Invariant and positivity properties of moving mesh operators for nonlinear evolution equations

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Abstract

It is shown that for a class of nonlinear one-dimensional evolution equations, the moving mesh methods (MMMs) can preserve linear invariant subspaces of nonlinear differential operators by using special approximations of the spatial gradient operator. The corresponding discrete evolution on such subspaces may coincide with that on fixed meshes and even with the continuous ones regardless the choice of the moving mesh. The corresponding less dimensional reductions of the difference solutions on the invariant subspaces are constructed.

Sufficient conditions of the positivity properties of MMMs for second-order nonlinear parabolic equations such as the weak Maximum Principle and the comparison principle are established. This makes possible to compare general solutions with particular known exact solutions on invariant subspaces and describe some asymptotic stability properties of the MMMs.

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1 Introduction: positivity and nonlinear invariance

Consider a general one-dimensional nonlinear evolution partial differential equation (PDE)

$$v_t = \mathbf{A}(v), \quad x \in S \subseteq \mathbf{R}, \;\; t > 0, \tag{1}$$

where $\mathbf{A}(v) = A(v, v_x, ..., \partial_x^m v)$, $A \in C^\infty$, is an m-th order ordinary differential operator and S is an interval. We assume that being endowed with a suitable boundary condition on the lateral boundary of $Q = S \times \mathbf{R}_+$ (if $S \neq \mathbf{R}$), equation (1) defines a smooth flow on, say, $X = C^\infty$ for bounded initial data $u_0 \in X$. If $S = \mathbf{R}$ we consider the Cauchy problem.

The moving mesh methods (MMMs in short) are known to be an efficient approach to solve nonlinear equations in the case where the solutions exhibit essentially non-stationary and singular behaviour like blow-up, extinction or quenching which are typical features of the quasilinear parabolic equations from filtration and combustion theory. MMMs consider equation (1) on a moving mesh (MM) $\Omega = \{x = X_i(t), \; i \in I, t > 0\} \subset Q$, with a priori unknown curves $x = X_i(t)$ depending on the solution $v(\cdot, t)$. Using the identity

$$\frac{d}{dt} v(X_i, t) = v_t(X_i, t) + v_x(X_i, t) X_i',$$

the differential equation for $V_i(t) = v(X_i(t), t)$ reads

$$V_i' - v_x(X_i, t) X_i' = \mathbf{A}(v(X, t)), \quad t > 0; \quad i \in I. \tag{2}$$

By a discretisation of this equation, one obtains the dynamical system

$$V_i' - [D_h V]_i X_i' = \mathbf{A_h}(V), \quad t > 0, \tag{3}$$

where $\mathbf{A_h}$ is a suitable approximation of the nonlinear operator $\mathbf{A}$, and D_h approximates the gradient operator d/dx. The equations for the moving mesh Ω should be added which all together form a finite-dimensional dynamical system for the functions $\{V_i(t), \; i \in I\}$. See [13], [14]. The MM approaches make it possible to concentrate the MM eventually close to the most important singularities of the solutions under consideration and this helps to describe correctly the behaviour of such solutions. Nevertheless, giving such advantages in computing the singular behaviour, the extra linear discrete gradient operator D_h in (3) changes dramatically the discrete PDE even for lower-order equations.

Namely, even for semilinear or quasilinear second-order parabolic equations, in the general case, equation (3) no more obeys the Maximum Principle for arbitrary MMs. On the other hand, if $X_i' \equiv 0$, i.e., the mesh is fixed (stationary), the Maximum Principle holds provided that the approximation $\mathbf{A_h}$ preserves such positivity property. It is therefore clear that highly moving meshes can break down any positivity or monotonicity properties inherited by the discrete equation from the original PDE. Therefore, the mathematical analysis of MMMs is harder than the corresponding explicit or implicit difference schemes with stationary meshes. This includes the questions on convergence of MMMs and on the asymptotic behaviour. Comparison of solutions of MMMs for parabolic equations becomes a complicated procedure because the solutions are defined, in general, on distinct MMs.

We first will consider another important question on *invariant linear subspaces* generated by MM operators (MMOs). It is known that several quasilinear PDEs with polynomial nonlinearities from reaction-diffusion theory, combustion, filtration, flame propagation and water wave theory admit finite-dimensional subspaces or sets which are invariant under the corresponding nonlinear operators of the PDEs. Then the PDE restricted to such invariant subspace reduces to a finite-dimensional dynamical system. We consider the cases when such a subspace W is invariant under the nonlinear operator $\mathbf{A}$ in the sense that

$$\mathbf{A}(W) \subseteq W. \tag{4}$$

This is the *nonlinear invariance* unlike the invariance with respect to Lie or Lie-Bäcklund groups of transformations with linear infinitesimal generators. In general, for several nonlinear equations of Mathematical Physics with typical quadratic nonlinearities and especially of reaction-diffusion type, nontrivial symmetry groups occur less often than the nonlinear invariance, see examples and comments below. There are interesting relationships between nonlinear invariance and the linear group-invariance. Often group-invariant solutions can be treated as lower-dimensional manifolds M on the subspaces W invariant under nonlinear operators. We consider a few such examples. This means that the wider sets of nonlinear invariant solutions "bifurcate" from group-invariant one on special manifolds from W. See examples below. We also present an example where the manifolds of the linear and nonlinear invariant solutions on a subspace coincide (it is a rare case).

In Section 2 we prove necessary and sufficient condition on the gradient approximation D_h to preserve the nonlinear invariance of a given two-dimensional subspace W. This means that, under a suitable approximation of the operator $\mathbf{A}$, the invariant MMO admits the same invariant subspace and moreover the discrete dynamics on W exactly coincides with the differential one regardless the structure of the MM.

In Sections 3 and 4 we present examples of invariant MMOs for the second and fourth-order quasilinear parabolic equations.

In Section 5 we consider invariant MMOs for higher-order evolution equations in the normal form

$$\partial_t^k v = \mathbf{A}(v) \equiv A(v, v_t, ..., \partial_t v^{k-1}, v_x, ..., \partial_x^m v), \quad k \geq 2. \tag{5}$$

In particular, we consider the second-order hyperbolic-like equations

$$v_{tt} = A(v, v_x, v_{xx}). \tag{6}$$

In Section 6 we establish necessary and sufficient conditions for the positivity of the solutions (the weak Maximum Principle), and also prove the results on comparison of solutions of MMMs for parabolic PDEs. In general, it is proved that for typical semilinear parabolic equations, both important properties are available if *the MM does not move too fast.* We show applications to stability analysis of the invariant MMMs.

2 Invariant subspaces for nonlinear differential and discrete operators

We first introduce a number of differential and corresponding discrete quadratic operators which exhibit the (nonlinear) invariance property to be preserved by MMOs.

2.1 Invariant subspaces of differential operators

Let us state the main invariance hypothesis on the nonlinear operator $\mathbf{A}$.

Hypothesis (I). The operator $\mathbf{A}$ admits a *two-dimensional invariant subspace* of the form $W_2 = \mathbf{L}\{1, \phi(x)\}$, where $\mathbf{L}\{\cdot\}$ denotes the linear span and $\phi(x) \not\equiv \text{const}$, $\phi \in X$, is a given function. There holds

$$\mathbf{A}(W_2) \subseteq W_2. \tag{7}$$

As a straightforward consequence, we obtain that *under hypothesis* **(I)** *the evolution equation (1) on* W_2 *is equivalent to a nonlinear two-dimensional dynamical system* (DS). Indeed, (7) means that for any $v = C_0 + C_1\phi \in W_2$ there holds

$$\mathbf{A}(C_0 + C_1\phi) \equiv F_0(C_1, C_2) + F_1(C_0, C_1)\phi \in W_2. \tag{8}$$

Substituting (8) into (1), we arrive at the DS on the coefficients $\{C_0(t), C_1(t)\}$ of the expansion on W_2:

$$C_0' = F_0(C_0, C_1), \quad C_1' = F_1(C_0, C_1), \quad t > 0. \tag{9}$$

For several nonlinear PDEs, this property of finite-dimensional reduction of evolution equations on invariant subspaces plays an important role in the problems of existence, regularity and the asymptotic behaviour for more general solutions. Let us present some examples of such nonlinear operators from different fields of applications to be used below. Here the Greek letters α, $\beta, \dots$ denote different constants.

(i) *Quadratic operators with two and three-dimensional invariant subspaces:*

(10) $$\mathbf{A_1}(v) = v_{xx} + (v_x)^2 + v, \quad W_2 = \mathbf{L}\{1, x^2\},$$

(11) $$\mathbf{A_2}(v) = v_{xx} + (v_x)^2 + v^2, \quad W_2 = \mathbf{L}\{1, \cos x\},$$

(12) $$\mathbf{A_3}(v) = vv_{xx} + \frac{1}{\sigma}(v_x)^2 + \alpha v + \beta, \quad W_2 = \mathbf{L}\{1, x^2\},$$

(13) $$\mathbf{A_4}(v) = \ vv_{xx} + \frac{1}{\sigma}(v_x)^2 + \sigma v^2, \ \sigma > -1, \quad W_2 = \mathbf{L}\{1, \cos(\lambda x)\},$$

where $\lambda = \sigma/\sqrt{\sigma+1}$,

(14) $$\mathbf{A_5}(v) = -v_{xxxx} - (v_x)^2 + v^2, \quad W_2 = \mathbf{L}\{1, \cosh x\},$$

(15) $$\mathbf{A_6}(v) = -vv_{xxxx} + \alpha v_x v_{xxx} + \beta(v_{xx})^2 + \gamma v + \delta, \quad W_2 = \mathbf{L}\{1, x^4\}.$$

The operators $\mathbf{A_1} - \mathbf{A_4}$ correspond to the semilinear and quasilinear second-order parabolic equations of reaction-diffusion type. The last two ones contain the differential operator of the porous medium (PM, $\sigma > 0$) or fast diffusion ($\sigma < 0$) type $vv_{xx} + \sigma^{-1}(v_x)^2$. The operator $\mathbf{A_5}$ reproduces a fourth-order parabolic equation which is Kuramoto-Sivashinskii equation from the theory of flame propagation with an extra lower-order source term v^2. The equations

$$v_{tt} = \mathbf{A_3}(v) + \gamma v_{xxxx} + \delta v_{xx}, \quad v_{tt} = \mathbf{A_4}(v) + \gamma v_{xxxx} + \delta v_{xx},$$

(both with $\gamma = \delta = 0$ are hyperbolic in $\{v > 0\}$ and elliptic in $\{v < 0\}$) can be treated as Boussinesq-type equations from water wave theory. Let us consider some invariant examples in more detail.

Example 1. The quasilinear heat equation with a source term in the pressure form

(16) $$v_t = vv_{xx} + \frac{1}{\sigma}(v_x)^2 + \sigma v^2, \quad \sigma > 0,$$

plays a special role in the study of localization of blow-up in combustion, see Chapter 4 in [17] and earlier references therein. The quadratic operator $\mathbf{A_4}$ on the right-hand side admits the subspace W_2 given in (13), so that the PDE on W_2 reduces to a nonlinear DS on the expansion coefficients

(17) $$v(x,t) = C_0(t) + C_1(t)\cos(\lambda x) \in W_2.$$

This DS has the form

(18) $$C_0' = \sigma C_0^2 + \frac{\sigma}{\sigma+1}C_1^2, \quad C_1' = \frac{\sigma(\sigma+2)}{(\sigma+1)}C_0C_1, \ \ t > 0.$$

Such solutions on W_2 play a key role in the localization and in the asymptotic analysis of blow-up, [5]. In particular, it is proved that the interface $s(t)$ of any solution $u(x,t) \geq 0$ with connected compact support satisfies the localization estimate

$$|s(t) - s(0)| \leq \frac{1}{2} L_s = \pi \frac{\sqrt{\sigma+1}}{\sigma} \text{ for any } t \in [0, T)$$

(L_s is called the *fundamental length*), where T is the finite blow-up time of the solution u.

Let us describe the manifold $M \subset W_2$ of group-invariant solutions of (16). Only in the special case

(19) $$C_0(t) \equiv C_1(t),$$

so that

(20) $$C_0' = \alpha\, C_0^2, \quad \alpha = \sigma(\sigma+2)/(\sigma+1),$$

whence $C_0(t) = \alpha(T-t)^{-1}$ and

(21) $$v(x,t) = [\alpha(T-t)]^{-1}[1 + \cos(\lambda x)] \geq 0,$$

this separable solution is invariant of a Lie group of scaling transformations. If $C_0 \not\equiv C_1$ then solutions (17) are not group invariant, see a general group classification of quasilinear heat equations in [8]. We see that W_2 contains the one-dimensional manifold M (with the parameter $T \in \mathbf{R}$) of group-invariant solutions (21). This is an interesting situation where more general nonlinear invariant solutions on W_2 bifurcate (in a natural sense) from a manifold of group-invariant solutions when identity (19) is violated. For $C_0 \not\equiv C_1$, the two-dimensional DS (18) can be viewed as an extension of the one-dimensional DS (20) generated the group-invariant solutions.

Example 2. The semilinear parabolic equation

$$u_t = u_{xx} + u \ln u$$

is known to admit a four-parametric Lie group of transformations [8], and hence a wide manifold of group-invariant solutions described in [17], p. 94. This equation presents a rare example where both linear and nonlinear invariance give rise to the same manifold of exact solutions. Namely, setting $u = e^v$, we obtain equation

(22) $$v_t = \mathbf{A_1}(v) \equiv v_{xx} + (v_x)^2 + v.$$

The two-dimensional manifold of invariant solutions $v(x,t) = C_0(t) + C_1(t)x^2$ on the two-dimensional invariant subspace W_2, $\mathbf{A_1}(W_2) \subseteq W_2$, coincides with the manifold of the group-invariant solutions. Equation (22) on W_2 is the DS on the expansion coefficients

(23) $$C_0' = C_0 + 2C_1, \quad C_1' = C_1(1 + 4C_1), \ \ t > 0.$$

There are several other examples of such a bifurcation-like effect of nonlinear invariant solutions from group-invariant ones, see the differential dynamics of PDEs on a number of invariant subspaces given in [3], [4], [6], [9]. Consider the following example.

Example 3. The hyperbolic-elliptic equation

$$v_{tt} = \mathbf{A_4}(v) \equiv (vv_x)_x + v^2 \quad (\sigma = 1)$$

admits exact solutions $v(x,t) = C_0(t) + C_1(t)\cos(x/\sqrt{2})$ on $W_2 = \mathbf{L}\{1, \cos(x/\sqrt{2})\}$ invariant relative to $\mathbf{A_4}$. The coefficients solve the fourth-order DS

$$C_0'' = C_0 + \frac{1}{2}C_1^2, \quad C_1'' = \frac{3}{2}C_0C_1, \ \ t > 0.$$

Therefore, there exists a four-dimensional manifold of such exact solutions on W_2. On the other hand, if $C_0 \equiv C_1$ and

$$C_0'' = \frac{3}{2}C_0^2,$$

(this ODE is easily integrated in quadratures and, in general, is expressed in terms of elliptic functions), there exists a particular solution $C_0(t) = 4(T-t)^{-2}$ which gives the scaling group-invariant separable solutions

$$v(x,t) = 4(T-t)^{-2}[1 + \cos(x/\sqrt{2})] \quad (T \in \mathbf{R}).$$

They form a one-dimensional manifold. Observe that exept this case, other separable solutions with $C_0(t)$ of non-power form are not group–invariant [18].

For operators (10) - (14), the subspaces are extended to three-dimensional invariant ones. Namely, in (10) and (12) the extended subspace is

$$W_3 = \mathbf{L}\{1, x, x^2\}$$

while in (11) it is

$$W_3 = \mathbf{L}\{1, \cos x, \sin x\}.$$

For (13) and (14) these are

$$W_3 = \mathbf{L}\{1, \cos(\lambda x), \sin(\lambda x)\} \quad \text{and} \quad W_3 = \mathbf{L}\{1, \cosh x, \sinh x\},$$

respectively. For the first-order (in time) evolution equations (1), such extension means no new exact solutions in view of their translational invariance. On the other hand, the hyperbolic equation (6) with operators $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ and $\mathbf{A_4}$ on the corresponding invariant subspaces W_2 is equivalent to a fourth-order DS (cf. Example 3), while on W_3 we arrive at a sixth-order one. In the later case a one dimensional subset only can be dispensed with due to the translational invariance [4].

(iii) *Five-dimensional subspaces* exist for special quadratic operators:

$$\mathbf{A_7}(v) = vv_{xx} - \frac{3}{4}(v_x)^2, \quad W_5 = \mathbf{L}\{1, x, x^2, x^3, x^4\}, \tag{24}$$

(note that (15) admits this W_5)

$$\mathbf{A_8}(u) = \; vv_{xx} - \frac{3}{4}(v_x)^2 + \frac{4}{3}v^2, \quad W_5 = \mathbf{L}\{1, \cos(\lambda x), \sin(\lambda x), \cos(\lambda x/2), \sin(\lambda x/2)\}, \tag{25}$$

where $\lambda = 4/\sqrt{3}$. Note that $\mathbf{A_7}$, admits also two-dimensional subspaces of W_5:

$$W_2 = \mathbf{L}\{1, x^2\} \quad \text{and} \;\; W_2 = \mathbf{L}\{1, x^4\}.$$

The operator

$$\mathbf{A_9}(u) = \; vv_{xx} - \frac{2}{3}(v_x)^2 \tag{26}$$

admits the two-dimensional subspace

$$W_2 = \mathbf{L}\{1, x^3\},$$

extended to the four-dimensional one $W_4 = \mathbf{L}\{1, x, x^2, x^3\}$ [15].

These operators in (1) give the quasilinear heat equations. The most of operators are taken from [3] and [4]. More examples of nonlinear invariant solutions, including those in multi-dimensional geometry, can be found in [4], [6], [9], [10] and [15], [16]. As a natural generalisation, it is possible to consider invariant sets on linear subspaces (where the PDEs reduce to overdetermined DSs), which give new explicit solutions, [10], [11].

2.2 Discrete operators

It is proved [7] that invariant subspaces can be preserved and are, in fact, stable with respect to natural discretisations on fixed meshes. In other words, we showed that the natural discretisation $\mathbf{A_h}$, with small $h > 0$ being the scale of uniform discretisation, of the operators of the type $\mathbf{A_1}, ..., \mathbf{A_8}$, admits invariant subspaces with the basic functions looking like $O(h)$-perturbations of those for differential operators. On the two-dimensional subspace $W_2 = \mathbf{L}\{1, \phi(x)\}$ this means that

$$\mathbf{A_h}(C_0 + C_1\phi) \equiv F_{0h}(C_0, C_1) + F_{1h}(C_0, C_1)\phi \in W_2, \tag{27}$$

so that the fixed mesh discrete equation

$$V_t = \mathbf{A_h}(V), \quad t > 0, \tag{28}$$

has the solution $V(x,t) = C_0(t) + C_1(t)\phi(x)$ on W_2 with the coefficients satisfying the DS similar to (9):

$$C_0' = F_{0h}(C_0, C_1), \quad C_1' = F_{1h}(C_0, C_1), \quad t > 0. \tag{29}$$

The coefficients $F_{0h},\ F_{1h}$ converge as $h \to 0$ to the coefficients of the DS (9).

Assuming for a moment that the mesh is fixed and uniform, $h > 0$, we introduce the natural approximations of the space derivatives

$$v_x = \frac{1}{2h}(V_{i+1} - V_{i-1}), \quad v_{xx} = \frac{1}{h^2}(V_{i+1} + V_{i-1} - 2V_i), \dots . \tag{30}$$

By substituting those into $\mathbf{A_1}-\mathbf{A_9}$, we create the corresponding discrete operators $\mathbf{A_{1h}}-\mathbf{A_{9h}}$. Then [7]: for the discrete operator $\mathbf{A_{1h}}$ in (10) the invariant subspace W_2 stays the same; operator $\mathbf{A_{2h}}$ in (11) admits the subspace

$$W_{2h} = \mathbf{L}\{1, \cos(\lambda x)\}, \ \text{where } \lambda(h) \text{ is such that } \frac{1}{h^2}\sin^2(\lambda h) = 1; \tag{31}$$

$\mathbf{A_{3h}}$ in (12) admits $\mathbf{L}\{1, x^2\}$; operator $\mathbf{A_{4h}}$ in (13) admits the subspace

$$W_{2h} = \mathbf{L}\{1, \cos(\lambda x)\}, \ \text{where } \frac{4}{h^2}\sin^2(\lambda h/2)[1 + \sigma^{-1}\cos^2(\lambda h/2)] = \sigma. \tag{32}$$

The subspace

$$\mathbf{L}\{1, \cosh(\lambda x)\}, \quad \lambda = \lambda(h), \tag{33}$$

can be constructed for the fourth-order operator $\mathbf{A_{5h}}$ in (14), $\mathbf{A_{6h}}$ admits $\mathbf{L}\{1, x^4\}$. The five-dimensional subspaces exist for the quadratic operators $\mathbf{A_{6h}}$, $\mathbf{A_{7h}}$ and $\mathbf{A_{8h}}$, see Section 4 in [7]. The same two-dimensional subspaces are available for the discrete operator $\mathbf{A_{9h}}$.

Example 4 [7]. Consider the discrete equation (16). On W_{2h}, given by (31), it reduces to the following DS on the coefficients $\{C_0, C_1\}$ of the expansion (17):

$$C_0' = \sigma C_0^2 + \frac{1}{\sigma h^2}\sin^2(\lambda h)C_1^2, \ \ C_1' = \frac{2}{h^2}[\sigma h^2 - 2\sin^2(\lambda h/2)]C_0 C_1. \tag{34}$$

For $h \ll 1$ we have that the parameter λ of the basic function satisfies

$$\lambda(h) = \frac{\sigma}{\sqrt{\sigma+1}} + \frac{\sigma^3(\sigma+4)}{24(\sigma+1)^{5/2}}\,h^2 + O(h^4),$$

so that $\lambda(h) \to \sigma/\sqrt{\sigma+1}$ as $h \to +0$, and in the limit the discrete DS (34) coincides with the continuous one (18).

Note that for standard approximations (30), the subspaces like (31) - (33) can be invariant on uniform fixed meshes only. In the next sections, in order to preserve this invariance on non-uniform MMs, we have to use essentially different approximations.

2.3 Invariance of moving mesh operators: main results

We now prove that the above nonlinear invariant properties of evolution can be preserved by MMOs. Consider a general MMM for equation (1). Assume that given initial data v_0, we approximate the evolution orbit $\{v(t),\ t>0\}$ by the approximate solution $V(t)=\{V_i(t),\ i\in I=\{1,2,...,N\}\}$ defined on the finite moving mesh (MM) $\Omega(V)=\{X_i(t),\ i\in I;\ t>0\}$ and satisfying a system of the ODE's (the moving mesh equation, MME in short) which gives the moving mesh operator (MMO) $\mathbf{B_h}$ in the internal points $i\in I_m\subset I$:

$$(35)\qquad [\mathbf{B_h}(V)]_i = V_i' - D_hVX_i' - \mathbf{A_h}(V) = 0.$$

We assume that initial data are prescribed at $t=0$ and boundary conditions are given on the lateral boundary $\Gamma(V)$ of $\Omega(V)$, $i\in I_m\subset I$ (if $S\neq\mathbf{R}$), so that the solution V is well defined at least locally in time on $\Omega(V)$. For a fixed constant $\tau\geq 0$, we denote $\Omega_\tau(V)=\Omega(V)\cap\{t=\tau\}$. By $\partial\Omega(V)$ we denote the parabolic boundary of $\Omega(V)$ where both initial and boundary conditions are prescribed.

We mainly study invariant properties of the MMO $\mathbf{B_h}$. We do not take into account the mechanism of generation of the moving mesh $\Omega(V)$ which is an important step in construction of moving mesh methods, [13], [14], see also applications to blow-up problems in [1], [2]. We construct MME's with *arbitrary* MM's $\Omega(V)$, which admit solutions on the invariant subspaces of the discrete nonlinear operators $\mathbf{A_h}$. A special choice of MM's can improve invariant properties of the MMMs, and then we would say about invariant properties of the MMMs in a whole. There are several examples when the requirement of an optimal approximation of invariant subspaces (forming a stable attractor of the given flow) plays an important role in the correct description of evolution properties of PDEs, and can be the decisive principle of construction of the corresponding MMOs. See examples below.

We restrict our attention to the problem on invariant subspaces of the MMO (35). In general, we mainly will deal with two-dimensional invariant subspaces introduced in Section 2.1.

Theorem 1 *Assume that given moving mesh $\Omega(V)$, the discrete operator $\mathbf{A_h}$ on $\Omega(V)$ admits the two-dimensional subspace $W_{2h}=\mathbf{L}\{1,\phi(x)\}$, with a smooth function ϕ. Let the discrete gradient operator satisfy*

$$(36)\qquad [D_h\phi(X)]_i = \phi'(X_i) \quad \text{on } \Omega(V).$$

Then $\mathbf{B_h}: W_{2h}\to W_{2h}$ and the MME (35) on $\Omega(V)$ is equivalent to a nonlinear two-dimensional dynamical system which coincides with (29) for the corresponding fixed mesh.

A natural symmetric representation of the gradient operator satisfying (36) is

$$[D_h V]_i = \frac{V_{i+1} - V_{i-1}}{\phi(X_{i+1}) - \phi(X_{i-1})}\, \phi'(X_i). \tag{37}$$

On the mesh $|X_{i+1} - X_{i-1}| = O(h)$, it approximates dv/dx, with the order $O(h)$ at least.

Proof of Theorem 1. We are looking for a solution to (35) in the form

$$V_i = V(X_i(t), t) = C_0(t) + C_1(t)\phi(X_i(t)) \in W_{2h}. \tag{38}$$

Substituting (38) into (35), we obtain

$$C_0' + C_1'\phi(X_i) + C_1\{\phi'(X_i) - [D_h\phi(X)]_i\}X_i' = \mathbf{A_h}(C_0 + C_1\phi(X)) \in W_{2h}. \tag{39}$$

In view of (36), the term depending on X_i' on the left-hand side of (39) vanishes identically, and due to (27) we arrive at the DS (29). ■

In the case when group-invariant solutions of (1) are available, belong to W_{2h} and are preserved by the discretisation $\mathbf{A_h}$, Theorem 1 guarantees that they are correctly described by any MMO with the discrete gradient approximation (37).

In the next sections we consider evolution equations with the above nonlinear operators $\mathbf{A_1} - \mathbf{A_9}$ from (10) - (15) and show how to construct a suitable discretisation $\mathbf{A_h}$ of the operators to get solutions of the MMEs on the invariant subspaces.

3 Invariant subspace $\mathbf{L\{1, x^m\}}$

Examples of such invariant subspaces with $m = 2, 3, 4$ are given in Section 2.1. We begin with the simplest example of nonlinear invariance for MMOs.

3.1 Semilinear operator $\mathbf{A_1}$

Consider equation (1), (10):

$$v_t = v_{xx} + (v_x)^2 + v. \tag{40}$$

The corresponding MME (35) has the form

$$V_i' - D_h V X_i' = \mathbf{A_{1h}}(V) \equiv V_{XX} + (V_X)^2 + V_i \quad \text{on } \Omega(V), \tag{41}$$

with the approximation $\mathbf{A_{1h}}$ to be determined later. In view of (37) with $\phi(x) = x^2$, the gradient operator is given by

$$[D_h V]_i = \frac{V_{i+1} - V_{i-1}}{X_{i+1}^2 - X_{i-1}^2}\, 2X_i, \tag{42}$$

and by (36) $[D_h X^2]_i \equiv 2X_i$. Consider now the discrete operator $\mathbf{A_{1h}}$. One can see that it admits W_2 if the discrete derivatives are given by

$$V_X = D_h V, \quad V_{XX} = d_h D_h V, \tag{43}$$

where d_h denotes the standard symmetric gradient approximation

$$d_h V = \frac{V_{i+1} - V_{i-1}}{X_{i+1} - X_{i-1}}. \tag{44}$$

Then

$$d_h D_h X^2 \equiv 2. \tag{45}$$

Finally, we obtain that given the mesh function $V = C_0(t) + C_1(t)X^2 \in W_2$ we have

$$\mathbf{A_{1h}}(V) = C_0 + 2C_1 + C_1(1 + 4C_1)X^2 \in W_2, \tag{46}$$

so that (41) on W_2 is equivalent to the DS (23), which is easily solved explicitly. Then the discrete evolution on W_2 coincides with the differential one given by the PDE (40) on W_2.

Remark: on conditionally invariant MMOs. Observe that, in general, the natural approximation $V_{XX} = D_h^2 V$ does not satisfy (45). One can calculate that

$$[D_h^2 X^2]_i = \frac{4X_i}{X_{i+1} + X_{i-1}}, \tag{47}$$

and therefore taking the condition $D_h^2 X^2 \equiv 2$ as one of the generating of the MM's, we arrive at the hypotheses $X_i = (X_{i+1} + X_{i-1})/2$, i.e., the MM has to be uniform. Hence, on any uniform MM $\Omega(V)$ the MME (41) with $V_{XX} = D_h^2 V$ admits solutions on W_2. This MME is conditionally invariant (not for any MM). On the other hand, the invariance of the standard approximation $V_{XX} = d_h^2 V$ implies that

$$[d_h d_h X^2]_i \equiv \frac{X_{i+2} - X_{i-2}}{X_{i+1} - X_{i-1}} = 2, \tag{48}$$

which is true for uniform MMs $\Omega(V)$. Finally, the invariant MMO (41) leaves invariant the extended subspace $W_3 = \mathbf{L}\{1, x, x^2\}$ on any uniform mesh since the discrete gradient operator differentiates correctly the linear function x on such meshes:

$$[D_h X]_i = \frac{2X_i}{X_{i+1} - X_{i-1}} = 1.$$

3.2 Porous medium operators $\mathbf{A_3}$, $\mathbf{A_7}$ and $\mathbf{A_9}$

Consider the equation

$$v_t = vv_{xx} + \frac{1}{\sigma}(v_x)^2 + \alpha v + \beta \quad \text{on } W_2 = \mathbf{L}\{1, x^2\}, \quad t > 0. \tag{49}$$

Using (42) and (43), we introduce the MME

$$V_i' - D_h V X_i' = \mathbf{A_{3h}}(V) \equiv V d_h D_h V + \frac{1}{\sigma}(D_h V)^2 + \alpha V_i + \beta \quad \text{on } \Omega(V), \tag{50}$$

which admits the solution

$$V = C_0(t) + C_1(t) X^2, \tag{51}$$

where the coefficients $\{C_0, C_1\}$ solve the same DS as in the differential case:

$$C_0' = 2C_0 C_1 + \alpha C_0 + \beta, \quad C_1' = \frac{2(\sigma+2)}{\sigma} C_1^2 + \alpha C_1, \ \ t > 0. \tag{52}$$

The invariance result is true for the extension W_3 on any uniform mesh. The subspace W_2 remains invariant if we add any linear differential operator $\gamma v_{xx} + \delta v_x : W_2 \to W_2$ to the right-hand side of (49).

3.3 The fourth-order operator $\mathbf{A_6}$

For the quadratic operator (15), the invariant approximation of the gradient operator is

$$[D_h V]_i = \frac{V_{i+1} - V_{i-1}}{X_{i+1}^4 - X_{i-1}^4} \, 3X_i^3.$$

Then the discrete DS is the same as the PDE $v_t = \mathbf{A_6}(v)$ on W_2. Both the continuous and discrete DS have the form: $v = C_0 + C_1 x^4$, $V = C_0 + C_1 X^4$ and

$$C_0' = -24 C_0 C_1 + \gamma C_0 + \delta, \ \ C_1' = (-24 + 96\alpha + 144\beta) C_1^2 + \gamma C_1, \quad t > 0.$$

Any operator $\mathbf{A_6} + L$ with a linear higher-order operator L with constant coefficients admits the same invariant subspace. It can be extended for uniform MMs.

4 Invariant subspace $\mathbf{L}\{1, \cos x\}$

We continue to construct invariant MMOs which preserve continuous dynamics of the PDEs on such invariant subspaces.

4.1 Semilinear operator $\mathbf{A_2}$

The MME corresponding to the evolution equation

$$v_t = v_{xx} + (v_x)^2 + v^2 \quad \text{on } W_2 = \mathbf{L}\{1, \cos x\}, \ \ t > 0, \tag{53}$$

has the form

$$V_i' - D_h V X_i' = \mathbf{A_{2h}}(V) \equiv V_{XX} + (V_X)^2 + V_i^2 \quad \text{on } \Omega(V), \tag{54}$$

where by (37)

$$[D_h V]_i = \frac{V_{i+1} - V_{i-1}}{\cos(X_{i+1}) - \cos(X_{i-1})}\,(-\sin(X_i)). \tag{55}$$

Then, by (36) $[D_h \cos X]_i = -\sin(X_i)$, and we take $V_X = D_h V$. In order to keep invariant properties of the second derivative, we take the approximation

$$V_{XX} = \overline{D}_h D_h V, \quad \text{with } [\overline{D}_h V]_i = \frac{V_{i+1} - V_{i-1}}{\sin(X_{i+1}) - \sin(X_{i-1})} \cos(X_i). \tag{56}$$

Therefore, $[\overline{D}_h D_h \cos X]_i = -\cos(X_i)$. Finally, we conclude that the invariant MMO (54) preserves the continuous evolution on W_2. For the invariant solutions

$$V = C_0(t) + C_1(t) \cos X \in W_2, \tag{57}$$

the DS is exactly the same as for the PDE (53):

$$C_0' = C_0^2 + C_1^2, \quad C_1' = -C_1 + 2C_0C_1, \ \ t > 0. \tag{58}$$

The DS cannot be solved explicitly, but is easily studied on the phase-plane. Equation (53) describes regional blow-up where blow-up sets of bell-shaped solutions have the measure 2π, see p. 295 in [17] and [12]. On any uniform MM, there hold

$$[D_h \sin X]_i = \cos X_i, \quad [D_h \cos X]_i = -\sin X_i,$$

so that the operator $\mathbf{A_{2h}}$ with $V_X = D_h X$, $V_{XX} = D_h^2 V$ admits the extended invariant subspace $W_3 = \mathbf{L}\{1, \cos x, \sin x\}$.

4.2 Quasilinear operators $\mathbf{A_4}$ and $\mathbf{A_5}$

As the next example, we consider equation (1), (13) in the case when $\mathbf{A_4}$ admits the subspace $W_2 = \mathbf{L}\{1, \cos x\}$:

$$\lambda \equiv \frac{\sigma}{\sqrt{\sigma + 1}} = 1, \quad \text{i.e.,} \ \ \sigma = \sigma_{\pm} = \frac{1}{2}(1 \pm \sqrt{5}) \tag{59}$$

($-\sigma_-$ is the Golden Mean). The MME is

$$V_i' - D_h V X_i' = \mathbf{A_{4h}}(V) \equiv V_i \overline{D}_h D_h V + \frac{1}{\sigma}(D_h V)^2 + \sigma V_i^2 \quad \text{on } \Omega(V), \tag{60}$$

where the derivatives are given in (55), (56). This MME exactly reproduces the continuous solutions of the form (57) and the corresponding DS has the form

$$C_0' = \sigma C_0^2 + \frac{1}{\sigma} C_1^2, \quad C_1' = (2\sigma - 1) C_0 C_1, \ \ t > 0. \tag{61}$$

On uniform MMs the MMO approximates the corresponding extended subspace $W_3 = \mathbf{L}\{1, \cos x, \sin x\}$. A similar analysis applies to the Kuramoto-Sivashinskii equation $v_t = \mathbf{A_5}(v)$.

5 Higher-order evolution equations

We now study invariant properties of equation (5). Setting

$$\partial_t^j v = u_j, \quad j = 1, ..., k-1, \tag{62}$$

we reduce it to the system of PDEs

$$v_t = u_1, \quad (u_j)_t = u_{j+1}, \quad j = 1, ..., k-2, \tag{63}$$

$$(u_{k-1})_t = A(v, u_1, ..., u_{k-1}, v_x, ..., \partial_x^m v). \tag{64}$$

Introducing the variable $W = (v, u_1, ..., u_{k-1}) \in \mathbf{R}^k$, we arrive at the evolution equation

$$W_t = \tilde{\mathbf{A}}(W), \quad t > 0. \tag{65}$$

The corresponding MME for W takes the standard form in $\mathbf{R}^k$ similar to (35):

$$W_i' - D_h W X_i' = \tilde{\mathbf{A}}_{\mathbf{h}}(W). \tag{66}$$

Checking the invariant property of the gradient operator, we deduce that Theorem 1 is true and the gradient approximation (37) preserves invariance of subspace W_2.

Example 5. Consider the Boussinesq-type equation with the operator $\mathbf{A_3}$

$$v_{tt} = v v_{xx} + \frac{1}{\sigma}(v_x)^2 + \alpha v + \beta.$$

It is equivalent to the system

$$v_t = u, \quad u_t = \mathbf{A_3}(v) + \alpha v + \beta,$$

and the corresponding MME has the form

$$V_i' - D_h V_i X_i' = U_i, \quad U_i' - D_h U_i X_i' = \mathbf{A_{3h}}(V) + \alpha V_i + \beta.$$

We define the gradient operator by (42), and then by Theorem 1, the MMO admits the invariant subspace $W_2 = \mathbf{L}\{1, x^2\}$.

6 Positivity properties

6.1 Weak Maximum Principle

We return back to the MMO (35). We consider second-order equations of parabolic type and assume a natural three point approximation of the nonlinear operator $\mathbf{A}$:

$$[\mathbf{A_h}(V)]_i = A_h(V_{i-1}, V_i, V_{i+1}, i, X).$$

Theorem 2 *Let $V(t) \geq 0$ on $\partial\Omega(V)$. Then the inequality*

$$(67) \qquad [D_h V X_i' + \mathbf{A_h}(V)]|_{V_i=0} \geq 0, \quad i \in I_2 = \{2, ..., N-1\}, \quad V_j \geq 0, \; i \neq j, \; t > 0,$$

is necessary and sufficient for the positivity of $V(t)$ with arbitrary nonnegative data on $\partial\Omega(V)$.

Proof. From (35) we obtain that at any instant, when $V_i = 0$ at an internal point of $\Omega(V)$, we have to have that $V_i' \geq 0$, whence the result. ∎

The proof applies also to other homogeneous boundary conditions like the Neumann ones:

$$(68) \qquad V_X = 0 \quad \text{on the lateral boundary of } \Omega(V).$$

The criterion in the theorem can be easily checked for invariant or standard MMOs.

Example 6. Consider a semilinear parabolic equation

$$v_t = v_{xx} + f(x, t, v, v_x),$$

where we assume that $f(x, t, 0, v_x) \equiv 0$. We take the standard MME on the MM $\Omega(V)$:

$$V_i' - D_h V X_i' = V_{XX} + f(X_i, t, V_i, V_X).$$

The difference operators are given by: with $h_i = X_{i+1} - X_i > 0$

$$V_X \equiv [D_h V]_i = \frac{V_{i+1} - V_{i-1}}{h_i + h_{i-1}}, \quad V_{XX} = \frac{2}{h_i + h_{i-1}} \left(\frac{V_{i+1} - V_i}{h_i} - \frac{V_i - V_{i-1}}{h_{i-1}} \right).$$

Then the condition (67) reads

$$\frac{V_{i+1} - V_{i-1}}{h_i + h_{i-1}} X_i' + \frac{2}{h_i + h_{i-1}} \left(\frac{V_{i+1}}{h_i} + \frac{V_{i-1}}{h_{i-1}} \right) \geq 0$$

or

$$V_{i+1} \left(X_i' + \frac{2}{h_i} \right) + V_{i-1} \left(\frac{2}{h_{i-1}} - X_i' \right) \geq 0$$

for any $V_{i\pm1} \geq 0$. This gives the *criterion* of the (internal) positivity of the solution

$$-\frac{2}{h_i} \leq X_i' \leq \frac{2}{h_{i-1}}, \quad i \in I_2,\ t > 0. \tag{69}$$

Both inequalities are valid if

$$|X_i'(t)| \leq 2 \min \left\{ \frac{1}{h_i},\ \frac{1}{h_{i-1}} \right\}, \quad i \in I_2,\ t > 0, \tag{70}$$

i.e., the MM does not move very fast. Thus, any MM moving sufficiently slow preserves positivity of the solution as the stationary mesh does. This is a natural continuity property of discrete parabolic operators describing the transitional behaviour from fixed meshes to MMs.

6.2 Comparison principle

It is a consequence of the positivity result.

Theorem 3 *Let V and W be two solutions of the MME with $\Omega(V) = \Omega(W)$. Let $V \geq W$ on $\partial\Omega$. The inequality*

$$[D_h(V-W)X_i' + \mathbf{A_h}(V) - \mathbf{A_h}(W)]|_{V_i=W_i} \geq 0, \quad i \in I_2, \quad V_j \geq W_j,\ i \neq j,\ t > 0, \tag{71}$$

is necessary and sufficient for the comparison $V \geq W$ in Ω for arbitrary ordered data on $\partial\Omega$.

Proof is straightforward and the result is equivalent to the positivity of the difference $V - W \geq 0$ in Ω satisfying a linearised MME.

Since the proof is essentially of the interior nature, comparison is true for the homogeneous Neumann conditions or monotone nonlinear Neumann conditions preserving ordering of solutions on the lateral boundary.

Example 7. Consider the semilinear equation

$$v_t = v_{xx} + f(x,t,v).$$

Taking the standard MME as in Example 6, one obtains that (70) guarantees comparison of solutions of this MME on any such coinciding MMs.

6.3 On asymptotic properties of MMEs

The standard comparison of different solutions V and W assumes that the corresponding MMs, $\Omega(V)$ and $\Omega(W)$, coincide. Since the MMs essentially depends on the solutions, in general, this makes comparison to be a delicate subject. Nevertheless, using invariant MMOs makes such a comparison not only possible but gives also an effective way to study asymptotic properties of discrete solutions. By invariant MMOs, we actually compare general solutions with exact invariant solutions taken on the same MMs. We state the main conclusion based on both invariance Theorem 1 and comparison Theorem 3.

Theorem 4 *Let the hypotheses of Theorem 1 hold. Fix a solution V on $\Omega(V)$ and an exact invariant invariant solution $W(t) \in W_{2h}$ such that*

$$V \geq W \;\; (V \leq W) \quad \text{on } \partial\Omega(V). \tag{72}$$

Assume that the hypothesis (71) is true for these solutions defined on the same MM $\Omega = \Omega(V)$. Then

$$V \geq W \;\; (V \leq W) \quad \text{in } \Omega(V). \tag{73}$$

Proof is straightforward. Consider an example.

Example 8: comparison and the asymptotic behaviour. Consider the semilinear equation (22). Let V be a discrete solution of the MME on a MM $\Omega(V)$:

$$V_i' - D_h V X_i' = V_{XX} + (V_X)^2 + V_i,$$

with the gradient approximation D_h given in (42) and invariant approximations (43) of the derivatives V_X and V_{XX}. The quadratic operator $\mathbf{A_{1h}}$ on the right-hand side admits the same invariant subspace as in (10) with the basic function $\phi(X) = X^2$. Let $W(t) \in W_2$ be an exact invariant solution. By Theorem 1, it satisfies the same MME on the MM $\Omega(V)$. The comparison condition then reduces to

$$(V_{i+1} - W_{i+1}) \left[\Delta_i X_i' + \frac{1}{H_i h_i} + \frac{1}{2H_i}(V_X + W_X) \right]$$

$$+(V_{i-1} - W_{i-1}) \left[-\Delta_i X_i' + \frac{1}{H_i h_{i-1}} - \frac{1}{2H_i}(V_X + W_X) \right] \geq 0$$

for all $V_{i\pm1} \geq W_{i\pm1}$, where

$$\Delta_i = \frac{\phi'(X_i)}{\phi(X_{i+1}) - \phi(X_{i-1})}, \;\; h_i = X_{i+1} - X_i, \;\; H_i = \frac{1}{2}(h_i + h_{i-1}).$$

This gives the following condition of comparison:

$$|H_i \Delta_i X_i'(t)| \le \min\left\{\frac{1}{h_i}, \frac{1}{h_{i-1}}\right\} - \frac{1}{2}\sup_{\Omega_t}(|V_X + W_X|), \tag{74}$$

i.e., for sufficiently regular solutions with bounded derivatives V_X, W_X, the MM does not move faster than $O(1/h)$ (a condition similar to the positivity condition (70)).

The above comparison is true for any invariant solution W. Choose the exact solution

$$W(x,t) = C_0(t) + C_1(t)x^2 \in W_2,$$

where the coefficients $\{C_0, C_1\}$ satisfy the DS (23). Assume that $V \ge W$ on the lateral boundary of $\Omega(V)$ so that $V(X,0) \ge W(X,0) \equiv C_0(0) + C_1(0)X^2$ in $\Omega_0(V)$. Then by comparison

$$V(X,t) \ge W(X,t) \equiv C_0(t) + C_1(t)X^2 \quad \text{in } \Omega(V),$$

provided that the MM satisfies the above comparison condition. A similar estimate from below is also proved. Finally, solving the DS (23), we obtain the exact asymptotic behaviour as $t \to \infty$ of a wide class of solutions of this invariant MME. We then conclude that as $t \to \infty$, the discrete solutions converge to the principle separable self-similar solution as in the continuous case described in [17], p. 93.

More examples on comparison with invariant solutions and the asymptotic behaviour can be constructed for equations and invariant approximations of other quadratic operators.

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