. The same proof works for B.

Let us prove (7.10) for A. It will suffice to show that $|||A|||_{\beta,\beta,s} \le c\|a\|_{C^{s+\beta}(\Sigma)}$ for all $0\le\beta\le l-s$. For $\beta=0$ this follows from (7.9). Assume now that it is true for some $0\le\beta\le l-s-1$. We will prove it for $\beta+1$. It easy to check that $D_2Au = D_1A_1u + A_0u$, where the integral operators A_i are given by

$$A_iu(y) = \int_{-\pi}^{\pi}\int_0^1 a_i(y,s,\tau)u(y_1+\tau\omega(y,s),s)dsd\tau$$

with the kernels $a_1(y,s,\tau) = a(y,s,\tau)\chi(y,s,\tau)$ and

$$a_0(y,s,\tau) = D_2a(y,s,\tau) - D_1\{a(y,s,\tau)\chi(y,s,\tau)\},$$

where $\chi = \tau D_2\omega(1+\tau D_1\omega)^{-1}$. Next we have

$$\|G_{\beta+1,\beta+1}Au\|_s \le c(\|G_{\beta,\beta}(1+|D_1|)^{-1}Au\|_s + \|G_{\beta,\beta}(1+|D_1|)^{-1}D_2Au\|_s.$$

From this we conclude that

$$|||A|||_{\beta+1,\beta+1,s} \le c(|||A|||_{\beta,\beta,s} + |||(1+|D_1|)^{-1}D_2A|||_{\beta,\beta,s}). \tag{7.11}$$

The identity $D_2A = D_1A_1 + A_0$ implies

$$|||(1+|D_1|)^{-1}D_2A|||_{\beta,\beta,s} \le c(|||A_1|||_{\beta,\beta,s} + |||A_0|||_{\beta,\beta,s}). \tag{7.12}$$

The induction hypothesis gives

$$|||A_1|||_{\beta,\beta,s} + |||A_0|||_{\beta,\beta,s} \le \|a_i\|_{C^{s+\beta}(\Sigma)}$$

for any $\beta+s\le l-1$. Since $\|a_i\|_{C^{s+\beta}(\Sigma)} \le c\|a\|_{C^{s+\beta+1}(\Sigma)}$, we conclude from this and (7.11), (7.12) that $|||A|||_{\beta+1,\beta+1,s} \le c\|a\|_{C^{s+\beta+1}(\Sigma)}$. Thus (7.10) for A follows by induction. The proof (7.10) for B is similar to that for A. $\square$

7.3 Change of independent variables

We are now in a position to formulate the second main result of Section 7, on change of independent variable in the Neumann operator and Hilbert transform. Recall the notation of Theorem 7.2.

Theorem 7.9. *Suppose that $y_1 = P_1(t, \xi_1)$, $y_2 = P_2(\xi_1)$ defines a diffeomorphism of $\mathbb{R}^2$ onto itself with inverse $Q = P^{-1}$, $P = (P_1, P_2)$. Assume that, $\partial_t P_1 > 0$ and, for some $c_2 > 0$, that the mappings $P_0 = P - I$, $Q_0 = Q - I$ are 2π-periodic with $|P_0|_{r-2} + |Q_0|_{r-2} \le c_2\varepsilon^2$, where the constant c_2 does not depend on ε. Let*

$$d_i = (\partial_{\xi_1} P_i) \circ Q, \ \ i = 1,\, 2, \quad d_3 = D_1 Q_1.$$

Then, with $v = u \circ Q$ for any function $u \in H_s^-$,

$$(K(h_\lambda)u) \circ Q = K(h_\lambda)v + R_K v, \tag{7.13}$$

$$(\partial_{\xi_1} K(h_\lambda)u) \circ Q = (d_1 D_1 + d_2 D_2)K(h_\lambda)v + R_N v, \tag{7.14}$$

$$(\partial_t^{-1} K(h_\lambda)u)\circ = D_1^{-1}(d_3 K(h_\lambda)v) + R_T v + R_\Pi v. \tag{7.15}$$

The remainders R_K, R_N and R_T are Q1D operators in H_s^- with

$$|||R_K|||_{\beta+1,\beta,s} + |||R_N|||_{\beta+2,\beta,s} + |||R_T|||_{\beta+1,\beta,s} \le c\varepsilon^2, \tag{7.16}$$

and the smoothing operator R_Π satisfies the inequality

$$\|(1 - D_1^2)R_\Pi v\|_s \le c\varepsilon^2 \|v\|_s. \tag{7.17}$$

Here the exponents s, β are arbitrary non-negative numbers with $2\beta + s < r - 7$, and the constant c depends only on c_2 and r.

The proof is based on the following explicit representation of the composition of a finite-depth Hilbert transform and the mapping Q.

Lemma 7.10. *Under the assumptions of Theorem 7.9, for any $h > 0$*

$$[K(h)u] \circ Q = K(h)v + K_2 v + K_3 v + D_1 K_4 v, \tag{7.18}$$

$$K_2 v = \int_{-\pi}^{\pi} k_2(s, y_2)v(y_1 + \omega(y, s), s)ds,$$

$$K_j v = \int_{-\pi}^{\pi}\int_0^1 k_j(y, s, \tau)v(y_1 + \tau\omega(y, s), s)dsd\tau \quad \textit{for} \quad j = 3, 4,$$

where k_j and ω are 2π-periodic, $\|k_2\|_{C^{r-4}(\mathbb{R}^3)} + \|k_{3,4}\|_{C^{r-5}(\Sigma)} + \|\omega\|_{C^{r-2}(\mathbb{R}^3)} \le c\varepsilon^2$, and $1 + D_1\omega > c^{-1} > 0$ for some constant $c > 0$ depending only on c_2, h and r.

Proof. Calculation shows that

$$[K(h)u] \circ Q(y) = \int_{-\pi}^{\pi} k\big(Q_2(y_2) - Q_2(s), h\big)u\big(Q_1(y), Q_2(s)\big)Q_2'(s)ds, \tag{7.19}$$

where the kernel of $K(h)$ is $k(s, h) = k(s) + k_1(s, h)$ and $k(s)$, the kernel of the Hilbert transform, is given by $(2\pi)^{-1}\cot(s/2)$. Recall that $k_1(s, h)$ is real-analytic, odd and 2π-periodic in s. Next, we have

$$Q_2'(s)k(Q_2(y_2) - Q_2(s), h) = k(y_2 - s, h) + k_2(s, y_2), \tag{7.20}$$

where, with $Q = (Q_1, Q_2) = (y_1 + Q_{10}, y_2 + Q_{20})$,

$$k_2 = Q_2'(s)k(Q_2(y_2) - Q_2(s)) - k(y_2 - s) + Q_{20}'(s)k_1(Q_2(y_2) - Q_2(s), h) \\ + \{k_1(Q_2(y_2) - Q_2(s), h) - k_1(y_2 - s, h)\}.$$

From this we conclude that

$$\|k_2\|_{C^{r-4}(\mathbb{R}^2)} \le c(h)\|Q_{20}\|_{C^{r-2}(\mathbb{R})} \le c(c_2)\varepsilon^2. \tag{7.21}$$

Let us consider a functional equation for $\omega(y, s)$

$$\phi(\omega, y, s) \equiv \omega(y, s) + Q_{10}(y_1 + \omega(y, s), s) - Q_{10}(y) = 0.$$

Since $\partial_t P_1 = (D_1 Q_1)^{-1}$, the present hypotheses imply the inequalities $0 < (1 + c_2\varepsilon^2)^{-1} \le D_1 Q_1 \le (1 + c_2\varepsilon^2)$. From this and the identity $D_\omega \phi = D_1 Q_1(y_1 + \omega, s)$ we conclude that this equation has a unique solution $\omega(y, s)$ of a class $C^{r-2}(\mathbb{R}^3)$. The inequality $\|\phi(0, \cdot, \cdot)\|_{C^{r-2}(\mathbb{R}^3)} \le 2|Q_{10}|_{r-2} \le c_2\varepsilon^2$, and the implicit function theorem now give that $\|\omega\|_{C^{r-2}(\mathbb{R}^3)} \le c(c_2)\varepsilon^2$. The identity $1 + D_1\omega = D_1 Q_1(y)(D_1 Q_1(y_1 + \omega, s))^{-1}$ implies that $1 + D_1\omega > c(c_2) > 0$. With this choice of ω, we have

$$u(Q_1(y), Q_2(s)) = u(Q_1(y_1 + \omega(y, s), s), Q_2(s)) \equiv v(y_1 + \omega(y, s), s).$$

Substituting this relation and (7.20) into (7.19) implies

$$[K(h)u] \circ Q(y) = \int_{-\pi}^{\pi} \Big\{k(y_2 - s, h) + k_2(s, y_2)\Big\} v(y_1 + \omega(y, s), s)ds. \tag{7.22}$$

Next we have

$$v(y_1 + \omega(y, s), s) = v(y_1, s) + \omega(y, s)\int_0^1 (D_1 v)(y_1 + \tau\omega(y, s), s)d\tau. \tag{7.23}$$

Substituting (7.23) in (7.22) gives

$$[K(h)u] \circ Q(y) = K(h)v(y) + \int_{-\pi}^{\pi} k_2(s, y_2)v(y_1 + \omega(y, s), s)ds \\ + \int_{-\pi}^{\pi}\int_0^1 \omega(y, s)k(y_2 - s, h)(D_1 v)(y_1 + \tau\omega(y, s), s)dsd\tau. \tag{7.24}$$

Since $\omega \in C^{r-2}(\mathbb{R}^3)$ vanishes for $s = y_2$, the kernel $\omega(y, s)k(y_2 - s, h)$ is of class C^{r-4}.

Hence we obtain

$$\int_{-\pi}^{\pi}\int_0^1 \omega(y, s)k(y_2 - s, h)(D_1 v)(y_1 + \tau\omega(y, s), s)dsd\tau \tag{7.25}$$
$$= \int_{-\pi}^{\pi}\int_0^1 k_3 v(y_1 + \tau\omega(y, s), s)dsd\tau + D_1\int_{-\pi}^{\pi}\int_0^1 k_4 v(y_1 + \tau\omega(y, s), s)dsd\tau,$$

where

$$k_3 = -D_1 k_4 \text{ and } k_4(y, s, \tau) = (1 + \tau D_1\omega(y, s))^{-1}\omega(y, s)k(y_2 - s, h). \tag{7.26}$$

Substituting (7.25) into (7.24) gives the needed representation (7.18). Estimates on k_j follow from the hypotheses of Theorem 7.9 and (7.26). $\square$

We turn now to the proof of Theorem 7.9. Lemma 7.10 shows that (7.13) holds with the remainder $R_K = K_2 + K_3 + D_1 K_4$. Note by the remark preceding Proposition 7.7 that $|||D_1 K_4|||_{\beta+1,\beta,s} \le |||K_4|||_{\beta,\beta,s}$. Since the norm $|||\cdot|||_{\alpha,\beta,s}$ decreases in α, we conclude that $|||R_K|||_{\beta+1,\beta,s} \le \sum_i |||K_i|||_{\beta,\beta,s}$. It follows from Lemma 7.10 that the operators K_i satisfy all condition of Proposition 7.8 with the exponent $l = r-5$, which implies $|||K_i|||_{\beta,\beta,s} \le c\|k_i\|_{C^{r-5}(\Sigma)} \le c\varepsilon^2$ for any non-negative β with $\beta + s \le r-5$. This along with the estimates from Lemma 7.10 gives

$$|||R_K|||_{\beta+1,\beta,s} \le c\varepsilon^2 \quad \text{for any} \quad \beta + s \le r-5. \tag{7.27}$$

Now we proceed towards the representation (7.14). Since $\partial_{\xi_1} = \sum_1^2 d_i D_i$, the function $(\partial_{\xi_1} K(h)u)\circ Q$ satisfies the equality (7.14) with the remainder term $R_N = \sum_1^2 d_i D_i R_K v$. The present hypotheses imply that $|d_i|_{r-3} \le c$ with the constant c depending only on the mapping P. From this and Proposition 7.7 we conclude that $|||d_i D_i R_K|||_{\beta+2,\beta,s} \le c|||D_i R_K|||_{\beta+2,\beta,s}$ for any $2\beta + 2 + s < r-5$. On the other hand, by (7.27),

$$|||D_1 R_K|||_{\beta+2,\beta,s} \le |||R_K|||_{\beta+1,\beta,s} \le c\varepsilon^2 \quad \text{for any} \quad \beta + s \le r-5,$$

and

$$|||D_2 R_K|||_{\beta+2,\beta,s} \le |||R_K|||_{\beta+2,\beta+1,s} \le c\varepsilon^2 \quad \text{for any} \quad \beta + s \le r-6.$$

From this we obtain that R_N satisfy inequality (7.16) for any $2\beta + s < r-7$, which is desired conclusion.

To obtain (7.15) recall that $\Pi_1 D_1^{-1} \equiv \Pi_t \partial_t^{-1} \equiv 0$. Therefore, after change of independent variables,

$$(\partial_t^{-1} u)\circ Q = D_1^{-1}(d_3 v) - \Pi_1\{d_3 D_1^{-1}(d_3 v)\}.$$

Since, for $u \in H_s^-$, $K(h_\lambda)u$ is odd in t, we obtain from (7.13) a representation for $(\partial_t^{-1} K(h_\lambda)u)\circ Q$ in the form (7.15) with remainder

$$R_T v = D_1^{-1}(d_3 R_K v) - \Pi_1\{d_3 D_1^{-1}(d_3 R_K v)\},\ R_\Pi v = -\Pi_1\{d_3 D_1^{-1}(d_3 K(h_\lambda) v\}.$$

It follows from the conditions of Theorem 7.9 that $|d_3|_{r-3} \le c$. We conclude from this and Proposition 7.7 that $|||R_T|||_{\beta+1,\beta,s} \le c(|d_3|_{r-3}^2 + 1)|||R_K|||_{\beta+1,\beta,s} \le c\varepsilon^2$ if $2\beta + s + 1 < r-5$. The estimate on R_Π is obvious, since $\Pi_1(\cdot)$ does not depend on y_1. $\square$

8 Canonical form

In this section we show how a certain integro-differential operator equation can be reduced, by a specific change of the independent variables, to canonical form with constant coefficients in the main term. We then give resolvent estimates on that main term. Because of the applications in mind, we consider the class of integro-differential equations in form

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial u}{\partial t}\right) + \partial_{\xi_1} K(h_\lambda)u + BK(h_\lambda)u = f, \tag{8.1}$$

where $B = b_0 + b_{-1}\partial_t^{-1}$. We will assume that the functions $a(t,\xi_1)$, $b_i(t,\xi_1)$ are 2π-periodic, a is even in all variables, b_0 is even in t and odd in ξ_1, b_{-1} is odd in both. Suppose also that

$$|a - g_0|_{r-2} + |b_i|_{r-4} \le c_3\varepsilon^2, \quad a > c_3^{-1} > 0, \tag{8.2}$$

where constant c_3 is independent of ε.

8.1 Reduction to canonical form

Recall the context of Theorem 7.9 and definitions (3.10) and (3.11), and introduce the vector-valued function $P(t,\xi_1)$ with the components P_i, $i = 1, 2$ defined by the formulae

$$P_1 = \frac{1}{\Theta(\xi_1)}\int_0^t \sqrt{a(s,\xi_1)}ds, \quad P_2 = \int_0^{\xi_1} \frac{g_0 ds}{\Theta(s)^2}, \quad \Theta(\xi_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{a(t,\xi_1)}dt. \tag{8.3}$$

Lemma 8.1. *Suppose that $\Psi_1(a) = g_0$. Then the mapping P is a diffeomorphism of the plane onto itself. Let its inverse $Q = P^{-1}$ be denoted by $t = Q_1(y)$, $\xi_1 = Q_2(y_2)$. The mappings $Q_0 = Q - I$, $P_0 = P - I$ are 2π-periodic, $\partial_t P_1 > 0$ and $|P_0|_{r-2} + |Q_0|_{r-2} \le c\varepsilon^2$.*

Proof. By construction, P_{10} is 2π-periodic and the condition $\Psi_1(a) = g_0$ implies that $P_2(t,\xi_1+\pi) - P_2(t,\xi_1-\pi) \equiv g_0\int_{-\pi}^{\pi}\Theta(\xi_1)^{-2}d\xi_1 = 2\pi$. Since the derivative of P_2 is periodic, we conclude that P_{20} is also periodic, where $P_0 = (P_{10}, P_{20})$. The existence of the inverse Q and estimates on P_0, Q_0 follows from (8.2)) and (8.3). $\square$

Now we can formulate the main result on reducing the integro-differential equation (8.1) to canonical form.

Theorem 8.2. *Suppose that $\Psi_1(a) = g_0$. Then equation (8.1) in the space H_s^-, $s \ge 2$, is equivalent to the following canonical equation for the function $v(y) = u \circ Q(y)$:*

$$D_1^2 v + g_0 D_2 K(h_\lambda)v = \sum_{j=-1}^{1} D_1^j A_j v + R_2 v + R_3 v + f_0. \tag{8.4}$$

Here $f_0 = (\Theta^2 f) \circ Q$ and $A_j = a_j^1 + a_j^2 K$, where K is the Hilbert transform and the coefficients a_i^j are 2π-periodic and symmetric as follows: in y_1 they are even for even i and odd for odd i; in y_2 they are even for $j = 1$ and odd for $j = 2$. Moreover

$$|a_j|_{r-4} \le c\varepsilon^2, \qquad \|f_0\|_s \le c\|f\|_s, \quad s \le r-2, \tag{8.5}$$

and

$$\frac{1}{16\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{(a_1^1)^2 - (a_1^2)^2\Big\}dy = \Psi_2(a), \quad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} a_0^j dy = 0, \quad j = 1,\, 2. \tag{8.6}$$

The remainder R_2 is Q1D and R_3 is a smoothing operators with, for any non-negative β, s, $2\beta + s \le r - 7$,

$$|||R_2|||_{\beta+2,\beta,s} \le c\varepsilon^2, \quad \|(1 - D_1^2)R_3 v\|_s \le c\varepsilon^2\|v\|_s, \tag{8.7}$$

in which the constant c depends only on r and c_3 in (8.2).

Proof. Changing variables using (8.3) and the identities (7.13)-(7.15) gives the following equation for the function $v = u \circ Q$

$$d_{11}D_1^2 v + qD_1 v + (d_1 D_1 + d_2 D_2 + b_0)K(h_\lambda)v + b_{-1}D_1^{-1}(d_3 K(h_\lambda)v) = Rv + f \circ Q, \quad (8.8)$$

with coefficients

$$d_{11} = a^{-1}(\partial_t P_1 \circ Q)^2 = (\Theta \circ Q)^{-2}, \qquad q = \partial_t(a^{-1}\partial_t P_1) \circ Q,$$
$$d_i = \partial_{\xi_1} P_i \circ Q, \quad i = 1, 2, \qquad d_3 = D_1 Q_1 = (\partial_t P_1 \circ Q)^{-1},$$

and remainder $R = -R_N - b_0 R_K - b_{-1}(R_T + R_\Pi)$. Here the operators R_N, R_K, R_T, R_Π are given by Theorem 7.9. Note that relation $\Psi_1(a) = g_0$ along with formula (8.3) implies the identity $g_0 d_{11} = d_2 = g_0(\Theta \circ Q)^{-2}$. Hence multiplying both sides of equation (8.8) by $g_0 d_2^{-1}$ leads to the canonical form (8.4) in which $A_i = a_i^1 + a_i^2 K$ where

$$\begin{aligned}
a_1^1 &= -\Theta \frac{\partial}{\partial t}\frac{1}{\sqrt{a}}, & a_1^2 &= -\Theta^2 \frac{\partial}{\partial \xi_1}\Big\{\Theta^{-1}\int_0^t \sqrt{a}dt\Big\}, \\
a_0^1 &= -D_1 a_1^1, & a_0^2 &= -D_1 a_1^2 - \Theta^2 b_0, \\
a_{-1}^1 &= 0, & a_{-1}^2 &= -b_{-1}\Theta^3/\sqrt{a}, \quad f_0 = \Theta^2 f,
\end{aligned} \quad (8.9)$$

$$-g_0^{-1}d_2 R_2 = R_N + b_0 R_K + b_{-1}R_T + b_0 K_1(h_\lambda) + d_1 D_1 K_1(h_\lambda) + b_{-1}D_1^{-1}(d_3 K_1(h_\lambda)\cdot),$$
$$g_0^{-1}d_2 R_3 = -b_{-1}R_\Pi + [D_1^{-1}, b_{-1}](d_3 K(h_\lambda)\cdot).$$

It follows from this, (8.2) and Lemma 8.1 that, for ε sufficiently small, the coefficients on the right side of equation (8.4) satisfy inequality (8.5) and have the needed symmetry properties. From the oddness of b_0 in ξ_1 and (8.9), we conclude that a_0^j have zero mean over period which implies the second equality of (8.6). In order to prove the first note that $|\partial(y_1, y_2)/\partial(t, \xi_1)| = g_0\Theta^{-3}\sqrt{a}$. Therefore, by the change of variable formula and (8.9),

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{(a_1^1)^2 - (a_1^2)^2\Big\}dy = 16\pi^2\Psi_2(a), \text{ as in formula (3.11).}$$

It follows from Lemma 8.1 and inequalities (8.2) that, under the assumptions of the theorem, d_1, d_2^{-1} have the uniform C^{r-3} bounds and $|b_i|_{r-4} \le c\varepsilon^2$. Hence, by Proposition 7.7, the inequality

$$|||R_2|||_{\beta+2,\beta,s} \le c|||R_N|||_{\beta+2,\beta,s} + c\varepsilon^2\{|||R_K|||_{\beta+2,\beta,s} + |||R_T|||_{\beta+2,\beta,s} + |||K_1(h_\lambda)|||_{\beta+1,\beta,s}\} \quad (8.10)$$

holds for any non-negative β, s such that $2\beta + s \le r - 7$. Since $K_1(h_\lambda)$ is a Q1D operator of any order and $|||\cdot|||_{\beta+2,\beta,s} \le |||\cdot|||_{\beta+1,\beta,s}$, the needed inequality (8.7) for the operator R_2 is a consequence of (8.2), (8.10) and estimates on remainder terms from Theorem 7.9.

In order to estimate R_3 we will use the identity

$$D_1^2[D_1^{-1}, b_{-1}] = 2(D_1 b_{-1})\Pi_1 - (D_1 b_{-1}) - (D_1^2 b_{-1})D_1^{-1}.$$

From this and the C^{r-4} estimates on b_i, we conclude that

$$\|(1-D_1^2)[D_1^{-1}, b_{-1}](d_3K(h_\lambda)v)\|_s \le c|b_{-1}|_{s+2}|d_3|_s\|v\|_s \le c\varepsilon^2\|v\|_s$$

for any $s \le r-6$. Similarly,

$$\|(1-D_1^2)(d_2^{-1}b_{-1}R_\Pi v)\|_s \le c|d_2^{-1}b_{-1}|_{s+2}(\|R_\Pi v\|_s + \|D_1^2 R_\Pi v\|_s) \le c\varepsilon^2\|v\|_s$$

which implies inequality (8.7) for the operator R_3 and the theorem follows. □

8.2 Resolvent estimates for the main part

In this section we consider in more detail the properties of the operator L which forms the main part of the canonical equation (8.4). We observe that for almost every positive g_0 this operator has an unbounded inverse in the Hilbert space H_s^- and estimate the norm of the resolvent $(L-\mu)^{-1}$ for typical μ from the continuous spectrum of L. In particular, we prove that for any Q1D operator R of order $(\beta+2,\beta)$, $\beta \ge 4$, in H_s^- the operator $(L-\mu)^{-1} \circ R$ is bounded for μ in a set which has 0 as a Lebesgue point. Recall the notation of Section 2.3 and the statement of Proposition 2.4.

Proposition 8.3. *Let $\nu \in (1,2)$, $g_0 \in \mathcal{N}(\nu)$, $\mu \in \mathcal{M}(g_0,\tau)$ with $1-\nu^{-1} < \tau < 1$ and $h > 0$ arbitrary. Let $L \equiv D_1^2 + g_0 D_2 K$ where K is the Hilbert transform. Then there exist inverses $L^{-1} : H^-_{s+\nu/2} \to H_s^-$ and $(L-\mu)^{-1} : H^-_{s+1} \to H^-_s$ such that*

$$\|L^{-1}u\|_s \le c(g_0,\nu)\min\{\|(1+|D_1|)^\nu u\|_s, \|(1+|D_2|)^{\nu/2}u\|_s\}, \tag{8.11}$$

$$\|(L-\mu)^{-1}u\|_s \le c(g_0)|\mu|^{-\tau}\min\{\|(1-D_1^2)u\|_s, \|(1+|D_2|)u\|_s\}, \tag{8.12}$$

$$\|(L-\mu)^{-1}D_2K_1(h)u\|_s \le c(g_0,h)\|u\|_{s-2}. \tag{8.13}$$

If R is a Q1D operator of order $(2,0)$ and of order $(\beta+2,\beta)$ in the space H_s^- with $\beta \ge 4$, then

$$\|(L-\mu)^{-1}Ru\|_s \le c(g_0)|\mu|^{-\tau}|||R|||_{\beta+2,\beta,s}\|u\|_s. \tag{8.14}$$

Proof. The existence of inverses and estimates (8.11), (8.12) follow from the equalities $\mathbf{F}(L-\mu)^{-1}u(\zeta) = (d(\zeta)-\mu)^{-1}\mathbf{F}u(\zeta)$, $\zeta \in \mathbb{Z}^2 \setminus \{0\}$ and Proposition 2.4. In order to prove (8.13) note that

$$|\mathbf{F}(L-\mu)^{-1}D_2K_1(h)u(\zeta)| = |(d(\zeta)-\mu)^{-1}|\zeta_2|(1-\tanh h|\zeta_2|)\mathbf{F}u(\zeta)|.$$

Introduce the set

$$\mathcal{P} = \{\zeta \in \mathbb{Z}^2 : \ |d(\zeta)-\mu| \ge (1+|\zeta_1|^2)/2\}$$

and the orthogonal projection P defined in terms of the Fourier transform by the equalities $\mathbf{F}Pu(\zeta) = \mathbf{F}u(\zeta)$ for $\zeta \in \mathcal{P}$, and $\mathbf{F}Pu(\zeta) = 0$ otherwise.

From this and the previous inequality we conclude that

$$\begin{aligned}|\mathbf{F}P(L-\mu)^{-1}D_2K_1(h)u(\zeta)| &\le c(h)|1+\zeta_1^2|^{-1}\exp(-c|\zeta_2|)|\mathbf{F}u(\zeta)| \\ &\le c(h)(1+|\zeta|^2|)^{-1}|\mathbf{F}u(\zeta)|.\end{aligned} \tag{8.15}$$

Using the identity $(L-\mu)^{-1} = L^{-1} + \mu L^{-1}(L-\mu)^{-1}$, we obtain

$$|\mathbf{F}(I-P)(L-\mu)^{-1}D_2K_1(h)u(\zeta)| \le c(|1+\zeta_1^2|\mu^{1-\tau}+1)|1+\zeta_1^2|\exp(-c|\zeta_2|)|\mathbf{F}u(\zeta)|.$$

It easy to see that inequality $c(1+|\zeta_2|) \ge (1+|\zeta_1|)^2$ holds for any $\zeta \in \mathbb{Z}^2 \setminus \mathcal{P}$. From this and the previous inequality we conclude that

$$|\mathbf{F}(I-P)(L-\mu)^{-1}D_2K_1(h)u(\zeta)| \le c(1+|\zeta|^2|)^{-1}|\mathbf{F}u(\zeta)|,$$

which along with (8.15) implies (8.13).

Note that, by Definition 7.6, the operators $G_{\beta+2,\beta}R$ and $(1+|D_1|)^{-2}PR$ are bounded in H_s and

$$R = SG_{\beta+2,\beta}R + P(1+|D_1|)^2(1+|D_1|)^{-2}R,$$

where $S = (I-P)G^{-1}_{\beta+2,\beta}$. Therefore it is sufficient to prove that

$$\|(L-\mu)^{-1}Su\|_{H_s} + \|P(L-\mu)^{-1}(1+|D_1|)^2u\|_{H_s} \le c|\mu|^{-\tau}\|u\|_{H_s}. \tag{8.16}$$

Recall that in term of the Fourier transform

$$\mathbf{F}\{(L-\mu)^{-1}Su\}(\zeta) = (d(\zeta)-\mu)^{-1}(1+|\zeta_2|)^{-\beta}(1+|\zeta_1|)^{\beta+2}\mathbf{F}u(\zeta)$$

on $\mathbb{Z}^2 \setminus \mathcal{P}$ and 0 on $\mathcal{P}$, and that $c(1+|\zeta_2|) \ge (1+|\zeta_1|)^2$ holds for any $\zeta \in \mathbb{Z}^2 \setminus \mathcal{P}$. It follows that

$$|\mathbf{F}(L-\mu)^{-1}Su(\zeta)| \le c(g_0)|\mu|^{-\tau}(1+|\zeta_1|)^2(1+|\zeta_1|)^{2-\beta}|\mathbf{F}u(\zeta)| \le c|\mu|^{-\tau}|\mathbf{F}u(\zeta)|$$

for $\zeta \in \mathbb{Z}^2 \setminus \mathcal{P}$ and $|\mathbf{F}(L-\mu)^{-1}Su(\zeta)| = 0$ for $\zeta \in \mathcal{P}$, which gives the estimate on the first term in (8.16). Next

$$|\mathbf{F}P(L-\mu)^{-1}(1+|D_1|)^2u(\zeta)| \le c(g_0)(1+|\zeta_1|)^{-2}(1+|\zeta_1|)^2|\mathbf{F}u(\zeta)|$$

which along with the previous inequality imply (8.14) and the proposition follows. □

Lemma 8.4. *Let $\vartheta = \sin ny_1 \cos my_2$ where (m,n) is the unique solution to the dispersion equation (2.21) for some g_0, h_0. Then provided $g_0 \in \mathcal{N}_0(\nu)$, $f \in H_s^-$ and $\langle f, \vartheta\rangle = 0$, any solution $u \in H_{s-1}^-$ to equation*

$$L_{h_0}u \equiv D_1^2u + g_0D_2K(h_0)u = f, \quad \langle u, \vartheta\rangle = 0,$$

satisfies the inequality $\|u\|_{s-1} \le c\|f\|_s$. Here c depends only on g_0 and h_0.

Proof. Set $d_{h_0}(\zeta) = d(\zeta) + g_0|\zeta_2|(\tanh h_0|\zeta_2| - 1)$ It is clear that $\mathbf{F}u(\zeta) = d_{h_0}(\zeta)^{-1}\mathbf{F}f(\zeta)$ if $d_{h_0}(\zeta) \neq 0$ and $\mathbf{F}u(\zeta) = 0$ otherwise. It follows from the definition of $\mathcal{N}(\nu)$ that $|d_{h_0}(\zeta)| \ge c(1+|\zeta_2|)^{-1} - \exp(-c|\zeta_2|)$. Hence $|d_{h_0}(\zeta)| \ge c(1+|\zeta_2|)^{-1}$ for all ζ outside some strip $|\zeta_2| \ge \rho$. On the other hand, $|d_{h_0}(\zeta)| \ge c\zeta_1^2 - c_1(\rho)$ in this strip. We conclude that $|d_{h_0}(\zeta)|^{-1} \le c(1+|\zeta_2|)$ for all ζ outside some circle. Since any circle contains a finite set of integer points, this estimates holds at any point ζ with $d_{h_0}(\zeta) \neq 0$ and the assertion follows. □

9 Descent method and resolvent estimates

In this Section we develop a descent method for reducing integro-differential equations in canonical form to lower order in order to obtain resolvent estimates which lead to possible use of the contraction mapping principle.

9.1 Preliminaries

We deal with a collection of complex-valued functions, which we call 'symbols', having the representation $a(y,\zeta) = a^1(y) - i\,(\operatorname{sgn}\zeta_2)a^2(y)$, $y \in \mathbb{R}^2$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{Z}^2$, where the $a^j(y)$ are smooth 2π-periodic functions with a^1 even and a^2 odd in y_2. We define the product of symbols in a natural way by

$$ab(y,\zeta) = a^1(y)b^1(y) - a^2(y)b^2(y) - i\operatorname{sgn}\zeta_2(a^1(y)b^2(y) + a^2(y)b^1(y)), \tag{9.1}$$

and set $|a|_l = |a^1|_l + |a^2|_l$. To every symbol a assign the operator A given by

$$Au(y) = a^1(y)u(y) + a^2(y)Ku(y), \quad y \in \mathbb{R}^2,$$

and denote by $\mathcal{A}$ the class such operators. The following lemmas give properties of $\mathcal{A}$.

Lemma 9.1. *Let $f \in C^l$ be a smooth periodic function. Then for $s, r \geq 0$, $s + r < l - 2$,*

$$\||D_2|^r(fKu - K(fu))\|_s \leq c|f|_l\|u\|_s.$$

Proof. Set $\Phi = fKu - K(fu)$. It is clear that $\mathbf{F}\Phi(\zeta)$ is given by

$$\sum_{\eta\in\mathbb{Z}^2} i(\operatorname{sgn}\zeta_2 - \operatorname{sgn}\eta_2)\mathbf{F}f(\zeta-\eta)Fu(\eta) = 2i\operatorname{sgn}\zeta_2 \sum_{\{\zeta_2\eta_2\leq 0\}} \mathbf{F}f(\zeta-\eta)\mathbf{F}u(\eta).$$

It follows that

$$|\mathbf{F}\Phi(\zeta)| \leq c|f|_l \sum_{\{\zeta_2\eta_2\leq 0\}} \{1 + |\zeta_2|^l + |\eta_2|^l + |\zeta_1 - \eta_1|^l\}^{-1}|\mathbf{F}u(\eta)|$$

and hence that

$$(1+|\zeta|)^s|\zeta_2|^r|\mathbf{F}\Phi(\zeta)| \leq c|f|_l \sum_\eta (1+|\eta|)^s(1+|\zeta-\eta|)^{s+r-l}|\mathbf{F}u(\eta)|.$$

Hence, since $l - r - s > 2$,

$$\sum_\zeta((1+|\zeta|)^{2s}|\zeta_2|^{2r}|\mathbf{F}\Phi(\zeta)|)^2 \leq c(l,r,s)|f|_l^2 \sum_\eta (1+|\eta|)^{2s}|\mathbf{F}u(\eta)|^2$$

and the lemma follows. □

Lemma 9.2. *Let the symbols $a(\cdot,\zeta), b(\cdot,\zeta) \in C^l$. Then $AB = C + R$, where the operator C belongs to class $\mathcal{A}$ and has the symbol $c = ab$. For any $s, r \geq 0$, $s + r < l - 2$,*

$$\|(1 + |D_2|^r)Ru\|_s \leq c|a|_l|b|_l\|u\|_s.$$

Proof. This observation follows from Lemma 9.1 and the identity

$$(AB - C)u = a^2K(b^2Ku - K(b^2u)) + a^2(K(b^1u) - b^1Ku) + a^2(\Pi_2(b^2u)),$$

since $\Pi_2 u$ does not depend on y_2. □

9.2 Commutators, canonical operators and symbols

Fix operators $A_j \in \mathcal{A}$ with symbols $a_j(y,\zeta)$, $j \in \{-1, 0, 1\}$ such that $a_{\pm 1}$ are odd in y_1 and a_0 is even in y_1. It follows that $D_1 A_1 : H_s^- \to H_{s-1}^-$ and $A_0, D_1^{-1}A_{-1} : H_s^- \to H_s^-$ for $a_j(\cdot,\zeta)$ sufficiently smooth. For solutions u of the canonical equation

$$Lu \equiv D_1^2 u + g_0 D_2 K u = \sum_{j=-1}^{1} D_1^j A_j u + f, \quad f \in H_s^-, \tag{9.2}$$

our first job is to obtain an equation for the function Γu, where $\Gamma \in \mathcal{A}$ is arbitrary. To do so, we calculate commutators of Γ and the linear operators in equation (9.2) using two auxiliary lemmas. The first gives the action of the commutator of L and $\Gamma \in \mathcal{A}$ on a solution to equation (9.2).

Lemma 9.3. *Whenever $\Gamma \in \mathcal{A}$ and $\mu \in \mathbb{R}$, the equality*

$$(L-\mu)\Gamma u = D_1 M u + W u + R_\Gamma u + \Gamma(D_1^{-1}A_{-1}u + f) \tag{9.3}$$

holds for any solution $u \in H_s^-$, $s \geq 2$, of (9.2). The operators $M, W \in \mathcal{A}$ have symbols $m(y,\zeta)$, $w(y,\zeta)$ defined by the formulae

$$m = 2D_1\gamma + a_1\gamma, \quad w = -\{D_1^2\gamma + a_1 D_1\gamma + (\mu - a_0)\gamma + ig_0 \operatorname{sgn}\zeta_2\, D_2\gamma\}, \tag{9.4}$$

where $\gamma(y,\zeta)$ is the symbol of Γ and

$$\|(1+|D_2|^r)(1+|D_1|)^{-1}R_\Gamma u\|_s \leq c(r,s,l,g_0)\Big(\sum_{j=0}^{1}|a_j|_l\Big)|\gamma|_l\|u\|_s \tag{9.5}$$

for any $r, s \geq 0$, $s + r < l - 3$.

Proof. We have $[\Gamma, L] \equiv \Gamma L - L\Gamma = \Gamma_{,11} - 2D_1\Gamma_{,1} + g_0[\Gamma, D_2K]$. Here the notation $\Gamma_{,j}$ is used for the operator of class $\mathcal{A}$ having symbol $D_j\gamma(y,\zeta)$. Since $[\Gamma, D_2K] = D_2\Gamma K - D_2K\Gamma - \Gamma_{,2}K$,

$$\Gamma L - L\Gamma = -2D_1\Gamma_{,1} + \Gamma_{,11} - g_0\Gamma_{,2}K + g_0 D_2[\Gamma, K]. \tag{9.6}$$

On the other hand,

$$\Gamma(D_1A_1 + A_0) = D_1\Gamma A_1 + \Gamma A_0 - \Gamma_{,1}A_1. \tag{9.7}$$

Applying the operator Γ to both the sides of (9.2) we obtain $\Gamma(Lu - \sum_{j=-1}^{1} D_1^j A_j u - f) = 0$. Substituting (9.6), (9.7) into this equality gives

$$\begin{aligned}(L-\mu)\Gamma u = D_1\{2\Gamma_{,1} + \Gamma A_1\}u + \{\Gamma A_0 - \mu\Gamma - \Gamma_{,11} - \Gamma_{,1}A_1 + g_0\Gamma_{,2}K\}u \\ + g_0 D_2[K,\Gamma]u + \Gamma\{D_1^{-1}A_{-1}u + f\}.\end{aligned}$$

It follows from Lemma 9.2 that $2\Gamma_{,1} + \Gamma A_1 = M + R_1$, where the operator $M \in \mathcal{A}$ has the symbol $m = 2D_1\gamma + a_1\gamma$ and the remainder has the estimate

$$\|(1+|D_2|^r)R_1 u\|_s \leq c(r,s,l)|a_1|_l|\gamma|_l\|u\|_s \tag{9.8}$$

for any non-negative $s, r,\ s+r < l-2$. Similarly,

$$\Gamma A_0 - \mu\Gamma - \Gamma_{,11} - \Gamma_{,1}A_1 + g_0\Gamma_{,2}K = W + R_2$$

where the operator $W \in \mathcal{A}$ has symbol w in (9.4) and

$$\|(1+|D_2|^r)R_2 u\|_s \le c(r,s,l)(1+|a_1|_l+|a_0|_l)|\gamma|_l\|u\|_s \tag{9.9}$$

for any non-negative $s, r,\ s+r < l-3$. From this and the previous identity we obtain the needed relation (9.3) with $R_\Gamma = D_1R_1 + R_2 + g_0D_2[K,\Gamma]$. The estimate (9.5) follows from Lemma 9.1 and (9.8),(9.9) which completes the proof. □

Now the idea is to choose the symbol γ such that the corresponding operator M coincide with A_1. In this case subtracting (9.2) and (9.3) gives an equation which has no first order terms on the right. It is clear that the equality $M = A_1$ is equivalent to the ordinary differential equation for the symbol γ

$$m(y,\zeta) \equiv 2D_1\gamma(y,\zeta) + a_1(y,\zeta)\gamma(y,\zeta) = a_1(y,\zeta). \tag{9.10}$$

Note that the corresponding homogeneous equation $2D_1\gamma + a_1\gamma = 0$ has the non-trivial periodic solution $\Psi^-(y,\zeta)$ where

$$\Psi^\pm(y,\zeta) = \exp\{\pm\frac{1}{2}D_1^{-1}a_1(y,\zeta)\}. \tag{9.11}$$

The inhomogeneous equation $2D_1\gamma + a_1\gamma = f$ has a periodic solution if and only if the symbol $f(y,\zeta)$ satisfies the solvability condition

$$\int_{-\pi}^{\pi} \Psi^+(y,\zeta)f(y,\zeta)dy_1 = 0 \quad \text{for any} \quad y_2 \in \mathbb{R}, \quad \zeta \in \mathbb{Z}^2.$$

Recall that because a_1 is odd in y_1, $D_1D_1^{-1}a_1 = a_1$ and a_1 satisfies the solvability condition. Therefore we can eliminate the first-order term on the right side of (9.2). To repeat this argument and eliminate the zero-order term, we arrive at the following differential equation for a new symbol γ_1:

$$2D_1\gamma_1 + a_1\gamma_1 = a_0 - w - \mu.$$

Here w is defined by (9.4) in which γ is a solution to (9.10). In general the right side of this equation does not satisfy the solvability condition. But, miraculously, it does so if one real constant μ is chosen correctly. Here is the result.

Lemma 9.4. *Suppose the symbols $a_{\pm1}(\cdot,\zeta) \in C^l$ are odd and $a_0(\cdot,\zeta) \in C^l$ is even in y_1. Then there exists a symbol $\gamma(\cdot,\zeta) \in C^l(\mathbb{R}^2)$ and a constant μ such that*

$$2D_1\gamma(y,\zeta) + a_1(y,\zeta)\gamma(y,\zeta) = a_1(y,\zeta), \tag{9.12}$$

$$\int_{-\pi}^{\pi} \Psi^+(y,\zeta)(w(y,\zeta) + \mu - a_0(y,\zeta))dy_1 = 0 \quad \textit{for any} \quad y_2 \in \mathbb{R}, \tag{9.13}$$

where w is defined by (9.4). *They are given by the formulae*

$$\gamma = 1 - \exp\Big\{-\frac{1}{2}D_1^{-1}a_1 - \frac{i\,\mathrm{sgn}\,\zeta_2}{2\pi g_0}D_2^{-1}\int_{-\pi}^{\pi}\Big(\frac{(a_1)^2}{4} + a_0 - \mu\Big)dy_1\Big\}, \tag{9.14}$$

$$\mu = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\frac{(a_1)^2}{4} + a_0\Big)\,dy_1dy_2, \tag{9.15}$$

and, for some function c which is bounded on bounded sets,

$$|D_1\gamma|_l + |\gamma|_l \le c(|a_0|_l, |a_1|_l)(|a_0|_l + |a_1|_l). \tag{9.16}$$

Proof. Note that the general periodic solution to equation (9.12) is $\gamma(y,\zeta) = 1 - C(y_2,\zeta)\Psi^-(y,\zeta)$, where $C = C^1(y_2) - i\,\mathrm{sgn}\,\zeta_2\; C^2(y_2)$ is an arbitrary symbol depending only on y_2, ζ_2. Recall that C^1 must be even and C^2 odd in y_2 if $\gamma \in \mathcal{A}$. We choose C to satisfy the equation (9.13) which, by (9.4), means that

$$\int_{-\pi}^{\pi} \Psi^+\{D_1^2\gamma + a_1 D_1\gamma + (\mu - a_0)(\gamma - 1) + ig_0\mathrm{sgn}\zeta_2\; D_2\gamma\}\, dy_1 = 0.$$

Substituting for γ gives the following equation for C:

$$0 = C\int_{-\pi}^{\pi} \Big\{\Psi^-\Psi^+(\mu - a_0) + \Psi^+D_1^2\Psi^- + \Psi^+a_1D_1\Psi^-\Big\}\, dy_1$$
$$+ ig_0\,\mathrm{sgn}\,\zeta_2\; C\int_{-\pi}^{\pi} \Psi^+D_2\Psi^-\, dy_1 + ig_0\mathrm{sgn}\,\zeta_2\,\frac{\partial C}{\partial y_2}\int_{-\pi}^{\pi} \Psi^+\Psi^-\, dy_1.$$

Simple calculations show that

$$\int_{-\pi}^{\pi} \Psi^+D_1^2\Psi^-\, dy_1 + \int_{-\pi}^{\pi} \Psi^+a_1D_1\Psi^-\, dy_1 = -\frac{1}{4}\int_{-\pi}^{\pi} (a_1)^2\, dy_1,$$
$$\int_{-\pi}^{\pi} \Psi^+D_2\Psi^-\, dy_1 = 0, \quad \int_{-\pi}^{\pi} \Psi^+\Psi^-\, dy_1 = 2\pi.$$

Substituting these identities into the previous equality gives that C satisfies

$$D_2C + h(y_2,\zeta)C = 0 \quad \text{where} \quad h = \frac{i\,\mathrm{sgn}\,\zeta_2}{2\pi g_0}\int_{-\pi}^{\pi}\Big(\frac{(a_1)^2}{4} + a_0 - \mu\Big)\, dy_1.$$

Set

$$\mu = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\frac{(a_1)^2}{4} + a_0\Big)\, dy_1dy_2.$$

Then h has zero mean over the period and the equation for the symbol C has a periodic solution $C = \exp(-D_2^{-1}h)$. Substituting C into the formula $\gamma = 1 - C\Psi^-$ gives the needed formula for γ and lemma follows. □

9.3 The main estimate

Now we can formulate and prove the main result of this section.

Theorem 9.5. *Suppose that $1 < \nu < 2$, $1 - \nu^{-1} < \tau < 2^{-1}$ and $g_0 \in \mathcal{N}(\nu)$, $\mu \in \mathcal{M}(\tau, g_0)$. Suppose also that the symbols a_i are related to μ by (9.15) and that, for some $n \ge 14$,*

$$|a_j|_n \le c_0|\mu|^{1/2}, \quad a_j((-1)^jy_1, y_2, \zeta) = (-1)^ja_j(y_1, y_2, \zeta).$$

Then whenever $3 \le \rho < n - 9$, any solution $u \in H_\rho^-$ to equation (9.2) satisfies the equation

$$(I - T)u = Su + (L - \mu)^{-1}(I - T)f. \tag{9.17}$$

Here the bounded operator $S : H^-_\rho \to H^-_\rho$ satisfies the inequality $\|Su\|_\rho \le c|\mu|^{1/2-\tau}\|u\|_\rho$ and the operator T can be represented as

$$T = \sum_{j=0}^{2} D_1^{-j}\Gamma_j, \quad \Gamma_j \in \mathcal{A},$$

where the symbols γ_j of the operators Γ_j satisfy $|\gamma_j|_{n-j} \le c|\mu|^{1/2}$. The constant c depends only on c_0, ρ, n, τ, ν and g_0.

Conversely, there exists $\delta > 0$, depending only on c, such that when $\mu \le \delta$ any solution of (9.17) *gives a solution of* (9.2).

The proof falls naturally into three steps.

The first step. Introduce the symbol $\gamma_0 \in C^n$ defined by the formula (9.14) with γ replaced by γ_0, and denote by $M_0, W_0 \in \mathcal{A}$ the operators having the symbols $m_0 \in C^n$, $w_0 \in C^{n-1}$ given by the formula (9.4) with γ also replaced by γ_0.

Since (by Lemma 9.4) $m_0 = a_1 \in C^n$, we have $M_0 = A_1$. Now subtracting (9.3) from (9.2) gives

$$(L-\mu)(I-\Gamma_0)u = B_0u + (I-\Gamma_0)(D_1^{-1}A_{-1}u + f) - R_{\Gamma_0}u \tag{9.18}$$

where the symbol b_0 of B_0 is defined by

$$b_0(y,\zeta) = a_0(y,\zeta) - \mu - w_0(y,\zeta). \tag{9.19}$$

It follows from (9.4) and (9.16) that

$$|\gamma_0(\cdot,\zeta)|_n + |b_0(\cdot,\zeta)|_{n-1} \le c|\mu|^{1/2}. \tag{9.20}$$

It follows from this and Lemma 9.3 that the remainder R_{Γ_0} has the estimate

$$\|(1+|D_2|^r)(1+|D_1|)^{-1}R_{\Gamma_0}u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \tag{9.21}$$

for any $r, \rho \ge 0$, $r + \rho < n-3$. Note that, by (9.13) and (9.19),

$$\int_{-\pi}^{\pi} \Psi^+(y,\zeta)b_0(y,\zeta_2)\,dy_1 = 0 \quad \text{for any } \ y_2. \tag{9.22}$$

The second step. Since the symbol b_0 satisfies the solvability condition (9.22), the differential equation $2D_1\gamma_1 + a_1\gamma_1 = b_0$ has the periodic solution

$$\gamma_1(y,\zeta) = \frac{1}{2}\Psi^-(y,\zeta)D_1^{-1}(\Psi^+(y,\zeta)b_0(y,\zeta)), \quad |D_1\gamma_1|_{n-1} + |\gamma_1|_{n-1} \le c|\mu|^{1/2}.$$

Denote by $M_1, W_1 \in \mathcal{A}$ the integro-differential operators having the symbols m_1, w_1 which are defined by the equalities (9.4) with γ replaced by γ_1. It is then clear that $B_0 = M_1$, and $m_1 \in C^{n-1}$, $w_1 \in C^{n-2}$. Now Lemma 9.3 gives

$$(L-\mu)\Gamma_1u = D_1B_0u + W_1u + R_{\Gamma_1}u + \Gamma_1(D_1^{-1}A_{-1}u + f).$$

Applying the operator D_1^{-1} to both sides of this equality and using identity $D_1D_1^{-1} = I - \Pi_1$, we obtain, for $u \in H^-_\rho$, that

$$(L-\mu)D_1^{-1}\Gamma_1u = B_0u + D_1^{-1}W_1u + D_1^{-1}R_{\Gamma_1}u - \Pi_1B_0u + D_1^{-1}\Gamma_1(D_1^{-1}A_{-1}u + f).$$

Subtracting this equality from (9.18) yields that

$$(L-\mu)(I-\Gamma_0-D_1^{-1}\Gamma_1)u = -D_1^{-1}W_1u + (I-\Gamma_0-D_1^{-1}\Gamma_1)(D_1^{-1}A_{-1}u+f) \\ -(R_{\Gamma_0}u + D_1^{-1}R_{\Gamma_1}u - \Pi_1 B_0 u). \quad (9.23)$$

Now we extract the main part of the right side of this equality. To this end introduce the operator $B_1 \in \mathcal{A}$ with the symbol $b_1 = (1-\gamma_0)a_{-1} - w_1 \in C^{n-2}$. It is easy to check that $|b_1|_{n-2} \le c|\mu|^{1/2}$. Now introduce the operators

$$P_1 = -(R_{\Gamma_0} + D_1^{-1}R_{\Gamma_1}) + D_1^{-1}((I-\Gamma_0)A_{-1} - W_1 - B_1), \quad (9.24)$$

$$Q_1 = -D_1^{-1}\Gamma_1 D_1^{-1}A_{-1} - [\Gamma_0, D_1^{-1}]A_{-1} + \Pi_1 B_0. \quad (9.25)$$

In this notation, we can rewrite equality (9.23) as

$$(L-\mu)(I-\Gamma_0-D_1^{-1}\Gamma_1)u = D_1^{-1}B_1u + (I-\Gamma_0-D_1^{-1}\Gamma_1)f + P_1u + Q_1u. \quad (9.26)$$

Our next task is to estimate the remainder terms P_1u, Q_1u. We begin with P_1u. It follows from estimates of the symbols γ_j and the estimates of the remainder from Lemma 9.3 that

$$\|(1+|D_2|^r)(1+|D_1|)^{-1}(R_{\Gamma_0}u + D_1^{-1}R_{\Gamma_1}u)\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \quad (9.27)$$

for any $r, \rho \ge 0$, $r+\rho < n-4$. Next note that the operator $U \equiv B_1 + W_1 - A_{-1}$ has symbol $-\gamma_0 a_{-1}$. Since

$$(I-\Gamma_0)A_{-1} - W_1 - B_1 \equiv -\Gamma_0 A_{-1} - U$$

we obtain, from Lemma 9.2, that

$$\|(1+|D_2|^r)((I-\Gamma_0)A_{-1} - W_1 - B_1)u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \quad (9.28)$$

for any $r, \rho \ge 0$, $r+\rho < n-2$. It follows from (9.27) and (9.28) that

$$\|(1+|D_2|^r)(1+|D_1|)^{-1}P_1u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \quad (9.29)$$

for any $r, \rho \ge 0$, $r+\rho < n-4$.

Let us estimate the norm of the operator Q_1. Since the function $\Pi_1 B_0 u$ does not depend on the variable y_1, we obtain from (9.20) that

$$\|(1+|D_1|^2)\Pi_1 B_0 u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \quad (9.30)$$

for any $\rho \ge 0$, $\rho \le n-1$. Next, since

$$D_1^2(D_1^{-1}\Gamma_1 D_1^{-1}A_{-1}u) \equiv \Gamma_{1,1}D_1^{-1}A_{-1}u + \Gamma_1(I-\Pi_1)A_{-1}u$$

and $|\gamma_1|_{n-1} \le c|\mu|^{1/2}$, we obtain that

$$\|D_1^2(D_1^{-1}\Gamma_1 D_1^{-1}A_{-1}u)\|_\rho \le c|\mu|\|u\|_\rho \quad (9.31)$$

for any $\rho \le n-2$. Straightforward calculations give the following identity for the commutator $[\Gamma_0, D_1^{-1}]$:

$$D_1^2[\Gamma_0, D_1^{-1}] = D_1^2(\Gamma_0 D_1^{-1} - D_1^{-1}\Gamma_0) = \Gamma_{0,1} - 2\Gamma_{0,1}\Pi_1 + \Gamma_{0,11}D_1^{-1},$$

from which we conclude that

$$\|D_1^2[\Gamma_0, D_1^{-1}]u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \tag{9.32}$$

for any $\rho \le n-2$. The inequalities (9.30)-(9.32) imply the needed estimate for $Q_1 u$:

$$\|(1+|D_1|^2)Q_1 u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \text{ for any } 0 \le \rho \le n-2. \tag{9.33}$$

In conclusion: for μ given by the conditions of Theorem 9.5, any solution $u \in H_\rho^-$ to (9.2) is also a solution of equation (9.26). The symbol of the operator B_1 satisfies the inequality $|b_1|_{n-2} \le c|\mu|^{1/2}$. The operators P_1, Q_1 satisfy the inequalities (9.29), (9.33)).

The third step. Note that, by construction, the symbol $b_j(y,\zeta)$ is odd in y_1 if $j=1$ and even if $j=0$. Since $\Psi^\pm(y,\zeta)$ is even in y_1, we have

$$\int_{-\pi}^{\pi} \Psi^+(y,\zeta)b_1(y,\zeta_2)\,dy_1 = 0 \quad \text{for any } \; y_2.$$

Hence the differential equation $2D_1\gamma_2 + a_1\gamma_2 = b_1$ has the periodic solution

$$\gamma_2(y,\zeta) = \frac{1}{2}\Psi^-(y,\zeta)D_1^{-1}(\Psi^+(y,\zeta)b_1(y,\zeta)), \quad |D_1\gamma_2|_{n-2} + |\gamma_2|_{n-2} \le c|\mu|^{1/2}. \tag{9.34}$$

Denote by $M_2, W_2 \in \mathcal{A}$ the integro-differential operators having symbols m_2, w_2 defined by (9.4) with γ replaced by γ_2. It follows from (9.4) and our estimates of b_1 that

$$|m_2|_{n-2} + |w_2|_{n-3} \le c|\mu|^{1/2}.$$

It is clear that $B_1 = M_2$. Lemma 9.3 now implies that

$$(L-\mu)\Gamma_2 u = D_1 B_1 u + W_2 u + R_{\Gamma_2} u + \Gamma_2(D_1^{-1}A_{-1}u + f).$$

Applying the operator D_1^{-2} to both the sides of this equality we obtain

$$(L-\mu)D_1^{-2}\Gamma_2 u = D_1^{-1}B_1 u + D_1^{-2}(W_2 u + R_{\Gamma_2} u) + D_1^{-2}\Gamma_2(D_1^{-1}A_{-1}u + f).$$

Subtracting this equation from (9.26) gives

$$\begin{aligned}(L-\mu)(I - \sum_{j=0}^{2} D_1^{-j}\Gamma_j)u = -D_1^{-2}W_2 u + (I - \sum_{j=0}^{2} D_1^{-j}\Gamma_j)f +& \\ + (P_1+Q_1)u - D_1^{-2}\Gamma_2 D_1^{-1}A_{-1}u - D_1^{-2}R_{\Gamma_2}u.&\end{aligned}$$

Set

$$P_2 = P_1 - D_1^{-2}R_{\Gamma_2}, \quad Q_2 = Q_1 - D_1^{-2}W_2 u - D_1^{-2}\Gamma_2 D_1^{-1}A_{-1}u.$$

Using this notation, we can rewrite the previous equation in the form

$$(L-\mu)(I - \sum_{j=0}^{2} D_1^{-j}\Gamma_j)u = P_2 u + Q_2 u + (I - \sum_{j=0}^{2} D_1^{-j}\Gamma_j)f. \tag{9.35}$$

It follows from the estimates on the remainder given by Lemma 9.3, (9.29), (9.33) and (9.34) that

$$\|(1+|D_2|^r)(1+|D_1|)^{-1}P_2 u\|_\rho \le c|\mu|^{1/2}\|u\|_\rho \tag{9.36}$$

for any $r,\ \rho \geq 0$ with $r + \rho < n - 5$, and that

$$\|(1 + |D_1|^2) Q_2 u\|_\rho \leq c|\mu|^{1/2} \|u\|_\rho \tag{9.37}$$

for any $0 \leq \rho \leq n - 3$. Finally, we introduce the operators

$$S = (L - \mu)^{-1}(P_2 + Q_2), \quad T = I - \sum_{j=0}^{2} D_1^{-j} \Gamma_j$$

and prove that they satisfy all the requirements of Theorem 9.5.

We start with estimating the operator S. Note that, by Proposition 8.3 and (9.37),

$$\|(L - \mu)^{-1} Q_2 u\|_\rho \leq c|\mu|^{-\tau} \|(1 + |D_1|^2) Q_2 u\|_\rho \leq c|\mu|^{1/2 - \tau} \|u\|_\rho \tag{9.38}$$

for any $0 \leq \rho \leq n - 3$.

Let us estimates $(L - \mu)^{-1} P_2$. Inequality (9.36) shows that, for $r + \rho < n - 5$, P_2 is a Q1D operator of order $(1, r)$ in the space H_ρ^- and $|||P_2|||_{1,r,\rho} \leq c|\mu|^{1/2}$. It follows that $|||P_2|||_{r+2,r,\rho} \leq |||P_2|||_{1,r,\rho} \leq c|\mu|^{1/2}$, $0 \leq r \leq n - \rho - 5$.

From this and Proposition 8.3 we obtain the inequality

$$\|(L - \mu)^{-1} P_2 u\|_\rho \leq c|\mu|^{-\tau} |||P_2|||_{r+2,r,\rho} \|u\|_\rho \leq c|\mu|^{1/2 - \tau} \|u\|_\rho.$$

for any $r \geq 4$ and $r + \rho < n - 5$. Now let $r = 4$ and $\rho < n - 9$. This, along with (9.38), gives the needed estimate on S. The estimates on symbols γ_j follows from the construction of the operator T.

It follows from the above discussion that any solution $u \in H_\rho^-$, $3 \leq \rho < n - 9$, to equation (9.2) is also a solution to the equation (9.17).

It remains to show that any solution $u \in H_\rho^-$ of (9.17) with $3 \leq \rho \leq n - 9$ and $\mu > 0$ sufficiently small also satisfies equations (9.2). Suppose that u is a solution to (9.17). Set $q \equiv Lu - \sum_{j=-1}^{1} D_1^j A_j u - f$, so that u satisfies a canonical equation with $f + q$ on the right. Hence

$$(I - T)u = Su + (L - \mu)^{-1}(I - T)(f + q)$$

and $(L - \mu)^{-1}(I - T)q = 0$ which, for μ sufficiently small, implies $q = 0$. Thus Theorem 9.5 is proven. □

10 The main integro-differential equation

In this section we will prove unique solvability of the integro-differential equation

$$\frac{\partial}{\partial t}\Big(\frac{1}{a}\frac{\partial u}{\partial t}\Big) + \partial_{\xi_1} K(h_\lambda) u + R_0 u = \sigma q + f, \quad \langle u, \vartheta \rangle = 0, \tag{10.1}$$

where $K(h_\lambda)$ is a finite-depth Hilbert transform and R_0 is the integral operator

$$R_0 u(t, \xi_1) = \int_{-\pi}^{\pi} r_0(t, \xi_1, s) u(t, s) ds$$

with 2π-periodic smooth kernel r_0 and $\vartheta(t, \xi_1) = \sin nt \cos m\xi_1$. (See Problem B in Section 5 and Theorem 7.2.) Since, from (3.1), we are interested in $\varepsilon^2 = \kappa\mu(1 + \omega_2)^{1/2}$ and

$h_\lambda = (1+\varepsilon\lambda)^{-1}h_0$ it suffices, for present purposes and simplicity of presentation, to impose the following restrictions on the coefficients in the equation (10.1). Let $\bar{a} = a - g_0$.

(a) Suppose that $r \geq 25$, $5 \leq s \leq 18$ and that the functions a, r_0 are 2π-periodic and even in all variables, symmetric in the sense that $r_0(t, \xi_1, s) = r_0(t, s, \xi_1)$, and

$$|\bar{a}|_{r-2} + |r_0|_{r-4} \leq c_0\mu, \quad |h_0 - h_\lambda| \leq c_0\mu^{1/2}.$$

The function $q(t, \xi_1)$ is even in ξ_1, odd in t and, for all s with $5 \leq s \leq r - 18$,

$$|q|_s \leq c_0, \quad \langle q, \vartheta\rangle_{H_0} \geq c_0^{-1} > 0.$$

(b) The positive numbers g_0, ν, τ and μ are related by the the incommensurability condition $g_0 \in \mathcal{N}_0(\nu)$, with $\nu \in (1, 4/3)$, and the fact that $-\mu^2 \in \mathcal{M}(g_0, \tau)$, where $1 - \nu^{-1} < \tau < 4^{-1}$. Then $h_0 > 0$ is such that the dispersion equation $n^2 = g_0 m \tanh(h_0 m)$ has a unique positive integer solution (n, m). All this is possible because of Proposition 2.4.

(c) The function a and the parameters g_0, μ satisfy the relations

$$\Psi_1(a) = g_0, \ \Psi_2(a) = -\mu^2, \text{ with } \Psi_j, \ j = 1, 2, \text{ given by (3.10), (3.11).}$$

Our goal is to establish the solvability of equation (10.1) and to investigate how the regularity of its solutions depends on properties of its coefficients. Let H_s^- be defined as in Section 7 with (t, ξ_1) in the role of $y = (y_1, y_2)$. We start by proving the existence of solutions to a modified equation in the Hilbert space H_s^-.

10.1 Solvability of a modified equation

Introduce the linear operator L_a defined by

$$L_a u = \frac{\partial}{\partial t}\Big(\frac{1}{a}\frac{\partial u}{\partial t}\Big) + \partial_{\xi_1} K(h_\lambda)u + BK(h_\lambda)u, \tag{10.2}$$

where the operator $B = b_0 + b_{-1}\partial_t^{-1}$ (see (8.1)) has coefficient $b_0 = la^{-1}a^{(1)}$ and $b_{-1} = l\gamma^{(1)}$, which depend on the parameter l, and let $\gamma = a_t/a$. Now consider the integro-differential equation

$$L_a w - \sigma\vartheta_0 = f, \quad \langle w, \vartheta\rangle = 0, \tag{10.3}$$

where, in the notation of Section 8.1, $\vartheta_0 = (\Theta^{-2}\vartheta \circ Q^{-1})$.

Theorem 10.1. *Suppose that assumptions (a)-(c) are satisfied and $f \in H_s^-$. Then there exists $\delta_1 > 0$ and a constant c_1, depending only on c_0, r, such that whenever $\mu \in (0, \delta_1]$ and $-\mu^2 \in \mathcal{M}(g_0, \tau)$, equation* (10.3) *has a unique solution $w = \mathcal{R}_1 f \in H_{s-1}^-$, $\sigma = \mathcal{R}_2 f \in \mathbb{R}$. The linear operator $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) : H_s^- \to\in H_{s-1}^- \times \mathbb{R}$ is bounded and satisfies the inequality*

$$\|\mathcal{R}_1 f\|_{s-1} + |\mathcal{R}_2 f| \leq c_1\mu^{-2\tau}\|f\|_s. \tag{10.4}$$

Proof. Note that b_0 and b_{-1} have the correct parity for Theorem 8.2 and, by condition (a), $|b_j|_{r-4} \le c\mu$, with the constant c depending only on r, l and c_0 in (a). From this and the estimates on a we conclude that equation (10.3) meets all requirements of Theorem 8.2 and so the function $v = w \circ Q$ satisfies

$$Lv \equiv D_1^2 v + g_0 D_2 K v = \sum_{j=-1}^{1} D_1^j A_j v + R_2 v + R_3 v - g_0 D_2 K_1(h_\lambda) v + \sigma\vartheta + f^*. \qquad (10.5)$$

The coefficients of the operators $A_j = a_j^1 + a_j^2 K$ and the function $f^* = (\Theta^2 f) \circ Q$ are defined by Theorem 8.2, where the remainders are shown to satisfy the bounds

$$|||R_2|||_{\beta+2,\beta,s-1} \le c\mu, \quad \|(1 - D_1^2) R_3 v\|_{s-1} \le c\mu \|v\|_{s-1} \qquad (10.6)$$

for any $2\beta + s - 1 \le r - 7$, and c depends only on c_0 and r.

Now our goal is to solve (10.5) using the descent method, Theorem 9.5. To check that the operators A_j satisfy the required conditions, note from (8.9) that $A_0 = a_0^1 + a_0^2 K$ where $a_0^1 = -D_1 a_1^1$ and $a_0^2 = -D_1 a_1^2 - l(\Theta^2 a^{-1} a^{(1)}) \circ Q$. By hypothesis (a), the function Θ and a are even in all variables. Hence $a^{-1}a^{(1)}$ is odd in ξ_1. Since the diffeomorphism Q given by Lemma 8.1 respects symmetry properties, the function $(\Theta^2 a^{-1} a^{(1)}) \circ Q$ is odd in y_2 and a_0 has zero mean on the periodic rectangle $[-\pi, \pi]^2$. Similarly the product $a_1^1 a_1^2$ has zero mean. From this and Theorem 8.2 we conclude that

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big(\frac{(a_1)^2}{4} + a_0 \Big) dy = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big(\frac{(a_1^1)^2 - (a_1^2)^2}{16\pi^2} \Big) dy = \Psi_2(a) = -\mu^2.$$

Furthermore, by hypothesis (b) of this Theorem, $|a_j^i|_{r-4} \le c\mu$ with the constant c depending on c_0 and r only.

Hence the canonical equation (10.5) meets all the requirements of Theorem 9.5 with μ replaced by $-\mu^2$ and $n = r - 4$. Since $s < r - 13 = n - 9$, we conclude that (10.3) is equivalent to the equation

$$v = (S + T)v + (L + \mu^2)^{-1}(I - T)(R_2 v + R_3 v - g_0 D_2 K_1(h_\lambda) v + \sigma\vartheta + f^*), \qquad (10.7)$$

and the constraint $\langle w, \vartheta \rangle = 0$ becomes $\chi(v) = \langle v \circ Q^{-1}, \vartheta \rangle = 0$.

Hence the problem of solving (10.3) is reduced to that for equation (10.7). We now prove that the operators in this equations can be arranged as the sum of a compact self-adjoint operator and an operator having small norm in the Hilbert space H_{s-1}^-. Introduce the operators

$$U_1 = (S + T) + (L + \mu^2)((I - T)(R_2 + R_3) + g_0 T D_2 K_1(h_\lambda)),$$
$$U_2 = (L + \mu^2)^{-1} D_2 \{K_1(h_0) - K_1(h_\lambda) - \mu^2 L^{-1} K_1(h_0)\},$$

and set $V_1 = -g_0 L^{-1} D_2 K_1(h_0)$, $V_2 = U_1 + g_0 U_2$. In this notation equation (10.7) and the orthogonality condition become

$$(I - V_1)v = V_2 v + \sigma q^* + (L + \mu^2)^{-1}(I - T) f^*, \quad \chi(v) = 0 \qquad (10.8)$$

where $q^* = (L + \mu^2)^{-1}(I - T)\vartheta$. The rest of the proof is based on the following two lemmas. The proof of the first is an elementary application of the contraction mapping principle, and we omit it.

Lemma 10.2. *In the Hilbert space H, let V_1 be a compact self-adjoint operator having simple eigenvalue 1 with corresponding eigenvector ϑ, and let V_2 be a bounded linear operator. Suppose that V_2, $q^* \in H$ and a functional $\chi \in H^*$ (all depending on $\mu \in (0, \delta]$) satisfy*

$$\|V_2\| \le c_1 |\mu|^{1-2\tau}, \quad \|q^*\| + \|\chi\| \le c_1, \quad |\langle q^*, \vartheta\rangle| \ge c_2, \quad \chi(\vartheta) \ge c_2,$$

with $c_i > 0$ independent of μ. Then there exists $\delta_1 > 0$, depending only on V_1 and c_i, such that equations

$$(I - V_1)v = V_2 v + \sigma q^* + f, \quad \chi(v) = 0,$$

for any $f \in H$ and $|\mu| \le \delta_1$, have a unique solution, v, σ, with $\|v\| + |\sigma| \le c\|f\|$. Here c depends only on c_i, the exponent τ and the operator V_1. □

Lemma 10.3. *Under the assumptions of Theorem 10.1, there exists c depending only on r,s l and c_0 such that*

$$\|U_i u\|_{s-1} \le c|\mu|^{1-2\tau}\|u\|_{s-1}, \quad i = 1, 2. \tag{10.9}$$

Proof. In Theorem 9.5 the operator T is given by

$$T = \sum_{j=0}^{2} D_1^{-j}\Gamma_j, \quad \Gamma_j = \gamma_j^1 + \gamma_j^2 K \in \mathcal{A}.$$

The symbols γ_j of the operators Γ_j have the bounds $|\gamma_j|_{r-6} \le c|\mu|$, and the inequality $\|Su\|_\rho \le c|\mu|^{1-2\tau}|u|_\rho$ holds for any $u \in H_\rho^-$ and $\rho < n - 9$. Here the constant c depends only on c_0 and r. Thus we get

$$\|(S + T)v\|_{s-1} \le c\mu^{1-2\tau}\|v\|_{s-1}. \tag{10.10}$$

In order to estimate the second term in the formula for U_1 we note that

$$|||TR_2|||_{\beta+2,\beta,s-1} \le c\sum_{j=0}^{2} |||\Gamma_j R_2|||_{\beta+2,\beta,s-1}.$$

Since $|||KR_2|||_{\beta+2,\beta,s-1} = |||R_2|||_{\beta+2,\beta,s-1}$, we conclude from the estimates on the symbols γ_j and from Proposition 7.7 that

$$|||\Gamma_j R_2|||_{\beta+2,\beta,s-1} \le c\|\gamma_j\|_{r-6}|||R_2|||_{\beta+2,\beta,s-1} \le c\mu|||R_2|||_{\beta+2,\beta,s-1},$$

for any $\beta > 0$ with $2\beta + s + 1 < r - 8$. Since $D_2K_1(h_\lambda)$ is a Q1D operator of arbitrary non-negative order, we have that

$$|||\Gamma_j D_2K_1(h_\lambda)|||_{\beta+2,\beta,s-1} \le c\mu|||D_2K_1(h_\lambda)|||_{\beta+2,\beta,s-1} \le c\mu$$

for the same β. These inequalities along with (10.6) give the estimate

$$|||(I - T)R_2 + g_0 T D_2 K_1|||_{\beta+2,\beta,s-1} \le c\mu$$

for any $\beta \ge 0$, $2\beta + s < r - 9$. On the other hand, estimates of the resolvent from Proposition 8.3 imply when $\beta \ge 4$ that

$$\|(L + \mu^2)^{-1}Rv\|_{s-1} \le c\mu^{-2\tau}|||R|||_{\beta+2,\beta,s-1}\|v\|_{s-1}$$

for any Q1D operator R of order $(2,0)$ and $(\beta+2,\beta)$, where $-\mu^2 \in \mathcal{M}(g_0,\tau)$. By hypotheses (a), for $\beta = 4$ we have $2\beta + s \le r - 9$, which along with the previous inequalities yields the estimate

$$\|(L+\mu^2)^{-1}\left((I-T)R_2 + g_0 T D_2 K_1\right) v\|_{s-1} \le c\mu^{1-2\tau}\|v\|_{s-1}. \tag{10.11}$$

Next, estimate (10.6) gives

$$\|(1-D_1^2)(I-T)R_3 v\|_{s-1} \le c\|(1-D_1^2)R_3 v\|_{s-1} \le c\mu\|v\|_{s-1}.$$

Estimates of the resolvent from Proposition 8.3 imply

$$\|(L+\mu^2)^{-1}(1-D_1^2)^{-1}v\|_{s-1} \le c\mu^{-2\tau}\|v\|_{s-1}.$$

From this and the previous inequality we get

$$\|(L+\mu^2)^{-1}(I-T)R_3 v\|_{s-1} \le c\mu^{1-2\tau}\|v\|_{s-1} \tag{10.12}$$

which, along with (10.10), (10.11) implies the estimate (10.9) for U_1.

Now we estimate the norm of the operator U_2. In terms of Fourier transforms it is represented as $\mathbf{F}U_2 v(\zeta) = U_2(\zeta)\mathbf{F}v(\zeta)$, where

$$|U_2(\zeta)| = |d(\zeta)+\mu^2|^{-1}|\zeta_2||\tanh(\zeta_2 h_0) - \tanh(\zeta_2 h_\lambda) - \mu^2 d(\zeta)^{-1}(\tanh(\zeta_2 h_0) - 1)|,$$

for $\zeta \ne 0$, and $U_2(0) = 0$. From estimates (2.22) and (2.25) and condition (a)

$$|U_2(\zeta)| \le c|\mu|^{-2\tau}(1+|\zeta_2|^2)(|h_0-h_\lambda| + \mu^2)e^{-c(h_0)|\zeta_2|} \le c|\mu|^{1-2\tau}e^{-c(h_0)|\zeta_2|} \le c|\mu|^{1-2\tau}.$$

So U_2 satisfies (10.9) and the lemma follows. □

Now we can return to the proof of Theorem 10.1.

Note that, in terms of Fourier transforms, V_1 is the operator of multiplication by

$$V_1(\zeta) = -g_0 d(\zeta)^{-1}|\zeta_2|(\tanh|h_0\zeta_2| - 1) \text{ for } \zeta \in \mathbb{Z}^2 \setminus \{0\} \quad V_1(0) = 0.$$

Estimates of d from (2.22) imply that, in the parabolic sector $|\zeta_1|^2 \le 2g_0|\zeta_2| + 1$,

$$|V_1(\zeta)| \le c(1+|\zeta_2|)^2 e^{-c|\zeta_2|} \le ce^{-c|\zeta|} \to 0 \text{ as } |\zeta| \to \infty,$$

and outside this sector the definition of d gives that

$$|V_1(\zeta)| \le c(1+|\zeta_1|)^{-2}e^{-c|\zeta_2|} \to 0 \text{ as } |\zeta| \to \infty.$$

Since $V_1(\zeta)$ vanishes at infinity and takes real values, V_1 is a compact self-adjoint operator in every space H_s^-. It follows from condition (b) that ϑ is the unique eigenfunction of V_1 in the space H_{s-1}^- corresponding to the eigenvalue 1. It remains to show that the function q^* also satisfies the hypotheses of Lemma 10.2. Using the identity $(L+\mu^2)^{-1} = L^{-1} - \mu^2(L+\mu^2)^{-1}L^{-1}$ we obtain

$$\|q^*\|_{s-1} \le \|(L^{-1} - \mu^2(L+\mu^2)^{-1}L^{-1})\vartheta\|_{s-1} + \|(L+\mu^2)^{-1}T\vartheta\|_{s-1}.$$

From estimates on the resolvent of L from Proposition 8.3 and estimates on Γ_j from Theorem 9.5 this gives

$$\|q^*\|_{s-1} \le c\{\|\vartheta\|_s + |\mu|^{2-2\tau}\|\vartheta\|_{s+1} + |\mu|^{1-2\tau}\|\vartheta\|_s\} \le c.$$

Hence the coefficient $\|q^*\|_{s-1}$ has a bound which depends only on r and the constant c_0 from condition (a). Next,

$$\langle q^*, \vartheta\rangle_{s-1} = \langle (I-T)\vartheta, (1+|\nabla|)^{2s-2}(L+\mu^2)^{-1}\vartheta\rangle_0.$$

It is obvious that $(1+|\nabla|)^{2s-2}(L+\mu^2)^{-1}\vartheta = p(\mu)\vartheta$ with the function

$$|p(\mu)| = (1+|(n,m)|)^{2s-2}|g_0 m(\tanh(mh_\lambda)-1) - \mu^2|^{-1},$$

where $p(\mu) \to p(0) \neq 0$ as $\mu \to 0$. Hence, for $\mu > 0$ sufficiently small,

$$|\langle q^*, \vartheta\rangle_{s-1}| \ge |p(\mu)|\{\langle \vartheta, \vartheta\rangle_0 - \langle T\vartheta, \vartheta\rangle_0\} \ge c > 0.$$

Hence, for sufficiently small μ, the function q^* satisfies the conditions of Lemma 10.2. Therefore, for $\mu \in (0, \delta_1]$, with $-\mu^2 \in \mathcal{M}(g_0, \tau)$ for suitably small $\delta_1 > 0$, equation (10.8) has a unique solution $\sigma \in \mathbb{R}$, $v \in H^-_{s-1}$, and this solution satisfies

$$|\sigma| + \|v\|_{s-1} \le c\|(L+\mu^2)^{-1}(I-T)f^*\|_s.$$

Estimates on T in the descent method, Theorem 9.5 and estimates on the resolvent of L from Proposition 8.3 imply

$$\|(L+\mu^2)^{-1}(I-T)f^*\|_{s-1} \le c|\mu|^{-2\tau}\|f^*\|_s.$$

Next we note that $w = v \circ P$ and $f^* = (\Theta^2 f) \circ Q$, where the diffeomorphism $Q = P^{-1}$ is given by Lemma 8.1. Under the assumption (a), P and Q are uniformly C^{r-2}-bounded. Hence $\|w\|_{s-1} \le c\|v\|_{s-1}$ and $\|f^*\|_s \le c\|f\|_s$ which, along with the previous inequality, gives the needed estimate on w and the theorem follows. □

10.2 Extended system

Our next goal is to prove unique solvability of the integro-differential equation (10.1) in the space H^-_{s+l-1} for s with $5 \le s \le r-18$ where $r \ge 25$ and $l \ge 0$. To this end we replace it by a equivalent extended system of operator equations for derivatives of u and a parameter σ. Multiplying both the sides of equation (10.1) by a we rewrite it in the form

$$0 = M(u,\sigma) \equiv u_{tt} - \gamma u_t + a\partial_{\xi_1}K(h_\lambda)u + a(R_0 u - \sigma q - f). \tag{10.13}$$

Let $\chi_0 = \chi_1 = 0$ and for integers $l \ge 0$ define linear operators by

$$\Psi_l(u,\sigma) = \psi_l + l\gamma^{(1)}\partial_t^{-1}(\psi_{l-1}) + l\gamma^{(1)}\varphi_l, \quad l \ge 0, \tag{10.14}$$

$$\Phi_l(u,\sigma) = \partial_t^{l-2}(\gamma u_t - a\partial_{\xi_1}K(h_\lambda)u - aR_0 u + aq\sigma), \tag{10.15}$$

$$\psi_l(u,\sigma) = \chi_l(u) + R_l(u,\sigma), \tag{10.16}$$

where

$$R_l = a^{-1}(-aR_0u + \sigma aq)^{(l)}, \quad l \geq 0,$$
$$\varphi_l(u) = (l-1)\partial_t^{-1}\{a^{-1}(\gamma^{(1)}u_t^{(l-2)} - a^{(1)}K(h_\lambda)u^{(l-1)})\} - \Pi_t(\frac{a_t}{a^2}u^{(l-1)}), \quad l \geq 1,$$
$$\chi_l(u) = \frac{1}{a}\sum_{k=0}^{l-2} C_k^l\big(\gamma^{(l-k)}u_t^{(k)} - a^{(l-k)}K(h_\lambda)u^{(k+1)}\big), \quad \text{for } l \geq 2, \quad C_k^l = \frac{l!}{(l-k)!k!}.$$

Here superscript (l) stands for the l^{th} derivative with respect to ξ_1. Henceforth let l denote an arbitrary, but fixed, even integer and denote by Δ_l and G_l the differential operators given by the formulae

$$\Delta_l = \rho_0\partial_{\xi_1}^l + \partial_t^l, \quad G_l = \Delta_l + g_0K(h_\lambda)\partial_{\xi_1}\partial_t^{l-2} \quad \text{with} \quad \rho_0 = (2g_0)^{l/2}.$$

With L_a given by (10.2), consider the following system of the operator equations for functions w, u and a parameter σ

$$L_aw - \Psi_l(u,\sigma) = f_l \equiv a^{-1}(af)^{(l)} + l\gamma^{(1)}\partial_t^{-1}(a^{-1}(af)^{(l-1)}), \quad \langle w, \vartheta\rangle = 0, \tag{10.17}$$
$$\Delta_lu - \Phi_l(u,\sigma) - \rho_0w = F_l \equiv \partial_t^{l-2}(af). \tag{10.18}$$

Note that, using Theorem 10.1, we can eliminate the function w and rewrite the extended system as an equation for the function u and a parameter σ

$$u = \Delta_l^{-1}\Big(\rho_0\mathcal{S}_1(u,\sigma) + \Phi_l(u,\sigma)\Big) + \mathcal{F}_1, \quad \sigma = \mathcal{S}_2(u,\sigma) + \mathcal{F}_2. \tag{10.19}$$

Here the linear operators $\mathcal{S}_i$ and the functions $\mathcal{F}_i$ are defined by the equalities

$$\mathcal{S}_i = \mathcal{R}_i\Big(\Psi_l(u,\sigma) - \sigma\vartheta_0\Big), \quad \mathcal{F}_1 = \Delta_l^{-1}(\rho_0\mathcal{R}_1f_l + F_l), \quad \mathcal{F}_2 = \mathcal{R}_2f_l.$$

The operators $\mathcal{R}_i$ and the function ϑ_0 are given in Theorem 10.1.

Proposition 10.4. *There exists $\delta_2 > 0$, depending only on r, l and the constant c_0 from condition (a), such that whenever $\mu \in (0, \delta_2]$ and $u \in H_{s+l-1}^-$, $w \in H_{s-1}^-$, σ satisfy the equations* (10.17), (10.18), *u and σ solve the* (10.1) *and $w = u^{(l)}$.*

Proof. It suffices to show, under the assumptions of the proposition, that $M(u,\sigma) = 0$ and u satisfies the orthogonality conditions. This is based on the following two identities.

Lemma 10.5. *Whenever $u, f \in H_{2+l}$ and $\sigma \in \mathbb{R}$,*

$$L_au^{(l)} - \Psi_l(u,\sigma) - f_l = a^{-1}M(u,\sigma)^{(l)} + l\gamma^{(1)}\partial_t^{-1}(a^{-1}M(u,\sigma)^{(l-1)}), \tag{10.20}$$
$$\partial_t^lu - \Phi_l(u,\sigma) - F_l \equiv \partial_t^{l-2}M(u,\sigma). \tag{10.21}$$

Proof. Differentiate the expression for M in (10.13) l times with respect to ξ_1 to obtain

$$w_{tt} - \gamma w_t + a\partial_{\xi_1}K(h_\lambda)w = (-aR_0u + \sigma aq + af)^{(l)} + \sum_{k=0}^{l-1} C_k^l\big(\gamma^{(l-k)}u_t^{(k)} - a^{(l-k)}K(h_\lambda)u^{(k+1)}\big) + M^{(l)}$$

where $w = u^{(l)}$. After dividing by a this equation becomes

$$\left(\frac{1}{a}w_t\right)_t + \partial_{\xi_1}K(h_\lambda)w + la^{-1}\left(a^{(1)}K(h_\lambda)w - \gamma^{(1)}u_t^{(l-1)}\right) = \psi_l + a^{-1}((af)^{(l)} + M^{(l)}). \tag{10.22}$$

Now our task is to eliminate the term $a^{-1}u_t^{(l-1)}$. Replacing l by $l-1$ in (10.22) gives

$$\left(\frac{1}{a}u_t^{(l-1)}\right)_t + K(h_\lambda)w = \psi_{l-1} + a^{-1}((af)^{(l-1)} \\ + M^{(l-1)}) + (l-1)a^{-1}(\gamma^{(1)}u_t^{(l-2)} - a^{(1)}K(h_\lambda)u^{(l-1)}).$$

Note that

$$\partial_t^{-1}\partial_t\left(\frac{1}{a}u_t^{(l-1)}\right) = \frac{1}{a}u_t^{(l-1)} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a_t}{a^2}u^{(l-1)}(t,\xi_1)dt.$$

From this we obtain that

$$\frac{1}{a}u_t^{(l-1)} = \partial_t^{-1}[-K(h_\lambda)w + \psi_{l-1} + a^{-1}((af)^{(l-1)} + M^{(l-1)})] + \varphi_l,$$

which, substituted in (10.22), yields (10.20).

Differentiation of M (given in (10.13)) $l-2$ times with respect to t gives the identity (10.21) and the lemma follows. $\square$

Lemma 10.6. *For any $k \geq 0$ the operator Δ_l has a bounded inverse $\Delta_l^{-1} : H_k^- \to H_{k+l}^-$ and*

$$\|u\|_{k+l} \leq c(k,l)\|\Delta_l u\|_k, \quad \|\Delta_l u\|_0 \leq 2\|G_l u\|_0. \tag{10.23}$$

Proof. In terms of Fourier transforms, Δ_l and G_l are operators of multiplication by

$$\mathcal{D}_l(\zeta) = (-1)^{l/2}(|\zeta_1|^l + \rho_0|\zeta_2|^l) \quad \text{and} \quad \mathcal{G}_l(\zeta) = \mathcal{D}_l(\zeta) + (-1)^{(l-2)/2}g_0\zeta_2|\zeta_1|^{l-2}\tanh(h_\lambda\zeta_2).$$

The first estimate follows from the representation

$$\mathbf{F}\Delta_l^{-1}u(\zeta) = \mathcal{D}_l^{-1}\mathbf{F}u(\zeta) \text{ for } \zeta \neq 0, \quad \mathbf{F}\Delta_l^{-1}u(0) = 0.$$

In order to prove the second note that

$$|\mathcal{G}_l(\zeta)| \geq |\mathcal{D}_l(\zeta)| - g_0|\zeta_2||\zeta_1|^{l-2}.$$

Applying the Young inequality gives

$$g_0|\zeta_2||\zeta_1|^{l-2} \leq 2^{-1}(|\zeta_1|^l + 2^{l/2}g_0^{l/2}|\zeta_2|^{l/2}) \leq 2^{-1}|D_l(\zeta)|,$$

which implies $2^{-1}|\mathcal{D}_l(\zeta)| \leq |\mathcal{G}_l(\zeta)|$ and the lemma follows. $\square$

Now we can complete the proof of Proposition 10.4 as follows. Identities (10.20) and (10.21) along with equations (10.17) and (10.18) imply

$$L_a(w - u^{(l)}) = -a^{-1}M^{(l)} - l\gamma^{(1)}\partial_t^{-1}(a^{-1}M^{(l-1)}), \quad \rho_0(w - u^{(l)}) = \partial_t^{l-2}M.$$

From this we conclude that

$$L_a \partial_t^{l-2} M = -\rho_0 a^{-1} M^{(l)} - \rho_0 l \gamma^{(1)} \partial_t^{-1} (a^{-1} M^{(l-1)}).$$

Multiplying both sides by a, we can rewrite this equality in the form $G_l M = R$, where

$$R = \gamma \partial_t^{l-1} M - \bar{a} \partial_{\xi_1} K(h_\lambda) \partial_t^{l-2} M - aBK(h_\lambda) \partial_t^{l-2} M - \rho_0 a l \gamma^{(1)} \partial_t^{-1} (a^{-1} M^{(l-1)}).$$

It is easy to check that$\|R\|_0 \leq |\mu| c(c_0) \|M\|_{l-1}$. From this the previous identity and (10.23) we obtain

$$\|M\|_l \leq \|\Delta_l M\|_0 \leq 2c \|G_l M\|_0 = 2c\|R\|_0 \leq \mu c \|M\|_{l-1}.$$

Hence $M = 0$ for μ sufficiently small, which along with (10.13) implies (10.1) and Proposition 10.4 follows. □

10.3 Unique solvability of the extended system

We are now in a position to formulate and prove the main result of this section on the unique solvability of the system (10.17), (10.18) and integro-differential equation (10.1).

Theorem 10.7. *Suppose the assumptions (a)-(c) are satisfied and $f \in H^-_{s+l}$, with even $l \geq 0$. Then there exist $\delta > 0$ and a constant c_1, both depending only on c_0, r, l, such that whenever $\mu \in (0, \delta]$ with $-\mu^2 \in \mathcal{M}(g_0, \tau)$, equations* (10.17), (10.18), *and hence equation* (10.1), *has a unique solution satisfying*

$$\|u\|_{s+l-1} + |\sigma| \leq c_1 \mu^{-4\tau} \{\|f\|_{s+l} + \|f\|_s (|\bar{a}|_{r+l-2} + |q|_{s+l} + |r_0|_{r+l-4})\}. \tag{10.24}$$

The proof will be divided into the three steps.

The First step is to show that all operators involved in equation (10.19) are well-defined and to estimates their norms. To deal with $\Psi_l(u, \sigma)$ and $\Phi_l(u, \sigma)$ in the formulae (10.14) and (10.15) we need a technical observation which is proved in the Appendix.

Lemma 10.8. *Let P be the integral operator*

$$Pu(t, \xi_1) = \int_{-\pi}^{\pi} p(t, \xi_1, s) u(t, s) ds$$

with a smooth 2π-periodic kernel $p \in C^\rho(\mathbb{R}^3)$. Then for any non-negative integer ρ

$$\|Pu\|_\rho \leq c(\|p\|_{C(\mathbb{R}^3)} \|u\|_\rho + \|p\|_{C^\rho(\mathbb{R}^3)} \|u\|_0) \leq c \|p\|_{C^\rho(\mathbb{R}^3)} \|u\|_\rho$$

with the the constant c depending only on ρ. □

Lemma 10.9. *Under the assumptions of Theorem 10.7*

$$\begin{aligned}
&\|\Psi_l\|_s \leq c\mu \|u\|_{s+l-1} + c|\sigma|\, |aq|_{s+l} + \|u\|_{s-1} \left(|\bar{a}|_{r+l-2} + |r_0|_{r+l-4}\right), \\
&\|\Phi_l(u, \sigma)\|_s \leq c\|u\|_{s+l-1} + c\|u\|_1 (|\bar{a}|_{s+l-1} + |r_0|_{r+l-4}) + |\sigma| \|aq\|_{s+l-2}, \\
&\|f_l\|_s \leq c\|f\|_{s+l} + c\|f\|_0 |\bar{a}|_{s+l}, \quad \|F_l\|_s \leq c\|f\|_s + c\|f\|_0 |a|_s,
\end{aligned}$$

where the constant c depends only on c_0, r, and l.

Proof. Note from condition (a) that the function a is close to the constant g_0 and for any $\rho \geq 0$, if a sufficiently smooth,

$$|\gamma^{(1)}|_\rho + |a^{-2}a_t|_\rho \leq c|\bar{a}|_{\rho+2} \tag{10.25}$$

in which $\bar{a} = a - g_0$, ρ is an arbitrary non-negative integer, and c depends only on ρ, r and c_0. We estimate step-by-step the terms involved in Ψ_l starting with φ_l. Since $\|fu\|_s \leq c(s)|f|_s\|u\|_s$, which is valid for every non-negative integer s,

$$\|\varphi_l\|_s \leq c|a^{-1}\gamma^{(1)}|_s\|u_t^{(l-2)}\|_s + c|a^{-1}a^{(1)}|_s\|u^{(l-1)}\|_s + |a^{-2}a_t|_s\|u^{(l-1)}\|_s.$$

We conclude from this and (10.25) that

$$\|\varphi_l\|_s \leq c\|u\|_{s+l-1}|\bar{a}|_{r-2} \leq c\mu\|u\|_{s+l-1}. \tag{10.26}$$

Let us now estimate the function χ_l. First observe that

$$\|\chi_l\|_s \leq c(l)\sum_{k=0}^{l-2}|\bar{a}|_{s+1+l-k}\|u\|_{s+k+1},$$

and recall the interpolation inequalities

$$|\bar{a}|_{s+l+1-k} \leq c|\bar{a}|_{l+s+3}^{1-\alpha}|\bar{a}|_{s+3}^{\alpha}, \quad \|u\|_{s+k+1} \leq c\|u\|_{s-1}^{1-\alpha}\|u\|_{s+l-1}^{\alpha},$$

with the exponent $\alpha = (k+2)/l$. Hence by Young's inequality

$$\begin{aligned}|\bar{a}|_{s+l+1-k}\|u\|_{s+k+1} &\leq c(|\bar{a}|_{l+s+3}\|u\|_{s-1})^{1-\alpha}(|\bar{a}|_{s+3}\|u\|_{s+l-1})^{\alpha}\\ &\leq c|\bar{a}|_{l+s+3}\|u\|_{s-1} + |\bar{a}|_{s+3}\|u\|_{s+l-1}.\end{aligned}$$

Since $s \leq r-18$, we have $|\bar{a}|_{s+3} \leq c|\bar{a}|_{r-2} \leq c\mu$ which, along with the previous inequality, gives the needed estimate on χ_l

$$\|\chi_l\|_s \leq c\mu\|u\|_{s+l-1} + c|\bar{a}|_{r+l-2}\|u\|_{s-1}. \tag{10.27}$$

Next, inequalities (10.25), (10.26) and (10.27) yield the following estimate on the group of terms involving χ_l, χ_{l-1} and φ_l

$$\begin{aligned}&\|\chi_l + l\gamma^{(1)}(\partial_t^{-1}\chi_{l-1} + \varphi_l)\|_s\\ &\qquad\leq c(\|\chi_l\|_s + \|\chi_{l-1}\|_s + \|\varphi_l\|_s) \leq c\mu\|u\|_{s+l-1} + c|\bar{a}|_{r+l-2}\|u\|_{s-1}.\end{aligned} \tag{10.28}$$

Now our task is to estimate R_l. Recall that

$$aR_l = \sigma(aq)^{(l)} - \int_{-\pi}^{\pi}(ar_0)^{(l)}u(t,s)ds.$$

Since $|a^{-1}|_{r-2} \leq c$, Lemma 10.8 gives

$$\|R_l\|_s \leq c(|\sigma||aq|_{l+s} + |ar_0|_{l+s}\|u\|_s).$$

We examine the last term in more detail. By interpolation,

$$|ar_0|_{s+l} \leq c|ar_0|_{l+s+1}^{\alpha}|ar_0|_{s+1}^{1-\alpha}, \quad \|u\|_s \leq c\|u\|_{s-1}^{\alpha}\|u\|_{s+l-1}^{1-\alpha}$$

with the exponent $\alpha = (l-1)/l$. Hence, by Young's inequality and condition (a) of the theorem,

$$|ar_0|_{s+l}\|u\|_s \le c(|ar_0|_{s+l+1}\|u\|_{s-1} + |ar_0|_{s+1}\|u\|_{s+l-1}) \\ \le c(|ar_0|_{l+r-4}\|u\|_{s-1} + c\mu\|u\|_{s+l-1}). \quad (10.29)$$

From this we obtain

$$\|R_l\|_s \le c\mu\|u\|_{s+l-1} + c|\sigma||aq|_{s+l} + c|ar_0|_{l+r-4}\|u\|_{s-1}, \quad (10.30)$$

which gives an estimate on the group of terms in Ψ_l which involve R_l, R_{l-1},

$$\|R_l + l\gamma^{(1)}\partial_t^{-1}R_{l-1}\|_s \le c\mu\|u\|_{s+l-1} + c|\sigma||aq|_{s+l} + c|ar_0|_{l+r-4}\|u\|_{s-1}.$$

This inequality, with (10.28) and estimates on a product of smooth functions, implies the needed estimate on Ψ_l.

Let us estimate $\Phi_l(u,\sigma)$. It follows from formula (10.15) that

$$\|\Phi_l(u,\sigma)\|_s \le \|\gamma u_t - a\partial_{\xi_1}K(h_\lambda)u - aR_0u + \sigma aq\|_{s+l-2}.$$

From Lemma 10.8 and the inequality $\|au\|_\rho \le c(\rho)(|a|_\rho\|u\|_0 + |a|_0\|u\|_\rho)$, which is valid for all functions a, u, and $\rho \ge 0$, we can now conclude that

$$\|\Phi_l(u,\sigma)\|_s \le c\|u\|_{s+l-1} + c\|u\|_1(|\bar a|_{s+l-1} + |r_0|_{r+l-4}) + |\sigma|\|aq\|_{s+l-2}.$$

This is the desired conclusion. It remains to note that the estimates on f_l and F_l are consequences of the estimates on products of smooth functions. □

The next lemma shows that the linear operators in equation (10.19) are well-defined in the space H^-_{s+l-1}

Lemma 10.10. *Let $\delta_3 = \min\{\delta_1,\delta_2\}$ where δ_j, $j=1,2$ are the constants from Theorem 10.1 and Proposition 10.4. Then for any $\mu \in (0,\delta_3]$ with $-\mu^2 \in \mathcal{M}(g_0,\tau)$ the operators $\Delta_l^{-1}\mathcal{S}_1$, $\Delta_l^{-1}\Phi_l$ and $\mathcal{S}_2$ are well-defined in the space $H_{s+l-1}\times\mathbb{R}$ and*

$$|\mathcal{S}_2(u,\sigma)| + \|\Delta_l^{-1}\mathcal{S}_1(u,\sigma)\|_{s+l-1} \\ \le c\mu^{-2\tau}\{\mu\|u\|_{s+l-1} + |\sigma|(1+|aq|_{s+l}) + \|u\|_{s-1}(|\bar a|_{r+l-2} + |r_0|_{r+l-4})\},$$

$$\|\Delta_l^{-1}\Phi_l(u,\sigma)\|_{s+l} \le c\|u\|_{s+l-1} + c\|u\|_1(|\bar a|_{s+l-1} + |r_0|_{r+l-4}) + c|\sigma|\|aq\|_{s+l-2},$$
$$\|\mathcal{F}_1\|_{s+l-1} + |\mathcal{F}_2| \le c\mu^{-2\tau}\{\|f\|_{s+l} + \|f\|_0|\bar a|_{s+l}\}.$$

Proof. Note that, under the assumptions of the lemma, the operator L_a meets all requirements of Theorem 10.1 from which it follows that the operators $\mathcal{R}_i$ are well defined and have bounds (10.4). From this and (10.23) we obtain

$$|\mathcal{R}_2\varphi| + \|\Delta_l^{-1}\mathcal{R}_1\varphi\|_{s+l-1} \le c\mu^{-2\tau}c\|\varphi\|_s, \quad (10.31)$$

which, with the estimate on Ψ_l from Lemma 10.9, implies the needed estimates on $\Delta_l^{-1}\mathcal{S}_1$ and $\mathcal{S}_2$. The estimate on $\Delta_l^{-1}\Phi_l$ is a consequence of (10.23) and the estimate on Φ_l from Lemma 10.9. It remains to note that the estimates on $\mathcal{F}_i$ follows directly from (10.31) and Lemma 10.9. □

The second step is to prove that solutions of the extended system (10.17)-(10.18) satisfy inequalities(10.24). First we estimate $\|u\|_0$ and σ.

Lemma 10.11. *The inequality*

$$\|u\|_0 + |\sigma| \leq c\mu^{1/2}\|u\|_{s-1} + \|f\|_s$$

holds for any $\mu \in (0, \delta_3]$, $-\mu^2 \in \mathcal{M}(g_0, \tau)$ *and solution* $u \in H_{s+l-1}$, σ *to equations* (10.17)-(10.18)*. Here* δ_3 *is given by Lemma 10.10.*

Proof. Note that u, σ satisfy equation (10.1). Rewrite it in the following form

$$u_{tt} + g_0\partial_{\xi_1}K(h_0)u = g_0\left(-R_0 + \partial_{\xi_1}(K(h_0) - K(h_\lambda))\right)u \\ + g_0\sigma q + g_0\left((g_0^{-1} - a^{-1})u_t\right)_t + g_0 f \equiv G.$$

Now, interpreting (t, ξ_1) as (y_1, y_2), it is easily seen that under the present hypotheses Lemma 8.4 and the preceding identity give the inequality $\|u\|_0 \leq c\|G\|_1$ which, along with estimates on a, q and r_0 from condition (a), implies that

$$\|u\|_0 \leq c|\sigma|\|q\|_{C^1(\mathbb{R}^2)} + c\mu\|u\|_3 + c\|\partial_{\xi_1}(K(h_0) - K(h_\lambda))u\|_1 + \|f\|_1.$$

Now suppose that the variables of the Fourier transform corresponding to (t, ξ_1) are (ζ_1, ζ_2). Then

$$|\mathbf{F}\partial_{\xi_1}(K(h_0) - K(h_\lambda))u(\zeta)| = |\zeta_2|\Big|\tanh h_0|\zeta_2| - \tanh h_\lambda|\zeta_2|\Big||\mathbf{F}u(\zeta)| \\ \leq c(h_0)|h_0 - h_\lambda||\mathbf{F}u(\zeta)| \leq c\mu^{1/2}|\mathbf{F}u(\zeta)|.$$

Hence $\|\partial_{\xi_1}(K(h_0) - K(h_\lambda))u\|_1 \leq c\mu^{1/2}\|u\|_1$, which implies

$$\|u\|_0 \leq c|\sigma| + c\mu^{1/2}\|u\|_{s-1} + \|f\|_1. \tag{10.32}$$

Next multiplying both the sides of (10.1) by ϑ and integrating over the square of the periods we obtain

$$\sigma\langle q, \vartheta\rangle = \langle u, \left((a^{-1} - g_0^{-1})\vartheta_t\right)_t + R_0\vartheta\rangle - \langle f, \vartheta\rangle$$

which, with condition (a), implies that $|\sigma| \leq c\mu\|u\|_0 + \|f\|_0$. It now follows by (10.32) that, for sufficiently small μ, u and σ satisfies the needed inequality. $\square$

We are now in a position to complete the proof of inequality (10.24).

Lemma 10.12. *Under the assumptions of Theorem 10.7, there exists* $\delta_4 \in (0, \delta_3]$ *and a constant* c_1*, depending only on* c_0*,* r*,* l*, such that inequality* (10.24) *holds for any solution* $u \in H_{s+l-1}$*,* $\sigma \in \mathbb{R}$*,* $w \in H_{s-1}$ *to equations* (10.17)-(10.18) *and* $\mu \in (0, \delta_4]$ *with* $-\mu^2 \in \mathcal{M}(g_0, \tau)$*.*

Proof. By Lemma 10.10, all terms in equation (10.19) are well defined and u, σ is a solution. It follows from Lemma 10.10 that

$$\|u\|_{s+l-1} \leq c\mu^{1-2\tau}\|u\|_{s+l-1} + c\|u\|_{s+l-2} + c\mu^{-2\tau}(\|f\|_{s+l} + |\sigma|(1 + |aq|_{s+l}) \\ + c\mu^{-2\tau}(\|u\|_{s-1} + \|f\|_s)\left(|r_0|_{r+l-4} + |\bar{a}|_{r+l-2}\right).$$

Note that for any $\varepsilon > 0$

$$\|u\|_{s+l-2} \leq \varepsilon\|u\|_{s+l-1} + c\|u\|_0,$$

where the constant c depends only on ε, s and l. Choosing δ_4 and ε sufficiently small we conclude that the inequality

$$\|u\|_{s+l-1} \le c\|u\|_0 + c\mu^{-2\tau}(\|f\|_{s+l} + |\sigma|(1 + |aq|_{s+l}) \\ + c(\|u\|_{s-1} + \|f\|_s)\mu^{-2\tau}\left(|r_0|_{r+l-4} + |\bar{a}|_{r+l-2}\right) \quad (10.33)$$

holds for any $\mu \in (0, \delta_4)$, $-\mu^2 \in \mathcal{M}(g_0, \tau)$, with the constant c depending only on r, l and c_0, g_0, h_0.

Setting $l = 0$ and using estimates from condition (a) we obtain

$$\|u\|_{s-1} \le c\|u\|_0 + c\mu^{-2\tau}(\|f\|_s + |\sigma|) + c\mu^{1-2\tau}\|u\|_{s-1}.$$

From this and estimates on σ and $\|u\|_0$ from Lemma 10.11 we get the inequality $(1 - c\mu^{1/2-2\tau})\|u\|_{s-1} \le c\mu^{-2\tau}\|f\|_s$. Since $1 > 4\tau$, we conclude that for $\mu \in (0, \delta_4]$ with sufficiently small δ_4, $\|u\|_{s-1} \le c\mu^{-2\tau}\|f\|_s$. From Lemma 10.11 we then get the estimate $|\sigma| \le c\|f\|_s$, and (10.24) follows from (10.33). This completes the proof of Theorem 10.7. □

The third step is to prove the solvability of the extended system. Recall from Lemma 10.10 that (10.17)-(10.18) for $\mu \in (0, \delta_4]$ is equivalent to (10.19). which can be rewritten in the form

$$(I - \mathcal{B})(u, \sigma) = \mathcal{F} \equiv (\mathcal{F}_1, \mathcal{F}_2) \in H^-_{s+l-1} \times \mathbb{R}, \quad (10.34)$$

where the operator $\mathcal{B} : H^-_{s+l-1} \times \mathbb{R} :\to H^-_{s+l-1} \times \mathbb{R}$ is defined by

$$\mathcal{B}(u, \sigma) = \Big(\Delta_l^{-1}(\rho_0 \mathcal{S}_1(u, \sigma) + \Phi_l(u, \sigma)), \mathcal{S}_2(u, \sigma)\Big).$$

Estimates from Lemma 10.10 now imply the inequality

$$\|\mathcal{B}(u, \sigma)\|_{H_{s+l-1}\times\mathbb{R}} \le c|\mu|^{1-2\tau}\|(u, \sigma)\|_{H_{s+l-1}\times\mathbb{R}} + e\|(u, \sigma)\|_{H_{s+l-2}\times\mathbb{R}},$$

where the constant c depends only on the exponent r, l and the constant c_0 from the conditions (a)-(c) and the constant e depends also on μ, the function a and the kernel r_0.

Since the embedding $H_{s+l-1} \times \mathbb{R} \to H_{s+l-2} \times \mathbb{R}$ is compact, the operator $\mathcal{B}$ is a k-set contraction on $H^-_{s+l-1} \times \mathbb{R}$ with $0 \le k \le c|\mu|^{1-2\tau}$. (See [3, Section 9, Chapter 2].) Therefore [3, Theorem 9.9, Chapter 2], the operator $I - \mathcal{B}$ is a Fredholm operator for any $\mu \le \delta_4$ satisfying the inequality $c|\mu|^{1-2\tau} < 1$. But from Lemma 10.12 we know that equation (10.34) has at most one solution for $\mu \le \delta_4$, $-\mu^2 \in \mathcal{M}(g_0, \tau)$. From this and the Fredholm alternative we conclude that, for sufficiently small μ with $-\mu^2 \in \mathcal{M}(g_0, \tau)$, equation 10.34 has a unique solution for any $\mathcal{F} \in H_{s+l-1} \times \mathbb{R}$. This completes the proof of Theorem 10.7. □

The proof of Theorem 6.4. Note that the conditions of Theorem 7.2 coincide with condition (a) of Theorem 6.4. It follows that under the assumptions of Theorem 6.4 the Neumann operator N has representation $\partial_{\xi_1} K(h_\lambda) + R_0$, in which R_0 is a Fredholm operator with the kernel r_0. By Theorem 7.2 this kernel is 2π-periodic, symmetric, and $|r_0|_{r+l-4} \le c\mu(1 + \|\vec{U}\|_{E_{r+l}})$. Hence the equation (5.15) can be re-written in the form (10.1) and its coefficient meet all requirements of Theorem 10.1. It follows from Theorem 10.1, that for any $\mu \in (0, \delta_2]$, $-\mu^2 \in \mathcal{M}$ and $f \in H^-_{s+l}$, $5 \le s \le r - 18$, this equation has a unique solution which is orthogonal to ϑ and satisfies the inequality (10.24) for even l. To obtain this inequality for all $l \ge 0$ replace r, s and l with $r - 1$, $s - 1$ and $l + 1$. It remains to note that $\|f\|_{s+l} \le c|f|_{s+l}$, $|u|_{s+l-3} \le c\|u\|_{s+l-1}$ and the theorem follows. □

A Appendix

A.1 Proof of Theorem 3.2 (Nash-Moser theorem)

We obtain the implicit function theorem as a consequence of the following proposition. Choose $a \geq 0$ and $b \geq 1$ satisfying the inequalities

$$b\Big(\frac{1}{2} - \frac{d+1}{a}\Big) > 3d + 2\varrho \text{ and } a > 3(d+b+2\varrho) + 4 \quad \text{where } d = r + k - \rho. \tag{A.35}$$

In what follows, (i)-(iii) refers to properties of scales of Banach spaces from Section 3.2 and (a)-(c) refers the hypotheses of Theorem 3.2. The proof depends on the following preliminary observation.

Theorem A.1. *Suppose that (a), (b) and (c) hold and that, for some $\mu_1 > 0$, equation $\Phi(u,\mu) = 0$ has a family of approximate solutions $w(\mu)$, $\mu \in [0, \mu_1]$, satisfying the inequalities*

$$\|w(\mu)\|_{r+a} \leq R/2, \quad |\Phi(w(\mu),\mu)|_{\rho+a} \leq \mu^{b+1+2\varrho}.$$

Then there exists positive μ_2 such that for $\mu \in \mathcal{M} \cap [0, \mu_2]$ the equation $\Phi(u,\mu) = 0$ has a solution satisfying the inequality $\|u\|_r \leq R$.

Proof. Consider the sequences $\{u_n\}$, $\{v_n\}$ and $\{\vartheta_n\}$ defined by

$$\begin{aligned}
&\vartheta_0 = \mu, && \vartheta_{n+1} = \vartheta_n^{3/2},\\
&u_0 = w(\mu), && u_{n+1} = u_n + S_{\vartheta_{n+1}} v_n,\\
&\Lambda(u_n,\mu) v_n = -\Phi(u_n,\mu).
\end{aligned} \tag{A.36}$$

Our goal is to prove that the sequence $\{u_n\}$ is well-defined and converges to the required solution of $\Phi(u,\mu) = 0$. The proof is based on two auxiliary lemmas. For simplicity of notations we write Λ_n, Φ_n, Φ_n' instead $\Lambda(u_n,\mu)$, $\Phi(u_n,\mu)$, $\Phi'(u_n,\mu)$. Now note from (iii), (c), the hypotheses of Theorem A.1 and the choice of a, that

$$\begin{aligned}
\|S_{\vartheta_1} v_0\|_r \leq c\|v_0\|_r &\leq c\mu^{-\varrho}(|\Phi_0|_{r+k} + \|u_0\|_{2r+k-\rho}|\Phi_0|_\rho)\\
&\leq c\mu^{-\varrho}(|\Phi_0|_{\rho+a} + \|u_0\|_{\rho+a}|\Phi_0|_\rho) \leq c(1+R/2)\mu^{b+1+\varrho}.
\end{aligned}$$

Here c is independent of μ. Hence, for μ sufficiently small, $\|u_1\|_r \leq R$, and the hypotheses of the next lemma are satisfied for $n = 1$ when the hypotheses of Theorem A.1 are satisfied.

Lemma A.2. *Suppose that for some $n > 0$ and $\mu \in \mathcal{M} \cap [0, \hat{\mu}] \subset [0, \mu_0]$*

$$\|u_n\|_r \leq R, \quad \|u_{n-1}\|_r \leq R, \quad \|u_{n-1}\|_{r+a} \leq \vartheta_{n-1}^{-a}, \quad |\Phi_{n-1}|_\rho \leq 1. \tag{A.37}$$

Then there exists $\mu' > 0$ and $c_0 > 0$ such that v_{n-1}, u_n and Φ_n are well defined and

$$|\Phi_n|_\rho \leq \frac{c_0}{\mu^\varrho}\Big(\vartheta_n^{a-d}\vartheta_{n-1}^{-a} + \frac{\vartheta_n^{-2d}}{\mu^\varrho}|\Phi_{n-1}|_\rho^{2-\frac{d+1}{a}}\Big), \quad \mu \in (0,\mu'] \cap \mathcal{M}, \tag{A.38}$$

where the constants c_0 and μ' do not depend on n.

Proof. In what follows we use c to denote various constants which depend on R, a, b, d, ρ, k and r but are independent of n and μ. Then by (c), v_{n-1} exists for all $\mu \in (0, \mu(0)] \cap \mathcal{M}$ and it is easy to check that

$$\Phi_n = \Lambda_{n-1}(S_{\vartheta_n} - 1)v_{n-1} + (\Phi'_{n-1} - \Lambda_{n-1})(u_n - u_{n-1}) + D(u_n, u_{n-1})$$

where

$$D(u, v) = \Phi(u, \mu) - \Phi(v, \mu) - \Phi'(v, \mu)(u - v). \tag{A.39}$$

It follows from (iii), (a) and (b) that $|\Phi_n|_\rho \le c(Q_1 + Q_2 + Q_3)$ where

$$Q_1 = \vartheta_n^{a-d}\|v_{n-1}\|_{\rho+a-k}, \quad Q_2 = |\Phi_{n-1}|_{\rho+1}\|u_n - u_{n-1}\|_r, \quad Q_3 = \|u_n - u_{n-1}\|_r^2.$$

Estimating v_{n-1} from (c), $|\Phi_{n-1}|_{\rho+a}$ from (a) and $|\Phi_{n-1}|_\rho$ from the hypotheses of the Lemma yields

$$Q_1 \le c\mu^{-\varrho}\vartheta_n^{a-d}(\|u_{n-1}\|_{r+a} + |\Phi_{n-1}|_{\rho+a}) \le c\mu^{-\varrho}\vartheta_n^{a-d}\vartheta_{n-1}^{-a}, \ \mu \in (0, \mu(a)] \cap \mathcal{M}.$$

It follows from the interpolation inequalities (ii) that

$$\|u_n - u_{n-1}\|_r \le c\|u_n - u_{n-1}\|_{\rho-k}^\alpha \|u_n - u_{n-1}\|_{a+\rho-k}^{1-\alpha} \text{ where } \alpha = 1 - d/a.$$

Also from (iii), (c) and (A.36) it follows that

$$\|u_n - u_{n-1}\|_{\rho-k} \le c\|v_{n-1}\|_{\rho-k} \le c\mu^{-\varrho}|\Phi_{n-1}|_\rho, \quad \mu \in (0, \mu(0)] \cap \mathcal{M},$$

and similarly for $\mu \in (0, \mu(a)] \cap \mathcal{M}$,

$$\|u_n - u_{n-1}\|_{\rho-k+a} \le c\|v_{n-1}\|_{\rho-k+a} \le c\mu^{-\varrho}(\|u_{n-1}\|_{r+a} + |\Phi_{n-1}|_{\rho+a}) \le c\mu^{-\varrho}\vartheta_{n-1}^{-a}.$$

These inequalities give the following estimate for Q_2 when $\mu \in (0, \mu(a)] \cap \mathcal{M}$:

$$\begin{aligned} Q_2 &= |\Phi_{n-1}|_{\rho+1}\|u_n - u_{n-1}\|_r \le c|\Phi_{n-1}|_{\rho+1}\|u_n - u_{n-1}\|_{\rho-k}^\alpha \|u_n - u_{n-1}\|_{a+\rho-k}^{1-\alpha} \\ &\le \frac{c}{\mu^{\varrho\alpha}}|\Phi_{n-1}|_{\rho+1}|\Phi_{n-1}|_\rho^\alpha \frac{\vartheta_{n-1}^{-a(1-\alpha)}}{\mu^{(1-\alpha)\varrho}} = \frac{c}{\mu^\varrho}|\Phi_{n-1}|_{\rho+1}|\Phi_{n-1}|_\rho^\alpha \vartheta_{n-1}^{-a(1-\alpha)} \\ &= \frac{c}{\mu^\varrho}|\Phi_{n-1}|_{\rho+1}|\Phi_{n-1}|_\rho^{1-(d/a)}\vartheta_{n-1}^{-d}. \end{aligned}$$

Now the interpolation inequality gives

$$|\Phi_{n-1}|_{\rho+1} \le c|\Phi_{n-1}|_\rho^{\frac{a-1}{a}}|\Phi_{n-1}|_{\rho+a}^{\frac{1}{a}}.$$

Hence by (a) and the present hypothesis,

$$Q_2 \le \frac{c}{\mu^\varrho}|\Phi_{n-1}|_\rho^{2-\frac{d+1}{a}}\vartheta_{n-1}^{-(d+1)}, \quad \mu \in (0, \mu(a)] \cap \mathcal{M}.$$

Now (c) and (iii) imply that

$$\|u_n - u_{n-1}\|_r \le c\vartheta_n^{-d}\|v_{n-1}\|_{\rho-k} \le \frac{c}{\mu^\varrho\vartheta_n^d}|\Phi_{n-1}|_\rho, \quad \mu \in (0, \mu(0)] \cap \mathcal{M}.$$

Hence for $\mu \in (0, \mu'] \cap \mathcal{M}$, $\mu' = \min\{\mu(0), \mu(a)\}$,

$$Q_1 + Q_2 + Q_3 \le \frac{c}{\mu^\varrho} \left\{ \vartheta_n^{a-d} \vartheta_{n-1}^{-a} + |\Phi_{n-1}|_\rho^{2-\frac{d+1}{a}} \vartheta_{n-1}^{-(d+1)} + \frac{1}{\vartheta_n^{2d} \mu^\varrho} |\Phi_{n-1}|_\rho^2 \right\}$$
$$= \frac{c}{\mu^\varrho} \left\{ \vartheta_n^{a-d} \vartheta_{n-1}^{-a} + |\Phi_{n-1}|_\rho^{2-\frac{d+1}{a}} \vartheta_n^{\frac{-2(d+1)}{3}} + \frac{1}{\vartheta_n^{2d} \mu^\varrho} |\Phi_{n-1}|_\rho^2 \right\}.$$

and the result follows. □

Lemma A.3. *Suppose the hypotheses of Lemma A.2 are satisfied on* $(0, \hat\mu]$ *for every* $1 \le n \le m$ *and* $|\Phi_0|_\rho \le \mu^{b+2\varrho+1}$ *for* $\mu \in (0, \hat\mu]$. *Then there exists* $\tilde\mu \in (0, \hat\mu] \cap \mathcal{M}$, *independent of* m, *such that*

$$|\Phi_n|_\rho \le \vartheta_n^b \mu^{2\varrho+1} \le 1, \quad n \le m \text{ for all } \mu \in (0, \tilde\mu] \cap \mathcal{M}. \tag{A.40}$$

Proof. For $n = 0$ the inequality (A.40) is true by hypothesis since $\vartheta_0 = \mu$. Now suppose (A.40) holds for $n = j - 1 \le m - 1$. Let $\delta_j = |\Phi_j|_\rho \vartheta_j^{-b}$. Lemma A.2 implies

$$|\Phi_j|_\rho \le \frac{c_0}{\mu^\varrho} \left\{ \vartheta_j^{a-d} \vartheta_{j-1}^{-a} + \frac{\vartheta_j^{-2d}}{\mu^\varrho} |\Phi_{j-1}|_\rho^{2-\frac{d+1}{a}} \right\}, \ \mu \in (0, \mu'] \cap \mathcal{M}.$$

Therefore

$$\delta_j \le \frac{c_0}{\mu^\varrho} \left\{ \vartheta_j^{a-b-d} \vartheta_{j-1}^{-a} + \frac{\vartheta_j^{-2d-b}}{\mu^\varrho} |\Phi_{j-1}|_\rho^{2-\frac{d+1}{a}} \right\} = \frac{c_0}{\mu^\varrho} \left\{ \vartheta_{j-1}^{\frac{a}{2} - \frac{3}{2}(b+d)} + \frac{1}{\mu^\varrho} \delta_{j-1}^{2-\frac{d+1}{a}} \vartheta_{j-1}^{b(\frac{1}{2} - \frac{d+1}{a}) - 3d} \right\}.$$

Now the choice of a, b gives that

$$\frac{a}{2} - \frac{3}{2}(b+d) > 3\varrho + 2, \quad \frac{d+1}{a} < \frac{1}{3}, \quad 2 - \frac{d+1}{a} > \frac{5}{3} \text{ and } b(\frac{1}{2} - \frac{d+1}{a}) - 3d > 2\varrho.$$

Therefore, since $\vartheta_{j-1} \le \mu$, the inductive hypothesis gives that

$$\delta_j \le \frac{c_0}{\mu^\varrho} \left\{ \vartheta_{j-1}^{3\varrho+2} + \delta_{j-1}^{5/3} \right\} \le \frac{c_0}{\mu^\varrho} \left\{ \mu^{3\varrho+2} + \mu^{(10\varrho+5)/3} \right\} \le \mu^{2\varrho+1}$$

for all non-zero $\mu \in \mathcal{M}$ sufficiently small. The lemma follows. □

Proof of Theorem A.1. Recall the hypotheses of Theorem A.1. We will now show that there exists a constant $\mu_3 > 0$ such that the inequalities (A.37) hold for all $n \in \mathbb{N}$ and $\mu \in (0, \mu_3] \cap \mathcal{M}$, and in the process prove Theorem A.1.

We have already observed that these inequalities follow from the hypotheses of Theorem A.1 when $n = 1$. Suppose (A.37) holds for every n, $1 \le n \le m$. Then, by our hypothesis and Lemma A.3, μ_3 can be chosen such that

$$|\Phi_0|_\rho \le \mu^{b+2\varrho+1}, \quad |\Phi_n|_\rho \le \vartheta_n^b \mu^{2\varrho+1} \le 1, \quad \text{if } n \le m, \ \mu \in (0, \mu_3] \cap \mathcal{M}. \tag{A.41}$$

To estimate $\|u_{m+1}\|_r$ and $\|u_m\|_{r+a}$ we note that (iii) and (c) imply for $i \in \{0, 1\}$ and $0 \le m' \le m$ that

$$\|u_{m+1} - u_{m'}\|_{r+ia} \le \sum_{n=m'}^{m} \|u_{n+1} - u_n\|_{r+ia} = \sum_{n=m'}^{m} \|S_{\vartheta_{n+1}} v_n\|_{r+ia}$$
$$\le c \sum_{n=m'}^{m} \vartheta_{n+1}^{-(r+k-\rho+ia)} \|v_n\|_{\rho-k} \le \frac{c}{\mu^\varrho} \sum_{n=m'}^{m} \vartheta_{n+1}^{-d-ia} |\Phi_n|_\rho.$$

Therefore, by the induction hypothesis,

$$\begin{aligned}\|u_{m+1}-u_{m'}\|_{r+ia} &\le \frac{c}{\mu^{\varrho}}\sum_{n=m'}^{m}\vartheta_{n+1}^{-d-ia}\vartheta_n^{b}\mu^{2\varrho+1} = c\mu^{\varrho+1}\vartheta_{m+1}^{-ia}\sum_{n=m'}^{m}\vartheta_{m+1}^{ia}\vartheta_{n+1}^{-d-ia}\vartheta_n^{b}\\ &\le c\mu^{\varrho+1}\vartheta_{m+1}^{-ia}\sum_{n=m'}^{m}\vartheta_m^{\frac{3}{2}ia}\vartheta_n^{-\frac{3}{2}(d+ia)+b} \le c\mu^{\varrho+1}\vartheta_{m+1}^{-ia}\sum_{n=m'}^{m}\vartheta_n^{b-\frac{3}{2}d} \le c\mu^{\varrho+1}\vartheta_{m+1}^{-ia}\vartheta_{m'}^{\frac{1}{2}},\end{aligned}\tag{A.42}$$

since $b-3d/2>1/2$. Hence $i=0$ gives

$$\|u_{m+1}\|_r \le \|u_0\|_r + c\mu^{\varrho+3/2} \le R/2 + c\mu^{\varrho+3/2} \le R$$

for all non-zero $\mu\in\mathcal{M}$ sufficiently small. Similarly, $i=1$ gives

$$\|u_m\|_{r+a} \le \|u_0\|_{r+a} + c\mu^{\varrho+3/2}\vartheta_m^{-a} \le \vartheta_m^{-a}$$

for all non-zero $\mu\in\mathcal{M}$ sufficiently small. Hence u_n, Φ_n satisfy (A.37) for all $n\le m+1$. Therefore, by Lemma A.3, $|\Phi_n|_\rho \le \vartheta_n^b\mu^{2\varrho+1} < 1$ for $n\le m+1$ for μ sufficiently small independent of n. Thus (A.41) holds, with μ_3 independent of n, for all $n\in\mathbb{N}$.

Now observe from (A.42) with $i=0$ that $\{u_n\}$ is a Cauchy sequence in E_r which, by the continuity of Φ, converges to a solution to the equation $\Phi(u,\mu)=0$. Theorem A.1 follows. □

Proof of Theorem 3.2 (Nash-Moser theorem). It suffices to establish the existence of an approximate solutions $w(\mu)$ satisfying the hypotheses of Theorem A.1. We construct such a solution by the direct Newtonian method, without using smoothing operators, in a finite number of steps.

Let N be a fixed natural number such that $(N+1)(\gamma-\varrho) > b+1+2\varrho$ and for $j\in\{0,1,\cdots,N\}$ let $s_j = 2(N-j)d+r+a$. (Note that s_j is decreasing in j and $r+a\le s_{j+1}\le s_j-d\le s_j-1$.) Then, by (a) and (c), for non-zero $\mu\in\mathcal{M}$ sufficiently small, there exists $v_0\in E_{s_0}$ which satisfies $\Lambda(0,\mu)v_0 = -\Phi(0,\mu)$. We seek to show that, for all non-zero $\mu\in\mathcal{M}$ sufficiently small, $w_j\in E_{s_j}$, $0\le j\le N$, can be defined recursively by

$$w_{j+1} = w_j + v_j \quad\text{where}\quad \Lambda(w_j,\mu)v_j = -\Phi(w_j,\mu), \quad w_0 = 0.$$

To do this we will show by induction that

$$\|w_j\|_{s_j} \le R/2, \quad |\Phi(w_j,\mu)|_{s_j+\rho-r} \le \text{const.}\mu^{(j+1)(\gamma-\varrho)} \tag{A.43}$$

holds for all $\mu>0$ sufficiently small and all $j\le N$. For $j=0$ the inequalities (A.43) follows from condition (a). Suppose that $j\le N-1$ and that (A.43) holds for $0\le i\le j$. It follows from (c) and the definition of d that for non-zero $\mu\in\mathcal{M}$ sufficiently small

$$\|v_i\|_{s_i-d} \le \frac{\text{const.}}{\mu^{\varrho}}|\Phi(w_i,\mu)|_{s_i+\rho-r} \le \text{const.}\mu^{(i+1)(\gamma-\varrho)-\varrho}. \tag{A.44}$$

Hence since $w_0=0$

$$\begin{aligned}\|w_{j+1}\|_{s_{j+1}} \le \sum_{i=0}^{j}\|v_i\|_{s_{j+1}} \le \sum_{i=0}^{j}\|v_i\|_{s_{i+1}} \le \sum_{i=0}^{j}\|v_i\|_{s_i-d}&\\ \le c\sum_{i=0}^{j}\mu^{(i+1)(\gamma-\varrho)-\varrho} \le c\mu^{\gamma-2\varrho} \le R/2&\end{aligned}$$

for all non-zero $\mu \in \mathcal{M}$ sufficiently small. Set $p_i = |\Phi(w_i, \mu)|_{s_i+\rho-r}$. Now we estimate p_{j+1} by noting, with D defined in (A.39), that

$$\Phi(w_{j+1}, \mu) = (\Phi'(w_j, \mu) - \Lambda(w_j, \mu))v_j + D(w_{j+1}, w_j),$$

where, by (a) and (A.44),

$$\|D(w_{j+1}, w_j)\|_{s_{j+1}+\rho-r} \le \text{const.}\|v_j\|^2_{s_{j+1}} \le \text{const.}\mu^{-2\varrho}p_j^2,$$

and, by (b) (since $\|w_j\|_{s_{j+1}} \le \|w_j\|_{s_j} \le R/2$) and (A.44),

$$\begin{aligned}|(\Phi'(w_j, \mu) - \Lambda(w_j, \mu))v_j|_{s_{j+1}+\rho-r} &\le \text{const.}|\Phi(w_j, \mu)|_{s_{j+1}+\rho-r+1}\|v_j\|_{s_{j+1}} \\ &\le \text{const.}|\Phi(w_j, \mu)|_{s_j+\rho-r}\|v_j\|_{s_j-d} \le \text{const.}\mu^{-\varrho}p_j^2\end{aligned}$$

for all non-zero $\mu \in \mathcal{M}$ sufficiently small. Hence $p_{j+1} \le \text{const.}\mu^{-2\varrho}p_j^2$. Putting $j = 0$ gives $p_1 \le \text{const.}\mu^{-2\varrho}p_0^2 \le \text{const.}\mu^{2(\gamma-\varrho)}$ and (A.43) follows. For $j > 0$, by the obvious induction hypotheses, $p_{j+1} \le \text{const.}\mu^{-2\varrho}p_j^2 \le \text{const.}\mu^{2(j+1)(\gamma-\varrho)-2\varrho} \le \text{const.}\mu^{(j+2)(\gamma-\varrho)}$, since $\gamma \ge 3\varrho$. Hence (A.43) holds for $j+1$ and it follows that

$$\|w_N\|_{s_N} \le R/2 \text{ and } |\Phi(w_N, \mu)|_{S_N+\rho-r} \le \text{const.}\mu^{(N+1)(\gamma-\varrho)}.$$

Since $s_N = r + a$ and $(N+1)(\gamma - \varrho) > b + 1 + 2\varrho$, the proof is complete. □

A.2 Proof of Lemma 6.1

In this section we prove existence and establish estimates of solution to the boundary-value problem

$$\Omega: \quad \operatorname{div}(Q\nabla v) = G, \quad \Gamma_0: \quad v = G_0, \quad \Gamma_{h_0}: \quad Q\nabla v \cdot \vec{e}_2 = G_{h_0}. \tag{A.45}$$

Under the assumptions of the lemma, the C^6-norm of the matrix Q and the C^0-norm of Q_{22}^{-1} are bounded by a constant $c(c_Q)$, and Q, G, G_i are 2π-periodic with respect to t, ξ_1. Without loss of generality we can assume that they are all infinitely differentiable. It is then well known that this problem has a unique infinitely differentiable solution and the goal is to investigate how its regularity depends on that of the matrix Q.

Let φ be the unique solution to the problem

$$\Omega: \quad \Delta\varphi = G, \quad \Gamma_0: \quad \varphi = 0, \quad \Gamma_{h_0}: \quad \nabla\varphi \cdot \vec{e}_2 = G_{h_0}. \tag{A.46}$$

Then, for any smooth solution v to problem (A.45), there exists a unique $u(t, \xi)$ such that u vanishes at Γ_{h_0} and $\nabla u = S\nabla\varphi - SQ\nabla v$ on Ω.

Hence problem (A.45) is equivalent to an elliptic boundary problem for the vector-function $w = (u, v)$

$$\begin{gathered}\Omega: \quad Q\nabla v + S^{-1}\nabla u = \nabla\varphi, \\ \Gamma_{h_0}: u = 0, \quad \Gamma_0: \quad v = G_0,\end{gathered} \tag{A.47}$$

and the equation for w can be re-written in a more convenient form as

$$w_{\xi_2} = Aw_{\xi_1} + F, \quad F = B\nabla\varphi, \tag{A.48}$$

where the matrices A and B are defined by

$$A = \frac{1}{Q_{22}} \begin{pmatrix} -Q_{12} & |Q| \\ -1 & -Q_{12} \end{pmatrix}, \quad B = \frac{1}{Q_{22}} \begin{pmatrix} -Q_{22} & Q_{12} \\ 0 & 1 \end{pmatrix}.$$

Since $Q_{22}^{-1} > c(c_Q) > 0$, it follows for $r \le s+6$ that

$$\|A\|_{C^r(\Omega)} + \|B\|_{C^r(\Omega)} \le c(r)(1 + \|Q\|_{C^r(\Omega)}). \tag{A.49}$$

Equations (A.47)-(A.48) form an elliptic boundary-value problem which has a unique smooth solution which is 2π-periodic with respect to t and ξ_1 and satisfies the inequality

$$\|w(t)\|_{C^{1+\sigma}(\Omega_0)} \le c(\sigma)(\|F(t)\|_{C^\sigma(\Omega_0)} + \|G_0(t)\|_{C^{1+\sigma}(\Gamma_0 \cap (\{t\}\times\mathbb{R}))}), \quad 0 < \sigma < 1. \tag{A.50}$$

We will consider these solutions as elements of Banach spaces defined as follows. Recall that Ω_0 is the projection of Ω onto the ξ-plane. For non-negative integers s, l, and $\sigma \in (0,1)$, denote by $C_{s,l+\sigma}$ the closed subspace of the Banach space $\cap_{k=0}^{s} C^{s-k}(\mathbb{R}; C^{l+k+\sigma}(\Omega_0))$, which consists of functions which are 2π-periodic with respect to t and ξ_1. The norm in the space $C_{s,l+\sigma}$ is defined by

$$|u|_{s,l+\sigma} = \sum_{k=0}^{s} \|u\|_{C^{s-k}(\mathbb{R};C^{l+k+\sigma}(\Omega_0))} = \sum_{k=0}^{s} \sum_{k+|\beta|\le s} \sup_{t\in\mathbb{R}} \|\partial_t^\beta u(t)\|_{C^{l+k+\sigma}(\Omega_0)}.$$

Since the embedding $C_{s+1,l+\sigma} \to C_{s,l+\sigma}$ is compact, for any $\varepsilon > 0$ there exists $c(\varepsilon, s, l, \sigma)$ such that

$$|u|_{s,l+\sigma} \le \varepsilon |u|_{s+1,l+\sigma} + c|u|_{0,l+\sigma}. \tag{A.51}$$

To estimate the right side of (A.48) in terms of G and Q, differentiate (A.46) and use standard *a priori* estimates on the solutions of elliptic equations to obtain, for $\sigma \in (0,1)$ and $r \le s+1$, that

$$|\partial_t^k \varphi|_{0,r-k+2+\sigma} \le c(|\partial_t^k G|_{0,r-k+\sigma} + |\partial_t^k G_{h_0}|_{0,r-k+1+\sigma}), \quad 0 \le k \le r,$$

and hence that

$$\|\nabla\varphi\|_{C^r(\Omega)} \le c(r)(\|G\|_{C^{r+1}(\Omega)} + \|G_{h_0}\|_{C^{r+2}(\Gamma_{h_0})}). \tag{A.52}$$

From this we get, for $r \le s+1$, that

$$\begin{aligned}\|F\|_{C^r(\Omega)} \le c\{&\|G\|_{C^{r+1}(\Omega)} + \|G_{h_0}\|_{C^{r+2}(\Gamma_{h_0})} \\ &+ \|Q\|_{C^r(\Omega)}(\|G\|_{C^1(\Omega)} + \|G_{h_0}\|_{C^2(\Gamma_{h_0})}\}.\end{aligned} \tag{A.53}$$

To estimate the solution of problem (A.47) and (A.48) let

$$\mathcal{G}_s = \|G\|_{C^{s+3}(\Omega)} + \sum_{i=0,h_0} \|G_i\|_{C^{s+3}(\Gamma_i)} + \|Q\|_{C^{s+6}(\Omega)} \Big\{ \|G\|_{C^3(\Omega)} + \sum_{i=0,h_0} \|G_i\|_{C^3(\Gamma_i)} \Big\}.$$

First we estimate lower order norms of the solution of (A.47)-(A.48). In the present notation (A.50) and (A.53) give, for $\sigma \in (0,1)$, that

$$|w|_{0,1+\sigma} \le c(|F|_{0,\sigma} + |G_0|_{0,1+\sigma}) \le c(\|G\|_{C^2(\Omega)} + \|G_{h_0}\|_{C^3(\Gamma_{h_0})} + \|G_0\|_{C^2(\Gamma_0)}). \tag{A.54}$$

To estimate $|w|_{s,1+\sigma}$ note that

$$|w|_{s,1+\sigma} = \sum_{k=0}^{s} I_k, \quad I_k = \sum_{|\gamma|\le s-k} |\partial_{\xi_2}^k D^\gamma w|_{0,1+\sigma},$$

where the last sum is taken over all 'tangential' derivatives $D^\gamma = \partial_t^{\gamma_0}\partial_{\xi_1}^{\gamma_1}$ of order $|\gamma| \le s-k$. (D^γ is a tangential derivative because ξ_2 is normal to $\partial\Omega$.) We will prove that for any positive ε there exists a constant c depending only on ε, s, σ and the constant c_Q, such that

$$I_k \le \varepsilon|w|_{s,1+\sigma} + c\mathcal{G}_s, \quad 0 \le k \le s. \tag{A.55}$$

First we estimate I_0. Denote by $w^{(\gamma)}$ the tangential derivative $D^\gamma w$ of order $|\gamma| = s$ which satisfies

$$w_{\xi_2}^{(\gamma)} = Aw_{\xi_1}^{(\gamma)} + F^{(\gamma)} + \psi_\gamma, \quad \psi_\gamma = \sum_{\substack{\alpha+\beta=\gamma \\ |\beta|\le s-1}} c_{\alpha\beta} A^{(\alpha)} w_{\xi_1}^{(\beta)}, \tag{A.56}$$

$$\Gamma_{h_0} : u^{(\gamma)} = 0, \quad \Gamma_0 : \quad v^{(\gamma)} = G_0^{(\gamma)}, \tag{A.57}$$

where the constants $c_{\alpha\beta}$ depend on $|\gamma| = s$. We split ψ_γ into two parts:

$$\psi_\gamma = \sum_{\substack{s-4\le|\beta|\le s-1 \\ \alpha+\beta=\gamma}} c_{\alpha\beta} A^{(\alpha)} w_{\xi_1}^{(\beta)} + \sum_{\substack{|\beta|\le s-5 \\ \alpha+\beta=\gamma}} c_{\alpha\beta} A^{(\alpha)} w_{\xi_1}^{(\beta)} = J_1 + J_2.$$

Applying estimate (A.50) to this problem gives

$$|w^{(\gamma)}|_{0,1+\sigma} \le c(\sigma)(|J_1|_{0,\sigma} + |J_2|_{0,\sigma} + \|F\|_{C^{s+1}(\Omega)} + \|G_0\|_{C^{s+2}(\Gamma_0)}). \tag{A.58}$$

It follows from the boundedness of the C^6-norm of Q and (A.51) that

$$|J_1|_{0,\sigma} \le c|w|_{s-1,1+\sigma} \le \varepsilon|w|_{s,1+\sigma} + c(\varepsilon)|w|_{0,1+\sigma}. \tag{A.59}$$

On the other hand since $|\alpha| + |\beta| = s$ and $|\cdot|_{0,\sigma} \le c\|\cdot\|_{C^1(\Omega)}$,

$$|J_2|_{0,\sigma} \le c\sum_{k=2}^{s-3} \|A\|_{C^{s-k+3}(\Omega)} \|w\|_{C^k(\Omega)}.$$

Recall the interpolation inequalities

$$\|A\|_{C^{s-k+3}(\Omega)} \le c\|A\|_{C^{s+3}(\Omega)}^{1-\lambda} \|A\|_{C^3(\Omega)}^{\lambda}, \quad \|w\|_{C^k(\Omega)} \le c\|w\|_{C^s(\Omega)}^{\lambda} \|w\|_{C(\Omega)}^{1-\lambda}$$

with the exponent $\lambda = k/s$. Hence

$$\|A\|_{C^{s-k+3}(\Omega)} \|w\|_{C^k(\Omega)} \le c\|w\|_{C^s(\Omega)}^{\lambda} (\|w\|_{C(\Omega)} \|A\|_{C^{s+3}(\Omega)})^{1-\lambda}$$

Since $k \le s-1$ we conclude, by Young's inequality, that

$$|J_2|_{0,\sigma} \le \varepsilon\|w\|_{C^s(\Omega)} + c(\varepsilon)\|w\|_{C(\Omega)} \|A\|_{C^{s+3}(\Omega)}. \tag{A.60}$$

A substitution of (A.59) and (A.60) into (A.58), and use of estimates (A.49) and (A.54), gives the needed inequality (A.55) for I_0.

Assuming (A.55) holds for I_k, $k \le n-1 < s$, we will now prove it for I_n. Denote by D^γ an arbitrary tangential derivative of order $|\gamma|$ and let L^α be the differential operator of order $|\alpha| = m + |\gamma|$ written in the form $\partial^m_{\xi_2} D^\gamma$. With $|\gamma| = s-n$, apply the particular operator $L^\alpha = \partial^{n-1}_{\xi_2} D^\gamma$ of order $|\alpha| = s-1$ to (A.48) to obtain

$$D^n_{\xi_2} D^\gamma w = \psi_\alpha + L^\alpha F, \quad \psi_\alpha = \sum_{\beta+\lambda=\alpha} c_{\beta\lambda}(L^\beta A)(L^\lambda \partial_{\xi_1} w). \tag{A.61}$$

Now split ψ_α into two parts

$$\psi_\alpha = \sum_{\substack{s-4\le|\lambda|\le s-1\\ \lambda+\beta=\alpha}} c_{\beta\lambda}(L^\beta A)(L^\lambda w_{\xi_1}) + \sum_{\substack{|\lambda|\le s-5\\ \lambda+\beta=\alpha}} c_{\beta\lambda}(L^\beta A)(L^\lambda w_{\xi_1}) = J_3 + J_4.$$

Note that the differential operators $L^\lambda \partial_{\xi_1}$ involve the normal derivative of order not exceeding $n-1$. Since the C^6-norm of A is bounded by a constant depending only on c_Q, we conclude from the induction hypotheses that

$$|J_3|_{0,\sigma} \le c(c_Q) \sum_{k=0}^{n-1} I_k \le \varepsilon |w|_{s,1+\sigma} + c\mathcal{G}_s. \tag{A.62}$$

On the other hand since $|\beta| + |\lambda| = s-1$, we have

$$|J_4|_{0,\sigma} \le c(c_Q) \sum_{k=2}^{s-3} \|A\|_{C^{s-k+3}(\Omega)} \|w\|_{C^k(\Omega)}.$$

Analysis similar to that which was used for estimating J_2 shows that the right side does not exceed $\varepsilon\|w\|_{C^s(\Omega)} + \|Q\|_{C^{s+6}(\Omega)}\|w\|_{C(\Omega)}$, which gives $|J_4|_{0,\sigma} \le \varepsilon |w|_{s,1+\sigma} + c\mathcal{G}_s$. From this (A.62) and estimate (A.53) on F we conclude, for $n \le s$ and $|\gamma| = s-n$, that $|\partial^n_{\xi_2} D^\gamma w|_{0,1+\sigma} \le \varepsilon |w|_{s,1+\sigma} + c\mathcal{G}_s$ and (A.55) follows.

The inequality (A.55) now yields the estimate $(1-s\varepsilon)|w|_{s,1+\sigma} \le c\mathcal{G}_s$, since $|w|_{s,1+\sigma} = \sum_0^s I_k$. From this and the fact that $\|v\|_{C^s(\Omega)} \le |w|_{s,1+\sigma}$ we obtain $\|v\|_{C^s(\Omega)} \le c\mathcal{G}_s$, by choosing ε sufficiently small. This completes the proof. □

A.3 Proof of Lemmas 7.3, 7.5 and 10.8

Proof of Lemma 7.3. Consider the integral equation

$$(I + P_0)\psi \equiv \psi(t,\xi_1) + \int_{-\pi}^{\pi} p_0(\xi_1 - s)\psi(t,s) = f(t,s) \tag{A.63}$$

for a continuous, 2π-periodic function f, where $f(t,s) = \sum\limits_{n=-\infty}^{\infty} f_n(t)e^{isn}$, $f_{-n} = \bar{f}_n$ and P_0 is the operator in (7.4). Let

$$\psi(t,s) = \sum_{n=-\infty}^{\infty} \psi_n(t)e^{isn} \text{ and } p_0(s) = \frac{1}{2\pi} Re\{3 + 2\sum_{k=1}^{\infty} e^{-2kh_\lambda + iks}\},$$

where $\psi_{-n} = \bar{\psi}_n$. Substituting these expressions into (A.63) gives

$$\sum_{n=-\infty}^{\infty} \psi_n e^{in\xi_1} + \frac{1}{2\pi} Re\Big\{ \sum_{n=-\infty}^{\infty} \psi_n \int_{-\pi}^{\pi} (3 + 2\sum_{k\geq 1} e^{-2kh_\lambda + ik(\xi_1 - s)}) e^{ins} ds \Big\} = \sum_{n=-\infty}^{\infty} f_n e^{in\xi_1}.$$

It is easy to check that

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \psi_n \int_{-\pi}^{\pi} (3 + 2\sum_{k\geq 1} e^{-2kh_\lambda + ik(\xi_1 - s)}) e^{ins} ds = 3\psi_0 + 2\sum_{n\geq 1} e^{-2nh_\lambda} \psi_n e^{in\xi_1}.$$

Since $Re\{2\psi_n e^{in\xi_1}\} = \psi_n e^{in\xi_1} + \psi_{-n} e^{-in\xi_1}$, we conclude that

$$4\psi_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} (1 + e^{-2|k|h_\lambda}) \psi_k e^{ik\xi_1} = \sum_{k=-\infty}^{\infty} f_k e^{ik\xi_1}.$$

Since $\psi = (I + P_0)^{-1} f$,

$$(I + P_0)^{-1} f - f = -\frac{3}{4} f_0 - \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-2|k|h_\lambda}}{1 + e^{-2|k|h_\lambda}} f_k e^{ik\xi_1} = \int_{-\pi}^{\pi} p_2(\xi_1 - s) f(s) ds,$$

where

$$p_2(s) = -\frac{3}{8\pi} - \frac{1}{2\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-2|k|h_\lambda}}{1 + e^{-2|k|h_\lambda}} e^{iks},$$

and the lemma follows. □

Proof of Lemmas 10.8, 7.5. We obtain both lemmas as a consequence of the following proposition. Denote by H_r the Sobolev space of functions $u(z)$, $z = (z_1, .., z_d)$, which are 2π-periodic in each of the variables z_i and have square-integrable derivatives of order r; then denote by C^r the Banach space of 2π-periodic functions having continuous derivatives of the same order. We denote by $\|\cdot\|_r$ and $|\cdot|_r$ the norm in H_r and C^r respectively. Consider the integral operator

$$Pu(z) = \int_{-\pi}^{\pi} p(z, s) u(z_1, ., z_{d-1}, s) ds$$

with a smooth 2π-periodic kernel $p \in C^\rho$

Lemma A.4. *Let $u \in H_\rho$, $(u \in C^\rho)$. Then*

$$\|Pu\|_\rho \leq c(|p|_0 \|u\|_\rho + |p|_\rho \|u\|_0), \quad \Big(|Pu|_\rho \leq c(|p|_0 |u|_\rho + |p|_\rho |u|_0)\Big),$$

with the the constant c depending only on ρ.

Proof. We will prove the first estimate. For $\rho = 0$ the Cauchy inequality implies $\|Pu\|_0 \leq \sqrt{2\pi} |p|_0 \|u\|_0$. Assuming the bound holds for exponents $\hat{\rho} < \rho$, we will prove it for $\hat{\rho} + 1$. We have

$$D_{z_i} Pu = \int_{-\pi}^{\pi} ((1 - \delta_{id})\, p\, D_{z_i} u + u\, D_{z_i} p) ds,$$

which along with the induction hypothesis implies

$$\|\nabla Pu\|_{\hat\rho} \le c(|p|_{\hat\rho+1}\|u\|_0 + |p|_0\|u\|_{\hat\rho+1} + |p|_{\hat\rho}\|u\|_1 + |p|_1\|u\|_{\hat\rho}). \tag{A.64}$$

Note the interpolation inequalities

$$|p|_1 \le c|p|_{\hat\rho+1}^{\alpha}|p|_0^{1-\alpha}, \quad \|u\|_{\hat\rho} \le c\|u\|_{\hat\rho+1}^{1-\alpha}\|u\|_0^{\alpha},$$

with the exponent $\alpha = 1/(\hat\rho+1)$. Hence, by Young's inequality,

$$|p|_1\|u\|_{\hat\rho} \le c(|p|_{\hat\rho+1}\|u\|_0)^{\alpha}(|p|_0\|u\|_{\hat\rho+1})^{1-\alpha} \le c|p|_{\hat\rho+1}\|u\|_0 + |p|_0\|u\|_{\hat\rho+1}.$$

The same argument bounds the term $|p|_{\hat\rho}\|u\|_1$. Substituting these into (A.64) completes the proof of the first inequality for $\hat\rho+1$, and hence for ρ, by induction. The proof of the second is similar. □

It is clear that Lemma 10.8 follows from the assertion of Lemma A.4. Let us prove Lemma 7.5. Consider the integral equation

$$(I + A_1)v = v(t,\xi_1) + \int_{-\pi}^{\pi} a_1(t,\xi_1,s)v(t,s)ds = f(t,\xi_1).$$

Note that, under the assumptions of this lemma, for $\varepsilon^2 \le \delta_0 < (\sqrt{2\pi}c_0)^{-1}$ the operator A_1 is a contraction on C^0. From the contraction mapping principle and the Fredholm theory of integral equations it now follows that

$$v(t,\xi_1) = f(t,\xi_1) + \int_{-\pi}^{\pi} r_1(t,\xi_1,s)f(t,s)ds,$$

where the periodic continuous function r_1 has the bound $|r_1|_0 \le c(\delta_0)$ and satisfies the integral equation

$$(I + A)r_1 \equiv r_1(t,\xi_1,s) + \int_{-\pi}^{\pi} a_1(t,\xi_1,s_1)r_1(t,s_1,s)ds_1 = -a_1(t,\xi_1,s). \tag{A.65}$$

Consider the operator equations

$$(I + \lambda A)v = F \in C^{r+l-3}, \quad \lambda \in [0,1]. \tag{A.66}$$

The operator A meets all requirements of Lemma A.4 which, with the inequality $|a_1|_0 \le \varepsilon^2 c_0$, implies that

$$|v|_{r+l-3} \le c(\varepsilon^2|v|_{r+l-3} + |v|_0|a_1|_{r+l-3}) + |F|_{r+l-3}, \quad |v|_0 \le \varepsilon^2 c|v|_0 + |F|_0.$$

Here c depends only on r, l and the constant c_0. It follows from this that there exists δ_1, depending only on r, l and c_0, such that the inequality

$$|v|_{r+l-3} \le c(|F|_0|a_1|_{r+l-3} + |F|_{r+l-3}) \tag{A.67}$$

holds for any solution $v \in C^{r+l-3}$ of equation (A.66) and $\varepsilon^2 \le \delta_1$. Hence there exist constants M_1 and M_2, which do not depend on λ but depend on a_1, such that $|v|_{r+l-3} \le M_1|F|_{r+l-3}$ and $|Av|_{r+l-3} \le M_1|v|_{r+l-3}$.

Now we observe that, for all $\lambda \in [0,1]$, the inverse $(I + \lambda A)^{-1} : C^{r+l-3} \to C^{r+l-3}$ exists. For small λ this follows from the contraction mapping principle. Suppose that the inverse exists for some λ_0 and let $\delta > 0$. Note that

$$I + (\lambda_0 + \delta)A = (I + \lambda_0 A)(I + \delta(I + \lambda_0 A)^{-1}A).$$

It now follow that $|(I + \lambda_0 A)^{-1}Av|_{r+l-3} \leq M_1 M_2 |v|_{r+l-3}$. Hence the mapping $\delta(I + \lambda_0 A)^{-1}A$ is a contraction on C^{r+l-3} for $\delta < (M_1 M_2)^{-1}$, which implies the existence $(I + (\lambda_0 + \delta)A)^{-1}$. Since δ does not depend on λ this implies the existence of $(I + A)^{-1}$. Hence the equation (A.66) has a unique solution satisfying inequality (A.67) for any $F \in C^{r+l-3}$. Setting $F = -a_1$ and $v = r_1$, an appeal to (A.65) completes the proof. □

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