Controlled Functional Differential Equations and Adaptive Stabilization*

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Abstract. Adaptive control of a class $\mathcal{N}$ of single-input, single-output systems described by nonlinear functional differential equations is considered. An $\mathcal{N}$-universal stabilizer, in the form of an adaptive discontinuous output feedback strategy incorporating gains of Nussbaum type, is developed within an analytic framework of functional differential inclusions.

Keywords. Adaptive control, functional differential inclusions, nonsmooth feedback, universal stabilization.

*Based on work supported, in part, by the UK EPSRC (Grant GR/L78086)

1 Introduction

By way of motivation, consider the class $\mathcal{L}$ of finite-dimensional, linear, minimum-phase, relative-degree-one, single-input $u(\cdot)$, single-output $y(\cdot)$ systems,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad CB \neq 0$$

with $x(t) \in \mathbb{R}^n$ and $u(t), y(t) \in \mathbb{R}$. Under a suitable coordinate transformation (see, for example, Ilchmann (1993, Lemma 2.1.3)), every system in $\mathcal{L}$ can be expressed in the form of two coupled subsystems

$$\left.\begin{array}{ll} \Sigma_1: & \dot{y}(t) = A_1 y(t) + A_2 z(t) + bu(t) \\ \Sigma_2: & \dot{z}(t) = A_3 y(t) + A_4 z(t) \end{array}\right\} \tag{1}$$

with $y(t), u(t) \in \mathbb{R}$, $z(t) \in \mathbb{R}^{n-1}$, $b := CB \neq 0$ and with A_4 having spectrum in the open left half complex plane.

As is well known (see, for example, Nussbaum (1983), Morse (1984) and Willems & Byrnes (1984)), the following output feedback strategy is an $\mathcal{L}$-universal stabilizer (in the sense that for each member of $\mathcal{L}$, every solution of the feedback control system is such that $(y(t), z(t)) \to (0,0)$ as $t \to \infty$ and $k(\cdot)$ is bounded):

$$u(t) = \nu(k(t))y(t), \quad \dot{k}(t) = y^2(t), \quad k(0) = k^0, \tag{2}$$

where $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function (for example, $\nu : w \mapsto w^2 \cos w$) with the properties

$$\text{(a)} \quad \limsup_{\eta \to \infty} \frac{1}{\eta} \int_0^\eta \nu = +\infty, \qquad \text{(b)} \quad \liminf_{\eta \to \infty} \frac{1}{\eta} \int_0^\eta \nu = -\infty\,. \tag{3}$$

System (1) can be represented by the schematic shown in Figure 1.

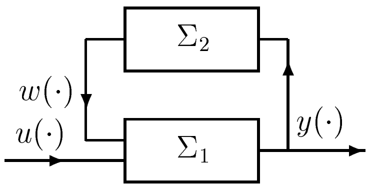

Figure 1: The generic system

The dynamic block Σ_1, which can be influenced directly by the controller, is also driven by the output $w(\cdot) = A_2 z(\cdot)$ from the dynamic block Σ_2. Viewed abstractly, the block Σ_2 can be regarded as a linear operator $\hat{T}_{\text{lin}}$ which maps $y(\cdot)$ to $w(\cdot) = A_2 z(\cdot)$, and hence system (1) can be interpreted as

$$\dot{y}(t) = A_1 y(t) + (\hat{T}_{\text{lin}} y)(t) + bu(t),$$

which, under adaptive feedback control (2), may be expressed as

$$\dot{x}(t) - (Tx)(t) = f(x(t)), \quad x(t) := (y(t), k(t)) \in \mathbb{R}^2, \tag{4}$$

where $(Tx)(t) = ((\hat{T}_{\text{lin}}y)(t), 0)$ and $f : x = (y, k) \mapsto (A_1 y + b\nu(k)y, y^2)$.

These observations provide the prototype for the class of nonlinear systems to be studied in this paper. In particular, we consider a class $\mathcal{N} \supset \mathcal{L}$ of nonlinear, single-input $u(\cdot)$, single-output $y(\cdot)$ systems $(b, g, p, \hat{T})$, having the same structure as in Figure 1, given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = g(p(t), y(t)) + (\hat{T}y)(t) + bu(t). \tag{5}$$

For such systems, we address the stabilization problem of feedback control to ensure $y(t) \to 0$ as $t \to \infty$. The nature of admissible nonlinearities g and nonlinear operators $\hat{T}$ will be made precise in due course: however, as a simple illustration, the nonlinear delay system

$$\dot{y}(t) = p(t)y^{1/3}(t) + q_1(t)y(t-h_1) + q_2(t)y^2(t-h_2) + \int_{-h_3}^{0} q_3(s)y^3(t+s)ds + bu(t),$$

with unknown real parameters $h_i \geq 0$, $b \neq 0$, and unknown functions $p, q_i \in L^\infty(\mathbb{R})$, falls within the scope of our results.

In order to achieve the stabilization objective in the context of nonlinear systems of class $\mathcal{N}$, we will investigate nonsmooth generalizations of (2) which incorporate a discontinuous feedback control action. Lack of smoothness places a functional differential *equation* model of the feedback system outside the scope of the classical theory: instead, an analytical framework of functional differential *inclusions* is developed in which the nonsmooth counterpart of (4) takes the form

$$\dot{x}(t) - (Tx)(t) \in F(x(t)).$$

In Section 2 we present an existence theorem for initial-value problems for functional differential inclusions of the above type. This result provides a basis for the analysis and design of the adaptive stabilizer of Section 3 (an extension, to controlled functional differential equations, of the adaptive stabilizer developed in Ryan (1998, Section 3.1) in a context of controlled ordinary differential equations).

We conclude this introduction with some remarks on notation. For $I \subset \mathbb{R}$, $C(I; \mathbb{R}^N)$ (respectively, $AC(I; \mathbb{R}^N)$) denotes the set of continuous (respectively, absolutely continuous) functions $I \to \mathbb{R}^N$. For $x : I \to \mathbb{R}^N$, the restriction of x to $J \subset I$ is denoted by $x|_J$, an extension of x to $\mathbb{R}$ is denoted by x^e. The open ball of radius $r > 0$, centred at $c \in \mathbb{R}^N$ is written as $\mathbb{B}_r(c)$. $L^p_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$ denotes the space of locally L^p-functions $x : \mathbb{R} \to \mathbb{R}^N$ (that is, for every compact $I \subset \mathbb{R}$, $x|_I \in L^p(I; \mathbb{R}^N)$). We write $\mathbb{R}_+ := [0, \infty)$. $\mathcal{K}$ denotes the class of continuous and strictly increasing functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, with $\gamma(0) = 0$. $\mathcal{K}_\infty$ will denote the subclass of functions $\gamma \in \mathcal{K}$ for which $\gamma(t) \to \infty$ as $t \to \infty$. $\mathcal{KL}$ is the class of functions $\gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that for each $t \in \mathbb{R}_+$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$ and for each $s \in \mathbb{R}_+$, $\gamma(s, \cdot)$ is decreasing with $\gamma(s, t) \to 0$ as $t \to \infty$.

2 Functional differential inclusions

With a view to providing a framework appropriate to the analysis of the adaptive stabilizer of Section 3, this section constitutes a study of functional differential inclusions of the form

$$\dot{x}(t) - (Tx)(t) \in F(x(t)), \quad \text{for a.a. } t \in \mathbb{R}_+ , \tag{6}$$

with prescribed initial data

$$x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right). \tag{7}$$

Definitions 1 and 4 below describe the underlying classes of maps F and operators T.

Definition 1. *For $N \in \mathbb{N}$, let $\mathcal{F}^N$ denote the space of set-valued maps $x \mapsto F(x) \subset \mathbb{R}^N$ that (i) are upper semicontinuous on $\mathbb{R}^N$, and (ii) take non-empty, convex, compact values.*

The following facts are well known (see, for example, Aubin & Cellina (1984, Proposition 3, page 41) and Aubin & Cellina (1984, Theorem 1, page 84)).

Proposition 2. *Let $F \in \mathcal{F}^N$. For each $\epsilon > 0$, there exists a locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $\operatorname{graph}(f) \subset \operatorname{graph}(F) + \mathbb{B}_\epsilon(0)$. (Here, $\mathbb{B}_\epsilon(0)$ is the open ball of radius ϵ centred at $0 \in \mathbb{R}^N \times \mathbb{R}^N$.)*

Proposition 3. *Let $F \in \mathcal{F}^N$. If $K \subset \mathbb{R}^N$ is compact, then $F(K) := \cup_{x\in K} F(x)$ is compact.*

Definition 4. *For $h \geq 0$ and $N \in \mathbb{N}$, let $\mathcal{T}_h^N$ denote the space of operators $T : L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ with the following properties.*

1. *For every $R > 0$ and every bounded interval I, there exists $M > 0$ such that, for all $x \in L^\infty(\mathbb{R};\mathbb{R}^N)$,*

$$\|x\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} < R \quad \Longrightarrow \quad \|(Tx)(t)\| < M \quad \textit{for a.a. } t \in I .$$

2. *For all $\alpha \geq 0$, the following hold:*

   (a) for all $x, \xi \in L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$,

$$x(s) = \xi(s) \textit{ for a.a. } s \in [-h,\alpha] \quad \Longrightarrow \quad (Tx)(s) = (T\xi)(s) \textit{ for a.a. } s \in [0,\alpha];$$

   (b) for all continuous $\zeta : [-h,\alpha] \to \mathbb{R}^N$, there exist $\tau > 0$, $r > 0$ and $c > 0$ such that, for all $x, \xi \in L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,\alpha]} = \zeta = \xi|_{[-h,\alpha]}$ and $x(t), \xi(t) \in \mathbb{B}_r(\zeta(\alpha))$ for all $t \in [\alpha, \alpha+\tau]$,

$$\text{ess-sup}_{t\in[\alpha,\alpha+\tau]} \|(Tx)(t) - (T\xi)(t)\| \leq c\, \text{ess-sup}_{t\in[\alpha,\alpha+\tau]} \|x(t) - \xi(t)\| .$$

Remarks 5.
(i) The essence of Property 2(a) in Definition 4 is that every $T \in \mathcal{T}_h^N$ is causal and has bounded memory.

(ii) Let $\alpha \geq 0$ and $x \in L^\infty([-h,\alpha);\mathbb{R}^N)$. Let x^e be any class $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ extension of x. The function $Tx^e|_{[0,\alpha)}$ is uniquely determined by x in the sense that, by virtue of Property 2(a), the former is independent of the extension x^e chosen for the latter. On this basis, we will adopt the following notational convention: for $t \in [0,\alpha)$, we simply write $(Tx)(t)$ in place of the notationally cumbersome $(Tx^e)(t)$ (where x^e is any class $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ extension of x).

(iii) $\mathcal{T}_h^N$ is a linear space: $T_1, T_2 \in \mathcal{T}_h^N \implies (a_1T_1 + a_2T_2) \in \mathcal{T}_h^N \quad \forall\, a_1, a_2 \in \mathbb{R}$.

(iv) If $h_1 > h_2$ then $\mathcal{T}_{h_1}^N \supset \mathcal{T}_{h_2}^N$.

(v) For $s \in \mathbb{R}$, let S_s denote the shift operator on functions $\mathbb{R} \to \mathbb{R}^N$ given by $(S_sx)(t) := x(t+s)$ for all $t \in \mathbb{R}$. Then

$$T \in \mathcal{T}_h^N \quad \implies \quad S_sTS_{-s} \in \mathcal{T}_{h+s}^N \quad \forall\, s \geq 0\,. \tag{8}$$

2.1 Examples

We provide some concrete examples which serve to illustrate that the class $\mathcal{T}_h^N$ subsumes many commonly encountered systems and devices.

2.1.1 Nonlinear delay elements

We first introduce a notational convenience.

Definition 6. *Let $\mathcal{D}^N$ denote the class of functions $(t,x) \mapsto \Psi(t,x)$, $\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ that are measurable in t and locally Lipschitz in x uniformly with respect to t: precisely, (i) for each fixed x, $\Psi(\cdot,x)$ is measurable and (ii) for every compact $K \subset \mathbb{R}^N$ there exists a constant k such that*

$$\text{for a.a. } t, \quad \|\Psi(t,x) - \Psi(t,y)\| \leq k\|x-y\| \quad \forall\, x,y \in K\,.$$

For $i = 0,...,n$, let $\Psi_i \in \mathcal{D}^N$ and $h_i \in \mathbb{R}_+$. Define $h := \max_i h_i$. For $x(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$, let

$$(Tx)(t) := \int_{-h_0}^0 \Psi_0(s, x(t+s))\, ds + \sum_{i=1}^n \Psi_i(t, x(t-h_i))\,. \tag{9}$$

The operator $T : L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$, so defined, is evidently of class $\mathcal{T}_h^N$.

2.1.2 Regular linear systems with bounded observation operator

Let $\mathbf{G}$ be the transfer function of a regular (in the sense of Weiss (1994)), linear system with state space X (a Hilbert space), with generating operators (A, B, C, D) and with $\mathbb{R}^N$-valued input and output. This means, in particular, that (i) A generates a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t \geq 0}$ of bounded linear operators on X, (ii) the control operator B is a bounded linear operator from $\mathbb{R}^N$ to X_{-1}, (iii) the observation operator C is a bounded linear operator from X_1 to $\mathbb{R}^N$, and (iv) the feedthrough operator D is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$. Here X_1 denotes the space $\mathrm{dom}(A)$ (the domain of A) endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - A)^{-1} z\|$, where s_0 is any fixed element of the resolvent set of A and $\|\cdot\|$ denotes the norm on X. As a regular linear system, the transfer function $\mathbf{G}$ is holomorphic and bounded on every half-plane $\mathbb{C}_\alpha$ with $\alpha > \omega(\mathbf{S}) := \lim_{t\to\infty} t^{-1} \ln \|\mathbf{S}_t\|$. Moreover,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} \mathbf{G}(s) = D .$$

The system is said to be exponentially stable if the semigroup $\mathbf{S}$ is exponentially stable, that is, if $\omega(\mathbf{S}) < 0$.

We assume that the observation operator C can be extended to a bounded linear operator from X to $\mathbb{R}^N$: this extended operator is again denoted by C and is the bounded observation operator referred to in the subsection title.

In terms of the generating operators (A, B, C, D), the transfer function $\mathbf{G}$ is given by

$$G(s) = C(sI - A)^{-1} B + D .$$

For any $z^0 \in X$ and input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^N)$, the state $z(\cdot)$ and the output $w(\cdot)$ of the regular system (with bounded observation operator) satisfy the equations

$$\dot{z}(t) = Az(t) + Bu(t) , \quad z(0) = z^0 , \tag{10}$$

$$w(t) = Cz(t) + Du(t) \tag{11}$$

for almost all $t \geq 0$. The derivative on the left-hand side of (10) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (10) in the space X_{-1}, then for any $z^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^N)$, (10) has unique strong solution given by the variation of parameters formula (see Theorem 2.9 (Chapter 4) in Pazy (1983))

$$z(t) = \mathbf{S}_t z^0 + \int_0^t \mathbf{S}_{t-s} Bu(s)\, ds . \tag{12}$$

Moreover, for every $\tau > 0$, there exists a constant $a_\tau > 0$ such that, for all $z^0 \in X$ and all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^N)$,

$$\|z(t)\| \leq a_\tau \left[\|z^0\| + \left(\int_0^t \|u(s)\|^2\, ds \right)^{\frac{1}{2}} \right] \quad \forall\, t \in [0, \tau] . \tag{13}$$

If the semigroup $\mathbf{S}$ is exponentially stable, then there exist constants $a, c_1, c_2 > 0$ such that

$$\|z(t)\| \le a \left[\|z^0\| + \left(\int_0^t \|u(s)\|^2 \, ds \right)^{\frac{1}{2}} \right], \quad \forall \, (t, z^0, u) \in \mathbb{R}_+ \times X \times L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N). \tag{14}$$

$$\|z\|_{L^2(\mathbb{R}_+;X)} \le c_1 \left[\|z^0\| + \|u\|_{L^2(\mathbb{R}_+;\mathbb{R}^N)} \right] \quad \forall \, (z^0, u) \in X \times L^2(\mathbb{R}_+; \mathbb{R}^N) \, . \tag{15}$$

$$\|z\|_{L^\infty(\mathbb{R}_+;X)} \le c_2 \left[\|z^0\| + \|u\|_{L^\infty(\mathbb{R}_+;\mathbb{R}^N)} \right] \quad \forall \, (z^0, u) \in X \times L^\infty(\mathbb{R}_+; \mathbb{R}^N) \, . \tag{16}$$

Define the operator $T : L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$ by

$$(Tu)(t) := \begin{cases} C \int_0^t \mathbf{S}_{t-s} B u(s) \, ds + D u(t), & t \ge 0 \\ 0, & t < 0 \end{cases} \tag{17}$$

and so $w(t) = C\mathbf{S}_t z^0 + (Tu)(t)$ for all $t \ge 0$. Note that, in view of (13) and boundedness of C, for each $\tau > 0$ there exists a_τ such that, for all $u \in L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$,

$$\|(Tu)(t)\| \le a_\tau \|C\| \left(\int_0^t \|u(s)\|^2 ds \right)^{\frac{1}{2}} + \|D\| \|u(t)\| \quad \text{for a.a. } t \in [0, \tau]$$
$$\text{and } \|(Tu)(t)\| = 0 \quad \forall t < 0. \tag{18}$$

In view of (18), Property 1 of Definition 4 evidently holds; setting $h = 0$, we see that Property 2(a) also holds and Property 2(b) is a consequence of linearity of T and (18). Therefore, the operator T is of class $\mathcal{T}_0^N$.

2.1.3 Input-to-state stable systems

Let $k : \mathbb{R}^P \times \mathbb{R}^N \to \mathbb{R}^P$ be locally Lipschitz with $k(0, 0) = 0$. For $u \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$, let $z(\cdot, z^0, u)$ denote the unique maximal solution of the initial-value problem

$$\dot{z}(t) = k(z(t), u(t)), \quad z(0) = z^0 \in \mathbb{R}^P. \tag{19}$$

Assume that the system is input-to-state stable (ISS) (Sontag 1989), that is, there exist functions $\theta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all $(z^0, u) \in \mathbb{R}^P \times L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^N)$,

$$\|z(t, z^0, u)\| \le \theta(\|z^0\|, t) + \text{ess-sup}_{s \in [0,t]} \gamma(\|u(s)\|) \quad \forall \, t \ge 0. \tag{20}$$

Let $W : \mathbb{R}^P \to \mathbb{R}^N$ be locally Lipschitz and such that there exists $L > 0$ such that $\|W(z)\| \le L\|z\|$ for all z. Assume system (19) has output w given by

$$w(t) = W(z(t, z^0, u)).$$

Fix $z^0 \in \mathbb{R}^P$ arbitrarily. Define the operator $T : L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N) \to L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N)$ by

$$(Tu)(t) := \begin{cases} W(z(t,z^0,u)), & t \ge 0 \\ 0, & t < 0\,. \end{cases} \tag{21}$$

In view of (20) and properties of W, there exists $c > 0$ such that, for all $u \in L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N)$,

$$\|(Tu)(t)\| \le c\left[1 + \text{ess-sup}_{s\in[0,t]}\gamma(\|u(s)\|)\right] \quad \forall\, t \ge 0\,. \tag{22}$$

To show that this operator* is of class $\mathcal{T}_0^N$ we argue as follows. Property 1 and Property 2(a) of Definition 4 evidently hold. To establish that Property 2(b) also we proceed as follows. Let $\alpha \ge 0$ and $\zeta \in C([-h,\alpha];\mathbb{R}^N)$. Let $\tau > 0$ and $r > 0$ and define $M := \|\zeta\|_{L^\infty([-h,\alpha];\mathbb{R}^N)} + r$. By input-to-state stability there exists a compact set $K \subset \mathbb{R}^P$ such that, for all u with $\|u\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} \le M$, we have $z(t,z^0,u) \in K$ for all $t \ge 0$. Let $\lambda > 0$ be a Lipschitz constant for $k(\cdot,\cdot)$ on the set $K \times \mathbb{B}_M(0)$. For all $x, \xi \in L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,\alpha]} = \zeta = \xi|_{[-h,\alpha]}$ and $\|u(t)\|, \|v(t)\| \le M$ for almost all $t \in [0, \alpha+\tau]$,

$$\begin{aligned}\|z(t,z^0,u) - z(t,z^0,v)\| &\le \int_0^t \|k(z(s,z^0,u),u(s)) - k(z(s,z^0,v),v(s))\|ds \\ &\le \lambda\int_\alpha^t [\|z(s,z^0,u) - z(s,z^0,v)\| + \|u(s)-v(s)\|]ds \quad \forall\, t \in [\alpha,\alpha+\tau]\,.\end{aligned}$$

By a version of Gronwall's Lemma (see, for example, Curtain & Pritchard (1977, Lemma 8.1)), it follows that

$$\|z(t,z^0,u) - z(t,z^0,v)\| \le \lambda\int_\alpha^t \exp(\lambda(t-s))\|u(s)-v(s)\|ds \quad \forall\, t \in [\alpha,\alpha+\tau]. \tag{23}$$

We may now conclude that there exists a constant $c_M > 0$ such that, for all $u, v \in L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,\alpha]} = \zeta = \xi|_{[-h,\alpha]}$ and $\|u(t)\|, \|v(t)\| \le M$ for almost all $t \in [\alpha, \alpha+\tau]$,

$$\|(Tu)(t) - (Tv)(t)\| \le c_M \text{ess-sup}_{t\in[\alpha,\alpha+\tau]}\|u(t)-v(t)\| \quad \forall\, t \in [\alpha,\alpha+\tau]. \tag{24}$$

Therefore, Property 2(b) of Definition 4 holds.

2.2 Existence of maximal solutions

We now focus on the question of existence of a solution of the initial-value problem

$$\left.\begin{array}{l} \dot{x}(t) - (Tx)(t) \in F(x(t)) \\ x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right) \end{array}\right\} \tag{25}$$

with $T \in \mathcal{T}_h^N$ and $F \in \mathcal{F}^N$. The special case wherein F is singleton valued, that is, $F(x) = \{f(x)\}$ for all x, yields an initial-value problem for a functional differential equation:

$$\left.\begin{array}{l} \dot{x}(t) - (Tx)(t) = f(x(t)) \\ x|_{[-h,0]} = x^0 \in C\left([-h,0];\mathbb{R}^N\right) \end{array}\right\} \tag{26}$$

*In effect, (21) defines a family of operators parameterized by $z^0 \in \mathbb{R}^P$.

By a *solution* of (25) (respectively, (26)) we mean a function $x \in C([-h,\omega);\mathbb{R}^N)$ for some $\omega > 0$, such that $x|_{[-h,0]} = x^0$, and $x|_{[0,\omega)} \in AC([0,\omega);\mathbb{R}^N)$ with $\dot{x}(t) - (Tx)(t) \in F(x(t))$ (respectively, $\dot{x}(t) - (Tx)(t) = f(x(t))$) for almost all $t \in [0,\omega)$ (recall our notational convention: for $t \in [0,\omega)$, $(Tx)(t)$ should be interpreted as $(Tx^e)(t)$ where x^e is any class $L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ extension of x). A solution is said to be *maximal* if it does not have a proper right extension which is also a solution.

We first present an existence result for the initial-value problem (26); this result, which plays a crucial rôle in the subsequent proof of existence of solutions of the initial-value problem (25), is proved via a fixed point argument akin to that used in Logemann & Owens (1988).

Theorem 7. *Let $h \geq 0$, $N \in \mathbb{N}$ and $T \in \mathcal{T}_h^N$. Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz. Let $x^0 \in C([-h,0];\mathbb{R}^N)$.*
There exists a unique maximal solution $x : [-h,\omega) \to \mathbb{R}^N$ of the initial-value problem (26); moreover, if x is bounded, then $\omega = \infty$.

Proof. See Appendix A1.

We now arrive at the main result of this section.

Theorem 8. *Let $h \geq 0$, $N \in \mathbb{N}$, $F \in \mathcal{F}^N$, $T \in \mathcal{T}_h^N$ and $x^0 \in C([-h,0];\mathbb{R}^N)$.*
(i) There exists a solution to the initial-value problem (25) on some interval $[-h,\omega)$ with $\omega > 0$.
(ii) Every solution can be maximally extended.
(iii) If a maximal solution $x : [-h,\omega) \to \mathbb{R}^N$ is bounded, then $\omega = \infty$.

Proof. See Appendix A2.

2.3 Asymptotic behaviour of solutions

Next, we present a result on asymptotic behaviour of solutions of (25): this result, which is reminiscent of Barbălat's Lemma (Barbălat 1959), plays a rôle in the analysis of the adaptive control strategy developed in the following section and is essentially Lemma 9(ii) of Desch *et al.* (1998) in the context of solutions of the system (25).

Lemma 9. *Let $h \geq 0$, $N \in \mathbb{N}$ and $F \in \mathcal{F}^N$. Let $T \in \mathcal{T}_h^N$ be bounded-input, bounded-output stable in the following sense:*

$$\forall R > 0 \ \exists M > 0 : \ \forall x \in L^\infty(\mathbb{R};\mathbb{R}^N) \ \|x\|_\infty < R \implies \|(Tx)(t)\| < M \text{ for a.a. } t \in \mathbb{R}_+ . \tag{27}$$

Let $l : \mathbb{R}^N \to \mathbb{R}_+$ be lower semicontinuous.
If $x : [-h,\infty) \to \mathbb{R}^N$ is a bounded solution of (25) and, for some $\tau > 0$,

$$\lim_{t\to\infty} \int_t^{t+\tau} l(x(s))\, ds = 0\,, \tag{28}$$

then $x(t)$ tends, as $t \to \infty$, to the zero level set $l^{-1}(0)$ of l.

Proof. See Appendix A3.

3. Adaptive control

Henceforth, attention focusses on issues of adaptive control in the context of a class $\mathcal{N}$ of nonlinear, single-input, single-output, systems of the form (5): we denote a system of this form by $(b, g, p, \hat{T})$.

3.1 System class $\mathcal{N}$

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions, known to the controller. The class $\mathcal{N}$ $(= \mathcal{N}(\phi, \psi))$ is defined to be the family of all those systems $(b, g, p, \hat{T})$ of the form (5) that satisfy Assumption 10 below.

Assumption 10.

1. *$b \neq 0$.*
2. *$g : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}$ is continuous and, for each compact $K \subset \mathbb{R}^P$ there exists $\mu_K \geq 0$ such that*

$$|g(r, y)| \leq \mu_K \phi(|y|) \quad \forall\ (r, y) \in K \times \mathbb{R}. \tag{29}$$

3. *$p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$.*
4. *For some $h \geq 0$, $\hat{T} \in \mathcal{T}_h^1$ and is bounded-input, bounded-output stable in the sense of (27).*
5. *For some constant $\mu \geq 0$, the following holds: for each $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$, there exists a constant $c \geq 0$ such that*

$$\int_0^t y(s)(\hat{T}y)(s)\ ds \leq c + \mu \int_0^t \psi(|y(s)|)|y(s)|\ ds \quad \forall\ t \geq 0. \tag{30}$$

3.2 Examples of admissible operators $\hat{T}$

3.2.1 Delay elements

Let ψ be such that, for some $\alpha > 1$, $\psi^{\alpha-1}(r) \geq r$ for all $r \in \mathbb{R}_+$. Let $\Psi \in \mathcal{D}^1$ (recall Definition 6) be such that, for some constant $\hat{\mu} > 0$,

$$|\Psi(t, y)|^\alpha \leq \hat{\mu}\psi(|y|)|y| \quad \text{for a.a. } t \text{ and all } y\,.$$

We will consider cases of point and distributed delays.

(a) Point delay. Define $\hat{T} \in \mathcal{T}_h^1$ by

$$(\hat{T}y)(t) := \Psi(t, y(t-h)).$$

Clearly, Assumption 10(4) holds. Moreover,

$$\begin{aligned}\int_0^t |(\hat{T}y)(s)|^\alpha ds &= \int_{-h}^{t-h} |\Psi(s+h, y(s))|^\alpha ds \\ &\le \hat{\mu} \int_{-h}^0 \psi(|y(s)|)|y(s)|\, ds + \hat{\mu} \int_0^t \psi(|y(s)|)|y(s)|\, ds\,.\end{aligned}$$

(b) Distributed delay. Define $\hat{T} \in \mathcal{T}_h^1$ by

$$(\hat{T}y)(t) := \int_{-h}^0 \Psi(s, y(t+s))\; ds\,.$$

Again, Assumption 10(4) holds. Moreover,

$$\begin{aligned}\int_0^t |(\hat{T}y)(s)|^\alpha ds &= \int_0^t \left|\int_{-h}^0 \Psi(\tau, y(s+\tau)) d\tau\right|^\alpha ds \\ &\le h^{\alpha-1} \int_0^t \int_{-h}^0 |\Psi(\tau, y(s+\tau))|^\alpha d\tau\, ds \le h^{\alpha-1}\hat{\mu} \int_0^t \int_{s-h}^s \psi(|y(\sigma)|)|y(\sigma)| d\sigma\, ds \\ &= h^{\alpha-1}\hat{\mu} \int_{-h}^t \int_{\max\{0,\sigma\}}^{\min\{t,\sigma+h\}} \psi(|y(\sigma)|)|y(\sigma)| ds\, d\sigma \\ &\le h^\alpha \hat{\mu} \left(\int_{-h}^0 \psi(|y(\sigma)|)|y(\sigma)|\, d\sigma + \int_0^t \psi(|y(\sigma)|)|y(\sigma)|\, d\sigma \right)\,.\end{aligned}$$

Thus in each case (point and distributed delay) and letting β denote the conjugate exponent of α (that is, $\beta^{-1} + \alpha^{-1} = 1$), there exists a constant μ such that

$$\begin{aligned}\int_0^t y(s)(\hat{T}y)(s)\; ds &\le \frac{1}{\beta} \int_0^t |y(s)|^\beta ds + \frac{1}{\alpha} \int_0^t |(\hat{T}y)(s)|^\alpha ds \\ &\le \frac{\mu}{2} \left(\int_{-h}^0 \psi(|y(s)|)|y(s)| ds + \int_0^t \left(|y(s)|^{\frac{1}{\alpha-1}} + \psi(|y(s)|) \right) |y(s)| ds \right) \\ &\le \text{constant} + \mu \int_0^t \psi(|y(s)|)|y(s)| ds \quad \forall\; t \ge 0\end{aligned}$$

and so Assumption 10(5) holds.

As a specific example, let $\psi(r) = 1 + r^3$. Let $h_1, h_2, h_3 > 0$ and $q_1, q_2, q_3 \in L^\infty(\mathbb{R})$. Then the operator $\hat{T} \in \mathcal{T}_h^1$, with $h := \max\{h_1, h_2, h_3\}$, given by

$$(\hat{T}y)(t) = q_1(t)y(t-h_1) + q_2(t)y^2(t-h_2) + \int_{-h_3}^0 q_3(s)y^3(t+s)ds$$

satisfies Assumption 10(4 & 5).

3.2.2 Regular linear systems

Let ψ be such that $\psi(r) \ge r$ for all $r \in \mathbb{R}_+$. Let (A, B, C, D) be the generating operators of a single-input, single-output, regular linear system with bounded observation operator

C. Assume that the semigroup $\mathbf{S}$ generated by A is exponentially stable. Similar to (17), define $\hat{T} \in \mathcal{T}_h^1$ by

$$(\hat{T}y)(t) := \begin{cases} C\int_0^t \mathbf{S}_{t-s}Bu(s)\, ds + Du(s), & t \geq 0 \\ 0, & t < 0\,. \end{cases}$$

In view of exponential stability of the semigroup and (15),(16), it is readily verified that the above operator satisfies Assumption 10(4 & 5).

As a specific example, consider a one-dimensional damped wave equation with Dirichlet boundary conditions and pointwise forcing (input) $y(t)$ applied at an interior point ξ_i of the spatial domain $\Omega := [0,1]$

$$\left.\begin{aligned} & z_{tt}(t,\xi) = a_0 z_{\xi\xi}(t,\xi) - a_1 z_t(t,\xi) + c_0\delta(\xi - \xi_i)y(t) \\ & z(0,\cdot) = z^0, \quad z_t(0,\cdot) = v^0 \\ & z(t,0) = 0 = z(t,1) \quad \forall\, t \geq 0 \end{aligned}\right\} \tag{31}$$

and with observation (output)

$$w(t) = \frac{c_1}{2\epsilon}\int_{\xi_o-\epsilon}^{\xi_o+\epsilon} z(t,\xi)\, d\xi \quad \xi_o \in (\epsilon, 1-\epsilon) \tag{32}$$

where $a_0, a_1 > 0$ and $\epsilon \in (0, \frac{1}{2})$. For $0 < \xi_o < \xi_i - \epsilon$ this system has input-output transfer function

$$\mathbf{G}(s) := \frac{c_0 c_1 \sinh\left((\xi_i - 1)\pi(s)\right)\left[\cosh\left((\xi_o - \epsilon)\pi(s)\right) - \cosh\left((\xi_o + \epsilon)\pi(s)\right)\right]}{2\epsilon(s^2 + a_1 s)\sinh\left(\pi(s)\right)},$$

where $\pi(s) := \sqrt{s(s+a_1)/a_0}$, and so qualifies as an exponentially stable regular linear system (with state space $X = H_0^1(\Omega) \times L^2(\Omega)$). The generator A of the exponentially stable semigroup $\mathbf{S}$ is given by

$$A = \begin{bmatrix} 0 & I \\ -P & -a_1 I \end{bmatrix}$$

where $Pz = -a_0 z_{\xi\xi}$ for all $z \in \mathrm{dom}(P) = H_0^2(\Omega)$. The control, observation and feedthrough operators are given by

$$Bu = \begin{bmatrix} 0 \\ c_0\delta(\cdot - \xi_i)u \end{bmatrix}, \quad C\zeta = C\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{c_1}{2\epsilon}\int_{\xi_o-\epsilon}^{\xi_o+\epsilon} \zeta_1(\xi)\, d\xi\,, \quad D = 0\,.$$

3.2.3 ISS systems

Let $w(t) := W(z(t,z^0,y)) \in \mathbb{R}$ be the output of a single-input, single-output initial-value problem of the form

$$\dot{z}(t) = k(z(t), y(t)), \quad z(0) = z^0, \quad w(t) = W(z(t))$$

where k and W are locally Lipschitz and W is such that $|W(z)| \leq L\|z\|$ for all z. Let $\alpha, \beta, \kappa \in \mathcal{K}_\infty$ and define $\rho, \eta, \gamma \in \mathcal{K}_\infty$ to be the indefinite integrals of α, its inverse α^{-1}, and β, respectively:

$$\rho(r) := \int_0^r \alpha, \quad \eta(r) := \int_0^r \alpha^{-1} \leq r\alpha^{-1}(r), \quad \gamma(r) := \int_0^r \beta \leq r\beta(r) \,.$$

Assume there exists some constant $\hat{\mu}$ such that, for all $(z^0, y) \in \mathbb{R}^P \times L^\infty_{\mathrm{loc}}(\mathbb{R})$,

$$\int_0^t \rho(\|z(s, z^0, y)\|) \leq \kappa(\|z^0\|) + \hat{\mu} \int_0^t \gamma(|y(s)|)ds \quad \forall\ t \geq 0\,,$$

and so the system under consideration is input-to-state stable (ISS), see Sontag (1998). Let $z^0 \in \mathbb{R}^P$ be arbitrary. Similar to (21), define $\hat{T} \in \mathcal{T}_h^1$ by

$$(\hat{T}y)(t) := \begin{cases} w(t), & t \geq 0 \\ 0, & t < 0\,. \end{cases}$$

By the ISS property, it follows that Assumption 10(4) holds. Invoking Young's inequality (Hewitt & Stromberg 1965, Theorem 13.2),

$$y(s)(\hat{T}y)(s) \leq L|y(s)|\|z(s, z^0, y)\| \leq L\left[\rho(|z(s, z^0, y)|) + \eta(|y(s)|)\right]$$

whence, for $\mu := \hat{\mu} + 1$

$$\int_0^t y(s)(\hat{T}y)(s)ds \leq L\left[\kappa(\|z^0\|) + \mu\int_0^t [\gamma(|y(s)|) + \eta(|y(s)|)]ds\right] \quad \forall\ t \geq 0.$$

Recalling that $\gamma(r) + \eta(r) \leq r[\alpha^{-1}(r) + \beta(r)]$, it follows that Assumption 10(5) holds provided that $\psi(r) \geq \alpha^{-1}(r) + \beta(r)$ for all $r \in \mathbb{R}_+$.

As a specific example, let $\psi(r) = 1 + r^3$. Consider the system (Sontag 1995)

$$\dot{z}(t) = -a_0 z(t) - a_1 z^3(t) + a_2(1 + z^2(t))y(t)$$

with $a_0, a_1 > 0$. Multiplying by $z(t)$ and invoking Young's inequality,

$$z(t)\dot{z}(t) \leq -\tfrac{1}{2}a_0 z^2(t) + a_3[y^2(t) + y^4(t)]$$

for some constant $a_3 > 0$. Therefore, Assumption 10(4) holds. Moreover, there exist constants $c_0, \mu > 0$ such that

$$\int_0^t z^2(s)ds \leq c_0(z^0)^2 + \mu\int_0^t [y^2(s) + y^4(s)]ds \quad \forall\ t \geq 0$$

and so Assumption 10(5) also holds.

3.3 Adaptive stabilizer

Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. The problem to be addressed is that of control design to ensure that, for all $(b, g, p, \hat{T}) \in \mathcal{N} = \mathcal{N}(\phi, \psi)$ and all initial data $y|_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R})$, every solution $y(\cdot)$ of (5) approaches zero.

Writing

$$\Phi : \mathbb{R}_+ \to \mathbb{R}_+, \quad r \mapsto \max\{\phi(r), \psi(r)\}$$

then the following adaptive feedback strategy (appropriately interpreted) will be shown to be a $\mathcal{N}$-universal stabilizer assuring that, for all $(b, g, p, \hat{T}) \in \mathcal{N}$ and all y^0, $y(t) \to 0$ as $t \to \infty$ whilst maintaining boundedness of the controller function $k(\cdot)$.

$$\left.\begin{aligned} u(t) &= \nu(k(t))\Phi(|y(t)|)\mathrm{sgn}(y(t)) \\ \dot{k}(t) &= \Phi(|y(t)|)|y(t)| \\ k|_{[-h,0]} &= k^0 \in C([-h, 0]; \mathbb{R}) \text{ with } k^0(0) > 0 \end{aligned}\right\} \tag{33}$$

where $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function with properties (3).

In view of the discontinuous nature of the feedback, the first of equations (33) is interpreted in the set-valued sense

$$u(t) \in \nu(k(t))\Phi(|y(t)|)\sigma(y(t)), \tag{34}$$

with $y \mapsto \sigma(y) \subset \mathbb{R}$ given by

$$\sigma(y) := \begin{cases} \{\mathrm{sgn}(y)\}, & y \neq 0 \\ [-1, 1], & y = 0\,. \end{cases}$$

Let $(b, g, p, \hat{T}) \in \mathcal{N}$. By properties of g and essential boundedness of p, there exists $\mu \in \mathbb{R}_+$ such that, for all $y \in \mathbb{R}$ and almost all t, $|g(p(t), y)| \le \mu\phi(|y|)$. Define $x \mapsto F(x) \subset \mathbb{R}^2$ by

$$F(x) = F(y, k) := \{v + bu : |v| \le \mu\phi(|y|), u \in \nu(k)\Phi(|y|)\sigma(y)\} \times \{\Phi(|y|)|y|\}\,.$$

$F(\cdot)$ is upper semicontinuous with non-empty, compact and convex values and so $F \in \mathcal{F}^2$. Define $T \in \mathcal{T}_h^2$ by

$$(Tx)(t) = (T(y, k))(t) := (\hat{T}y, 0)(t) \quad \forall\ t.$$

We now embed the feedback-controlled system in a functional differential inclusion:

$$\dot{x}(t) - (Tx)(t) \in F(x(t)), \quad x|_{[-h,0]} = (y^0, k^0) \in C\left([-h, 0]; \mathbb{R}^2\right). \tag{35}$$

Since $F \in \mathcal{F}^2$ and $T \in \mathcal{T}_h^2$ it follows, by Theorem 8, that, for all x^0, (35) has a solution and every solution can be maximally extended.

Theorem 11. *Let $x(\cdot) = (y(\cdot), k(\cdot)) : [-h, \omega) \to \mathbb{R}^2$ be a maximal solution of the initial-value problem (35). Then (i) $\omega = \infty$, (ii) $\lim_{t\to\infty} k(t)$ exists and is finite, (iii) $y(t) \to 0$ as $t \to \infty$.*

Proof. See Appendix A4.

4 Examples

Expanding on the specific examples given in Section 3.2, consider the following three systems

$$\text{System 1:} \quad \begin{cases} \dot{y}(t) = g(p(t), y(t)) \\ \qquad\qquad +q_1(t)y(t-h_1) + q_2(t)y^2(t-h_2) + \int_{-h_3}^{0} q_3(s)y^3(t+s)ds + bu(t) \end{cases} \tag{36}$$

$$\text{System 2:} \quad \begin{cases} z_{tt}(t,\xi) = a_0 z_{\xi\xi}(t,\xi) - a_1 z_t(t,\xi) + c_0\delta(\xi-\xi_i)y(t) \\ \dot{y}(t) = g(p(t), y(t)) + \frac{c_1}{2\epsilon}\int_{\xi_o-\epsilon}^{\xi_o+\epsilon} z(t,\xi)\, d\xi + bu(t) \end{cases} \tag{37}$$

$$\text{System 3:} \quad \begin{cases} \dot{z}(t) = -a_0 z(t) - a_1 z^3(t) + a_2(1+z^2(t))y(t) \\ \dot{y}(t) = g(p(t), y(t)) + W(z(t)) + bu(t) \end{cases} \tag{38}$$

with $q_1, q_2, q_3 \in L^\infty(\mathbb{R})$, $a_0, a_1 > 0$ and where the first of equations (37) has spatial domain $\Omega = [0,1]$ and is subject to Dirichlet boundary conditions as in Section 3.2.2. Each of these three systems may be expressed in the form (5), with the operator $\hat{T}$ defined as in Sections 3.2.1, 3.2.2 and 3.3.3, respectively. We remark that, in the case of System 2, the contribution of the initial state of the wave equation

$$t \mapsto \mathbf{S}_t \begin{bmatrix} z^0 \\ v^0 \end{bmatrix}$$

may, by exponential stability of the semigroup, be absorbed by the function g.

Defining

$$\phi = \psi : r \mapsto 1 + r^3$$

and assuming that $b \neq 0$ and $p \in L^\infty(\mathbb{R}; \mathbb{R}^P)$, then each of Systems 1, 2 and 3 is of class $\mathcal{N} = \mathcal{N}(\phi, \psi)$ provided that g satisfies Assumption 10(2). Therefore, by Theorem 11, the following control (appropriately initialized and with the discontinuity interpreted in the set-valued sense)

$$u(t) = k^2(t)\cos(k(t))\phi(|y(t)|)\mathrm{sgn}(y(t)), \quad \dot{k}(t) = \phi(|y(t)|)|y(t)|$$

stabilizes each of the three widely disparate uncertain systems: in all three cases, the function $y(t) \to 0$ and the monotone adapting parameter $k(\cdot)$ converges.

4.1 System 1

By way of illustration, consider System 1 with $b = \frac{1}{2}$, $g : (p, y) \mapsto y^{\frac{1}{3}} + y^3 + p$, $p : t \mapsto \sin(7t)$, $h_1 = 1$, $h_2 = h_3 = 2$, $q_1(t) = q_2(t) = 1$ for all t, $q_3 : t \mapsto \sin(-\frac{1}{2}\pi t)$, and with initial conditions $y(t) = \frac{1}{2}t + 1$ for $-2 \leq t \leq 0$ and $k(0) = 1$. Figure 2 depicts the system behaviour (computed using an improved Euler method within MATLAB) under the adaptive control.

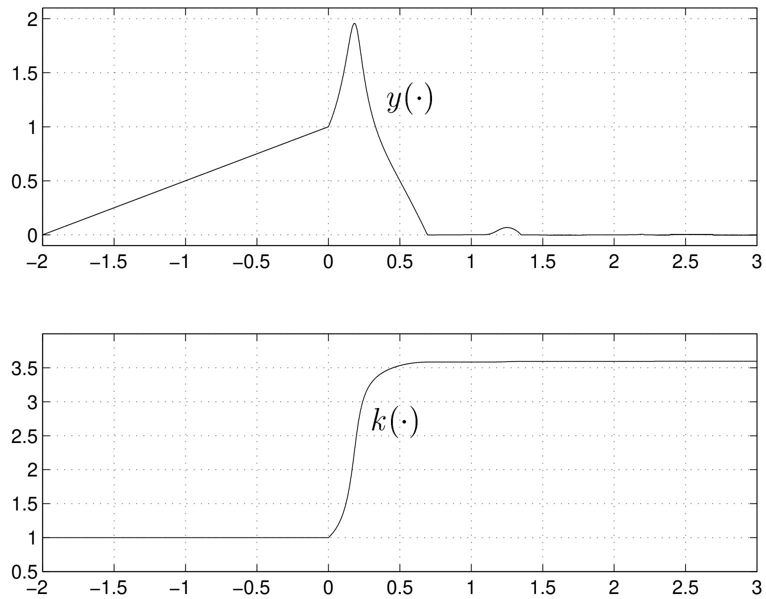

Figure 2: Typical behaviour of System 1 under adaptive control.

4.2 System 2

Consider System 2 with parameters $a_0 = 10$, $a_1 = 1$, $c_0 = 10$, $c_1 = 1$, $\xi_o = 1/3$, $\xi_i = 2/3$, $\epsilon = 0.01$, with Dirichlet boundary conditions ($z(t,0) = z(t,1) = 0$ for all $t \geq 0$) on the first of equations (37), with $b = 1/2$, $g : (p,y) \mapsto y^3 + p$, with initial conditions $z(0,\cdot) = 0 = z_t(0,\cdot)$, $y(t) = 1$ for all $t \in [-1,0]$, $k(0) = 0$, and with $p(\cdot) = p_1(\cdot)$ where $p_1(\cdot)$ is the first coordinate of the solution to the Lorenz system

$$\left.\begin{aligned} \dot{p}_1(t) &= p_2(t) - p_1(t) \\ \dot{p}_2(t) &= 2.8p_1(t) - 0.1p_2(t) - p_1(t)p_3(t) \\ \dot{p}_3(t) &= p_1(t)p_2(t) - \tfrac{8}{30}p_3(t) \\ (p_1(0), p_2(0), p_3(0)) &= (1,0,3) \end{aligned}\right\} \tag{39}$$

Figures 3 and 4 depict the system behaviour (computed - adopting a truncated eigenfunction expansion, of order 8, to model the wave equation - using SIMULINK within MATLAB) under the adaptive control; Figure 4 shows the state evolution of the first of equations (37).

4.3 System 3

Consider System 3 with $a_0 = 2$, $a_1 = 1$, $a_2 = 3$, $b = -\frac{1}{2}$, $p(\cdot) = (p_1(\cdot), p_2(\cdot), p_3(\cdot))$, $W : z \mapsto z$, and

$$g(p(t), y(t)) = p_1(t)y(t) + p_2(t)y^3(t) + p_3(t)\sqrt{|y(t)|}$$

where $(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ is the solution of the initial-value problem (39). For initial conditions $y(0) = \frac{1}{2}$, $z(0) = -2$ and $k(0) = 1$, Figure 5 depicts the system behaviour (computed using SIMULINK within MATLAB) under the adaptive control.

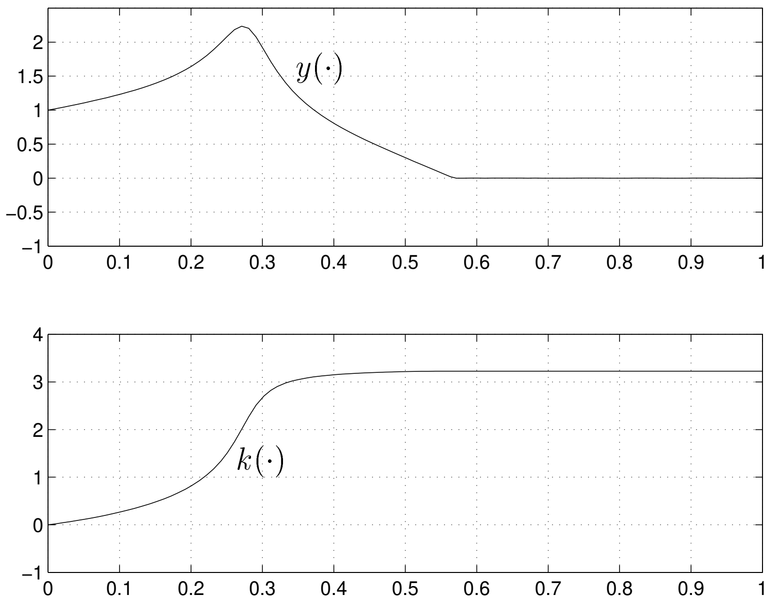

Figure 3: Typical behaviour of System 2 under adaptive control.

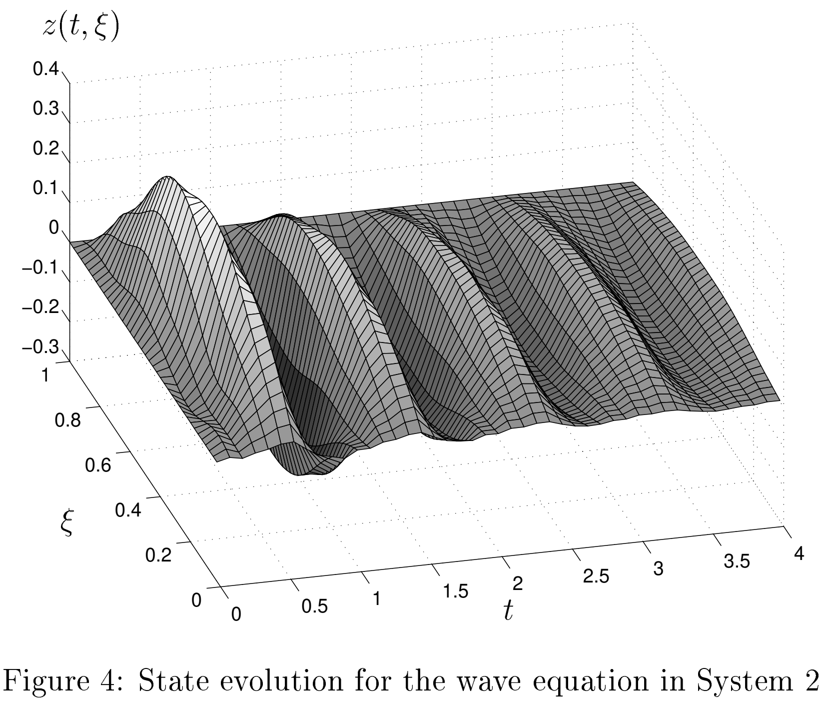

Figure 4: State evolution for the wave equation in System 2.

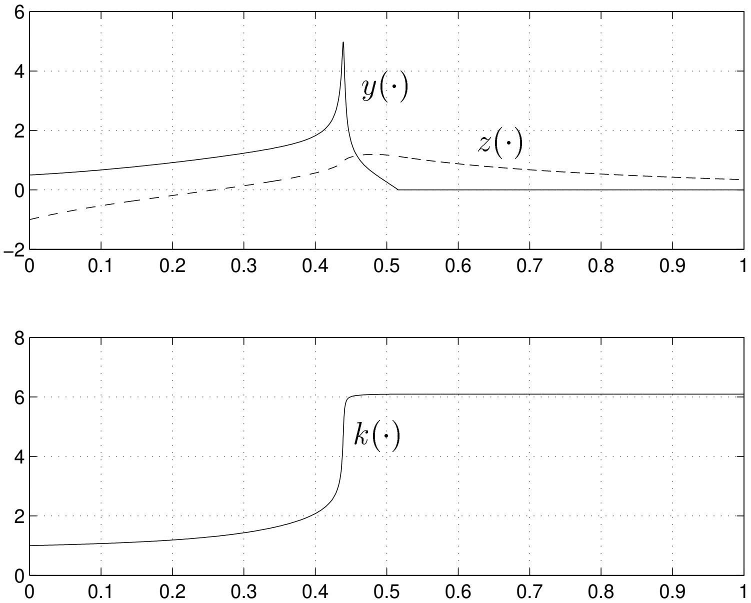

Figure 5: Typical behaviour of System 3 under adaptive control.

Appendix

A1 Proof of Theorem 7

Let $\mathcal{S}$ denote the set of all pairs $(\rho,\xi) \in (0,\infty] \times C([-h,\rho);\mathbb{R}^N)$ such that ξ is a solution of the initial-value problem (26):

$$\mathcal{S} := \{(\rho,\xi)|\ 0 < \rho \le \infty,\ \xi \in C([-h,\rho);\mathbb{R}^N) \text{ is a solution of (26)}\}.$$

Our first objective is to show that $\mathcal{S} \neq \emptyset$ (implying the existence of at least one solution of (26)) and, for all $(\rho,\xi),(\sigma,\eta) \in \mathcal{S}$, if $\rho \le \sigma$ then $\eta|_{[-h,\rho)} = \xi$ (that is, any two solutions of (26) must coincide on the intersection of their domains). These two facts immediately lead to the first assertion of the theorem (namely, the existence of a unique maximal solution $x : [-h,\omega) \to \mathbb{R}^N$), via the following construction: let $\omega := \sup\{\rho|\ (\rho,\xi) \in \mathcal{S}\}$ and define $x \in C([-h,\omega);\mathbb{R}^N)$ by the property $x|_{[-h,\rho)} = \xi$ for all $(\rho,\xi) \in \mathcal{S}$.

Since f is locally Lipschitz, there exists a constant $L > 0$ such that

$$\|f(u) - f(v)\| \le L\|u - v\| \quad \forall\ u,v \in \mathbb{B}_1(x^0(0)).$$

By Property 2(b) (Definition 4) of T, there exists $\tau > 0$, $r \in (0,1)$ and $c > 0$ such that for all $x,\xi \in L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$ with $x|_{[-h,0]} = x^0 = \xi|_{[-h,0]}$ and $x(t),\xi(t) \in \mathbb{B}_r(x^0(0))$ for almost all $t \in [0,\tau]$, the following holds

$$\|(Tx)(t) - (T\xi)(t)\| \le c\,\text{ess-sup}_{s\in[0,\tau]}\|x(s) - \xi(s)\|.$$

Define $R := \|x^0(0)\| + r$. By Property 1 of Definition 4 of T there exists $M > 0$ such that

$$\|x\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} < R \quad \Longrightarrow \quad \|(Tx)(t)\| < M \quad \text{for a.a. } t \in [0,\tau).$$

For $\epsilon > 0$, define

$$C_\epsilon := \{x \in C([-h,\epsilon),\mathbb{R}^N) \,|\, x|_{[-h,0]} = x^0,\ \|x(t) - x^0(0)\| \le r/2\ \forall\ t \in (0,\epsilon)\}$$

which, when equipped with the metric

$$(x,\xi) \mapsto \Delta_\epsilon(x,\xi) := \sup_{-h\le t<\epsilon} \|x(t) - \xi(t)\|$$

is a complete metric space.

For each $\epsilon > 0$, define the operator Γ_ϵ on C_ϵ by

$$(\Gamma_\epsilon x)(t) := \begin{cases} x^0(t), & t \in [-h,0] \\ x^0(0) + \int_0^t [(Tx)(s) + f(x(s))]\, ds, & t \in (0,\epsilon) \end{cases}$$

(Again, recall our notational convention: for $t \in [0,\epsilon)$, $(Tx)(t)$ should be interpreted as $(Tx^e)(t)$ where x^e is any class $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ extension of x.) Clearly, any fixed point of Γ_ϵ is a solution of (26). Next, we show that Γ_ϵ is a contraction for all sufficiently small $\epsilon > 0$.

Fix $\epsilon^* \in (0,\tau)$ sufficiently small so that

$$\epsilon^*(c+L) < 1 \quad \text{and} \quad 2\epsilon^*(M + \|f(x^0(0))\| + Lr) < r\,.$$

Let $\epsilon \in (0,\epsilon^*)$ and $x \in C_\epsilon$. By definition, $(\Gamma_\epsilon x)|_{[-h,0]} = x^0$ and

$$\begin{aligned} \|(\Gamma_\epsilon x)(t) - x^0(0)\| &= \left\| \int_0^t (Tx)(s) + f(x(s))\; ds \right\| \\ &\le \int_0^\epsilon \|(Tx)(s)\|\; ds + \int_0^\epsilon \|f(x(s))\|\; ds \\ &\le \epsilon M + \epsilon(\|f(x^0(0))\| + Lr) \\ &< r/2 \quad \forall\ t \in (0,\epsilon]\,. \end{aligned}$$

Therefore, $(\Gamma_\epsilon x)(\cdot) \in C_\epsilon$ and so $\Gamma_\epsilon : C_\epsilon \to C_\epsilon$. Furthermore, for all $\xi,\eta \in C_\epsilon$,

$$\begin{aligned} \Delta_\epsilon(\Gamma_\epsilon x, \Gamma_\epsilon \xi) &= \sup_{t\in(0,\epsilon)} \left\| \int_0^t (Tx)(s) - (T\xi)(s) + f(x(s)) - f(\xi(s))\, ds \right\| \\ &\le \int_0^\epsilon \|(Tx)(s) - (T\xi)(s)\|\, ds + \int_0^\epsilon \|f(x(s)) - f(\xi(s))\|\, ds \\ &\le \epsilon\, \text{ess-sup}_{s\in[0,\epsilon)} \|Tx(s) - T\xi(s)\| + L\int_0^\epsilon \|x(s) - \xi(s)\|\, ds \\ &\le \epsilon\, c\Delta_\epsilon(x,\xi) + \epsilon L \Delta_\epsilon(x,\xi) = \epsilon(c+L)\Delta_\epsilon(x,\xi) \end{aligned}$$

and, since $\epsilon(c+L) < 1$, it follows that Γ_ϵ is a contraction on C_ϵ.

We have now shown that, for all $\epsilon \in (0,\epsilon^*)$, $\Gamma_\epsilon : C_\epsilon \to C_\epsilon$ is a contraction. By the contraction mapping theorem, for each $\epsilon \in (0,\epsilon^*)$, Γ_ϵ has a unique fixed point. Therefore, (26) has a solution, equivalently, $\mathcal{S} \ne \emptyset$. Let $(\rho,\xi),(\sigma,\eta) \in \mathcal{S}$. By continuity of ξ and

η, $\xi|_{[-h,\epsilon)}, \eta|_{[-h,\epsilon)} \in C_\epsilon$ for all sufficiently small $\epsilon \in (0,\epsilon^*)$ and, since these restrictions are fixed points of the contraction Γ_ϵ, they must coincide.

The above argument has established the following facts, assembled for later referral.

$$\left.\begin{array}{l}\text{For every initial-value, (26) has a solution and}\\ \text{all solutions must coincide on some interval } [-h,\epsilon) \text{ with } \epsilon > 0\,.\end{array}\right\} \tag{40}$$

To show that any two solutions must coincide on the intersection of their domains, we argue as follows. Let $(\rho,\xi),(\sigma,\eta) \in \mathcal{S}$ and assume, without loss of generality, that $\rho \le \sigma$. Seeking a contradiction, suppose $\eta|_{[-h,\rho)} \ne \xi$. Then there exists $t \in (0,\rho)$ such that $\xi(t) \ne \eta(t)$. Let $t^* := \inf\{t \in (0,\rho) \,|\, \xi(t) \ne \eta(t)\}$. In view of (40), we have $t^* > 0$.

Define $z^0 \in C([-(h+t^*),0];\mathbb{R}^N)$ by $z^0(t) := \xi(t+t^*) = \eta(t+t^*)$ for all $t \in [-(h+t^*),0]$ and consider the initial-value problem

$$\left.\begin{array}{l}\dot z(t) - (S_{t^*}TS_{-t^*}z)(t) = f(z(t))\\ z|_{[-(h+t^*),0]} = z^0\,.\end{array}\right\} \tag{41}$$

In view of (8), together with (40) interpreted in the context of the initial-value problem (41), we conclude that, for some $\epsilon \in (0,\rho-t^*)$, (41) has precisely one solution z with domain $[-(h+t^*),\epsilon)$.

Now define $\xi^*,\eta^* \in C([-h,\epsilon);\mathbb{R}^N)$ by

$$\xi^*(t) = \xi(t+t^*), \quad \eta^*(t) := \eta(t+t^*) \quad \forall\, t \in [-(h+t^*),\epsilon)\,.$$

For almost all $t \in [0,\epsilon)$,

$$f(\xi^*(t)) = f(\xi(t+t^*)) = \dot\xi(t+t^*) - (T\xi)(t+t^*) = \dot\xi^*(t) - (S_{t^*}TS_{-t^*}\xi^*)(t)$$

and so $\xi^* = z$ (the unique solution of (41) with domain $[-(h+t^*),\epsilon)$). Similarly, we have $\eta^* = z$ and so

$$\xi(t) = \eta(t) \quad \forall\, t \in [t^*, t^*+\epsilon)$$

which contradicts the definition of t^*. Therefore, $\eta|_{[-h,\rho)} = \xi$. The existence of a unique maximal solution of (26) follows.

Assume that the unique maximal solution $x \in C([-h,\omega);\mathbb{R}^N)$ of (26) is bounded. Seeking a contradiction, suppose $\omega < \infty$. By boundedness of x, together with the continuity of $f(\cdot)$ and the Property 1 (Definition 4) of T, there exists $M > 0$ such that $\|\dot x(t)\| \le M$ for almost all $t \in [0,\omega)$. Therefore, x is uniformly continuous and so extends to a continuous function $\xi : [-h,\omega] \to \mathbb{R}^N$. Define $z^0 \in C([-(h+\omega),0];\mathbb{R}^N)$ by

$$z^0(t) := \xi(\omega+t) \quad \forall\, t \in [-(h+\omega),0]\,.$$

Consider the initial-value problem

$$\left.\begin{array}{l}\dot z(t) - (S_\omega TS_{-\omega}z)(t) = f(z(t))\\ z|_{[-(h+\omega),0]} = z^0\,.\end{array}\right\} \tag{42}$$

By (8) and the above existence result, there exists a solution $z : [-(h+\omega), \omega_z) \to \mathbb{R}^N$ ($\omega_z > 0$) of (42). Now define $x^* : [-h, \omega + \omega_z) \to \mathbb{R}^N$ as follows

$$x^*(t) := \begin{cases} x(t), & t \in [-h, \omega) \\ z(t-\omega), & t \in [\omega, \omega + \omega_z) \end{cases}$$

Noting that, for almost all $t \in [\omega, \omega + \omega_z)$,

$$\dot{x}^*(t) - (Tx^*)(t) = \dot{z}(t-\omega) - (S_\omega T S_{-\omega} z)(t-\omega) = f(z(t-\omega)) = f(x^*(t)),$$

it follows that x^* is a solution of the original initial-value problem (26) and is a proper right extension of the solution x. This contradicts maximality of x. Therefore, $\omega = \infty$.

A2 Proof of Theorem 8

(i) Let $(\epsilon_n) \subset (0,1)$ be a monotone sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$. By Proposition 2, for each n, there exists a locally Lipschitz function f_n with

$$\mathrm{graph}(f_n) \subset \mathrm{graph}(F) + \mathbb{B}_{\epsilon_n}(0). \tag{43}$$

By Proposition 3, there exists a compact set $B \subset \mathbb{R}^N$ such that

$$F(\mathbb{B}_1(x^0(0))) \subset B.$$

Hence, for each $n \in \mathbb{N}$,

$$\|f_n(x)\| < r := \sup_{v \in B} \|v\| + 1 \quad \forall\, x \in \mathbb{B}_1(x^0(0)). \tag{44}$$

By Theorem 7, for each $n \in \mathbb{N}$, the initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) - (Tx)(t) = f_n(x(t)) \\ &x|_{[-h,0]} = x^0 \in C\left([-h,0]; \mathbb{R}^N\right) \end{aligned}\right\} \tag{45}$$

has a unique maximal solution which we denote by $\xi_n : [-h, \omega_n) \to \mathbb{R}^N$. Let

$$\tau_n := \inf\{t \in (0, \omega_n) \mid \|\xi_n(t) - x^0(0)\| = 1/2\} \quad \forall\, n \in \mathbb{N},$$

with the convention that $\inf \emptyset := \infty$.

Evidently, for each $n \in \mathbb{N}$, $\xi_n(t) \in \mathbb{B}_1(x^0(0))$ for all $t \in [0, \tau_n)$ and so

$$\|\xi_n(t)\| < R := \max_{s \in [-h,0]} \|x^0(s)\| + 1 \quad \forall\, t \in [-h, \tau_n).$$

By Property 1 (Definition 4) of T, there exists $K > 0$ such that, for all $n \in \mathbb{N}$,

$$\|(T\xi_n)(t)\| < K \quad \text{for a.a. } t \in [0, \tau_n) \cap [0, 1).$$

(Once again, recall our notational convention: for $t \in [0,\omega_n)$, $(T\xi_n)(t)$ should be interpreted as $(T\xi_n^e)(t)$ where ξ_n^e is any class $L^\infty_{\rm loc}(\mathbb{R};\mathbb{R}^N)$ extension of ξ_n.)

Writing $M := K + r$, we may conclude that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\xi_n(t) - \xi_n(s)\| \le \int_t^s \|\dot{\xi}_n(\sigma)\| d\sigma \\ \le \int_t^s \|(T\xi_n)(\sigma)\| d\sigma + \int_t^s \|f_n(\xi_n(\sigma))\| d\sigma \\ < M|t-s| \quad \forall\ s,t \in [0,\tau_n) \cap [0,1),\ t \le s. \quad (46)\end{aligned}$$

Next, we define $\omega^* := \inf_{n\in\mathbb{N}} \tau_n \ge 0$ ($\omega^* = \infty$ is possible) and claim that $\omega^* \ne 0$. Suppose otherwise, then (τ_n) has a subsequence (which we do not relabel) with $\tau_n < 1$ for all n and $\tau_n \downarrow 0$ as $n \to \infty$, which, together with (46), yields the contradiction:

$$0 < 1/2 = \|\xi_n(\tau_n) - x^0(0)\| < M\tau_n \to 0 \text{ as } n \to \infty\,.$$

Therefore, $\omega^* > 0$.

Fix $\rho \in \mathbb{R}$ such that $0 < \rho < \min\{1, \omega^*\}$. For each n, define $x_n := \xi_n|_{[0,\rho)} \in AC([0,\rho);\mathbb{R}^N)$ and let $x_n^* \in C([-h,\rho);\mathbb{R}^N)$ denote the concatenation of x_n and x^0 given by

$$x_n^*(t) := \begin{cases} x^0(t), & t \in [-h,0] \\ x_n(t), & t \in [0,\rho) \end{cases}$$

For all $t \in [0,\rho)$, $(x_n(t))_{n\in\mathbb{N}} \subset \mathbb{B}_r(x^0(0))$ and, by (46), for all $n \in \mathbb{N}$,

$$\|\dot{x}_n(t)\| < M \quad \text{for a.a. } t \in [0,\rho)\,. \tag{47}$$

By Aubin & Cellina (1984, Theorem 4, page 13) and extracting a subsequence if necessary, we may assume that (x_n) converges uniformly to a function $x \in AC([0,\rho);\mathbb{R}^N)$, and $(\dot{x}_n)$ converges weakly in $L^1([0,\rho);\mathbb{R}^N)$ to $\dot{x}$. Let $x^* \in C([-h,\rho);\mathbb{R}^N)$ denote the concatenation of x and x^0. By Property 2(b) (Definition 4) of T, there exists $\omega \in (0,\rho)$ and $c > 0$ such that, for all n sufficiently large,

$$\|(Tx_n^*)(t) - (Tx^*)(t)\| \le c \sup_{s\in[0,\omega)} \|x_n^*(s) - x^*(s)\| \quad \text{for a.a. } t \in [0,\omega)\,. \tag{48}$$

Define $I := [0,\omega)$. We will show that $x^*|_I$ is a solution of the initial-value problem (25). By construction, $x^*|_{[-h,0]} = x^0$ and $x^*|_{[0,\omega)} = x$. Thus, it remains only to show that $\dot{x}(t) - (Tx^*)(t) \in F(x(t))$ for almost all $t \in I$.

For each $n \in \mathbb{N}$, define $y_n : I \to \mathbb{R}^N$ by $y_n(t) := (Tx_n^*)(t)$ for all $t \in I$ and let $y : I \to \mathbb{R}^N$ be given by $y(t) = (Tx^*)(t)$ for all $t \in I$. By (48),

$$\int_0^\omega \|y_n(t) - y(t)\| dt \le c\omega \sup_{s\in[0,\omega)} \|x_n^*(s) - x^*(s)\| \ \to\ 0 \quad \text{as } n \to \infty\,.$$

Therefore, (y_n) converges (strongly) in $L^1(I;\mathbb{R}^N)$ to $y \in L^1(I;\mathbb{R}^N)$.

Let $(z_n) \subset AC(I : \mathbb{R}^N)$ denote the sequence of functions z_n given by $z_n(t) = f_n(x_n(t))$ for all $t \in I$ (and so, for all n, $z_n(t) = \dot{x}_n(t) - (Tx_n^*)(t)$ for almost all $t \in I$). By weak convergence of $(\dot{x}_n)$ and (strong) convergence of (y_n), it follows that (z_n) converges weakly in $L^1(I; \mathbb{R}^N)$ to $z = \dot{x} - y \in L^1(I; \mathbb{R}^N)$.

Let $\epsilon > 0$ be arbitrary. By (43) and since $\epsilon_n \downarrow 0$ as $n \to \infty$, there exists n_ϵ such that, for all $t \in I$,

$$(x_n(t), z_n(t)) \in \text{graph}(F) + \mathbb{B}_\epsilon(0) \quad \forall\ n > n_\epsilon\,.$$

By Aubin & Cellina (1984, Theorem 1, page 60), we conclude that

$$(x(t), z(t)) \in \text{graph}(F) \quad \text{for a.a. } t \in I\,,$$

equivalently,

$$z(t) = \dot{x}(t) - (Tx^*)(t) \in F(x(t)) \quad \text{for a.a. } t \in I.$$

This establishes assertion (i).

(ii) Let $x : [-h, \omega) \to \mathbb{R}^N$ be a solution of (25). Define

$$\mathcal{A} := \big\{(\rho, \xi) | \omega \le \rho \le \infty,\ \xi : [-h, \rho) \to \mathbb{R}^N \text{ is a solution of (25) with } \xi|_{[-h,\omega)} = x\big\}$$

On this non-empty set define a partial order $\preceq$ by

$$(\rho_1, \xi_1) \preceq (\rho_2, \xi_2) \quad \iff \quad \rho_1 \le \rho_2 \text{ and } \xi_1(t) = \xi_2(t) \text{ for all } t \in [-h, \rho_1).$$

Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $P := \sup\{\rho | (\rho, \xi) \in \mathcal{O}\}$ and let $\Xi : [-h, \rho) \to \mathbb{R}^N$ be defined by the property that, for every $(\rho, \xi) \in \mathcal{O}$, $\Xi|_{[0,\rho)} = \xi$. Then (P, Ξ) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's Lemma, it follows that $\mathcal{A}$ contains at least one maximal element.

(iii) Assume that $x \in C([-h, \omega); \mathbb{R}^N)$ is a bounded maximal solution of (25). Seeking a contradiction, suppose $\omega < \infty$. By boundedness of x, together with Proposition 3 and Property 1 (Definition 4) of T, there exists $M > 0$ such that $\|\dot{x}(t)\| \le M$ for almost all $t \in [0, \omega)$. Therefore, x is uniformly continuous and so extends to a continuous function $\xi : [-h, \omega] \to \mathbb{R}^N$. The result now follows by an argument analogous to that used in the proof of Theorem 7.

A3 Proof of Lemma 9

By boundedness of x and properties of F and T, it follows from (25) that $\dot{x}$ is essentially bounded and so x is uniformly continuous. By boundedness, x has non-empty ω-limit set $\Omega(x)$. Since x approaches $\Omega(x)$, it suffices to show that $\Omega(x) \subset l^{-1}(0)$. Seeking a contradiction, suppose that there exists $z \in \Omega(x)$ with $z \notin l^{-1}(0)$. Define $\epsilon := l(z)/2 > 0$. Since $z \in \Omega(x)$, there exists a sequence (t_n) with $t_n \to \infty$ such that $x(t_n) \to z$ as $n \to \infty$. By (28), and passing to a subsequence if necessary, we may assume

$$t_{n+1} - t_n > \tau \quad \text{and} \quad \int_{t_n}^{t_n+\tau} l(x(s))ds \le \frac{\epsilon\tau}{n} \quad \forall\ n.$$

Therefore, for each n, there exists $s_n \in (t_n, t_n + \tau/n)$ such that $l(x(s_n)) < \epsilon$. Since $s_n - t_n \downarrow 0$ as $n \to \infty$, it follows by uniform continuity of x that $\|x(s_n) - x(t_n)\| \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} x(s_n) = \lim_{n\to\infty} x(t_n) = z\,.$$

Invoking lower semicontinuity of l, we arrive at a contradiction:

$$\epsilon \geq \liminf_{n\to\infty} l(x(s_n)) \geq l(z) = 2\epsilon\,.$$

□

A4 Proof of Theorem 11

For almost all $t \in [0, \omega)$, we have

$$y(t)\dot{y}(t) \leq y(t)(\hat{T}y)(t) + [\mu + b\nu(k(t))]\Phi(|y(t)|)|y(t)| = y(t)(\hat{T}y)(t) + [\mu + b\nu(k(t))]\dot{k}(t)\,,$$

which, on integration and invoking Assumption 10(5), yields

$$\begin{aligned} 0 \leq y^2(t) &\leq y^2(\tau) + c^* + 2\int_\tau^t [\mu^* + b\nu(k(s))]\,\dot{k}(s)\; ds \\ &= c^* + 2\mu^*[k(t) - k(\tau)] + 2b\int_{k(\tau)}^{k(t)} \nu(s)ds \quad \forall\ \tau, t \in [0, \omega),\ \tau \leq t \end{aligned} \tag{49}$$

for some constants c^* and μ^*. Seeking a contradiction, suppose k is unbounded and so, by monotonicity, $k(t) \to \infty$ as $t \uparrow \omega$. Fix $\tau \in [0, \omega)$ such that $k(\tau) \geq 1$. Dividing by $k(t) \geq k(\tau) \geq 1$ in (49) gives

$$0 \leq y^2(\tau) + c^* + 2\mu^* + \frac{2b}{k(t)}\int_{k(\tau)}^{k(t)} \nu(s)ds \quad \forall\ t \in [\tau, \omega). \tag{50}$$

Recalling that $b \neq 0$ and taking limit inferior as $t \uparrow \omega$ ($k(t) \uparrow \infty$) in (50), we arrive at a contradiction to one or the other of properties (3) of ν. Therefore, k is bounded. Boundedness of y follows by (50). By boundedness of $x = (y, k)$, we conclude that $\omega = \infty$, which is assertion (i) of the theorem. By boundedness and monotonicity of k, assertion (ii) holds. Finally, by boundedness of k,

$$\infty > \int_0^\infty \dot{k}(t)dt \geq \int_0^\infty y^2(t)dt$$

and so, by Lemma 9, we conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof. □

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