Exact Solution of Integro-Differential Equations of Diffusion Along a Grain Boundary

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Abstract

We analyze model problems of stress induced atomic diffusion from a point source or the surface of a material into an infinite or semi-infinite grain boundary, respectively. The problems are formulated in terms of a partial differential equations which involve singular integral operators. The self-similarity of these equations leads to singular integro-differential equations which are solved in closed form by reduction to an exceptional case of the Riemann boundary value problem of the theory of analytical functions on an open contour. We also give a series representation and a full asymptotic expansion of the solution in the case of large arguments.

1 Introduction

Stress induced atomic diffusion along surfaces and grain boundaries in polycrystalline solids is an important problem for materials science and related disciplines. Structural materials subjected to high temperature creep conditions often fail by the growth and coalescence of grain boundary cavities caused by stress induced grain boundary diffusion (Chuang et al. **(6)**, Spingarn and Nix **(23)**, Pharr and Nix **(20)**, Martinez and Nix **(15)**, **(16)**. More recently, stress induced grain boundary diffusion has also been found (Thouless **(25)**, Vinci et al. **(28)**, Thompson and Carel **(24)** to be an important mechanism of strain relaxation in thin film structures used in microelectronics, integrated optoelectronics, data storage technologies and micro-electro-mechanical systems.

Despite the importance of stress induced grain boundary diffusion and related phenomena, there has been relatively limited effort in rigorous mathematical modeling of such phenomena. For example, one might contemplate the following fundamental questions: What is the growth rate of a stress induced diffusion zone spreading from a point source along a grain boundary? How fast can atomic diffusion grow from a free surface into a semi-infinite grain boundary? What are the characteristic length scales and shapes of these diffusion processes? To the best knowledge of the authors, such problems have been neither formulated nor solved in the literature. Here we develop a mathematical model which gives closed form solutions to such problems.

Several mathematical methods form the background of the present work. Koiter **(12)** considered the problem of diffusion of load from a stiffener into a sheet and reduced it to a singular integro-differential equation on a semi-infinite axis. He applied the Mellin transform to the equation and solved the corresponding difference equation

$$T(s+1) + 2s \cot \pi s T(s) = 0 \tag{1.1}$$

in a strip of a complex variable using the Laplace transform. A linear difference equation of the first order

$$\omega(z)T(z+w) - \chi(z)T(z) = \psi(z) \tag{1.2}$$

was treated by Guichard, Appel, Mellin, Prym, Hurwitz, Barnes (see Barnes **(4)**). Barnes used infinite products to represent the solution of the equation. Bantsuri **(2)**, **(3)** constructed the solution of the difference equation

$$T(s+m) + (s+q)_k H(s)T(s) = g(s), \quad \Re(s) = \varepsilon, \tag{1.3}$$

where $(s+q)_k = (s+q)(s+q+1)\cdots(s+q+k-1)$; $q, H(s)$ and $g(s)$ are given, $H(s)$ is a Hölder function and additionally

$$k = m; \quad H(s) = 1 + O(e^{-\beta|s|}), \ \ s \to \infty, \ \ \Re(s) = \varepsilon, \ \ \beta > 0. \tag{1.4}$$

Bantsuri's method is often called the method of canonical solutions. Čerskiĭ **(7)** studied an equation of smooth transition and reduced it to a particular case of the Carleman boundary value problem for a strip (Carleman **(5)**). This case is equivalent to a difference equation in a strip. Čerskiĭ observed that the difference equation (1.2) may be transformed into an exceptional case of the Riemann boundary value problem for a semi-infinite axis. The theory of this case, namely the situation when the coefficient of the Riemann problem has a logarithmic singularity, was examined by Mel'nik **(17)**. Tikhonenko **(26)**, Popov and Tikhonenko **(21)** found exact solutions of some problems of thermal conductivity and contact mechanics for a wedge. Their solutions were based on the results of Čerskiĭ and Mel'nik. Additionally, Popov and Tikhonenko **(21)** established periodic properties of the solution. It is worth pointing out that the corresponding difference equations (1.3), in the cases they considered, satisfied conditions (1.4).

On the other hand, first-order difference equations in a strip arise in diffraction theory. The authors applied either Maliuzhenets' function **(14)**, the solution of the factorization problem

$$f(s+2\beta) = \cot\left(\frac{s}{2} + \frac{\pi}{4}\right) f(s-2\beta) \tag{1.5}$$

(see e.g. Abrahams and Lawrie **(1)**) or the Barnes double gamma functions (Lawrie and King **(13)**).

The present work involves the following integro-differential equation

$$uf'(u) = \int_0^\infty h\left(\frac{u}{v}\right) f'''(v)\frac{dv}{v}, \qquad 0 < u < \infty \tag{1.6}$$

which may be reduced into Carleman's problem with the boundary condition (1.3) when $k \neq m$, where m is the shift and k is the degree of the polynomial in (1.3). This situation leads to fractional singularities of the solution of the corresponding Riemann problem at the ends. We use Čerskiĭ's idea of reduction of Carleman's problem to Riemann's problem and Mel'nik's formulae to construct an exact solution of the difference equation. Periodic properties of the solution and the residue theorem allow us to find a series representation of the function $f(u)$. The power series of the $f(u)$ converges for any bounded u. A full asymptotic expansion for large u of the function $f(u)$ is found. Numerical procedure for calculation of the $f(u)$ is described.

2 Formulation of the problem and reduction to an integro-differential equation

We consider atomic diffusion from a point source or the surface of a material into an infinite or semi-infinite grain boundary, respectively, in response to an applied stress σ_0.

We begin by modeling an infinite grain boundary along a coordinate axis z with a point source at the origin. The atomic diffusion causes a wedge of material to spread along the grain boundary. The opening displacement of the diffusion wedge can be modeled as a continuous array of climb edge dislocations. The normal traction at a position z in the boundary due to a climb edge dislocation of Burgers vector b at ζ is (Hirth and Lothe **(11)**)

$$\sigma_{gb}(z, \zeta) = \frac{E^* b}{4\pi(z - \zeta)}, \tag{2.1}$$

where E^* is the plane strain elastic modulus, $E/(1-\nu^2)$. Using this solution as the Green function, the grain boundary traction σ_{gb} due to a diffusion wedge with the opening displacement function $2u(z,t)$ is

$$\sigma_{gb}(z, t) = \sigma_0 - \frac{E^*}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u(\zeta, t)}{\partial \zeta}\frac{d\zeta}{z - \zeta}, \tag{2.2}$$

where σ_0 is the applied stress.

We model stress driven atomic diffusion and assume that, relative to the point source, the chemical potential at any point on the grain boundary is (Rice and Chuang **(22)**)

$$\mu(z, t) = -\sigma_{gb}(z, t)\Omega. \tag{2.3}$$

With this assumption, the atomic flux in the boundary per unit out-of-plane thickness is

$$j(z,t) = -\frac{\delta_{gb}D_{gb}}{kT\Omega}\frac{\partial\mu(z,t)}{\partial z} = \frac{\delta_{gb}D_{gb}}{kT}\frac{\partial\sigma_{gb}(z,t)}{\partial z}, \tag{2.4}$$

where $\sigma_{gb}(z,t)$ is the normal traction on the boundary, $\delta_{gb}D_{gb}$ is the product of the grain boundary thickness and the atomic diffusivity in the boundary, Ω is the atomic volume and kT is the product of Boltzman's constant and temperature. Due to mass conservation, the flux divergence at any point on the boundary is related to the displacement rate at that point through

$$2\frac{\partial u(z,t)}{\partial t} = -\Omega\frac{\partial j(z,t)}{\partial z}, \tag{2.5}$$

which gives

$$\frac{\partial u(z,t)}{\partial t} = -\frac{\delta_{gb}D_{gb}\Omega}{2kT}\frac{\partial^2\sigma_{gb}(z,t)}{\partial z^2}. \tag{2.6}$$

Taking the time derivative of (2.2) and then inserting (2.6) lead to

$$\frac{\partial\sigma_{gb}(z,t)}{\partial t} = \frac{E^*D_{gb}\delta_{gb}\Omega}{4\pi kT}\int_{-\infty}^{\infty}\frac{\partial^3\sigma_{gb}(\zeta,t)}{\partial\zeta^3}\frac{d\zeta}{z-\zeta}. \tag{2.7}$$

This governing equation is supplemented by the initial condition

$$\sigma_{gb}(z,0) = \sigma_0 \tag{2.8}$$

and the boundary condition

$$\sigma_{gb}(\infty,t) = \sigma_0, \tag{2.9}$$

$$\sigma_{gb}(0,t) = 0. \tag{2.10}$$

The first boundary condition ensures consistency with the initial condition and the second boundary condition ensures the continuity of chemical potential near the point source.

Equation (2.7) is an unusual integro-differential diffusion equation. Assume the solution

$$\sigma_{gb}(z,t) = \sigma_0 f(u), \tag{2.11}$$

where

$$u = z/(ct)^m. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.7) and canceling the explicit dependence on t, we find that the parameters

$$m = 1/3, \qquad c = \frac{3E^*\delta_{gb}D_{gb}\Omega}{4\pi kT}, \tag{2.13}$$

transform (2.7) into

$$uf'(u) = \int_{-\infty}^{\infty}\frac{f'''(v)dv}{v-u} \tag{2.14}$$

with the boundary conditions expressed as

$$f(0) = 0, \qquad f(\infty) = 1. \tag{2.15}$$

A similar equation can be used to describe atomic diffusion from the surface of a material into a grain boundary in response to an applied stress σ_0 In this case, the grain boundary is assumed to occupy the interval $[0, \infty)$ of the z axis. Eshelby **(8)** has studied the elasticity problem of dislocations near a free surface. His solutions can be used as the Green function for diffusion along a semi-infinite grain boundary. According to Eshelby **(8)**, the normal traction at a point z in the boundary due to a climb edge dislocation of Burgers vector b at ζ is

$$\sigma_{gb}(z,\zeta) = \frac{E^* b}{4\pi} S(z,\zeta), \quad \text{where} \quad S(z,\zeta) = \frac{1}{z-\zeta} - \frac{1}{z+\zeta} - \frac{2\zeta(z-\zeta)}{(z+\zeta)^3}. \tag{2.16}$$

The grain boundary traction σ_{gb} due to a material wedge with the opening displacement function $2u(z,t)$ is

$$\sigma_{gb}(z,t) = \sigma_0 - \frac{E^*}{2\pi} \int_0^\infty S(z,\zeta) \frac{\partial u(\zeta,t)}{\partial \zeta} d\zeta, \tag{2.17}$$

where σ_0 is the applied stress.

Relative to a flat, free surface, the chemical potential along the grain boundary is

$$\mu(z,t) = -\sigma_{gb}(z,t)\Omega, \tag{2.18}$$

and similar to the infinite case, we have the grain boundary diffusion equation. This gives

$$\frac{\partial u(z,t)}{\partial t} = -\frac{\delta_{gb} D_{gb} \Omega}{2kT} \frac{\partial^2 \sigma_{gb}(z,t)}{\partial^2 z}. \tag{2.19}$$

Taking the time derivative of (2.17) and then inserting (2.19) yield

$$\frac{\partial \sigma_{gb}(z,t)}{\partial t} = \frac{E^* D_{gb} \delta_{gb} \Omega}{4\pi kT} \int_0^\infty S(z,\zeta) \frac{\partial^3 \sigma_{gb}(\zeta,t)}{\partial \zeta^3} d\zeta. \tag{2.20}$$

The initial condition is

$$\sigma_{gb}(z,0) = \sigma_0 \tag{2.21}$$

and the boundary conditions are

$$\sigma_{gb}(\infty,t) = \sigma_0, \tag{2.22}$$

$$\sigma_{gb}(0,t) = 0. \tag{2.23}$$

The second boundary condition ensures the continuity of chemical potential near the free surface.

Similarly, we consider solution of form

$$\sigma_{gb}(z,t) = \sigma_0 f(u), \tag{2.24}$$

where

$$u = x/(ct)^{1/3}, \qquad c = \frac{3E^* \delta_{gb} D_{gb} \Omega}{4\pi kT}. \tag{2.25}$$

These transformations lead to a semi-infinite integro-differential equation

$$-uf'(u) = \int_0^\infty S(u,v) f'''(v) dv \tag{2.26}$$

with boundary condition

$$f(0) = 0, \qquad f(\infty) = 1. \tag{2.27}$$

Equation (2.26) differs from (2.14) in both the integration interval and the kernel function. The semi-infinite kernel function

$$S(u,v) = \frac{1}{u-v} - \frac{1}{u+v} - \frac{2v(u-v)}{(u+v)^3} \tag{2.28}$$

consists of three terms. The first term is identical to the Cauchy kernel in the infinite case. The second and third ones are image terms due to the presence of a free surface.

3 Class of solutions

Let us start with the problem of diffusion along an infinite grain boundary. This problem is equivalent to the following integro-differential equation

$$uf'(u) = \int\limits_{-\infty}^{\infty} \frac{f'''(v)}{v-u} dv, \qquad -\infty < u < \infty \tag{3.1}$$

with the additional conditions

$$f(0) = 0, \qquad f(\infty) = 1. \tag{3.2}$$

Due to the symmetry of the original problem, the solution of equation (3.1) is an even function and its first and third derivatives are odd:

$$f(u) = f(-u), \quad f'(u) = -f'(-u), \quad f'''(u) = -f'''(-u). \tag{3.3}$$

Since there is the factor u in the left-hand side of equation (3.1), the Fourier transformation fails in this case. To apply the Mellin transform, we take into account relations (3.3) and reduce equation (3.1) to an integro-differential equation of the Mellin convolution type on a semi-infinite interval

$$uf'(u) = 2\int\limits_{0}^{\infty} \frac{f'''(v)}{1-(u/v)^2}\frac{dv}{v}, \qquad 0 < u < \infty. \tag{3.4}$$

Let us introduce the class of solutions. A function $\phi(u)$, which satisfies everywhere on each finite segment $(0, C)$, except possibly the point $u = 0$, the Hölder condition, behaves at 0 and at infinity as

$$|\phi(u)| < Au^{\alpha}, \quad u \to 0; \quad |\phi(u)| < Bu^{\beta}, \quad u \to \infty,$$

is said to belong to the class $\mathcal{H}^{\alpha,\beta}(0,\infty)$. Here A, B, C are positive constants; α, β are real parameters. The function $f(u)$ is sought in the class $\mathcal{H}^{2-\delta,0}(0,\infty)$, where $0 < \delta < 1$. Then it is reasonable to assume that

$$f'(u) \in \mathcal{H}^{1-\delta,0}(0,\infty), \quad f''(u) \in \mathcal{H}^{-\delta,-1}(0,\infty), \quad f'''(u) \in \mathcal{H}^{-1-\delta,-2}(0,\infty).$$

In other words, we demand that the function $f(u)$ and its derivatives can be estimated as $u \to 0$ and $u \to \infty$ as follows

$$|f(u)| < B_0 u^{2-\delta}, \quad |f^{(k)}(u)| < B_k u^{2-k-\delta} \quad (k = 1, 2, 3), \quad u \to 0,$$

$$f(u) \sim 1, \quad |f^{(k)}(u)| < C_k u^{1-k} \quad (k = 1, 2, 3), \quad u \to \infty. \tag{3.5}$$

Here $B_m \, (m = 0, \ldots 3)$ and $C_k \, (k = 1, 2, 3)$ are positive constants. It will be shown that in this class there is a unique solution of equation (3.1) with conditions (3.2).

4 Carleman boundary value problem for a strip.

Let us reduce the integro-differential equation (3.4) to a Carleman boundary value problem of the theory of analytical functions. Due to inequalities (3.5), the Mellin transform

$$F(s) = \int_0^\infty f'''(u) u^{s-1} du \tag{4.1}$$

of the third derivative of the function $f(u)$ is analytic in the strip $1 + \delta < \Re(s) < 2$. We assume notations $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of a complex value z. Using integration by parts and estimations (3.5), we get

$$F(s) = (s-1)(s-2) \int_0^\infty u^{s-4} [u f'(u)] du$$

$$= -(s-1)(s-2)(s-3) \int_0^\infty u^{s-4} f(u) du, \quad 1 + \delta < \Re(s) < 2. \tag{4.2}$$

The Mellin inversion formula yields

$$u f'(u) = \frac{1}{2\pi i} \int_\Omega \frac{F(s)}{(s-1)(s-2)} u^{3-s} ds, \quad \Omega = \{s : \ \Re(s) = c\} \tag{4.3}$$

uniformly with respect to c: $1 + \delta < c < 2$. Now let us represent the right-hand side of equation (3.4) in terms of a Mellin inverse integral. The Mellin transform of the kernel

$$\int_0^\infty \frac{2}{1-t^2} t^{s-1} dt = \pi \cot \frac{\pi s}{2}$$

is analytic in the strip $0 < \Re(s) < 2$ (see Gradshteyn and Ryzhik **(10)**, formula 3.241(3)). Due to the Mellin convolution theorem (Titchmarsh **(27)**, Theorem 44) we may write for the same contour Ω as in (4.3)

$$2 \int_0^\infty \frac{f'''(v)}{1-(u/v)^2} \frac{dv}{v} = \frac{1}{2\pi i} \int_\Omega F(s) \pi \cot \frac{\pi s}{2} u^{-s} ds. \tag{4.4}$$

Then we introduce a new function

$$\Phi(s) = \pi \cot \frac{\pi s}{2} F(s) \tag{4.5}$$

and assume that
(i) the function $\Phi(s)$ is analytically continuable into the strip $\Pi = \{-3 + c < \Re(s) < c\}$;
(ii) there exists a constant C such that

$$\int_{-\infty}^{\infty} |\Phi(x + it)|^2 dt \leq C \tag{4.6}$$

uniformly with respect to $x \in [-3 + c, c]$.

The last inequality provides vanishing of the function $\Phi(s)$ as $|s| \to \infty$ and $s \in \overline{\Pi}$, and gives us an opportunity to use the theory of the exceptional case of Riemann's boundary value problem. It will a posteriori be shown that the function $\Phi(s)$ really satisfies all these conditions.

Remark. The analyticity of the function $\Phi(s)$ in the strip Π and condition (4.6) mean (see Titchmarsh **(27)**, Theorem 97) that its Fourier original

$$\phi_{\mathfrak{F}}(\xi) = \mathfrak{F}^{-1}[\Phi(c + it)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(c + it) e^{-i\xi t} dt$$

possesses the properties

$$\phi_{\mathfrak{F}}(\xi) \in L_2(-\infty, +\infty), \quad e^{-3\xi} \phi_{\mathfrak{F}}(\xi) \in L_2(-\infty, +\infty)$$

and vice versa. The function $\Phi(c + it)$ is analytically continuable into the strip $0 < \Im(t) < 3$. In terms of the Mellin inverse integral of the function $\Phi(s)$

$$\phi_{\mathfrak{M}}(u) = \mathfrak{M}^{-1}[\Phi(s)] = \frac{1}{2\pi i} \int_{\Omega} \Phi(s) u^{-s} ds \tag{4.7}$$

the analyticity of the $\Phi(s)$ in the strip Π and (4.6) yield

$$u^c \phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0, \infty) \tag{4.8}$$

and

$$u^{c-3} \phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0, \infty), \tag{4.9}$$

where the weight space $L_{2,\frac{1}{u}}(0, \infty)$ consists of such functions $g(u)$ that

$$\int_{0}^{\infty} |g(u)|^2 \frac{du}{u} < \infty.$$

Now we substitute relations (4.3) and (4.4) into (3.4) and change the function $F(s)$ by the new one $\Phi(s)$. We get

$$\frac{1}{2\pi i} \int_{\Omega} \Phi(s) u^{-s} ds = \frac{1}{2\pi i} \int_{\Omega} \frac{\Phi(s) u^{-s+3}}{\pi (s - 1)(s - 2) \cot \frac{\pi s}{2}} ds, \quad 0 < u < \infty. \tag{4.10}$$

To equalise the powers of the u in the last equation, we may take into account that the function $\Phi(s)$ is analytic in the strip Π. Due to the Cauchy theorem, let us move the contour Ω in the first integral in (4.10) from the position $\Re(s) = c$ to another one $\Omega_{-1} = \{\Re(s) = c - 3\}$. Then putting $s = s_1 - 3$ we have

$$\frac{1}{2\pi i}\int\limits_{\Omega} \Phi(s)u^{-s}ds = \frac{1}{2\pi i}\int\limits_{\Omega_{-1}} \Phi(s)u^{-s}ds = \frac{1}{2\pi i}\int\limits_{\Omega} \Phi(s_1 - 3)u^{-s_1+3}ds_1. \tag{4.11}$$

Thus, equation (3.4) can be rewritten in the following equivalent form

$$\frac{1}{2\pi i}\int\limits_{\Omega} \left[\Phi(s-3) - \frac{\Phi(s)}{\pi(s-1)(s-2)\cot\frac{\pi s}{2}}\right] u^{-s+3}ds = 0 \tag{4.12}$$

for any $u \in (0, \infty)$. It means that we arrive at a particular case of Carleman's boundary value problem for a strip:
To find the function $\Phi(s)$, that is analytic in the strip $\Pi = \{-3 + c < \Re(s) < c\}$, satisfies condition (4.6) and the following boundary condition

$$\Phi(\sigma) + K(\sigma)\Phi(\sigma - 3) = 0, \quad \sigma \in \Omega, \tag{4.13}$$

where

$$K(s) = -\pi(s-1)(s-2)\cot\frac{\pi s}{2}. \tag{4.14}$$

5 Exceptional case of the Riemann boundary value problem.

5.1 Reduction of the Carleman problem to a Riemann problem.

Following the work of Čerskiĭ **(7)** we reduce the Carleman problem (4.6), (4.13) to a Riemann boundary value problem for a plane with the cut Γ along the positive part of the real axis. We map the strip Π on a complex plane with the cut $\Gamma = \{\zeta = \xi, \quad \arg\zeta = 0, \quad 0 < \xi < \infty\}$ using the "gluing" function

$$\zeta = \exp\left\{\frac{2\pi i}{3}(c - s)\right\}. \tag{5.1}$$

The upper side ($\zeta = \xi + i0$, $\arg\zeta = 0$) of the positive real axis of the ζ-plane corresponds to the contour Ω of the s-plane and the contour $\Omega_{-1} = \{\Re(s) = c - 3\}$ is mapped on to the lower side ($\zeta = \xi - i0$, $\arg\zeta = 2\pi$) of the contour Γ. The points $\sigma = c - i\infty$, $\sigma = c + i\infty$ are mapped on to the points $\xi = 0$, $\xi = \infty$, respectively. Therefore, when the point s moves along Ω from the point $c - i\infty$ to $c + i\infty$, the corresponding point ζ moves along the contour Γ in the positive direction from 0 to ∞. We introduce the function

$$\omega(\zeta) = \frac{1}{\zeta^{1/2}}\Phi\left(c + \frac{3i}{2\pi}\log\zeta\right), \tag{5.2}$$

where

$$s = c + \frac{3i}{2\pi}\log\zeta,$$

the function $\log\zeta$ and $\zeta^{1/2}$ are defined and analytic in the ζ-plane with the cut Γ. Moreover, on the upper side of the contour Γ, the logarithmic function is real and the $\zeta^{1/2}$ is positive. Thus, the limit values of the function $\omega(\zeta)$ are

$$\omega^+(\xi) = \omega(\xi + i0) = \frac{1}{\sqrt{\xi}}\Phi\left(c + \frac{3i}{2\pi}\log\xi\right) = e^{-\frac{\pi i}{3}(c-\sigma)}\Phi(\sigma),$$

$$\omega^-(\xi) = \omega(\xi - i0) = -\frac{1}{\sqrt{\xi}}\Phi\left(c - 3 + \frac{3i}{2\pi}\log\xi\right) = -e^{-\frac{\pi i}{3}(c-\sigma)}\Phi(\sigma - 3),$$

$$0 < \xi < \infty, \quad \sigma \in \Omega. \tag{5.3}$$

Inequality (4.6), formulated for the limit values of the function $\omega(\zeta)$, yields

$$\int_0^\infty |\omega^\pm(\xi)|^2 d\xi \leq C, \tag{5.4}$$

i.e. $\omega^\pm(\xi) \in L_2(0,\infty)$. Due to the Čerskiĭ theorem (Čerskiĭ **(7)**), the function $\Phi(s)$ satisfies the conditions (i), (ii) if and only if the function $\omega(\zeta)$ can be represented as a Cauchy integral

$$\omega(\zeta) = \frac{1}{2\pi i}\int_0^\infty \frac{\Sigma(\xi)}{\xi - \zeta}d\xi, \tag{5.5}$$

where $\Sigma(\xi) \in L_2(0,\infty)$. Thus, the Carleman problem (4.6), (4.13) is equivalent to the following exceptional case of the Riemann boundary value problem:
It is required to determine the function $\omega(\zeta)$ analytic in the entire ζ-plane, representable by a Cauchy integral with the L_2-density; its boundary values $\omega^+(\xi)$, $\omega^-(\xi)$ are L_2-integrable on the contour Γ functions and satisfy the boundary condition

$$\omega^+(\xi) = K_1(\xi)\omega^-(\xi), \quad \xi \in \Gamma, \tag{5.6}$$

where

$$K_1(\xi) = K\left(c + \frac{3i}{2\pi}\log\xi\right), \quad \arg\xi = 0, \quad \xi \in \Gamma. \tag{5.7}$$

The coefficient $K_1(\xi)$ of the problem (5.6) grows at 0 and at infinity as $\log^2\xi$.

Mel'nik **(17)** analysed the Riemann problem

$$\omega^+(t) = G(t)\omega^-(t), \quad t \in L \tag{5.8}$$

on the contour L that consists of a union of n simple, smooth, non-interesting curves $a_k b_k$. The coefficient of the problem $G(t)$ is proposed to be a Hölder function on each closed portion of the curves $a_k b_k$ except at the ends. Near the ends, the function $G(t)$ admits the representation

$$G(t) = G^*(t)\ln^{r_k}(t - c_k), \quad t \to c_k, \quad c_k = a_k, b_k,$$

where r_k are real and the function $G^*(t)$ satisfies the Hölder condition on each closed curve $a_k b_k$.

In the case of the problem (5.6), the coefficient $K_1(\xi)$ can be represented in a neighbourhood of the point $\xi = 0$ as follows

$$K_1(\xi) = \log^2 \xi \, K^*(\xi), \quad \xi \to 0, \tag{5.9}$$

where

$$K^*(\xi) = K^*(0) + \frac{h(\xi)}{\log \xi}, \quad \xi \to 0 \tag{5.10}$$

with such a function $h(\xi)$ that

$$h(\xi) = d + O(|\log \xi|^{-1}), \quad \xi \to 0, \quad d = \text{const},$$

and the Hölder condition for the function $K^*(\xi)$ fails near the end $\xi = 0$. However, in spite of this circumstance, we may apply Mel'nik's method of solution to the problem (5.6). This follows because the canonical function of this problem and, therefore, the desirable function $\omega(\zeta)$ keep the same form as it would be in the case of a Hölder factor in front of the $\log^2 \zeta$ in formula (5.9). An additional logarithmic factor in front of the main power term that defines the behaviour of the function $\omega(\zeta)$ at the ends of the contour, will appear. It will not change the picture of solvability of the problem (5.6). This follows from the results of Tikhonenko **(26)**, Popov and Tikhonenko **(21)** who used the conform map

$$w = \frac{i - \zeta}{i + \zeta}, \quad \zeta = i\frac{1 - w}{1 + w} \tag{5.11}$$

to reduce the original Riemann problem to the case analysed by Mel'nik and studied the properties of the solution of the problem in the case when the function $K^*(\xi)$ behaves at the ends as the function $K^*(\xi)$ in (5.10) at the point $\xi = 0$. We apply the same map (5.11) and transform problem (5.6) into a new Riemann boundary value problem on the upper semi-circle. The points $\xi = 0$ and $\xi = \infty$ are mapped on to the points $\omega = 1$ and $\omega = -1$ of the w-plane, respectively. The contour Γ is mapped on to the contour $\gamma = \{|w| = 1, \quad \Im(w) \geq 0\}$. The upper side $(\zeta = \xi + i0)$ of the contour Γ corresponds to the left-hand side $(|w| = 1 - 0)$ of the contour γ and the lower side of Γ is mapped on to the right-hand side $(\zeta = \xi - i0)$ of γ. The point $a = 1$ is the start point of the contour γ and $b = -1$ is the terminal point of γ.

Then we introduce a new function

$$\varphi(w) = \frac{1}{1 + w}\omega\left(i\frac{1 - w}{1 + w}\right) \tag{5.12}$$

and notice that

$$\omega^\pm(\xi) = \omega^\pm\left(i\frac{1 - \eta}{1 + \eta}\right) = (1 + \eta)\varphi^\pm(\eta), \quad \eta \in \gamma, \tag{5.13}$$

where $\varphi^\pm(\eta)$ are limit values of the function $\varphi(w)$ as $w \to \eta \in \gamma^\pm$, respectively. Here $\gamma^+ = \{|w| = 1 - 0, \quad \Im(w) > 0\}$, $\gamma^- = \{|w| = 1 + 0, \quad \Im(w) > 0\}$. The new unknown function $\varphi(w)$ behaves at infinity as $w^{-1}\omega(-i)$. This follows from definition (5.12). In terms of the function $\varphi(\eta)$, condition (5.4) gives

$$\left| \int_\gamma |\varphi^\pm(\eta)|^2 d\eta \right| \leq \text{const}. \tag{5.14}$$

The new problem can be formulated as follows:
To find the function $\varphi(w)$ analytic in the w-plane due to the boundary condition

$$\varphi^+(\eta) = G(\eta)\varphi^-(\eta), \quad \eta \in \gamma, \tag{5.15}$$

where

$$\begin{aligned} G(\eta) &= K_1\left(i\frac{1-\eta}{1+\eta}\right) \\ &= -\pi\left[c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right) - 1\right]\left[c + \frac{3i}{2\pi}\ln\left(i\frac{1-\eta}{1+\eta}\right) - 2\right] \\ &\quad \times \cot\left\{\frac{\pi}{2}\left[c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right)\right]\right\}. \end{aligned} \tag{5.16}$$

The function $\varphi(w)$ vanishes at infinity as ω^{-1}, is representable by a Cauchy integral and its limit values $\varphi^{\pm}(\eta)$ belong to the class $L_2(\gamma)$.

It is worth to say that since $\Im(\log\xi) = 0$ as $\xi \in \Gamma$ then

$$\Im\left(i\frac{1-\eta}{1+\eta}\right) = 0, \quad \eta \in \gamma, \tag{5.17}$$

and we stipulate that

$$0 < \arg(\eta + 1) < \pi, \quad 0 < \arg(\eta - 1) < \pi, \quad \eta \in \gamma.$$

For these branches of the logarithmic functions we get

$$\Im\left[\log(\eta - 1) - \log(\eta + 1) - \frac{i\pi}{2}\right] = 0, \quad \eta \in \gamma. \tag{5.18}$$

5.2 Solution of the Riemann problem.

The crucial step in the solution of each Riemann boundary value problem for an open contour (see Muskelishvili **(18)**, Gakhov **(9)**) is to determine the increment Δ of the argument of the problem coefficient $G(\eta)$ as the point η traverses the curve γ in the positive direction. For any concrete branch of the function $\arg G(\eta)$ we define Δ as follows

$$\Delta = [\arg G(\eta)]_\gamma. \tag{5.19}$$

Then we take into account that

$$\begin{aligned} \log\left(\frac{1-\eta}{1+\eta}i\right) &\to -\infty, \quad \eta \to 1 \in \gamma, \\ \log\left(\frac{1-\eta}{1+\eta}i\right) &\to +\infty, \quad \eta \to -1 \in \gamma. \end{aligned} \tag{5.20}$$

Moreover,

$$\log\left(\frac{1-\eta}{1+\eta}i\right) < 0, \quad \eta \in \gamma, \quad \Re(\eta) > 0,$$

$$\log\left(\frac{1-\eta}{1+\eta}i\right) > 0, \quad \eta\in\gamma, \quad \Re(\eta) < 0. \tag{5.21}$$

At the point $\eta = i$ we have

$$\log\left(\frac{1-\eta}{1+\eta}i\right) = 0. \tag{5.22}$$

Let us split the function $G(\eta)$ defined by (5.16) into three factors: $G(\eta) = g_1(\eta)g_2(\eta)g_3(\eta)$, where

$$\begin{aligned} g_1(\eta) &= c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right) - 1, \\ g_2(\eta) &= c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right) - 2, \\ g_3(\eta) &= \cot\left\{\frac{\pi}{2}\left[c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right)\right]\right\}. \end{aligned} \tag{5.23}$$

For each function we observe

$$\begin{aligned} &g_1(\eta)\to -i\infty, \quad g_2(\eta)\to -i\infty, \quad g_3(\eta)\to i \quad \text{as } \eta\to 1, \\ &g_1(\eta)\to c-1, \quad g_2(\eta)\to c-2, \quad g_3(\eta)\to \cot\frac{\pi c}{2} \quad \text{as } \eta\to i, \\ &g_1(\eta)\to i\infty, \quad g_2(\eta)\to i\infty, \quad g_3(\eta)\to -i \quad \text{as } \eta\to -1. \end{aligned} \tag{5.24}$$

It is also clear from the choice of the class of solutions, i.e. from the definition of the parameter c $1+\delta < c < 2, \quad (0<\delta<1)$, that

$$c-1>0, \quad c-2<0, \quad \cot\frac{\pi c}{2} < 0. \tag{5.25}$$

Therefore we may conclude that

$$[\arg g_1(\eta)]_\gamma = \pi, \quad [\arg g_2(\eta)]_\gamma = -\pi, \quad [\arg g_3(\eta)]_\gamma = \pi. \tag{5.26}$$

Hence the desired value is found

$$\Delta = [\arg g_1(\eta)]_\gamma + [\arg g_2(\eta)]_\gamma + [\arg g_3(\eta)]_\gamma = \pi. \tag{5.27}$$

Let us introduce the Cauchy integral

$$Y(w) = \frac{1}{2\pi i}\int_\gamma \frac{\log G(\tau)}{\tau - w}d\tau, \tag{5.28}$$

where the density is any fixed branch of the function $\log G(\tau)$. Due to Sokhotskii - Plemelj formulae, the function $\exp\{Y(w)\}$ satisfies the boundary condition (5.15). Moreover, any function

$$X(w) = (w-1)^p(w+1)^q e^{Y(w)} \tag{5.29}$$

with arbitrary integer numbers p and q is analytic in the entire w-plane, except at the points of the contour γ. Its limit values satisfy the boundary condition (5.15). It thus follows that formula (5.29) defines a family of solutions of equation (5.15) in the class of functions with at most algebraic growth at infinity. We have to choose the branch of the logarithmic function $\log G(\tau)$ and the parameters p and q in such a way that the conditions

$$\varphi(w) = O\left(\frac{1}{w}\right), \quad w \to \infty; \qquad \left| \int_\gamma |\varphi^\pm(\eta)|^2 d\eta \right| \le const \tag{5.30}$$

be held.

On the contour γ, the function $G(\eta)$ is expressible in terms of the limit values $X^+(\eta), X^-(\eta)$ of the function $X(w)$

$$G(\eta) = \frac{X^+(\eta)}{X^-(\eta)}, \qquad \eta \in \gamma. \tag{5.31}$$

We substitute this formula into the boundary condition (5.15)

$$\frac{\varphi^+(\eta)}{X^+(\eta)} = \frac{\varphi^-(\eta)}{X^-(\eta)}, \qquad \eta \in \gamma \tag{5.32}$$

and apply the analytical continuation principle and Liouville's theorem (see, e.g. Gakhov **(9)**). The expression $[X(w)]^{-1}\varphi(w)$ is an entire function in the w-plane. The rate at which this function may grow at infinity, is algebraic. Hence

$$\frac{\varphi(w)}{X(w)} = P_\varkappa(w), \tag{5.33}$$

where $P_\varkappa(w)$ is a polynomial of degree $\varkappa$ with arbitrary complex coefficients

$$P_\varkappa(w) = C_0 + C_1 w + \cdots + C_\varkappa w^\varkappa, \qquad \varkappa \ge 0. \tag{5.34}$$

The integer $\varkappa$ and the coefficients $C_0, \ldots, C_\varkappa$ are to be determined. Thus, the general representation of the function $\varphi(w)$ is

$$\varphi(w) = X(w) P_\varkappa(w). \tag{5.35}$$

5.3 Satisfaction the auxiliary conditions

Let us satisfy conditions (5.30) and find the parameters $p, q, \varkappa$ and choose the branch of the function $\log G(\tau)$. The solution must vanish at infinity. From (5.28),(5.29) and (5.35) we get

$$\varphi(w) = O(w^{p+q+\varkappa}), \qquad w \to \infty \tag{5.36}$$

and therefore

$$p + q + \varkappa = -1. \tag{5.37}$$

To satisfy condition (5.14), we study the behaviour of the function $\varphi(w)$ at the points $w = -1, w = 1$. Due to the definition of the function $G(\eta)$, we have the following relations in the vicinity of the ends of the contour γ

$$G(\eta) = G_1(\eta) \log^2(\eta - 1), \qquad \eta \to 1, \qquad \eta \in \gamma,$$

$$G(\eta) = G_2(\eta)\ln^2(\eta+1), \qquad \eta \to -1, \qquad \eta \in \gamma, \tag{5.38}$$

where

$$G_1(\eta) = \frac{9i}{4\pi}\left[1 - \frac{2\log 2 + \pi i + \frac{2\pi i}{3}(2c-3)}{\log(\eta-1)}\right] + O\left(\frac{1}{|\log(\eta-1)|^2}\right), \quad \eta \to 1,$$

$$G_2(\eta) = -\frac{9i}{4\pi}\left[1 - \frac{2\log 2 + \pi i - \frac{2\pi i}{3}(2c-3)}{\log(\eta+1)}\right] + O\left(\frac{1}{|\log(\eta+1)|^2}\right), \quad \eta \to -1. \tag{5.39}$$

Here we took into account formula (5.18) and

$$\cot\left\{\frac{\pi}{2}\left[c + \frac{3i}{2\pi}\log\left(i\frac{1-\eta}{1+\eta}\right)\right]\right\} \to \pm i, \quad \eta \to \pm 1, \quad \eta \in \gamma.$$

The behaviour of the function $G(\eta)$ in the vicinity of the points $\eta = 1$, $\eta = -1$ can be described as follows

$$G_1(\eta) = G_1(1) + \frac{d_1 + h_1(\eta)}{\log(\eta-1)}, \quad \eta \to 1,$$

$$G_2(\eta) = G_2(-1) + \frac{d_2 + h_2(\eta)}{\log(\eta+1)}, \quad \eta \to -1, \tag{5.40}$$

where

$$G_1(1) = \frac{9i}{4\pi}, \quad G_2(-1) = -\frac{9i}{4\pi},$$

$$d_1 = 3c - \frac{9}{4} - \frac{9i}{2\pi}\log 2, \quad d_2 = 3c - \frac{27}{4} + \frac{9i}{2\pi}\log 2. \tag{5.41}$$

The functions $h_1(\eta)$, $h_2(\eta)$ are estimated by

$$h_1(\eta) = O\left(\frac{1}{\log(\eta-1)}\right), \quad \eta \to 1, \qquad h_2(\eta) = O\left(\frac{1}{\log(\eta+1)}\right), \quad \eta \to -1, \tag{5.42}$$

We investigate the behaviour of the Cauchy integral

$$Y(w) = \frac{1}{2\pi i}\int_\gamma \frac{\log G(\tau)}{\tau - w}d\tau \tag{5.43}$$

in the neighbourhood of the ends of the contour γ. It is known (see Muskhelishvili **(18)**) that if L is an open contour with the ends a_- and a_+ (a_- is the start point), and $G_*(\tau)$ is a Hölder function on the whole curve, including the ends: $G_*(\tau) \in H(L)$, then

$$\frac{1}{2\pi i}\int_L \frac{G^*(\tau)}{\tau - w}d\tau = \mp\frac{G^*(a_\mp)}{2\pi i}\log(w - a_\mp) + D^{(1)}_\mp(w), \quad w \to a_\mp, \tag{5.44}$$

where $D^{(1)}_\mp(w)$ are bounded in the vicinity of the points $w = a_\mp$ and tend to a definite limit as $w \to a_\mp$.

The behaviour of the Cauchy integral

$$\Psi(w) = \frac{1}{2\pi i}\int\limits_L \frac{G^*(\tau)\log\log(\tau-b)}{\tau-b}d\tau \tag{5.45}$$

as $w\to b$ and b is one of the ends a_-, a_+, is described by Mel'nik formulae **(17)**

$$\Psi(w) = \pm\frac{G^*(a_\mp)}{2\pi i}\{\log(w-a_\mp)[1-\log\log(w-a_\mp)]$$

$$\pm\pi i\log\log(w-a_\mp)\} + D^{(2)}_\mp(w), \quad w\to a_\mp,\ w\notin\gamma, \tag{5.46}$$

$$\Psi(\eta) = \pm\frac{G^*(a_\mp)}{2\pi i}\log(\eta-a_\mp)[1-\log\log(\eta-a_\mp)] + D^{(3)}_\mp(\eta), \quad \eta\to a_\mp,\ \eta\in\gamma, \tag{5.47}$$

where the functions $D^{(2)}_\mp(w)$ are analytic in the vicinity of the points $a_\mp$, respectively, on the w-plane with the cut L. These functions are continuously continuable on L including the ends $a_\mp$. The functions $D^{(3)}_\mp(\eta)$ are continuous at the points $\eta = a_\mp$.

The Cauchy integral with the density $1/\log(\tau-b)$ admits the representation (Zverovich **(29)**)

$$\frac{1}{2\pi i}\int\limits_L \frac{d\tau}{(\tau-w)\log(\tau-a_\mp)} = \mp\frac{1}{2\pi i}\log\log(w-a_\mp) + D^{(4)}_\mp(w), \quad w\to a_\mp \tag{5.48}$$

with the functions $D^{(4)}_\mp(w)$ that are bounded as $w\to a_\mp$. Additionally, we need the following result

$$\frac{1}{2\pi i}\int\limits_L \frac{d\tau}{(\tau-w)\log^r(\tau-a_\mp)} = O(1), \quad w\to a_\mp, \quad r>1. \tag{5.49}$$

It follows from the Zverovich formula **(29)**

$$\frac{1}{2\pi i}\int\limits_a^\infty \frac{d\tau}{(\tau-w)\log^z(\tau-a)} = \begin{cases} \zeta(z, \frac{1}{2\pi i}\log(w-a)), & n>0 \\ \zeta(z, 1-\frac{1}{2\pi i}\log(w-a)), & n\le 0 \end{cases} \tag{5.50}$$

that is valid for $\Re(z) > 1$. Here $\zeta(z,\alpha)$ is the generalized zeta function; the branch of the logarithmic function is fixed in such a way that $\log 1 = 2\pi n i$.

We proceed to estimate integral (5.43) for the start point of the contour γ. Due to representations (5.38), (5.40), we have

$$\log G(\tau) = 2\log\log(\tau-1) + \log G_1(1) + \frac{d_1}{G_1(1)\log(\tau-1)}$$

$$+O(\frac{1}{|\log(\tau-1)|^2}) \quad \text{as} \quad \tau\to 1, \quad \tau\in\gamma. \tag{5.51}$$

Using formulae (5.44), (5.46), (5.48), (5.49) we may derive

$$Y(w) = \nu_1(w)\log(w-1) + \rho_1\log\log(w-1) + Y_1(w), \quad w\to 1,\ w\notin\gamma, \tag{5.52}$$

where

$$\nu_1(w) = \frac{1}{\pi i}[1 - \log\log(w-1)] - \frac{\log G_1(1)}{2\pi i},$$

$$\rho_1 = 1 + \frac{d_1 i}{2\pi G_1(1)}, \tag{5.53}$$

the function $Y_1(w)$ is bounded at $w = 1$ and tends to a definite limit as $w \to 1$. As $w \to -1$ and $w \notin \gamma$ we have

$$Y(w) = \nu_2(w)\log(w+1) + \rho_2 \log\log(w+1) + Y_2(w), \quad w \to -1,\ w \notin \gamma, \tag{5.54}$$

where

$$\nu_2(w) = -\frac{1}{\pi i}[1 - \log\log(w+1)] + \frac{\log G_2(-1)}{2\pi i},$$

$$\rho_2 = 1 - \frac{d_2 i}{2\pi G_2(-1)}, \tag{5.55}$$

the function $Y_2(w)$ is bounded as $w \to -1$ and $\lim Y_2(w)$ exists as $w \to -1$. The function $\log\log(w-b)$ $\quad(b = 1$ or $\ b = -1)$ possesses the following properties

$$\log\log(w-b) = \log|\log(w-b)| + i\arg Z(w)$$

with $Z(w) = \log(w-b)$. If $|w-b| < 1$ then

$$\arg Z(w) = \pi + \tan^{-1}\frac{\arg(w-b)}{\log|w-b|}.$$

Assuming that $w \to b$ along any path with a finite number of circuits, one may write

$$\lim_{w\to b}\arg Z(w) = \lim_{w\to b}\Im\{\log Z(w)\} = \pi.$$

This result was obtained by Mel'nik **(17)**. Therefore

$$\Im\{\log\log(w-1)\} \to \pi, \quad w \to 1,$$

$$\Im\{\log\log(w+1)\} \to \pi, \quad w \to -1. \tag{5.56}$$

Now it is straightforward to deduce the formulae that describe the behaviour of the function $\exp\{Y(w)\}$ at the ends

$$\exp\{Y(w)\} = (w-1)^{\Re\{\nu_1(1)\}}\log^{\mu_1}(w-1)\Xi_1(w), \quad w \to 1,$$

$$\exp\{Y(w)\} = (w+1)^{\Re\{\nu_2(-1)\}}\log^{\mu_2}(w+1)\Xi_2(w), \quad w \to -1, \tag{5.57}$$

where

$$\Re\{\nu_1(1)\} = -1 - \frac{1}{2\pi}\Im\{\log G_1(1)\},$$

$$\Re\{\nu_2(-1)\} = 1 + \frac{1}{2\pi}\Im\{\log G_2(-1)\},$$

$$\mu_1 = 1 - \frac{1}{2\pi}\Im\left\{\frac{d_1}{G_1(1)}\right\} = \frac{1}{2} + \frac{2c}{3},$$

$$\mu_2 = 1 + \frac{1}{2\pi}\Im\left\{\frac{d_2}{G_2(-1)}\right\} = -\frac{1}{2} + \frac{2c}{3}, \tag{5.58}$$

where $\Xi_1(w)$, $\Xi_2(w)$ are bounded as $w \to 1$, $w \to -1$, respectively, with definite limits $\Xi_1(1)$, $\Xi_2(-1)$.

Let us denote

$$\theta_1 = \arg G_1(1), \quad \theta_2 = \arg G_2(-1). \tag{5.59}$$

Due to formulae (5.38) and (5.56), we obtain for any fixed branch of the function $\log G(\tau)$

$$\lim_{\tau\to 1} \arg G(\tau) = \arg G_1(1) + 2\arg\log(\tau - 1) = \theta_1 + 2\pi. \tag{5.60}$$

Similarly,

$$\arg G(-1) = \theta_2 + 2\pi. \tag{5.61}$$

It is clear from (5.41) and (5.59) that

$$\theta_1 = \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{5.62}$$

Definition (5.19) of the increment Δ yields

$$\arg G(-1) = \arg G(1) + \Delta, \tag{5.63}$$

and $\Delta = \pi$ (see (5.27)). From (5.60) (5.61) we obtain the link between θ_1 and θ_2

$$\theta_2 = \pi + \theta_1 \tag{5.64}$$

and therefore

$$\theta_2 = \frac{3\pi}{2} + 2k\pi. \tag{5.65}$$

Now we may evaluate the values $\Re\{\nu_1(1)\}$, $\Re\{\nu_2(-1)\}$:

$$\Re\{\nu_1(1)\} = -1 - \frac{\theta_1}{2\pi} = -\frac{5}{4} - k,$$

$$\Re\{\nu_2(-1)\} = 1 + \frac{\theta_2}{2\pi} = \frac{7}{4} + k. \tag{5.66}$$

Formulae (5.57), (5.58) and (5.66) enable us to determine the desirable behaviour of the function $X(w)$ (5.29) at the end points $1, -1$

$$X(w) = 2^q (w-1)^{p-k-5/4} \log^{\mu_1}(w-1)\Xi_1(1), \quad w \to 1,$$

$$X(w) = (-2)^p (w+1)^{q+k+7/4} \log^{\mu_2}(w+1)\Xi_2(-1), \quad w \to -1, \tag{5.67}$$

and therefore $\varphi(w)$, the function defined in (5.35), behaves at the ends of the contour γ as follows

$$\varphi(w) = A_1 (w-1)^{p-k-5/4} \log^{\mu_1}(w-1), \quad w \to 1,$$

$$\varphi(w) = A_2(w+1)^{-p-\varkappa+k+3/4}\log^{\mu_2}(w+1), \quad w \to -1. \tag{5.68}$$

Here we used relation (5.37) and

$$A_1 = 2^q P_\varkappa(1)\Xi_1(1), \quad A_2 = (-2)^p P_\varkappa(-1)\Xi_2(-1).$$

In order the functions $\varphi^\pm(\eta)$ to be L_2-functions on the contour γ, i.e. to satisfy condition (5.14), it is necessary and sufficient that

$$p - k - \frac{5}{4} > -\frac{1}{2},$$

$$-p - \varkappa + k + \frac{3}{4} > -\frac{1}{2}. \tag{5.69}$$

Remark. The behaviour of the limit values $\varphi^\pm(\eta)$ of the function $\varphi(w)$ can be derived similarly if we take into account the Sokhotskii-Plemelj formulae and relation (5.47)

$$\varphi^\pm(\eta) = A_1^\pm(\eta-1)^{p-k-5/4}\log^{\mu_1^\pm}(\eta-1), \quad \eta \to 1, \quad \eta \in \gamma,$$

$$\varphi^\pm(\eta) = A_2^\pm(\eta+1)^{-p-\varkappa+k+3/4}\log^{\mu_2^\pm}(\eta+1), \quad \eta \to -1, \quad \eta \in \gamma,$$

where

$$\mu_1^\pm = -\frac{1}{2} + \frac{2c}{3} \pm \frac{1}{2}, \quad \mu_2^\pm = -\frac{3}{2} + \frac{2c}{3} \pm \frac{1}{2}.$$

and $A_1^\pm$, $A_2^\pm$ are constants. The system of inequalities (5.69) yields

$$\frac{3}{4} < p - k < \frac{5}{4} - \varkappa.$$

If we take into account that $p, q, \varkappa$ are integers and $\varkappa \geq 0$, then we obtain immediately

$$\varkappa = 0, \quad p - k = 1.$$

Now we may choose

$$k = -1, \quad p = 0 \tag{5.70}$$

and therefore from (5.37)

$$q = -1. \tag{5.71}$$

This shows that the polynomial $P_\varkappa(w) \equiv C_0$ and C_0 is an arbitrary constant. The solution of the Riemann problem (5.15), (5.30) is given by

$$\varphi(w) = C_0 X(w), \tag{5.72}$$

where

$$X(w) = (w+1)^{-1}e^{Y(w)}. \tag{5.73}$$

It behaves at the points $w = 1$, $w = -1$ as follows

$$\varphi(w) = A_1(w-1)^{-1/4}\{\log(w-1)\}^{1/2+2c/3}, \quad w \to 1,$$

$$\varphi(w) = A_2(w+1)^{-1/4}\{\log(w+1)\}^{-1/2+2c/3}, \quad w \to -1, \tag{5.74}$$

and decays at infinity as w^{-1}. The arguments $\theta_1,\ \theta_2$ (5.59) have been found

$$\theta_1 = -\frac{3\pi}{2}, \quad \theta_2 = -\frac{\pi}{2}$$

and therefore from (5.60), (5.61) we get

$$\arg G(1) = \frac{\pi}{2}, \quad \arg G(-1) = \frac{3\pi}{2}. \tag{5.75}$$

Now we may say that the branch of the logarithmic function $\log G(\tau)$ in the Cauchy integral (5.43) is chosen in the following way

$$\frac{\pi}{2} \le \arg G(\tau) \le \frac{3\pi}{2} \quad \text{as} \quad \tau \in \gamma. \tag{5.76}$$

6 Solution of the Carleman problem.

First, we define the intermediate function (5.12)

$$\omega(\zeta) = (1+w)\varphi(w) = C_0 \exp\left\{\frac{1+\zeta}{2\pi i}\int_0^\infty \frac{\log K_1(\eta)}{(\eta-\zeta)(i+\eta)}d\eta\right\}. \tag{6.1}$$

Then using the inverse map (5.1) we obtain from (5.2) the expression for the original function $\Phi(s)$

$$\Phi(s) = C_0 \exp\left\{\frac{\pi i}{3}(c-s)\right\} Q(s), \quad s \in \Pi,$$

$$\Phi(\sigma) = C_0 \exp\left\{\frac{\pi i}{3}(c-\sigma) + \frac{1}{2}\log K(\sigma)\right\} Q(\sigma), \quad \sigma \in \Omega,$$

$$\Phi(\sigma-3) = -C_0 \exp\left\{\frac{\pi i}{3}(c-\sigma) - \frac{1}{2}\log K(\sigma)\right\} Q(\sigma), \quad \sigma \in \Omega, \tag{6.2}$$

where

$$Q(s) = \exp\left\{-\frac{i+\exp(\frac{2\pi i}{3}(c-s))}{3}\int_\Omega \frac{\log K(\sigma)d\sigma}{[1-\exp\{\frac{2\pi i}{3}(\sigma-s)\}][i+\exp\{\frac{2\pi i}{3}(c-\sigma)\}}\right\}. \tag{6.3}$$

The boundary condition

$$\Phi(\sigma) + K(\sigma)\Phi(\sigma-3), \quad \sigma \in \Omega \tag{6.4}$$

of the Carleman problem (4.13), (4.6) is satisfied. It is verified by the direct substitution of expressions (6.2) into (6.4). Solution (6.2) possesses property (4.6). This is because the corresponding function $\varphi(w)$ satisfies the inequality

$$\left|\int_\gamma |\varphi(\eta+\tau)|^2 d\eta\right| \le const.$$

uniformly with respect to τ, where τ is an arbitrary point of the w-plane. This fact follows from (5.74) and (5.36), (5.37) directly.

Analysis of formulae (6.2), (6.3) shows that the function $\Phi(s)$ is analytic everywhere in the strip Π.

7 Analysis of the solution of the integro-differential equation.

The solution of the original integro-differential equation (3.4) is given by the inverse Mellin transform

$$f(u) = -\frac{1}{2\pi i}\int\limits_{\Omega}\frac{F(s)u^{3-s}ds}{(s-1)(s-2)(s-3)} = \frac{1}{2\pi i}\int\limits_{\Omega}\frac{\Phi(s)u^{3-s}ds}{(s-3)K(s)}. \tag{7.1}$$

Here we took into account relations (4.2), (4.5) and (4.14).

Let us transform the integral representation of the solution (7.1) into the form convenient for numerical calculations. Additionally, we have to verify conditions (3.5) and find the constant C_0.

7.1 Limit values of the solution of the Carleman problem.

To evaluate integral (7.1), we need the relations between the limit values (see Popov, Tikhonenko **(21)**)

$$\Phi_-(s_m) = \lim_{\epsilon\to+0}\Phi(3m+it+c-\epsilon), \quad s_m = 3m+\sigma \tag{7.2}$$

and

$$\Phi_+(s_m) = \lim_{\epsilon\to+0}\Phi(3m+it+c+\epsilon), \quad s_m = 3m+\sigma \tag{7.3}$$

on each contour $\Omega_m = \{s : \Re(s) = 3m+c\}$, where $m = 0, \pm1, \pm2, \ldots$. Analysis of integral (6.2) shows that the function $\Phi(s)$ is analytic in each strip

$$\Pi_m = \{s :\ c+3m-3 < \Re(s) < c+3m\} \tag{7.4}$$

and its limit values $\Phi_+(s_m)$ and $\Phi_-(s_m)$ do not coincide with each other. Directly from formulae (6.2) we get

$$\Phi_-(s_0) = \Phi(\sigma), \quad \Phi_+(s_{-1}) = \Phi(\sigma-3), \quad \sigma\in\Omega_0=\Omega. \tag{7.5}$$

Then

$$\Phi_+(s_0) = C_0\exp\{\frac{\pi i}{3}(c-\sigma)\}Q(\sigma)\exp\{-\frac{1}{2}\log K(\sigma)\} = -\Phi(\sigma-3),$$

$$\Phi_-(s_1) = C_0\exp\{\frac{\pi i}{3}(c-s_1)\}\lim_{s\to s_1,\, s\in\Pi_{-1}}Q(s). \tag{7.6}$$

Putting in the above formula $s_1 = \sigma+3$ we obtain

$$\Phi_-(s_1) = -C_0\exp\{\frac{\pi i}{3}(c-\sigma)\}\lim_{s\to\sigma,\, s\in\Pi}Q(s). \tag{7.7}$$

From the Sokhotskii-Plemelj formulae we have

$$\lim_{s\to\sigma,\, s\in\Pi}Q(s) = \exp\{\frac{1}{2}\log K(\sigma)\}Q(\sigma), \quad \sigma\in\Omega. \tag{7.8}$$

Thus,

$$\Phi_-(s_1) = -\Phi(\sigma), \quad \sigma\in\Omega. \tag{7.9}$$

In a similar way, one may deduce

$$\Phi_-(s_m) = (-1)^m \Phi(\sigma), \quad \Phi_+(s_m) = (-1)^{m+1}\Phi(\sigma - 3), \quad m = 0, \pm 1, \pm 2, \ldots. \tag{7.10}$$

Now we substitute formulae (7.10) into the boundary condition (4.13) and multiply with $(-1)^m$

$$(-1)^m \Phi(\sigma) = (-1)^{m+1}\Phi(\sigma - 3)K(\sigma), \quad \sigma \in \Omega.$$

Then we put $s_m - 3m$ instead of σ and use relations (7.10). The limit values of the function $\Phi(s)$ on each contour Ω_m are linked by

$$\Phi_-(s_m) = K(s_m - 3m)\Phi_+(s_m), \quad s_m \in \Omega_m, \quad m = 0, \pm 1, \pm 2, \ldots,$$

$$K(s) = -\pi(s-1)(s-2)\cot\frac{\pi s}{2}. \tag{7.11}$$

7.2 Series representation for the function $f(u)$.

First, we use the boundary condition (4.13) of the Carleman problem and transform integral (7.1) into the form

$$f(u) = -\frac{1}{2\pi i}\int\limits_{\Omega} \frac{\Phi(s-3)u^{3-s}ds}{s-3}. \tag{7.12}$$

Putting $s_{-1} = s - 3$, where $s \in \Omega$ and $s_{-1} \in \Omega_{-1}$, we get

$$f(u) = -\frac{1}{2\pi i}\int\limits_{\Omega_{-1}} \frac{\Phi_+(s)}{s}u^{-s}ds. \tag{7.13}$$

Relation (7.11) for $m = -1$ enable us to express the integrand in terms of the limit value $\Phi_{-1}(s)$, $s \in \Omega_{-1}$

$$f(u) = -\frac{1}{2\pi i}\int\limits_{\Omega_{-1}} \frac{\Phi_-(s)u^{-s}ds}{\pi s(s+1)(s+2)\tan\frac{\pi s}{2}}. \tag{7.14}$$

The function $\Phi_-(s)$ $(s \in \Omega_{-1})$ is analytically continuable into the strip $\Pi_{-1} = \{c - 6 < \Re(s) < c-3\}$, where $-5+\delta < c-6 < -4$ and $-2+\delta < c-3 < -1$ $(0 < \delta < 1)$. We integrate the integrand (7.14) round the rectangular contour $\mathcal{L}_{-1}$ consisting of the portions $|\Im(s)| \leq R$ of the contours Ω_{-1}, Ω_{-2} and the segments $c - 6 \leq |\Re(s)| \leq c-3$ of the lines $\Im(s) = \pm R$. The only poles within the contour $\mathcal{L}_{-1}$ are $s = -2$ and $s = -4$. The point $s = -2$ is a pole of the second multiplicity and $s = -4$ is a simple pole. Due to the residue theorem, letting $R \to \infty$, we obtain

$$f(u) = -\frac{1}{\pi^2}[\Phi'(-2) - \Phi(-2)\left(\log u - \frac{3}{2}\right)]u^2 + \frac{2}{\pi^2}\frac{\Phi(-4)}{4\cdot 3\cdot 2}u^4$$

$$-\frac{1}{2\pi i}\int\limits_{\Omega_{-2}} \frac{\Phi_+(s)u^{-s}ds}{\pi s(s+1)(s+2)\tan\frac{\pi s}{2}}. \tag{7.15}$$

Further, to pass to the next strip $\Pi_{-2} = \{c - 9 < \Re(s) < c - 6\}$, we use relation (7.11) for $m = -2$

$$\Phi_+(s) = -\frac{\Phi_-(s)}{\pi(s+4)(s+5)\cot\frac{\pi s}{2}}, \quad s \in \Omega_{-2}. \tag{7.16}$$

Additionally, we take into account the periodicity of the function $Q(s)$ (see (6.3))

$$Q(s) = Q(s+3m), \quad s \in \Pi, \quad m = 0, \pm 1 \pm 2, \dots \tag{7.17}$$

and therefore from (6.2)

$$\Phi(s) = (-1)^m \Phi(s+3m), \quad \Phi'(s) = (-1)^m \Phi'(s+3m),$$
$$s \in \Pi, \quad m = 0, \pm 1 \pm 2, \dots \tag{7.18}$$

In particular,

$$\Phi(-4) = -\Phi(-1), \quad \Phi(-2) = -\Phi(1), \quad \Phi'(-2) = -\Phi'(1). \tag{7.19}$$

We have

$$f(u) = -\frac{u^2}{\pi^2}\left[\Phi(1)\left(\log u - \frac{3}{2}\right) - \Phi'(1)\right] - \frac{2}{\pi^2}\frac{\Phi(-1)}{4!}u^4$$
$$+\frac{1}{2\pi i}\int\limits_{\Omega_{-2}} \frac{\Phi_-(s)u^{-s}ds}{\pi^2 s(s+1)(s+2)(s+4)(s+5)}, \tag{7.20}$$

Then, we repeat the previous procedure. The function $\Phi_-(s)$ can be continued analytically in the next strip Π_{-2}. The only pole within the strip Π_{-2} is $s = -5$ that is a simple pole. Therefore

$$f(u) = \frac{u^2}{\pi^2}u^2\left[\Phi(1)(-\log u + \frac{3}{2}) + \Phi'(1)\right] - \frac{2}{\pi^2}\frac{\Phi(-1)}{4!}u^4 + \frac{1}{\pi^2}\frac{2\Phi(1)}{5!}u^5 + W_3. \tag{7.21}$$

Here we replaced $\Phi(-5)$ by $\Phi(1)$ and

$$W_3 = \frac{1}{2\pi i}\int\limits_{\Omega_{-3}} \frac{\Phi_-(s)u^{-s}ds}{\pi^3 s(s+1)(s+2)(s+4)(s+5)(s+7)(s+8)\tan\frac{\pi s}{2}} \tag{7.22}$$

due to

$$\Phi_+(s) = \frac{\Phi_-(s)}{\pi(s+7)(s+8)\tan\frac{\pi s}{2}}, \quad s \in \Omega_{-3}. \tag{7.23}$$

In the strip Π_{-3} there are two poles $s = -8$ and $s = -10$ of the second and first multiplicity, respectively. The residue theorem yields for the integral W_3

$$W_3 = -\frac{2}{\pi^4}\frac{u^8}{\mu_8}\left[\Phi(1)(-\log u + \nu_8) + \Phi'(1)\right] + \frac{2}{\pi^4}\frac{u^{10}}{\mu_{10}}\Phi(-1) + W_4,$$

where

$$k^{-1}\mu_k = (k-1)(k-2)(k-4)(k-5)\dots 2 \qquad (k \neq 3m),$$
$$\nu_{6j+2} = \frac{1}{6j+2} + \left(\frac{1}{6j+1} + \frac{1}{6j}\right) + \left(\frac{1}{6j-2} + \frac{1}{6j-3}\right) + \dots + 1 \tag{7.24}$$

and

$$W_4 = -\frac{1}{2\pi i}\int\limits_{\Omega_{-4}} \frac{\Phi_-(s)u^{-s}ds}{\pi^4 s(s+1)(s+2)(s+4)(s+5)(s+7)(s+8)(s+10)(s+11)} \tag{7.25}$$

with a simple pole at the point $s = -11$ of the integrand in the strip Π_{-4}. Finally, this procedure leads to the following series representation of the function $f(u)$

$$f(u) = \sum_{j=0}^{\infty} \frac{(-1)^j u^{6j+2}}{\pi^{2j+2}}$$

$$\times \left\{ \frac{2}{\mu_{6j+2}}[\Phi(1)(\nu_{6j+2} - \log u) + \Phi'(1)] - \frac{2u^2}{\mu_{6j+4}}\Phi(-1) + \frac{u^3}{\mu_{6j+5}}\Phi(1) \right\}. \qquad (7.26)$$

The above series is absolutely convergent for all u: $|u| < \infty$. Indeed, for example,

$$\lim_{j\to\infty} \left| \frac{u^{6j+8}\nu_{6j+8}}{\pi^{2j+4}\mu_{6j+8}} \frac{\pi^{2j+2}\mu_{6j+2}}{u^{6j+2}\nu_{6j+2}} \right| = \lim_{j\to\infty} \left| \frac{u^6}{\pi^2(6j+8)(6j+7)(6j+6)(6j+4)} \right| = 0$$

and the radius of convergence of series (7.26) equals infinity. We note that, in particular,

$$f(u) \sim -\frac{1}{\pi^2}\Phi(1)u^2 \log u, \quad u \to 0 \qquad (7.27)$$

and therefore $f(0) = 0$, i.e. the first condition in (3.2) has been satisfied. The derivatives of the function $f(u)$ admit the following estimations as $u \to 0$

$$f'(u) \sim -\frac{2}{\pi^2}\Phi(1)u \log u, \quad f''(u) \sim -\frac{2}{\pi^2}\Phi(1) \log u,$$

$$f'''(u) \sim -\frac{2}{\pi^2}\Phi(1)\frac{1}{u}, \quad u \to 0. \qquad (7.28)$$

Thus, the first group of conditions (3.5) in the neighbourhood of the point $u = 0$ is satisfied. We note that, due to (7.28), the right-hand side in (3.1) is understood in the generalized sense.

7.3 Asymptotic expansion for large u.

Let us derive an asymptotic expansion for the function $f(u)$ convenient for large u. We start with the basic integral representation

$$f(u) = -\frac{1}{2\pi i} \int_{\Omega} \frac{\Phi(\sigma)u^{-\sigma+3}d\sigma}{(\sigma-1)(\sigma-2)(\sigma-3)\pi \cot \frac{\pi\sigma}{2}}. \qquad (7.29)$$

At first we "jump" over the contour Ω

$$\Phi_-(s) = -\pi(s-1)(s-2)\cot\frac{\pi s}{2}\Phi_+(s), \quad s \in \Omega.$$

Here we used relation (7.11) for $m = 0$. Then we transform integral (7.29) into the form

$$f(u) = \frac{1}{2\pi i} \int_{\Omega} \frac{\Phi_+(s)}{s-3} u^{-s+3} ds. \qquad (7.30)$$

The function $\Phi_+(s)$ can be continued analytically into the strip $\Pi_1 = \{c < \Re(s) < c+3\}$. We integrate the function $(s-3)^{-1}\Phi_+(s)u^{3-s}$ round the contour $\mathcal{L}_1$ that is

$$\mathcal{L}_1 = \{\Re(s) = c,\ \Im(s) \in (-R, R)\} \cup \{\Re(s) \in (c, c+3),\ \Im(s) = -R\}$$

$$\cup\{\Re(s) = c+3,\ \Im(s) \in (-R, R)\} \cup \{\Re(s) \in (c, c+3),\ \Im(s) = R\}.$$

The positive directions of the contours $\mathcal{L}_1$ and Ω_1 do not coincide. The only pole in the strip Π_1 is $s = 3$. The residue theorem yields

$$f(u) = -\Phi(3) + \frac{1}{2\pi i}\int\limits_{\Omega_1} \frac{\pi(s-4)(s-5)\tan\frac{\pi s}{2}}{s-3}\Phi_+(s)u^{-s+3}ds. \tag{7.31}$$

Here we took into account that

$$\Phi_-(s) = K(s-3)\Phi_+(s), \quad s \in \Omega_1.$$

In the next strip, $\Pi_2 = \{c+3 < \Re(s) < c+6\}$, there is a simple pole at the point $s = 7$. Therefore

$$f(u) = -\Phi(3) + \frac{3\cdot 2}{2}\Phi(7)u^{-4}$$

$$-\frac{1}{2\pi i}\int\limits_{\Omega_2} \frac{\pi^2}{s-3}(s-4)(s-5)(s-7)(s-8)\Phi_+(s)u^{-s+3}ds. \tag{7.32}$$

The integrand is an analytical function in the strip Π_3. We use the first relation in (7.18) and again, continue analytically the integrand into the next strip Π_4 where there is a simple pole $s = 13$. We get

$$f(u) = \Phi(0) + \frac{3\cdot 2}{2}\Phi(1)u^{-4} - \frac{\pi^2}{5}9\cdot 8\cdot\ 6\cdot 5\cdot 3\cdot 2\,\Phi(13)u^{-10}$$

$$-\frac{1}{2\pi i}\int\limits_{\Omega_4} \frac{\pi^3\Phi_-(s)}{(s-3)\cot\frac{\pi s}{2}}(s-4)(s-5)(s-7)(s-8)(s-10)(s-11)u^{-s+3}ds. \tag{7.33}$$

Here we took into account that $\Phi(3) = -\Phi(0),\ \Phi(7) = \Phi(1)$. The next pole, $s = 19$, is in the strip Π_6. Finally we may write the following expansion when u is large

$$f(u) \sim \Phi(0) + \frac{3!}{2}\Phi(1)u^{-4} - \frac{\pi^2}{5}\frac{9!\Phi(1)u^{-10}}{7\cdot 4\cdot 1} + \frac{\pi^4}{8}\frac{15!\Phi(1)u^{-16}}{13\cdot 10\cdot 7\cdot 4\cdot 1} - \ldots, \quad u \to \infty. \tag{7.34}$$

Due to the second condition in (3.2), we have $\Phi(0) = 1$ and

$$f(u) \sim 1 + \frac{\Phi(1)}{2u^4}\sum_{j=0}^{\infty}\frac{(-1)^j\mu_{6j+4}}{(3j+2)^2}u^{-6j}, \quad u \to \infty. \tag{7.35}$$

The series in the right-hand side of (7.35) diverges but it is a Poincare asymptotic expansion for large u (see, e.g. Olver **(19)**). The condition $\Phi(0) = 1$ defines the constant C_0. From (6.2) we obtain

$$C_0 = \frac{e^{-\pi i c/3}}{Q(0)}. \tag{7.36}$$

7.4 Verification of the conditions (4.8), (4.9).

Let us show that the inverse Mellin transform $\phi_{\mathfrak{M}}(u)$ of the function $\Phi(s)$ satisfies conditions

$$u^c\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0,\infty), \quad u^{c-3}\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0,\infty). \tag{7.37}$$

Due to definition (4.7) of the function $\phi_{\mathfrak{M}}(u)$, the analyticity of the function $\Phi(s)$ in the strip Π and relation (7.11), we may write

$$\phi_{\mathfrak{M}}(u) = \frac{1}{2\pi i}\int\limits_{\Omega} \Phi_-(s)u^{-s}ds = \frac{1}{2\pi i}\int\limits_{\Omega_{-1}} \Phi_+(s)u^{-s}ds$$

$$= \frac{1}{2\pi i}\int\limits_{\Omega_{-1}} \frac{\Phi_-(s)u^{-s}ds}{\pi(s+1)(s+2)\tan\frac{\pi s}{2}}. \tag{7.38}$$

Using the procedure of Section 7.2 we obtain

$$\phi_{\mathfrak{M}}(u) = -\frac{2}{\pi^2}\Phi(1)u^2\log u + O(u^2), \quad u \to 0. \tag{7.39}$$

The above estimation yields

$$u^c\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0,A), \quad u^{c-3}\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(0,A). \tag{7.40}$$

These inequalities are valid for any positive $A < \infty$ and for $c > 1$. On the other hand, if we continue analytically the function $\Phi(s)u^{-s}$ into the strip Π_1 we get

$$\phi_{\mathfrak{M}}(u) = -\frac{1}{2\pi i}\int\limits_{\Omega} \pi(s-1)(s-2)\cot\frac{\pi s}{2}\Phi_+(s)u^{-s}ds. \tag{7.41}$$

In this strip, there is a simple pole at the point $s = 4$. The next pole will be in the strip Π_3 at the point $s = 10$. The residue theorem leads to the following representation for large u

$$\phi_{\mathfrak{M}}(u) = 12\Phi(1)u^{-4} + O(u^{-10}), \quad u \to \infty. \tag{7.42}$$

It is clear that

$$u^c\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(a,\infty), \quad u^{c-3}\phi_{\mathfrak{M}}(u) \in L_{2,\frac{1}{u}}(a,\infty) \tag{7.43}$$

for any small positive a and $c < 4$. Therefore, from (7.40) and (7.43) we may conclude, that the solution satisfies conditions (4.8) and (4.9).

8 Numerical results

To find the numerical values of $f(u)$, we may use the series representation (7.26) and the asymptotic expansion (7.35) (for large u). The following values

$$\Phi(1) = \frac{Q(1)}{e^{\pi i/3}Q(0)}, \quad \Phi(-1) = \frac{e^{\pi i/3}Q(-1)}{Q(0)},$$

$$\Phi'(1) = \frac{1}{e^{\pi i/3} Q(0)} \left[-\frac{\pi i}{3} Q(1) + Q'(1) \right] \tag{8.1}$$

are involved into formulae (7.26), (7.35) for the function $f(u)$. Let us calculate the derivative $Q'(s)$. We take into account the integral representation (6.3) and relations (5.11), (5.1). Then the function $Q(s)$ will be described by

$$Q(s) = \exp\left\{ \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\log G(\tau)}{\tau - w} d\tau \right\} \tag{8.2}$$

with

$$w = i \tan\left\{ \pi \left(\frac{1}{4} + \frac{s-c}{3} \right) \right\}. \tag{8.3}$$

Due to

$$\frac{dw}{ds} = \frac{i\pi}{3}(1 - w^2)$$

we may obtain

$$\frac{dQ}{ds} = Q(s) Q_0(s), \quad Q_0(s) = \frac{1-w^2}{6} \int\limits_{\gamma} \frac{\log G(\tau)}{(\tau - w)^2} d\tau. \tag{8.4}$$

To get formulae convenient for numerical calculations, we transform the integral representations for $Q(s)$, $Q_0(s)$ in such a way. We put $\tau = e^{i\theta}$. Then

$$Q(s) = \exp\left\{ \frac{1}{2\pi} \int\limits_0^{\pi} \frac{\log G_0(\theta)}{e^{i\theta} - w} e^{i\theta} d\theta \right\}, \quad w \notin \gamma, \tag{8.5}$$

where

$$G_0(\theta) = -\pi(\sigma - 1)(\sigma - 2) \cot \frac{\pi\sigma}{2} \tag{8.6}$$

with

$$\sigma = c + \frac{3i}{2\pi} \log\left(\tan \frac{\theta}{2} \right). \tag{8.7}$$

Due to (5.76), the branch of the function $\log G_0(\theta)$ must be chosen in the following manner

$$\arg G_0(0) = \frac{\pi}{2}, \quad \arg G_0(\pi) = \frac{3\pi}{2},$$

$$\frac{\pi}{2} < \arg G_0(\theta) < \frac{3\pi}{2}, \quad 0 < \theta < \pi. \tag{8.8}$$

This means that

$$\log G_0(\theta) = |\log G_0(\theta)| + i \tan^{-1}\left(\frac{\Im\{G_0(\theta)\}}{\Re\{G_0(\theta)\}} \right) + i\pi. \tag{8.9}$$

The function $\log G_0(\theta)$ behaves at the ends of the segments $[0, \pi]$ as follows

$$\log G_0(\theta) = O(\log\log\theta), \quad \theta \to 0,$$

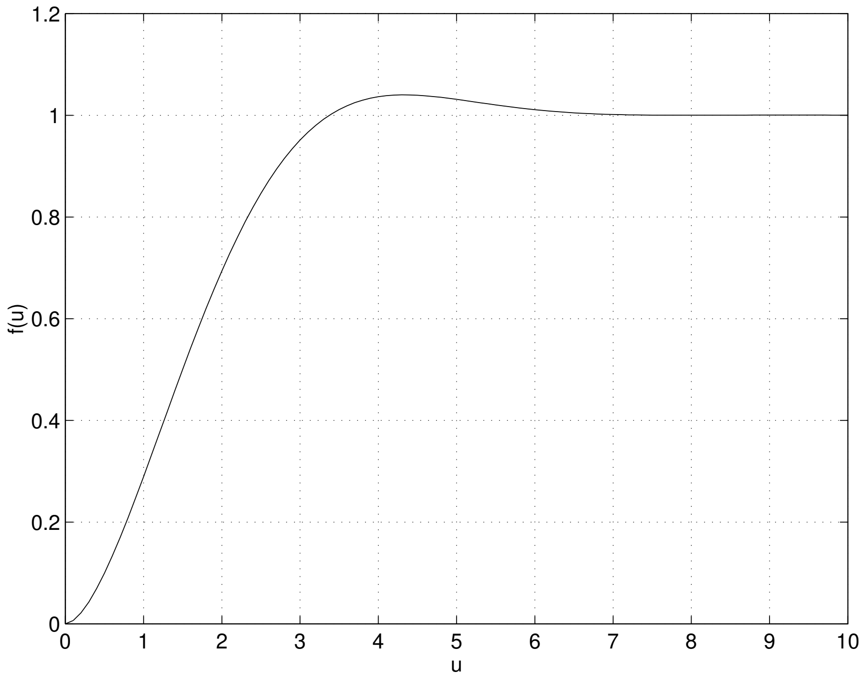

Figure 1: Function $f(u)$.

$$\log G_0(\theta) = O(\log\log(\pi - \theta)), \quad \theta \to \pi. \tag{8.10}$$

The procedure for the integral $Q_0(s)$ is the same

$$Q_0(s) = \frac{w^2 - 1}{6i} \int_0^\pi \frac{\log G_0(\theta)}{(e^{i\theta} - w)^2} e^{i\theta} d\theta. \tag{8.11}$$

It is worth to say that we need the values of functions (8.5) and (8.11) at the points $w = -1, 0, 1$ only, and the expression $e^{i\theta} - w$ in (8.5) and (8.11) does not vanish at these points. Numerical evaluation of the integrals (8.5), (8.11) yields the following values of the functions $\Phi(1)$, $\Phi(-1)$ and $\Phi'(1)$

$$\Phi(1) = 1.477258, \quad \Phi(-1) = 1.142318, \quad \Phi'(1) = 0.712287.$$

In 1 we present the graph of the function $f(u)$.

It was constructed on the base of the series representation (7.26). The maximum of the function $f(u)$ occurs at the point $u = 4.330$ and

$$\max f(u) = 1.04027.$$

The asymptotic expansion (7.35) for large u is in a good agreement with the series representation (7.26) for $u > 6$.

2 illustrates the asymptotic expansion of function $f(u)$ for different numbers of the first terms of the representation (7.35). The graphs of the first, second and third derivatives of the function $f(u)$ are shown in 3.

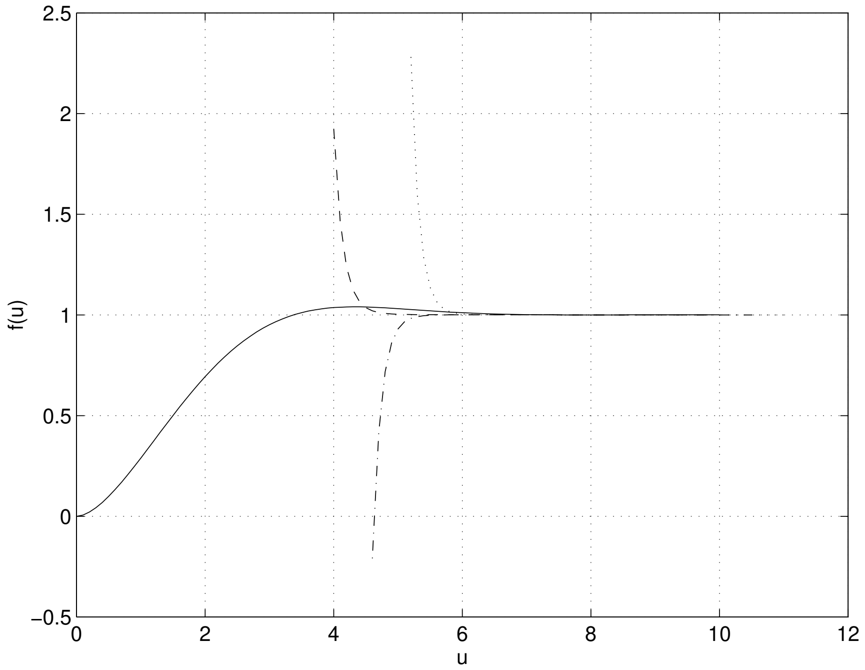

Figure 2: Function $f(u)$: series representation (7.26) (—); asymptotic expansion (7.35) with the 4 first terms (- - -), 5 terms ($- \cdot -\cdot$), 6 terms ($\cdots$).

9 Integro-differential equation of diffusion along a semi-infinite grain boundary

To solve the integro-differential equation (2.14) we modify the previous technique. The kernel $S(u,v)$ is representable in the form

$$S(u,v) = \frac{1}{v} h\left(\frac{u}{v}\right), \quad h(t) = \frac{1}{t-1} - \frac{1}{t+1} - \frac{2(t-1)}{(t+1)^3}. \tag{9.1}$$

Due to formulae 3.222(2) and 3.194(3) (Gradshteyn and Ryzhik **(10)**), its Mellin transform is an analytic function in the strip $-1 < \Re(s) < 3$ and it is given by

$$H(s) = \int_0^\infty h(t) t^{s-1} dt = -\pi \cot \pi s \left(1 + \frac{4s - 2s^2 - 1}{\cos \pi s}\right). \tag{9.2}$$

We seek the solution in the same class as before, namely, the function $f(u)$ and its derivatives are assumed to satisfy conditions (3.5). The boundary condition of the corresponding Carleman problem has the form

$$\Phi(\sigma) + d(\sigma) K(\sigma) \Phi(\sigma - 3) = 0, \quad \sigma \in \Omega, \tag{9.3}$$

where $K(\sigma)$ is the same function that (4.14) and

$$d(s) = \cot \pi s \tan \frac{\pi s}{2} \left(1 + \frac{4s - 2s^2 - 1}{\cos \pi s}\right). \tag{9.4}$$

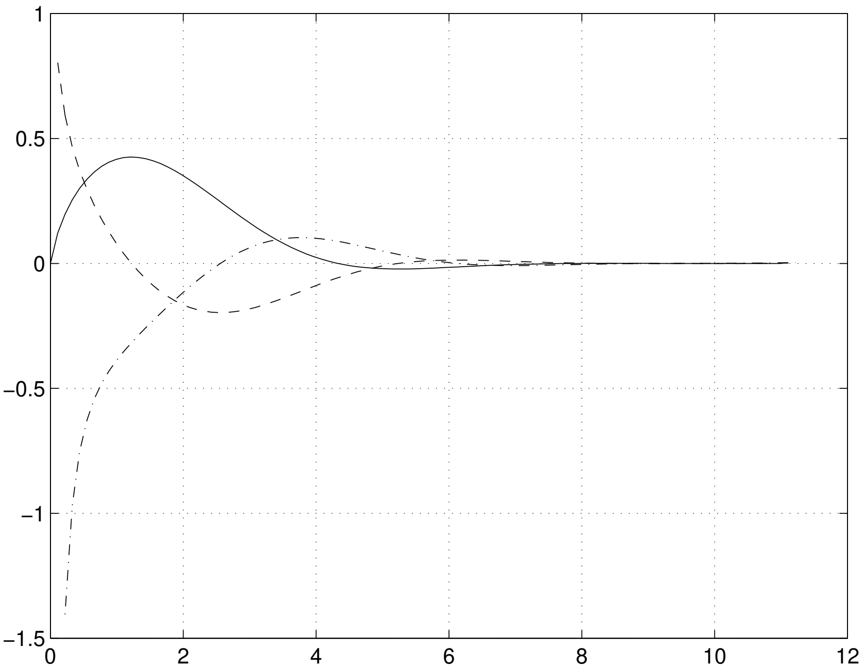

Figure 3: Derivatives of the function $f(u)$: $f'(u)$ (—), $f''(u)$ (- - -), $f'''(u)$ (— · —·)

The next step of the algorithm is to define the increment Δ introduced in (5.19). In our case, due to (5.27), we have

$$\Delta = \pi + [\arg d_0(w)]_\gamma, \tag{9.5}$$

with

$$d_0(w) = d\left(c + \frac{3i}{2\pi}\log i\frac{1-w}{1+w}\right). \tag{9.6}$$

Having regard to the estimations

$$d_0(w) \sim 1, \quad \text{as} \quad w \to \pm 1, \quad w \in \gamma,$$

$$d_0(i) \sim 2(2-c) > 0 \quad \text{as} \quad c \to 2-0, \tag{9.7}$$

we get

$$[\arg d_0(w)]_\gamma = 0. \tag{9.8}$$

We note that the parameter c may be an arbitrary number: $1+\delta < c < 2$ and $0 < \delta < 1$. Additionally, we can verify formula (9.8) numerically. Calculating the values of the $\Re\{d_0(\eta)\}$ and $\Im\{d_0(\eta)\}$ as the point η traverses the contour γ in the positive direction, we find that the increment of the argument of the function $d_0(\eta)$ is equal to zero. Therefore the solution of the Carleman problem, the complex potential $\Phi(s)$, referred to the semi-infinite grain boundary can be obtained from (6.2) and (6.3) by replacing the function $K(\sigma)$ by $d(\sigma)K(\sigma)$. The solution of the original integro-differential equation, the desirable function $f(s)$, is given by the inverse Mellin transform

$$f(u) = \frac{1}{2\pi i}\int\limits_\Omega \frac{\Phi(s)u^{3-s}ds}{(s-3)d(s)K(s)}. \tag{9.9}$$

In this case the limit values (7.2) and (7.3) of the solution of Carleman problem (9.3) satisfy the conditions

$$\Phi_-(s_m) = d(s_m - 3m)K(s_m - 3m)\Phi_+(s_m), \quad m = 0, \pm 1, \pm 2, \ldots \tag{9.10}$$

The first step of the procedure of Section 7.2 yields

$$f(u) = \frac{1}{2\pi i} \int\limits_{\Omega_{-1}} \frac{\sin \pi s \Phi(s) u^{-s} ds}{\pi s(s+1)(s+2)[\cos \pi s - 4(s+3) + 2(s+3)^2 + 1]}. \tag{9.11}$$

There are three poles of the integrand in the strip Π_{-1}: $s^{(0)} = -2$ of the second multiplicity and two complex-conjugate simple poles

$$s^{(1)} = -4.7396 + 1.1190i, \quad s^{(2)} = -4.7396 - 1.1190i.$$

Thus, the function $f(u)$ behaves at the point $u = 0$ as follows

$$f(u) = O(u^2 \log u), \quad u \to 0.$$

At infinity, as in the case of the infinite boundary we have $f(u) \sim 1$. The zeros of the function $d(s)$ are not periodic and therefore the final series representation and asymptotic expansion can not be derived so straightforward as in the case of the infinite grain boundary. However the function $d(s)K(s)$ is meromorphic and the procedure of Section 7 may be modified for integral (9.11).

10 Conclusion

The authors have analysed model problems of mass transport (i) from a point source into an infinite grain boundary of a material and (ii) from the surface of a material into a semi-infinite grain boundary. The new unusual partial integro-differential equations of atomic diffusion have been derived. The self-similarity of the solution allowed to reduce these problems to one-dimensional singular integro-differential equations with two auxiliary conditions at zero and at infinity.

The class of functions where the solution exists and it is unique has been established. The equations have been reduced to a particular case of the Carleman boundary value problem for a strip (a difference equation of the first order in a strip of a complex variable).The coefficient of the problem growths at infinity as polynomial of degree which differs from the shift of the Carleman problem. Such a case, as the authors know, has not been met neither in contact mechanics, nor in diffraction theory. The solution of the Carleman problem has been found by reducing to an exceptional case of the Riemann boundary value problem for an open contour with the coefficient of logarithmic growth at the ends of the contour.

The authors have constructed the exact solution of the integro-differential equations by quadratures. In the case of the infinite grain boundary, the solution, the function $f(u)$, is given in the form of a power series with infinite radius of convergence. Additionally, for large arguments, a full asymptotic expansion is derived. The final formulae involve the values of the solution of Carleman problem at two points only. The graphs for the function $f(u)$ and its first three derivatives have been represented.

Acknowledgements

The work of HG was supported by National Science Foundation through Grant MSS-9358093.

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