BEHAVIOUR OF INTERFACES IN A DIFFUSION-ABSORPTION EQUATION WITH CRITICAL EXPONENTS

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Abstract. We consider the Cauchy problem for the porous medium equation with strong absorption

$$u_t = (u^m)_{xx} - u^p \qquad \text{for } x \in \mathbf{R},\ t > 0, \tag{0.1}$$

with continuous compactly supported initial data $u(x,0) = u_0(x) \geq 0$ in the critical case $m + p = 2$ of the range of parameters $m > 1$, $p < 1$. We study the regularity of solutions and interfaces and compare the results with the purely diffusive case $u_t = (u^m)_{xx}$. Important differences are found in the interface behaviour and equations, in the occurrence of turning points and inflection points of the interfaces and in the fact that bounded solutions extinguish in finite time. All these phenomena are examined and described. The critical case studied here allows for a comparatively simpler and richer analysis of the qualitative behaviour of nonlinear parabolic equations involving finite propagation ($m > 1$) and strong absorption ($p < 1$).

1. **Introduction**

This paper is devoted to the investigation of the question of behaviour and higher regularity of solutions and interfaces of the Cauchy problem for the equation

$$u_t = (u^m)_{xx} - u^p, \tag{1.1}$$

with initial function

$$u(x,0) = u_0(x) \geq 0, \quad x \in \mathbf{R}. \tag{1.2}$$

We consider the range of parameters

$$m > 1, \quad p < 1, \quad m + p = 2, \tag{1.3}$$

so that we combine slow diffusion ($m > 1$), which originates finite propagation and interfaces, with strong absorption for small intensities ($p < 1$), which means that bounded solutions extinguish in finite time and interfaces can move forward or backward in space, thus exhibiting *heating* and *cooling fronts*. Such names derive from the typical application

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of this equation to the description of thermal propagation in absorbing media. For definiteness, we assume that the initial function u_0 is continuous, nonnegative and bounded. We also assume compact support in order to have interfaces. The equation

$$u_t = a(u^m)_{xx} - bu^p, \qquad a, b > 0, \tag{1.4}$$

apparently more general than (1.1), can be reduced to it by a simple rescaling. It is interesting to insert these coefficients in the formulae to see how the results for the diffusion-absorption equation bifurcate from the purely diffusive case $a = 1, b = 0$, i.e., the porous medium equation. These equations are model cases in the general type of diffusion-absorption equations of the form $u_t = \Phi(u)_{xx} - F(u)$ with suitable monotone nondecreasing functions Φ, F, cf. [26].

A basic analysis of regularity and interface behaviour has been performed in previous papers of the authors [17], [18] for $m + p \geq 2$ and $0 < m + p < 2$ respectively. We address here the special properties of the critical case $m + p = 2$, which offers a simpler mathematical treatment of the typical phenomena associated to slow diffusion with strong absorption, and at the same time marks the transition between quite different types of interface dynamics occurring for $m + p > 2$ and $m + p < 2$.

Main results. We begin the study by the question of higher regularity of solutions and interfaces. We establish analytic smoothness of the interfaces of compactly supported solutions. We also prove smoothness of the pressure of the solution, a function defined as

$$v = \frac{m}{m-1}\, u^{m-1}. \tag{1.5}$$

We remark that for the purely diffusive case $b = 0$ (i.e., for the Porous Medium Equation), a similar regularity result holds for solutions with suitable initial data, but only for large times if the data are general, cf. [2, 7]. Indeed, it can also happen for certain initial configurations that the interface is steady during a finite time interval (the *waiting time*), at the end of which an angle can be formed in the interface, so that the optimal global regularity is then Lipschitz continuity, cf. [6]. We show here that the waiting-time phenomenon is absent for $b > 0$, so that no singularity arises in the interface evolution.

Interesting phenomena occur at the critical points of the interface, which consist of *turning points*, where the monotonicity of the interface changes, and *inflection points*, where the concavity changes. We describe a countable set of the special patterns exhibiting turning and inflection-like behaviour of the interfaces. The construction of the patterns (they are not of any self-similar or a group invariant structure) consists in matching of three different regions: inner, intermediate and outer region. We show that in the inner region close to the interface the local inner spatial structure of all the turning/inflection patterns is governed by Kummer's polynomials $K_l(\zeta)$ for arbitrary integers $l = 2, 3, \ldots$, where ζ is the appropriate spatial rescaled variable. Each pattern is shown to match with the outer (x, t)-domain via the intermediate region governed by a first-order Hamilton-Jacobi equation.

2. **Preliminaries. Interface equation**

A study of the question of regularity of solutions and interfaces of the solutions to this problem is included in the paper [17]. It is known (see the survey [26] and [17]) that the Cauchy problem (1.1)-(1.2) for $p > 0$ has a unique continuous weak solution in a finite time interval $0 < t < T_e$, and that it vanishes identically as t approaches the *extinction time* T_e. The Lipschitz continuity of solutions and interfaces is established in the exponent range $m + p \geq 2$. If $-m < p \leq 0$ then, at least, it admits a unique proper (maximal) solution which is constructed by monotone decreasing regular approximations, see [24] and [23]. If $p \leq -m$ then no weak or maximal solution exists and any monotone approximation gives the trivial function $u(x,t) \equiv 0$ for $t > 0$ [24] *(complete extinction).*

THE PRESSURE. Taking $a, b > 0$ not necessarily 1, the natural variable to study regularity of solutions with interfaces is the *pressure* $v(x,t)$ defined as

$$v = a\frac{m}{m-1}u^{m-1}. \tag{2.1}$$

It satisfies the equation

$$v_t = (m-1)vv_{xx} + (v_x)^2 - abm\chi_{\{v>0\}}. \tag{2.2}$$

It is a C^∞-function in the positivity domain where $u > 0$ since the equation is uniformly parabolic on any set $\{u \geq \delta\}$ with arbitrarily small $\delta > 0$. The regularity as $u \to 0$, when the equation degenerates, is one of the questions under study in this paper.

ESTIMATES. Under the assumption that u_0 is nonnegative, it is known (see [20]) that the pressure v admits an a priori bound for the second space derivative of the form

$$v_{xx} \geq -\frac{1}{(m+1)t}, \tag{2.3}$$

so-called semiconvexity inequality. It follows that if v is bounded then it also has a bounded derivative in x for $t > 0$:

$$|v_x(x,t)|^2 \leq \frac{2}{m+1}\frac{\|v(\cdot,t)\|_\infty}{t}. \tag{2.4}$$

Moreover, (2.3) implies that for every $x_0 \in \mathbf{R}$ and $t_0 > 0$ there exist the lateral derivatives $D_x^- v(x_0,t_0)$ and $D_x^+ v(x_0,t_0)$ and

$$D_x^- v(x_0,t_0) \leq D_x^+ v(x_0,t_0) \tag{2.5}$$

(this is a the type of inequality typically found in conservation laws as a form of the entropy conditions). See also [26] and [17].

FINITE PROPAGATION AND INTERFACE EQUATION. On the other hand, if the initial data have compact support so does the solution and there appear two outer interfaces defined by

$$\eta(t) = \inf\{x \in \mathbf{R} :\ u(x,t) > 0\}, \qquad \zeta(t) = \sup\{x \in \mathbf{R} :\ u(x,t) > 0\}. \tag{2.6}$$

According to the results of [17], for bell-shaped solutions both interfaces, $x = \eta(t)$ and $x = \zeta(t)$ are Lipschitz continuous curves on $(\varepsilon, T_e - \varepsilon)$, $\varepsilon > 0$. The present equation represents a simple case (maybe the simplest one) of nonlinear diffusion-absorption equation exhibiting interfaces which do not follow the standard dynamical equation called *Darcy's law*, which is the well-known interface equation for pure diffusion, i.e., $b = 0$. For the right-hand interface Darcy's law reads

$$D^+\zeta(t) = -D_x^- v(\zeta(t), t).$$

However, for equation (1.1) with $m > 1 > p$, $m + p = 2$, the movement of the right-hand interface obeys the equation

$$D^+\zeta(t) = -\left(D_x^- v - \frac{c}{D_x^- v}\right)(\zeta(t), t), \quad c = abm, \tag{2.7}$$

cf. [17, Theorem 6.2]. Much of the analysis of [17] is based on comparison with the set of travelling wave solutions (TW's), which for $m + p = 2$ are given by

$$v(x,t) = \mu\,[\lambda t - x]_+, \tag{2.8}$$

where the spatial slope $\mu > 0$ and the wave velocity $\lambda \in \mathbf{R}$ are related by the formula

$$\lambda = F(\mu) \equiv \mu - \frac{abm}{\mu}, \tag{2.9}$$

which agrees with (2.7). Clearly F is a one-to-one map from $(0, \infty)$ onto $\mathbf{R}$ so that there exists the explicit inverse function $\mu = \mu(\lambda) > 0$ for all $\lambda \in \mathbf{R}$. We call T_λ the resulting solution with parameter λ. Observe that we obtain a heating wave ($\lambda > 0$) if $\mu > \mu_0 = \sqrt{abm}$, while for $\mu = \mu_0$ we get a stationary solution, and for $0 < \mu < \mu_0$ we have a cooling wave with speed $\lambda < 0$. In the limit of small absorption, $b \to 0$, we see that the branch of cooling wave solutions disappears and the stationary solution becomes flat. In the sequel we put $a = b = 1$.

FURTHER REGULARITY. Other results proved in [17] are as follows: the slope $D_x^- v$ is separated from zero and infinity in the whole interval $(0, T_e)$, uniformly on compact subsets. Moreover, for every $0 < t_0 < T_e$ the left derivative $D^-\zeta(t_0)$ exists and

$$D^-\zeta(t_0) = \lim_{t \uparrow t_0} D^+\zeta(t), \tag{2.10}$$

while also $D^+\zeta(t_0) = \lim_{t \downarrow t_0} D^+\zeta(t)$. Such a result holds in the more general parameter range $m > 1 > p$, $m + p \geq 2$. We will show in the next two sections how this regularity can be improved in the critical case $m + p = 2$.

3. C^1-regularity of the interface

Consider solutions $u(x,t)$ which have exactly two interfaces $\eta(t) < \zeta(t)$ for all $t \in (0, T_e)$ with $u(x,t) > 0$ for $\eta(t) < x < \zeta(t)$. For instance, this is true for solutions with bell-shaped initial data having a single maximum. The bell-shapedness is preserved in time since new maximum points cannot occur by the Maximum Principle.

We start the analysis of higher regularity of the interfaces by showing that the solution behaves in an approximately linear way near any interface point for $0 < t < T_e$. Let us choose a point (x_0, t_0) located on the right-hand interface, $x_0 = \zeta(t_0)$, for $0 < t_0 < T_e$. We know that v_x is bounded and that (2.3) holds. We prove the following result.

Theorem 3.1. *For every $t_0 \in (0, T_e)$*

$$D^+\zeta(t_0) = D^-\zeta(t_0). \tag{3.1}$$

Moreover, near (x_0, t_0) we have the estimate

$$v(x,t) = T_{\lambda_0}(x - x_0, t - t_0) + o(|x - x_0| + |t - t_0|), \tag{3.2}$$

with $\lambda_0 = D^+\zeta(t_0)$. In particular, the interface slope does not vanish on $(0, T_e)$

$$D_x^- v(\zeta(t_0), t_0) = -\mu(\lambda_0) \neq 0. \tag{3.3}$$

Observe that estimate (3.2) of the asymptotic shape comparison with the TWs is true for more general equations [17], [18].

Proof. (i) Following the ideas of [5], we perform the scaling

$$v_\delta(x,t) = \frac{1}{\delta} v(x_0 + \delta x, t_0 + \delta t) \tag{3.4}$$

with parameter $0 < \delta < 1$. The idea is to perform an invariant scaling transformation with the parameter δ around an interface point (x_0, t_0) and then let $\delta \to 0$ (blow-up) in order to study the situation there. It is easy to see that v_δ satisfies the same equation (2.2) in the domain $\mathbf{R} \times (-t_0/\delta, (T_e - t_0)/\delta)$. It has the same uniform estimate for the first derivatives since $v_{\delta,x} = v_x$ and $v_{\delta,t} = v_t$. As for the second derivative, we have

$$v_{\delta,xx}(x,t) \geq -\frac{\delta}{(m+1)(\delta t + t_0)}, \tag{3.5}$$

where the right-hand side goes to zero as $\delta \to +0$, an important fact in what follows. Moreover, by [17] the right-hand interface $x = s_\delta(t)$ for v_δ passes through the origin with (the right-hand) velocity

$$D^+ s_\delta(0) = \lambda_0.$$

We also know that there exists the left velocity $D^- s_\delta(0) = D^-\zeta(t_0) = \lambda_1$, which can be in principle different from λ_0. Moreover,

$$\lim_{t \uparrow 0} D^+ s_\delta(t) = D^- s_\delta(0) = \lambda_1, \quad \lim_{t \downarrow 0} D^+ s_\delta(t) = \lambda_0. \tag{3.6}$$

The interface equation (2.7) relates these velocities with the slopes at the interfaces.

(ii) Let us now pass to the limit as $\delta \to +0$. We use the Lipschitz continuity of solutions and interfaces together with known interior regularity to conclude that the family v_δ is compact in a Hölder (x,t)-space on compact subsets, so that along a sequence it converges to a certain limit

$$V(x,t) = \lim_{\delta_n \to 0} v_{\delta_n}(x,t). \tag{3.7}$$

It is clear that V is a continuous, weak and nonnegative solution of equation (2.2) in the whole plane $\{-\infty < x, t < \infty\}$, and that it has at $t = 0$ the linear profile

$$V(x, 0) = [-\mu(\lambda_0)x]_+. \tag{3.8}$$

On the other hand, the second derivative estimate passes to the limit as

$$V_{xx} \geq 0. \tag{3.9}$$

We want to conclude that $V(x, t)$ coincides with $T_{\lambda_0}(x, t)$ which ends the proof. The conclusion is immediate for $t > 0$ by uniqueness of weak solutions, so we only have to check what happens for $t < 0$.

(iii) Let us discuss the limit V for $t < 0$. We apply a simple geometric approach based on the semiconvexity estimate. Using (3.6) and the convexity it is easy to see that for $t < 0$ the solution V is positive in the region $x - \lambda_1 t < 0$ with interface $x = \lambda_1 t$ and it has there a slope

$$D_x^- V(\lambda_1 t, t) = -\mu(\lambda_1),$$

and in view of the convexity a lower estimate holds

$$V(x, t) \geq T_{\lambda_1}(x, t), \quad t < 0. \tag{3.10}$$

This immediately implies in the limit $t \to -0$ that $\lambda_1 \leq \lambda_0$, since the slope μ is an increasing function of λ. We will prove later that equality holds. For the moment we continue as follows. It is known (and proved by approximation) that the derivative v_x of the solutions of equation (2.2) satisfies the Maximum Principle. Since V_x attains the value $-\mu(\lambda_0)$ at $t = 0$, it follows that $\|V_x(\cdot, t)\|_\infty$ was equal or larger than this value for all $t < 0$. Now, the function $V(x, t)$ is convex in x for every t so that the maximum value of $-V_x$ is taken at $x = -\infty$. It must be precisely λ_0, not larger, otherwise V would resemble a travelling wave with a larger speed and the interface would eventually move faster than $x = \lambda_0 t$, a contradiction. It follows that

$$\lim_{x \to -\infty} V_x = -\mu(\lambda_0). \tag{3.11}$$

Suppose now that equality holds, $\lambda_1 = \lambda_0$. Since V_x starts with the value $-\mu(\lambda_1)$ at the interface and it is monotone and ends with value $-\mu(\lambda_0)$, we conclude that V_x is constant on the positivity set of V when $\lambda_0 = \lambda_1$, so that $V = T_{\lambda_0}$ and the proof is complete when we undo the scaling.

(iv) Next, let us examine the possibility $\lambda_1 < \lambda_0$ and let us arrive at a contradiction. In that case we still have (3.10) and (3.11), and

$$\mu(\lambda_1) \leq -v_x \leq \mu(\lambda_0)$$

for $x < -\lambda_1 t$, $t < 0$. Let us prove next that

$$V(x, t) \geq T_{\lambda_0}(x, t) \tag{3.12}$$

for $t < 0$. We need another estimate, this time on $v_{tt}(x, t)$. It takes the form

$$|v_{tt}| \leq \frac{C}{|x - x_0|},$$

valid on small triangular regions of the form

$$R = \{(x,t) : \; x_0 - \delta < x < x_0, \; 0 < 2\lambda_1(t_0 - t) < (x_0 - x)\}$$

with $\delta > 0$. This comes from the scaling arguing by contradiction on a sequence of points $(x_n, t_n) \to (x_0, t_0)$ in R, putting $\delta_n = x_0 - x_n$, using the uniform convergence of the v_δ and their derivatives and the equality

$$v_{\delta,tt} = \delta v_{tt}.$$

Once this is established, we can derive the estimate

$$|v_{\delta,tt}| \le \frac{C}{|x|}$$

on growing triangles, and in the limit the same inequality is true for V in an infinite triangle backwards in time. Using Taylor's formula at a point (x,t) with $x < 2\lambda_1 t < 0$, we get

$$V(x,t) = V(x,0) + tV_t(x,0) + \frac{t^2}{2} V_{tt}(x,s).$$

The last term is bounded by $C\,t^2/x$. The other two can be computed explicitly at $t = 0$ and amount to the travelling wave. We conclude that

$$V(x,t) - T_{\lambda_0}(x,t) \ge -\frac{C}{|x|}.$$

Taking the limit as $x \to -\infty$ we see that it goes to zero. Since the function is convex, it must be nonnegative everywhere.

The final step consists in proving that equality holds in (3.12). This is an easy consequence of the Strong Maximum Principle since both members are solutions of equation (2.2), which is uniformly parabolic with smooth coefficients in the positivity domain $\{v \ge \varepsilon > 0\}$. Since both solutions coincide for $t \ge 0$ and they are ordered, they must also coincide before. As a summary, this means that the "initial" data (3.8) prescribe the unique TW solution

$$V(x,t) \equiv [-\mu(\lambda_0)(x - \lambda_0 t)]_+$$

for any $t \approx 0$. □

Theorem 3.2. *The interface functions $\eta(t)$ and $\zeta(t)$ are C^1 functions for $0 < t < T_e$. Moreover, the pressure is C^1 in the positivity set and up to the interfaces, i.e., in*

$$C = \{(x,t) : \; 0 < t < T_e, \; \eta(t) \le x \le \zeta(t)\}.$$

Finally, $v\,v_{xx} \to 0$ as $(x,t) \in \{v > 0\}$ approaches a point on one of the interfaces, so that the equation

$$v_t = (v_x)^2 - m \tag{3.13}$$

holds on both interfaces.

Proof. The C^1 regularity of the interfaces comes from the equality $D^+\zeta(t_0) = D^-\zeta(t_0)$ which shows differentiability at every time t_0 for the right-hand interface (the same holds for the left-hand one by symmetry). The continuity of $\zeta'(t)$ follows then from the two partial statements

$$D^-\zeta(t_0) = \lim_{t\uparrow t_0} D^+\zeta(t), \qquad D^+\zeta(t_0) = \lim_{t\downarrow t_0} D^+\zeta(t).$$

The C^1 smoothness of the pressure away from the interfaces is standard since the equation is parabolic non-degenerate there. Near the interfaces it follows from the blow-up argument of the previous theorem, as does the fact that $v\, v_{xx} \to 0$ as (x,t) approaches the interface, hence the limit equation (3.13). It represents a more classical form of the interface dynamics (2.7). We recall that the pressure has discontinuous gradient across the interfaces, the values involved in formula (3.13) are the inner limits, taken from positivity set $\{v > 0\}$. $\square$

4. **Analyticity of the interfaces**

As in the previous section we consider solutions with exactly two interfaces defined for $0 < t < T_e$, T_e being the extinction time.

Theorem 4.1. *The interfaces are real analytic functions for* $0 < t < T_e$.

We adapt the arguments earlier used in [2] to establish real analyticity of interfaces in solutions of the one-dimensional Porous Medium Equation. There is a difference in the result: in our present case the interfaces are real analytic functions throughout the interval of existence of the solution $(0, T_e)$, while in the PME case there is a possible exception in the form of a corner point at the waiting time.

Let $v(x,t)$ be the function defined by (1.5). Using notations from [2], we introduce the new unknown

$$V(\xi,t) = v\left(\frac{\zeta_+ + \zeta_-}{2} + \frac{\zeta_- - \zeta_+}{2}\cos\xi, t\right), \quad \xi \in \boldsymbol{R},\, t \in [0, T_e - \varepsilon],$$

where we use the more convenient notation $x = \zeta_+(t)$ (resp. $x = \zeta_-(t)$) for the right-hand (resp. left-hand) interface. The new function $V(\xi,t)$ can be extended as a 2π-periodic function of ξ which solves the equation

$$\begin{aligned} V_t = {} & \left(\frac{2}{\zeta_+ - \zeta_-}\right)^2 \left[\frac{(m-1)V}{\sin^2\xi}\left(V_{\xi\xi} - \frac{\cos\xi}{\sin\xi}V_\xi\right) + \left(\frac{V_\xi}{\sin\xi}\right)^2\right] \\ & + \frac{(1+\cos\xi)Z_- + (1-\cos\xi)Z_+}{\zeta_+ - \zeta_-}\frac{V_\xi}{\sin\xi} - m, \end{aligned} \tag{4.1}$$

where we have put $Z_\pm(t) = \zeta'_\pm(t)$, which are continuous functions according to the results of the previous section. Equation (4.1) differs from that for the PME [2] by the last constant term on the right-hand side. Step by step we will show that this term plays no essential role in both the local and global analyticity analysis.

It follows from the definition of the change of variables $x \mapsto \xi$ that near the line $\xi = 0$ (i.e., $x = \zeta_-(t)$)

$$V(\xi, t) = v_x(\zeta_-, t)\frac{\zeta_+ - \zeta_-}{4}\xi^2 + o(\xi^2).$$

Therefore,

$$\begin{aligned} &V_\xi(0,t) = 0, \\ &\frac{V_\xi(\xi,t)}{\sin\,\xi} = v_x(\zeta_-, t)\frac{\zeta_+ - \zeta_-}{2} + o(1), \\ &\frac{V(\xi,t)}{\sin^2\xi} = v_x(\zeta_-, t)\frac{\zeta_+ - \zeta_-}{4} + o(1). \end{aligned} \tag{4.2}$$

Similar expansions hold at $\xi = \pi$ ($x = \zeta_+(t)$).

Hence, since $v_x(\zeta_\pm(t), t)$ is nonzero by (3.3) and therefore $V(\xi, t)/\sin^2\xi > 0$ uniformly, equation (4.1) is a nondegenerate parabolic equation with singularities in the coefficients of the first-order terms. Moreover, the free boundaries $x = \zeta_\pm(t)$ become the vertical lines $\xi_+ = \pi$, $\xi_- = 0$ in the plane of the new variables (ξ, t).

The equations of the free boundaries can now be derived from equations (2.7), (4.1) and (4.2). We have that

$$\zeta_\pm'(t) \equiv Z_\pm(t) = -\left[v_x(\zeta_\pm, t) - \frac{m}{v_x(\zeta_\pm, t)}\right], \tag{4.3}$$

$$V_{\xi\xi}(0,t) = \frac{\zeta_+ - \zeta_-}{2}v_x(\zeta_-, t), \qquad V_{\xi\xi}(\pi,t) = \frac{\zeta_+ - \zeta_-}{2}v_x(\zeta_+, t), \tag{4.4}$$

whence the equations

$$\begin{cases} Z_-(t) = -Y_- + \dfrac{m}{Y_-}, & Z_+(t) = -Y_+ + \dfrac{m}{Y_+}, \\ Y_-(t) = \dfrac{2V_{\xi\xi}(0,t)}{\zeta_+ - \zeta_-}, & Y_+(t) = \dfrac{2V_{\xi\xi}(\pi,t)}{\zeta_+ - \zeta_-}. \end{cases} \tag{4.5}$$

Problem (4.1), (4.5) is considered as an initial-value problem by defining the triple $w(t) = (V, \zeta_+, \zeta_-)$. This function is a solution of a dynamical system of the form

$$w'(t) = F(w(t)) \qquad \text{with} \qquad F(V, \zeta_+, \zeta_-) = (V_t, Z_+, Z_-). \tag{4.6}$$

In order to consider the problem as an abstract parabolic initial-value problem in the sense of [29], as in [2], define the following Banach spaces:

$$E_{0,\theta} = \{v \in h^\theta(\mathbf{R}) : V(\xi) \equiv V(\xi + 2\pi) \equiv V(-\xi),\, V(0) = V(\pi) = 0\},$$

$$E_{1,\theta} = h^{2+\theta}(\mathbf{R}) \cap E_{0,\theta},$$

where $\theta \in (0,1)$, and $h^s(\mathbf{R})$ denotes the little Hölder-space with non-integer exponent s. Next, set

$$X_{k,\theta} = E_{k,\theta} \oplus \mathbf{R}^2, \qquad k = 0, 1,$$

and consider the open subset $\mathcal{O}_\theta \subset X_{1,\theta}$ defined by

$$\mathcal{O}_\theta = \{w \in X_{1,\theta} \mid \zeta_- < \zeta_+,\, V(\xi) > 0 \text{ for } 0 < \xi < \pi,\, V_{\xi\xi}(0) > 0,\, V_{\xi\xi}(\pi) > 0\}.$$

We thus consider the dynamical system (4.6) with $F : \mathcal{O}_\theta \mapsto X_{0,\theta}$. The local analyticity result is obtained by using the general approach to abstract evolution equations [4]. As in [3], it can be shown that $F : \mathcal{O}_\theta \mapsto X_{0,\theta}$ is a real analytic map, and that for every $w \in \mathcal{O}_\theta$ the Fréchet derivative $dF(w) : X_{1,\theta} \mapsto X_{0,\theta}$, viewed as an unbounded operator in $X_{0,\sigma}$ ($0 < \sigma \leq \theta$), generates an analytic semigroup. Since the derivative dF does not depend on the last constant term $-m$ in equation (4.1), the arguments of [2] apply in this case. Namely, this problem reduces to showing that the operator $A = a(x)(d/dx)^2 + (b(x)/x)(d/dx)$, $x > 0$ with coefficients $a, a^{-1}, b \in h^\theta([0,\infty))$, $a > 0$, generates an analytic semigroup in $h^\theta([0,\infty))$ if $\mathrm{dom}(A) = \{u \in h^{2+\theta}([0,\infty)) | \, u'(0) = 0\}$. This is done in [3] by explicitly constructing the resolvent of the operator A.

The next step is to check that $w(t)$ is (locally) real analytic. This can be done by literal repetition of the arguments of [2] based on the approach [29]. If the mapping F smoothly depends on a parameter λ, so does the solution of the equation $w'(t) = F(w)$ [29]. Let us consider the following dependence of F on a real positive parameter λ: $(\lambda, w) \mapsto \lambda F(w)$. This is a real analytic map from $\mathbf{R}_+ \times \mathcal{O}_\theta$ to $X_{0,\theta}$. By virtue of the equation for $w(t)$ we also have that $w(\lambda t)$ is a solution of the equation $w'(\tau) = \lambda F(w(\tau))$. Thus, $w(\lambda t)$ analytically depends on λ in the topology of $C([0,T]; X_{1,\theta})$ with some $T > 0$.

The global analyticity analysis on $(0, T_e)$ is performed with a slight modification of the transformations and main arguments in [2]. Let us introduce the local variables near the interface $x = \zeta(t)$: the new independent space variable will be y defined by $v(x,t) = y^2/2$. This transformation renders the interface $x = \zeta(t)$ the vertical line $y = 0$ in the plane of the new variables. It is easy to see that the function $p(y,t) \equiv v_x$ solves the problem

$$p_t = \frac{m-1}{2} p^2 \left(p_{yy} + \frac{m+1}{m-1} \frac{p_y}{y} \right) + \frac{m p_y}{y} \qquad \text{in } (-\eta, \eta) \times (0, t'], \tag{4.7}$$

$$p(t,\eta) = \phi(t), \quad p(y,0) \text{ is given}, \;\; p(y,t) \text{ is an even function of } y. \tag{4.8}$$

In this problem $\phi(t)$ is a prescribed C^∞ function: $\phi(t) = v_x$ along the line $v = \eta$. Notice that $\phi(t)$ is separated away from zero and infinity [17, Theorem 6.2]. In order to rewrite this problem as a problem with homogeneo us boundary conditions, we set $p = P + \psi$ where the function P is the new unknown and ψ is an even function satisfying the equation

$$\psi_t = \frac{m-1}{2} \psi^2 A(\psi) \quad \text{for } t > 0,\, y \in (-\eta, \eta),$$

where

$$A = \frac{m-1}{2} \left[\left(\frac{d}{dy} \right)^2 + \frac{m+1}{m-1} \frac{1}{y} \frac{d}{dy} \right],$$

and the same boundary conditions that $p(y,t)$. Such a function $\psi(y,t)$ can be constructed explicitly [2]. For the function P we now have the problem

$$P_t = \frac{m-1}{2}(P+\psi)^2 A(P) + B(P) + f(P,t) \quad \text{in } (-\eta,\eta)\times(0,t'],$$
$$P(\pm\eta,t)=0, \quad P(0,t) \text{ is given}, \quad P(-y,t)=P(y,t),$$

with the notations

$$B = \frac{m}{y}\frac{d}{dy}, \quad f(P,t) = \frac{m-1}{2}(P+\psi)^2 A(\psi) + B(\psi) - \psi_t.$$

Finally, we need to prove that any local solution p of problem (4.7)-(4.8) is bounded in h^θ. It suffices to show that v_{xx} is bounded because

$$p_y = (v_x)_y = v_{xx}x_y, \quad \text{and} \quad p = v_x = yy_x = \frac{y}{x_y}, \qquad \text{whence} \quad p_y = \frac{yv_{xx}}{v_x}.$$

Thus, p is Hölder continuous in y if, say, v_{xx} is bounded. The lower estimate on v_{xx} is given by (2.3).

Lemma 4.1. *For every $t_0 \in (0,T_e)$, there exists an $\varepsilon > 0$ such that in the domain $D_\varepsilon = (l_\varepsilon(t),\eta(t))\times(t_0,t_0+\varepsilon)$*

$$v_{xx} \le \frac{\delta}{t-t_0}, \tag{4.9}$$

where $x = l_\varepsilon(t)$ is the ε-level curve of $v(x,t)$.

Proof. First of all, it can be seen from Theorem 3.2 that for any $t \in (0,T_e)$

$$\lim_{x\to\zeta(t)} vv_{xx} = 0. \tag{4.10}$$

The function $v_{xx} = W$ satisfies the equation

$$\mathcal{L}W \equiv W_t - (m-1)v\,W_{xx} - 2mv_x\,W_x - (m+1)W^2 = 0.$$

Let us consider the function

$$\omega = \frac{C}{v} + \frac{\delta}{t-t_0+\gamma v}$$

depending on positive constants C , γ, δ to be defined. A straightforward calculation gives

$$\begin{aligned}\mathcal{L}\omega \;\equiv\; & \frac{mC}{v^2} + \frac{C\,v_x^2}{v^2} + \frac{\delta(m\gamma-1)}{(t-t_0+\gamma v)^2} + \frac{(2m-1)\delta\gamma v_x^2}{(t-t_0+\gamma v)^2} \\ & - \frac{2(m-1)\delta\gamma^2 v\,v_x^2}{(t-t_0+\gamma v)^3} - (m+1)\left(\frac{C}{v} + \frac{\delta}{t-t_0+\gamma v}\right)^2.\end{aligned}$$

Making use of the obvious inequality $\gamma v \le t - t_0 + \gamma v$, we arrive at the inequality

$$\begin{aligned}\mathcal{L}\omega &> \frac{\delta(m\gamma - 1)}{(t-t_0+\gamma v)^2} + \frac{\delta\gamma v_x^2}{(t-t_0+\gamma v)^2} - \frac{2(m+1)C^2}{v^2} - \frac{2\delta^2(m+1)}{(t-t_0+\gamma v)^2} + \frac{mC}{v^2} \\ &> \frac{1}{(t-t_0+\gamma v)^2}\left[\delta(m\gamma-1) - 2\delta^2(m+1)\right] + \frac{C(m-2(m+1)C)}{v^2}.\end{aligned}$$

To satisfy the condition $\mathcal{L}\omega > 0$, it suffices to claim

$$m\gamma - 2\delta(m+1) \ge 1, \qquad C \le \frac{m}{2(m+1)}.$$

Let us now take two level curves of $v(x,t)$, $x = l_\varepsilon(t)$ and $x = l_\sigma(t)$, $0 < \sigma < \varepsilon$, and define the curvilinear domain $R_{\varepsilon,\sigma} = (l_\sigma, l_\varepsilon) \times (t_0, t_0+\varepsilon)$ so that

$$\omega \ge W \quad \text{on the parabolic boundary of } R_{\varepsilon,\sigma}. \tag{4.11}$$

This choice of ε (independent of C) and σ is always possible. Indeed, once the parameters C, γ and δ are chosen, to satisfy (4.11), it amounts fulfilling the inequalities

$$vv_{xx}\big|_{x=l_\varepsilon(t)} \le \frac{\delta}{1+\gamma}, \qquad vv_{xx}\big|_{x=l_\sigma(t)} \le C, \qquad vv_{xx}\big|_{t=t_0} \le \frac{\delta}{\gamma}.$$

By the Maximum Principle we conclude that

$$W \le \frac{C}{v} + \frac{\delta}{t-t_0+\gamma v} \quad \text{in } R_{\varepsilon,\sigma},$$

where the constants C , γ, δ are independent of σ. Hence, this inequality holds in the domain D_ε (the limit of $R_{\varepsilon,\sigma}$ when $\sigma \to 0$). Next, the constant C is subject to the sole condition $C \le m/2(m+1)$. Letting $C \to +0$, we arrive at the desired inequality:

$$W \le \frac{\delta}{t-t_0+\gamma v} \le \frac{\delta}{t-t_0}.$$

□

Repeating the arguments of [3], we now check that problem (4.7)-(4.8) has a local solution p bounded in h^θ which can be continued on $(0,t']$ for every $t' < T_e$. It follows then that the local orbit $\{V(t),\, 0 \le t \le T_e - \varepsilon\}$ is precompact in $h^{2+\theta}$ which gives the crucial estimate [2] to extend the analyticity property to any interval $(0,t')$ with $t' < T_e$. As we know [20], at $t = T_e^-$ when the interfaces meet, they create a singularity like $\sqrt{T_e - t}$. This means Hölder continuity at $t = T_e^-$ with the exponent $1/2$ which is supposed to be the best uniform regularity of the interfaces on the closed intervals like $[\varepsilon, T_e]$.

5. **Immediate consequences of analyticity**

5.1. **No waiting time effect.** As an immediate consequence of the analyticity we arrive at the following conclusion.

Lemma 5.1. *The interfaces do not contain vertical parts in the* $\{x,t\}$*-plane.*

Proof. Indeed, let us assume, for contradiction, that the right-hand interface satisfies $\zeta_+(t) = A$ for $t \in [t_0, t_1]$ with some constant A and $t_0 \ge 0$. Set

$$T' = \sup\{\tau \in (t_0, T_e) : \zeta(t) = A \text{ on } [t_0, \tau]\}.$$

Since the interface is flat near the extinction point [20], we have $T' < T_e$. Consider Taylor's expansion of $\zeta_+(t)$ in a neighbourhood of the point T':

$$\zeta(t) = A + \sum_{i \ge 1} \frac{\zeta^{(i)}(T')}{i!}(t - T')^i.$$

Since $\zeta(t) = A$ in a left neighbourhood of the point $t = T'$, $\zeta^{(i)}(T' - 0) = 0$ for all $i \ge 1$ and, correspondingly, $\zeta^{(i)}(T') = 0$. On the other hand, this means that $\zeta(t) = A$ in a right neighbourhood of the point $t = T'$, which contradicts the choice of T'. $\square$

Indeed, this proof shows that the interfaces cannot have straight-line pieces unless they are globally straight lines. This happens in the case of travelling waves, but cannot happen for bounded solutions because of the extinction property.

5.2. **Finite number of turning and inflection points.** The critical points of $\zeta_+(t)$, i.e., the zeroes of $\zeta_+'(t)$ are classified in two groups. When the zero, $t = t_0$, is of odd order the sign of $\zeta_+'(t)$ changes in a neighbourhood of t_0 we call it a *turning point*. On the other hand, for an zero of even order, ζ is monotone in a neighbourhood of t_0 and it is called an *inflection*.

Lemma 5.2. *The number of critical points of the function $\zeta_\pm(t)$ is finite on every interval $[\tau, T_e]$, $\tau > 0$.*

Proof. On the one hand, we check that the zeros of the non-trivial analytic function $\zeta_\pm'(t)$ are isolated. This proves the result in intervals of the form $[\tau, T_e - \tau]$. Near the extinction point we use the results of [20]. $\square$

The result implies the the number of critical points is at most countable on the whole time interval $[0, T_e]$. The number is finite if the initial function is well behaved. For instance, due to the Maximum Principle this is true if the initial pressure is C^1 inside the support and its derivarive has a finite number of oscillations.

6. Countable sets of turning and inflection patterns

We now study some delicate asymptotic aspects of the interface dynamics. Consider the quasilinear quadratic equation (2.2) with $a = b = 1$ for the pressure (1.5):

(6.1) $$v_t = \mathbf{A}(v) - m\chi_{\{v>0\}} \equiv (m-1)vv_{xx} + (v_x)^2 - m\chi_{\{v>0\}}.$$

For convenience, we will fix the left-hand interface that we call again $x = \eta(t)$. We begin with an exact solution.

6.1. **Example: the first turning pattern.** The first exact solutions describing a turning point of interfaces was constructed by Kersner in [27]. It has a simple truncated inverse parabolic shape

$$v_1(x,t) = [C_0(t) - C_1(t)x^2]_+, \tag{6.2}$$

where the coefficients $\{C_0(t), C_1(t)\}$ solve the following dynamical system:

$$C_0' = -2(m-1)C_1C_0 - m, \quad C_1' = -2(m+1)C_1^2, \quad t > 0. \tag{6.3}$$

This system is integrated explicitly which gives the solution

$$v_1(x,t) = \left[\frac{m+1}{2}t^{1-k}(T_e^k - t^k) - \frac{x^2}{2(m+1)t}\right]_+, \quad k = \frac{2m}{m+1} \in (1,2), \tag{6.4}$$

where the finite extinction time $T_e > 0$ is arbitrary. It is worth mentioning that the exact solution (6.4) is not invariant under any Lie or Lie-Bäcklund group of transformations. See a survey on the results of a general group-classification of quasilinear heat equations in [15] and for this equation in [31]. On the other hand, it is a solution of the equation (6.1) on a two-dimensional subspace Span$\{1, x^2\}$ which is invariant under the quadratic operator $\mathbf{A}$ and can be constructed by a nonlinear method of separation of variables. See [14] and other examples therein.

Consider some other properties of this explicit solution. The most striking property is as follows: its left-hand interface, which is analytic for $t \in (0, T_e)$

$$\eta_2(t) = -(m+1)\sqrt{t^{2-k}(T_e^k - t^k)} < 0, \tag{6.5}$$

is non-monotone and attains the minimum value X_1 at the moment T_1

$$T_1 = T_e\left(\frac{2-k}{2}\right)^{1/k}, \quad X_1 = \eta_2(T_1) = -(m+1)T_1\sqrt{\frac{k}{2-k}}. \tag{6.6}$$

Therefore, $t = T_1$ is the turning point of the interface. We will show that this turning point corresponds to the first, generic and stable, K_2-turning pattern of the interface for general solutions, where K_2 denotes the second Kummer polynomial introduced later on.

At $t = T_e^-$ a new singularity of finite time extinction occurs: the solution vanishes identically. It is curious that the asymptotic behaviour near the extinction time as $t \to T_e^-$, exhibited by the above exact solution, is also the stable and generic one. This is proven in [20], [21].

6.2. **Asymptotic analysis of patterns in the inner region.** Let us study the situation around a critical point of the left-hand analytic interface of the solution, $x = \eta(t)$. This means that $\eta'(t_0) = 0$. Using the translational invariance of the PDE, we can put the origin of coordinates at that point and assume that the interface, which is well-defined for all $t \approx 0$, $\eta(0) = 0$, exhibits a special *turning* or *inflection* behaviour as $t \to -0$. In order to classify such a behaviour as $t \to -0$, we introduce the standard self-similar rescaled variables

$$v(x,t) = (-t)\theta(\xi,\tau), \quad \xi = x/(-t), \tag{6.7}$$

where $\tau = -\ln(-t)$ is the new time variable, so that the finite limit $t \to -0$ corresponds to the infinite time behaviour, $\tau \to \infty$, of the rescaled function $\theta(\xi, \tau)$. It satisfies the following quasilinear parabolic equation

$$\theta_\tau = \mathbf{B}(\theta), \quad \tau \gg 1, \tag{6.8}$$

where $\mathbf{B}$ is the quadratic operator

$$\mathbf{B}(\theta) = \mathbf{A}(\theta) - m\chi_{\{\theta>0\}} + \mathbf{C}\theta, \quad \mathbf{C}\theta = -\theta_\xi \xi + \theta.$$

Obviously, the autonomous equation (6.8) admits the unique nontrivial weak stationary piece-wise linear solution with the interface at the origin $x = 0$:

$$S(\xi) = (\sqrt{m}\,\xi)_+ \equiv (\sqrt{m}\,x)_+. \tag{6.9}$$

Let us first study the "nonlinear" eigenfunctions which are stationary solutions of the rescaled equation (6.8):

$$\mathbf{B}(h) = 0. \tag{6.10}$$

It is known that in several blow-up problems nonlinear eigenfunctions can form finite (see Chapter 4 in [30]) or even infinite countable subset (see [16], [12] and references therein). We show that for the present focusing-like problem, this is not the case.

Proposition 1. *Finite interface solutions of equation (6.10) form a one-dimensional set of stationary profiles*

$$h_A(\xi) = [A\xi + m - A^2]_+, \quad A \in \mathbf{R}, \tag{6.11}$$

with straight monotone interfaces of the travelling-wave type on the $\{x, t\}$-plane. No other solutions exist.

It follows from scaling (6.7), that for any $A \ne \sqrt{m}$ the self-similar solutions describe propagation with a non-zero velocity:

$$v(x,t) \equiv [Ax + (m - A^2)(-t)]_+, \quad \eta'(t) = (m - A^2)/A,$$

and indeed these solutions are the TWs with constant speeds. Thus, nonlinear self-similar turning/inflection patterns do not exist.

We now present a new crucial geometric property of such patterns to be studied.

Proposition 2. *If $\eta'(0) = 0$, then for all $\tau \gg 1$ the profile $\theta(\cdot, \tau)$ intersects $S(\xi)$ at least at two points and the corresponding two continuous intersection curves converge to* 0 *as $\tau \to \infty$ (i.e., as $t \uparrow 0$).*

Proof. We give a simple geometric explanation of the above property. Assume for a contradiction, that, for some $\tau_0 < 0$, the profile $\theta(\xi, \tau_0)$ with, say, $\eta(\tau_0) > 0$ (the turning case) has a single intersection with $S(\xi) \equiv h_0(\xi)$, so that the number of intersections satisfies

$$I(\tau_0; h_0) = 1. \tag{6.12}$$

Then by the Strong Maximum Principle in the positivity domain, this intersection is transversal, i.e., stable upon small C^1-perturbation of the profiles. By continuity of the family (6.11) upon parameter A, for any $A \approx \sqrt{m}$, $A > \sqrt{m}$, there holds $I(\tau_0; h_A) = 1$. Therefore, at $\tau = \infty$ (i.e., $t = 0$) the interfaces of both solutions $u(x,t)$ and the corresponding self-similar (TW) solution $V(x,t) = (-t)h_A(x/(-t))$ coincide. By a typical intersection comparison conclusion (see Lemma 8.2 in [20]), this means that the unique intersection disappears at $t = 0$ so that the solutions are ordered. Namely, $u(x,0) \geq V(x,0)$, $x \in \mathbf{R}$. By the usual comparison of weak solutions, this implies that for $t \geq 0$

$$\eta(t) \leq \frac{m - A^2}{A}\, t,$$

so that $\eta'(0) \leq (m - A^2)/A < 0$. This contradicts the assumption. $\square$

In view of Proposition 1 (which actually means nonexistence of nonlinear eigenfunctions), we now begin the construction of the "semilinear" patterns based on the linearized analysis and a matching procedure. In fact, it is expected that a subset of all the nonlinear and semilinear patterns gives a complete description of all the possible asymptotic structures. We do not discuss this question for the problem under consideration. The completeness question is quite actual in blow-up problems for quasilinear reaction-diffusion equations, though it is far away from being solved entirely. We mention that the so-called *transition phenomena* between nonlinear and semilinear spectra for a typical blow-up problem in the delicate case of gradient dependent diffusivity (where, unlike the current problem, the nonlinear spectrum can be countable [16]) is considered in [12].

In the *inner domain*, to be defined later on according to the appropriate scaling, the turning and inflection patterns can be considered as perturbations about the stationary profile. We perform the linearization transformation

$$\theta(\xi, \tau) = S(\xi) + Y(\xi, \tau),$$

where the perturbation $Y(\xi, \tau) \to 0$ as $\tau \to \infty$ solves the quasilinear equation

$$Y_\tau = \mathbf{D}Y + \mathbf{A}(Y), \quad \tau \gg 1, \tag{6.13}$$

and $\mathbf{D}$ is the linear operator

$$\mathbf{D}Y = (m-1)\sqrt{m}\,\xi Y_{\xi\xi} + (2\sqrt{m} - \xi)Y_\xi + Y. \tag{6.14}$$

Setting

$$\xi = \gamma\zeta, \quad \gamma = (m-1)\sqrt{m}, \tag{6.15}$$

we obtain the equation

$$Y_\tau = \mathbf{D_1}Y + \gamma^{-2}\,\mathbf{A}(Y), \tag{6.16}$$

where $\mathbf{D_1}$ is the hypergeometric operator

$$\mathbf{D_1}Y = \zeta Y_{\zeta\zeta} + (N - \zeta)Y_\zeta + Y, \quad N = \frac{2}{m-1}. \tag{6.17}$$

As $Y(\cdot, \tau) \to 0$, the quadratic operator $\mathbf{A}$ in (6.16) forms an asymptotically negligible perturbation of the linear degenerate (at $\zeta = 0$) parabolic flow

$$Y_\tau = \mathbf{D_1}Y.$$

Observe that this perturbation is non-singular and the perturbed equation inherits a natural interior parabolic regularity. Therefore, the Ω-limit set of equation (6.16) can be studied by standard asymptotic techniques.

The operator $\mathbf{D_1}$ is a standard singular Sturm-Liouville operator of the form

$$\mathbf{D_1} Y = \frac{1}{\rho}\,(pY')' + Y, \quad \eta \in (0, \infty), \tag{6.18}$$

with positive coefficients

$$p(\zeta) = \zeta^N e^{-\zeta} > 0, \quad \rho(\zeta) = \zeta^{N-1} e^{-\zeta} > 0, \quad \zeta > 0.$$

It is not difficult to see that both points $\zeta = 0$ and $\zeta = \infty$ are in the limit-point case of singular endpoints [28] and $-\mathbf{D_1} + 1$ is semibounded below. Therefore, it admits a unique Friedrichs extension [10] which is a self-adjoint operator in the weighted Hilbert space $L^2_\rho(\mathbf{R}_+)$ with the discrete spectrum (see [28] and [8])

$$\lambda_l = l - 1, \quad l = 0, 1, 2, \dots . \tag{6.19}$$

Observe that after a natural transformation, the value $N = 2/(m-1)$ can be treated as the artificial dimension of an Euclidean space $\mathbf{R}^N$ of radial functions and then $L^2_\rho(\mathbf{R}_+)$ is identified with the weighted space $L^2_\mu(\mathbf{R}^N)$, $\mu = e^{-\zeta}$.

The eigenfunctions $\{K_l(\eta)\}$ of $\mathbf{D_1}$ are Kummer's functions (k-th order polynomials)

$$K_0(\zeta) = c_0, \ \lambda_0 = -1; \quad K_1(\zeta) = c_1(-\zeta + N), \ \lambda_1 = 0;$$

$$K_2(\zeta) = c_2[\zeta^2 - 2(N+1)\zeta + N(N+1)], \ \lambda_2 = 1, \tag{6.20}$$

$$K_3(\zeta) = c_3[-\zeta^3 + 3(N+2)\zeta^2 - 3(N+1)(N+2)\zeta + N(N+1)(N+2)], \ \lambda_3 = 2, \ \text{etc.}$$

They form an orthonormal basis in $L^2_\rho(\mathbf{R}_+)$, $\{c_l > 0\}$ are normalization constants. The unstable subspace and the centre one are one-dimensional: $E^u = \text{Span}\{K_0\}$, $E^c = \text{Span}\{K_1\}$, and the stable subspace E^s has codimension 2. Denote

$$K_l(0) = a_l > 0.$$

Each polynomial $K_l(\zeta)$ has exactly l zeros for $\zeta > 0$ [8].

As usual in the study of singular nonlinear phenomena, the unstable manifold for the rescaled equation (6.13) corresponds to the unstabilities related to the translational invariance of the original parabolic PDE. The pattern on the centre manifold $W^c(0)$ which is supposed to be tangent to the centre subspace E^c at $Y = 0$ does not correspond to any turning or inflection pattern since by Proposition 2, Kummer's function $K_1(\eta)$ does not provide us with the minimal number two of zeros for $\zeta > 0$ (has a unique root at $\zeta = N$). Obviously, the centre manifold corresponds to the behaviour of strictly monotone interfaces which do not exhibit any kind of turning or inflections singularities.

Thus, the turning/inflection patterns are generated by the evolution orbits on the stable manifold which is spanned by Kummer's polynomials $K_l(\zeta)$ with any $l \geq 2$. As the first result, we easily check the following

Proposition 3. *Kersner's exact solution (6.4) forms at $t \approx T_1$ the minimal shape K_2-turning behaviour.*

The proof is straightforward and relies on the representation (6.4) where we perform translations $x \mapsto -X_1 + x$ and $t \mapsto T_1 + t$. In fact, in the rescaled variables this exact solution belongs to the subspace

$$W_3 = \mathrm{Span}\{1, \zeta, \zeta^2\},$$

which is invariant under the quadratic operator $\mathbf{D_1} + \mathbf{A_1}$. Then we arrive at a Dynamical System for the coefficients of the expansion of $\theta(\cdot, \tau) \in W_3$ (it is equivalent to (6.3)). Solving this dynamical system (or using (6.4)), one can see that the corresponding pattern has the form

$$Y_2(\zeta, \tau) = -Ce^{-\tau} K_2(\zeta) + O(e^{-2\tau}), \tag{6.21}$$

where $C = C(T) > 0$ is a constant. Observe an exponential convergence as $\tau \to \infty$ to the linear K_2-pattern. The corresponding interface is given by

$$\zeta_2(\tau) = c\, e^{-\tau} + O(e^{-\tau}).$$

In the original spatial variable x this gives the following interface behaviour:

$$\eta_2(t) \equiv \gamma \zeta_2(\tau) e^{-\tau} = c_1\, e^{-2\tau} + O(e^{-3\tau}) \equiv c_1(-t)^2 + O((-t)^3), \quad t \to -0. \tag{6.22}$$

The analysis for $t > 0$ is the same and gives a similar expansion with $(-t)$ replaced by t. The constant c_1 in (6.22) is the same for $t < 0$ and for $t > 0$. Of course, in view of analyticity of the function (6.5), all the left-hand and ri ght-hand derivatives coincide at the turning moment $t = T_1$. The quadratic structure (6.4) of minimal complexity represents the generic stable turning pattern. Other higher mode patterns to be constructed below are expected to be unstable.

Remark. It is curious that the spatial structure of Kummer's functions plays also a key role in the construction of finite nonlinear spectra of blow-up patterns for the quasilinear heat equation $u_t = (u^\sigma u_x)_x + u^\beta$ in the parameter range $\sigma > 0$, $\beta > \sigma + 1$ of single point blow-up. See [13] and also [30], p. 192.

Turning patterns in the inner region. In order to obtain the l-th turning pattern we set

$$Y = e^{-\lambda_l \tau} W \quad (\lambda_l = l - 1),$$

where W solves the equation

$$W_\tau = (\mathbf{D_1} + \lambda_l) W + e^{-\lambda_l \tau} \gamma^{-2} \mathbf{A}(W)$$

with an exponentially small perturbation for $\tau \gg 1$. Therefore, passing to the limit $\tau \to \infty$, one can expect that the ω-limit set of any bounded orbit consists of bounded stationary solutions of the unperturbed equation $(\mathbf{D_1} + \lambda_l) w = 0$. Hence, the l-th pattern corresponds to the asymptotic behaviour close to the one-dimensional eigenspace of $\mathbf{D_1}$ spanned by K_l: $W(\cdot, \tau) \sim \mathrm{const}\, K_l(\cdot)$ as $\tau \to \infty$. This gives the following asymptotic expansion of t he solutions to (6.16)

$$Y_l(\zeta, \tau) = -Ce^{-(l-1)\tau} K_l(\zeta) + o(e^{-(l-1)\tau}), \quad l = 2, 3, ..., \tag{6.23}$$

where $C > 0$ and $l - 1 \equiv \lambda_l > 0$. In terms of the rescaled variable θ this reads

$$\theta_l(\zeta, \tau) = [\alpha\, \zeta - Ce^{-(l-1)\tau} K_l(\zeta) + o(e^{-(l-1)\tau})]_+, \quad \alpha = \sqrt{m}\, \gamma = m(m-1). \tag{6.24}$$

Therefore, the corresponding interface $\zeta_l(\tau) \to 0$ as $\tau \to \infty$ is obtained from the equation

$$\alpha\zeta = Ce^{-(l-1)\tau} a_l(1 + o(1))$$

and has the form

$$\zeta_l(\tau) = c\, e^{-(l-1)\tau}(1 + o(1)).$$

Finally, in the original spatial x-variable the interface of the l-th turning pattern takes the form

$$\eta_l(t) = c_1 e^{-l\tau}(1 + o(1)) \equiv c_1\, (-t)^l(1 + o(1)), \quad t \to -0, \;\; c_1 > 0. \tag{6.25}$$

Inflection patterns in the inner region. The construction is similar. They correspond to the opposite sign in the linearized solution (6.23) so that the rescaled l-th inflection pattern has the form

$$\theta_l(\zeta, \tau) = \alpha\, \zeta + Ce^{-(l-1)\tau} K_l(\zeta) + o(e^{-(l-1)\tau}), \tag{6.26}$$

where $C > 0$. This linearized representation makes sense for $\zeta > 0$ only. In order to find the corresponding interface $\zeta_l(\tau) < 0$ we apply a linear interpolation for $\zeta < 0$. Namely, bearing in mind that $K_l(\zeta)$ is an l-th order polynomial with finite derivative $K_l'(0)$, we can use the following linear interpolation for $\zeta < 0$:

$$\theta_l(\zeta, \tau) \approx \alpha\zeta + Ce^{-(l-1)\tau} a_l(1 + o(1)). \tag{6.27}$$

Thus the interface of the inflection pattern is similar to the corresponding turning one (cf. (6.25))

$$\eta_l(t) = -c_1\, e^{-l\tau}(1 + o(1)) \equiv -c_1\, (-t)^l(1 + o(1)), \quad t \to -0, \;\; c_1 > 0. \tag{6.28}$$

The existence of such turning/inflection patterns on the stable manifolds tangent to the corresponding one-dimensional stable eigenspaces, can be established by a topological method of the so-called conditional invariance (isolating blocks) introduced in [11], see blow-up applications in [1] and in [9]. In this case we need to establish special bounds on the solutions in the intermediate and outer regions (which are governed by lower-order Hamilton-Jacobi equations), which are quite similar to those on the blow-up profiles for semilinear and quasilinear heat equations, [11], [1], [9].

It is worth mentioning that the analysis above is assumed to be complete in the sense that it gives all possible patterns. Such a conclusion is typical for dynamical systems with such invariant manifolds and is true for some less degenerate blow-up problems, see references in [30], Chapter 4. A general result of such nature is known to play an important role in the zero-set analysis of parabolic equations, see Theorem 4 in [25].

6.3. **Intermediate Hamilton-Jacobi region.** It follows from the expansion in the inner region with sufficiently large $\zeta \gg 1$ for which

$$K_l(\zeta) = (-1)^l b_l \zeta^l (1 + o(1)), \quad b_l > 0,$$

that the linearized turning/inflection patterns take the form

$$Y_l(\xi, \tau) = \mp C_1 e^{-(l-1)\tau} \zeta^l + o(e^{-(l-1)\tau} \zeta^l), \quad C_1 > 0. \tag{6.29}$$

We thus define the *intermediate region* with the new spatial rescaled variable

$$y = \zeta e^{-(l-1)\tau/l} \equiv \gamma^{-1} \xi e^{-(l-1)\tau/l} \equiv \gamma^{-1} x e^{\tau/l}. \tag{6.30}$$

The linearized solution $Y = Y(y, \tau)$, instead of (6.16), solves the quasilinear equation

$$Y_\tau = -\frac{1}{l} Y_y y + Y + e^{-(l-1)\tau/l} (y Y_{yy} + N Y_y) + e^{-2(l-1)\tau/l} \gamma^{-2} \, \mathbf{A}(Y), \tag{6.31}$$

which is a perturbation of the linear Hamilton-Jacobi equation

$$Y_\tau = -\frac{1}{l} Y_y y + Y.$$

This nonlinear singularly perturbed dynamical system is similar to that studied in [20] and [21]. Passage to the limit $\tau \to \infty$ is based on the stability theorem [19] for uniformly Lyapunov-stable Ω-limit sets. Then the stationary Hamilton-Jacobi equation

$$-\frac{1}{l} H_y y + H = 0$$

gives the desired family of profiles

$$H(y) = \mp C_1 y^l, \tag{6.32}$$

which represent the reduced Ω-limit set of the perturbed equation (6.31).

6.4. **Outer region.** Finally, using a standard compactness argument, see e.g. Section 8 in [22], we can extend the expansion in the intermediate region to the *outer domain* corresponding to the original spatial variable x which is supposed to be sufficiently small: $x \approx +0$. Namely, we now calculate the limit-time profile $v(x, -0)$ ($\tau = \infty$). We need to pass to the limit $\tau \to \infty$ in the expansion

$$v(x, t) \equiv (-t)\theta(\xi, \tau) = e^{-\tau} [\sqrt{m}\, \xi \mp C_1 y^l + ...].$$

Using the spatial scaling variables, one can see that as $\tau \to \infty$ for any fixed $x \approx +0$

$$v(x, t) = e^{-\tau} [\sqrt{m} x e^\tau \mp C_1 (\gamma^{-1} x e^{\tau/l})^l + ...] \to \sqrt{m} x \mp C_2 x^l + ..., \tag{6.33}$$

where $C_2 = C_1 \gamma^{-l}$.

6.5. **Extension beyond focusing.** In order to finish our construction of all the turning/inflection patterns, we need to extend the expansions for small $t > 0$. Having (6.33), for $t > 0$, we arrive at the Cauchy problem with the initial pressure

$$v(x,0) = \sqrt{m}x \mp C_2 x^l (1 + o(1)), \;\; x \geq 0; \quad l = 2, 3, \ldots . \tag{6.34}$$

We see that as $x \to +0$ the initial data $v(x,0)$ is a perturbation of order $O(x^l)$ of the stationary profile $S(x) = (\sqrt{m}\,x)_+$. This determines a slow motion of the interface at $t \approx +0$ (remind that $\eta'(0)$ must vanish for all the patterns).

The asymptotic analysis as $t \to +0$ of this initial singularity is easier and can be studied by direct construction of suitable super and subsolutions. Nevertheless, we present a scheme of analysis which is similar to the above one in order to indicate the main difference between the evolutions singularity and the initial one.

The self-similar scaling (cf. (6.7) and (6.15))

$$v(x,t) = t\theta(\zeta,\tau), \;\; \zeta = x/\gamma t, \quad \tau = \ln t \to -\infty,$$

with further linearization $\theta(\tau) = S + Y(\tau)$ lead to the quasilinear equation (6.16) with a different hypergeometric operator (cf. (6.17))

$$\mathbf{D_1} Y = \zeta Y_{\zeta\zeta} + (N + \zeta) Y_\zeta - Y.$$

Unlike the focusing case of the evolution singularity as $t \to -0$, this operator has a continuous spectrum. The required expansion with exponential decay as $\tau \to -\infty$

$$Y(\zeta,\tau) = \mp C e^{\mu\tau} \Psi(\zeta) + o(e^{\mu\tau}), \quad \mu > 0, \tag{6.35}$$

follows from the initial condition (6.34):

$$\Psi(\zeta) = \mp C_1 \zeta^l + \ldots, \quad \zeta \gg 1.$$

In the linear approximation, as above, substituting the expansion (6.35) into the equation, we get the equation on the “eigenvalue” μ:

$$\zeta \Psi'' + (N + \zeta)\Psi' - (\mu + 1)\Psi = 0.$$

Among exponentially decaying solutions, it admits solutions with polynomial growth as $\zeta \to \infty$ of the form $\Psi(\zeta) \sim \zeta^{\mu+1}$. Hence $\mu = l - 1 > 0$. Finally, denoting $\Psi(0) = a_l$, the interface behaviour for $t \approx +0$ is described by the approximation (linear interpolation for the inflection patterns)

$$\alpha\zeta = \pm C e^{(l-1)\tau} a_l (1 + o(1)), \quad \tau \to -\infty,$$

whence

$$\eta_l(t) = \pm c_1 t^l (1 + o(1)), \quad t \to +0, \quad l = 2, 3, \ldots . \tag{6.36}$$

The constants c_1 in (6.25), (6.28) and (6.36) coincide.

This completes the construction of K_l-th turning/inflection patterns for any $l = 2, 3, \ldots$.

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