Frictional contact of a fibre and an elastic solid

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Abstract

Motivated by an asymptotic model of pullout test for a frictional fibre in a solid, we consider a boundary layer model problem for an elastic space containing an infinite cylindrical fibre with a frictional interface. In the region where frictional sliding occurs, the transfer of load across the interface is governed by a Coulomb friction law. Outside the slipping region the fibre and the surrounding matrix are perfectly bonded. The problem is reduced to a singular integral equation which is analyzed in detail. We study two cases: (i) the case when the slipping region is finite, and (ii) the case when this region is semi-infinite. We assume that the Mode-II stress intensity factor vanishes at the end of the slipping zone. The latter assumption is used to determine the boundary between the two regions. Generalizations include the case of fibre debonding accompanied by opening and frictional sliding at the fibre-matrix interface.

1 Introduction

1.1 Background

Mathematical models of fibre-reinforced composites and analysis of fibres with frictional interfaces have many attractive applications in engineering problems dealing with fibre-bridged cracks. Systematic theoretical study of fracture of fibrous composites began more than twenty five years ago with the work of Kelly (1970) and Aveston & Kelly (1973) who analysed the load transfer between fibres and matrix, and discussed the energy balance for cracks in composites containing fibres isotropically arranged in two or in three dimensions. The approach of Aveston and Kelly was extended further by Budiansky, Hutchinson & Evans (1986). One of their key results is the analysis of the shear lag model for a cylindrical fibre and matrix, which was required to compute the energy changes associated with crack propagation. Dollar & Steif (1988, 1989, 1991) presented two-dimensional analytical and numerical models for two-phase composites with Coulomb friction interfaces. Instead of assuming that a constant shear stress acts at the fibre-matrix interface, Dollar and Steif gave a more rigorous analysis, based on a dislocation distribution method, where the friction stress may depend on the normal stress at the fibre surface. The dislocation distribution method was also used by Meda & Steif (1994a,b) who derived a system of integral equations and obtained numerical solutions for three-dimensional cracks bridged by cylindrical fibres. The asymptotic algorithm for modelling of reinforcement of cracks by frictional fibres has been developed by Movchan & Willis (1997); the stress and displacement fields and the length of the slipping region on the fibre surface were obtained by solving a model problem for a fibre in an elastic half-space.

A model problem for a single fibre embedded into an elastic matrix has proved to be an important ingredient of asymptotic and numerical schemes used for description of fibre-reinforced composites. Mathematical models of the pullout test for a single fibre have been widely discussed in the literature. The work of Muki & Sternberg (1970) presents a simplified integral equation model for analysis of load transfer from an elastic rod partially embedded into a half-space. The integral equation approach has been developed by Atkinson & Avila (1982) for analysis of stress components near the debonding cracks on the fibre-matrix interface. The load transfer from a rigid cylindrical inclusion into an elastic half-space was analysed by Selvadurai & Rajapakse (1985) who obtained numerical solution for the case of axially symmetric load. The integral equation model for the two-dimensional fibre pullout test was introduced and studied numerically by Steif & Hoysan (1986). Freund (1992) studied a model describing sliding of circular fibre along a hole in an elastic solid, and obtained asymptotic solutions for the cases when the fibre is very stiff or very complaint compared to the matrix material.

1.2 Main results

All the papers listed above involve certain conjectures about the behaviour of the physical fields in the vicinity of the end of the frictional contact region. The boundary layer is described by a model problem posed for an infinite elastic solid with an infinite frictional fibre. No exact analytical solution has been published so far for an elasticity problem describing three-dimensional frictional elastic fibre in an elastic matrix.

- In the present work we obtain this analytical solution for a model describing a frictional contact of an elastic solid and a fibre. This problem is reduced to a singular integral equation. We analyse the properties of the integral operator and present the algorithm based on the method of orthogonal polynomials, which leads to an infinite system of linear algebraic equations.

- In the particular case when the friction region is semi-infinite the problem is reduced to an equation of the Wiener-Hopf type. The Wiener-Hopf problem is solved explicitly, and the position of the boundary of the friction region is specified from the analysis of a certain transcendental equation. This solution of the model problem describes a boundary layer that occurs in the junction of a thin frictional fibre and a finite elastic solid.

- At the end of the frictional region the shear mode stress intensity factor is supposed to vanish. Combined with the contact/boundary conditions, and the equations of equilibrium, it specifies the "junction problem" for a cylindrical domain (a frictional fibre) and a conical domain (surrounding elastic matrix). The asymptotics at infinity for the displacement and stress components within the fibre and the elastic matrix are different, and they are not available in the literature. In the present paper, we obtain the asymptotic formulae of the displacement field at infinity is specified within the fibre and the surrounding matrix.

- Generalizations are considered to the case of fibre debonding with opening and frictional sliding.

We note that the model problem for an infinite elastic solid with a frictional fibre is considered for the case of an axi-symmetric point force load; also it is assumed that the material constants of the frictional fibre and the matrix are equal. We emphasize that the method presented here still works for the case of a general load and different elastic materials; however, we intend to keep the technical part of analysis at a reasonable level, so that a reader can easily follow the main ideas and see the new analytical results.

1.3 The structure of the paper

We begin (Introduction) with a simple asymptotic example dealing with the displacement field around a thin frictional fibre in an elastic solid. The asymptotic scheme is easy to follow, and it is sufficiently accurate for evaluation of the length of the slippage zone on the fibre-matrix interface. However, it does not allow one to study the elastic fields in the neighbourhood of the region of transition from the frictional contact to perfect bonding. It gives us enough motivation to look at the model problem associated with the boundary layer. Sections 2 and 3 include formulation of the model problem and analysis of the corresponding integral equation. The case of a semi-infinite slippage region is studied in Section 4. We apply the Wiener-Hopf method to solve the problem, and analyze the behaviour of the solution at infinity and in the vicinity of the transition region.

Further generalizations, discussed in Appendix, include the case of frictional sliding combined with opening at the fibre-matrix interface. Also, Appendix includes technical details with analysis of discrepancy terms for the asymptotic problem about a thin frictional fibre, analysis of the integral equation derived for the main model problem on a frictional fibre in an infinite elastic matrix, and the derivation of the asymptotic formulae characterizing the behaviour of the solution at infinity.

1.4 Practical motivation: a pullout test for a thin fibre with a frictional interface

Here, we consider a simple example of an asymptotic approximation for a solution of the model describing a pullout test for a thin frictional fibre. Without going into technical details, we present the structure of the asymptotic solution (see (1.3)) and indicate the presence of the boundary layer field near the end region of the frictional fibre. The lower-dimensional model given below is simple, and the corresponding numerical estimates for the length of the friction are straightforward to implement. However, analysis of the boundary layer near the end of the region of frictional contact is still a challenge, which will be discussed in the main text of the paper.

1.4.1 Governing equations

Consider a boundary value problem of linear elasticity in the domain $\Omega_\varepsilon = G_\varepsilon \cup \Pi_\varepsilon$, where Π_ε is a thin cylinder of radius εa, $0 < \varepsilon \ll 1$, and G_ε is the remaining part of an elastic solid.

Let $\mathbf{u}$ be the displacement vector which satisfies the equilibrium equation in Ω_ε

$$\mathbf{L}\mathbf{u} := G\Delta\mathbf{u} + G(1-2\nu)^{-1}\text{grad div }\mathbf{u} = 0.$$

The traction boundary condition on $\partial\Omega_\varepsilon \setminus \partial\Pi_\varepsilon$ is

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}) = \mathbf{p},$$

and the homogeneous Dirichlet boundary conditions at the end S_ε of the thin fibre Π_ε are of the form:

$$\mathbf{u} = \boldsymbol{\phi} = const \quad \text{on} \quad S_\varepsilon. \tag{1.1}$$

On some part S_0 of the boundary of the elastic cylinder Π_ε the fibre and the surrounding matrix are perfectly bonded, whereas on the remaining part of its lateral surface S_f the transfer of load across the interface is governed by a Coulomb friction law:

$$[\mathbf{u}] = 0, \quad [\boldsymbol{\sigma}^{(n)}(\mathbf{u})] = 0 \quad \text{on} \quad S_0,$$

$$[u_r] = 0, \quad [\boldsymbol{\sigma}^{(n)}(\mathbf{u})] = 0, \quad \tau_{rz} = -\varepsilon\mu\sigma_r, \quad \mu > 0 \quad \text{on} \quad S_f. \tag{1.2}$$

1.4.2 The structure of the asymptotic solution

The asymptotic approximation of the displacement field $\mathbf{u}$ is sought in the form

$$\mathbf{u}(\mathbf{x};\varepsilon) \sim \mathbf{u}^{(0)}(\mathbf{x}) + [\chi_f \mathfrak{U}_f(\mathbf{x},\varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x},\varepsilon)]\chi\left(\frac{R}{\varepsilon}\right) + \varepsilon\mathfrak{W}(\mathbf{x},\varepsilon) + \varepsilon\mathfrak{V}(\mathbf{x},\varepsilon), \tag{1.3}$$

where χ_f and χ_m are the characteristic functions of the fibre and matrix, $\mathbf{u}^{(0)}$ is the solution of the model problem associated with the homogeneous solid (without any defect); the terms $\mathfrak{U}_f$, $\mathfrak{U}_m$ compensate for a discrepancy produced by $\mathbf{u}^{(0)}$ at the lateral surface of the cylindrical fibre; the last two terms $\mathfrak{W}$ and $\mathfrak{V}$ are the boundary layer fields that exist in the neighbourhoods of the ends of the friction region. The subscript indices f, m in $\mathfrak{U}_f$, $\mathfrak{U}_m$ stand for "fibre" and "matrix", respectively, so that the support of $\chi_f\mathfrak{U}_f$ is located within the cylindrical fibre, and the support of $\chi_m\mathfrak{U}_m$ is in the surrounding matrix. These terms $\mathfrak{U}_m$, $\mathfrak{U}_f$ were derived by Movchan and Willis (1997) and have the following form

$$\mathfrak{U}_f = v_3(z)\mathbf{e}^{(3)} - \varepsilon\rho\frac{dv_3(z)}{dz}(z)(\zeta_1,\zeta_2,0)^T + \varepsilon v_3^{(1)}(z)\mathbf{e}^{(3)} + O(\varepsilon^2), \tag{1.4}$$

$$\mathfrak{U}_m = -\varepsilon\rho a^2\frac{dv_3(z)}{dz}(z)||\zeta||^{-2}(\zeta_1,\zeta_2,0)^T + O(\varepsilon^2),$$

with $||\zeta||^2 = \zeta_1^2 + \zeta_2^2$, $\rho = \frac{\nu}{2(1-\nu)}$, $R = l_s - z$; ν is the Poisson ratio, and l_s represents the length of the friction region.

1.4.3 Derivation of a one-dimensional simplified model

The function $v_3(z)$ satisfies the following differential equation

$$\frac{d^2v_3}{dz^2}(z) - \frac{\mu\nu}{a}\frac{dv_3}{dz}(z) = \frac{2\mu(1-\nu^2)}{aE}\tilde{\sigma}_r, \quad z \in (0, l_s), \tag{1.5}$$

where $\tilde{\sigma}_r$ is the compressive radial stress that exists in the solid without the fibre. We need to specify the boundary conditions for the function v_3 at the ends of the interval $(0, l_s)$.

Also, in (1.3) we use a sufficiently smooth cut-off function χ such that $\chi(t) = 1 \quad$ when $\quad t > 1$, and $\chi(t) = 0 \quad$ when $\quad t < \frac{1}{2}$. The product

$$[\chi_f \mathfrak{U}_f(\mathbf{x}, \varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x}, \varepsilon)]\chi\left(\frac{R}{\varepsilon}\right) \tag{1.6}$$

being substituted into the Lamé system and the boundary conditions gives the discrepancy terms containing the derivatives of χ. An explicit form of this discrepancy is given Appendix A.

Formal analysis (for details see (A.1)-(A.3)) shows that the leading-order discrepancy in the equilibrium equations and the boundary conditions is proportional to $v_3(0)$. We can set

$$v_3(0) = 0, \tag{1.7}$$

and hence, remove the error of order $O(\varepsilon^{-2})$ in the equilibrium equations. In a similar way, we can assume, without loss of generality, that $v_3^{(1)}(0) = 0$.

Note that the next-order discrepancy terms in the right-hand sides (see (A.4), (A.5)) vanish when

$$\frac{dv_3}{dz}(0) = 0. \tag{1.8}$$

In the text bellow the conditions $v_3(0) = 0$ and $\frac{dv_3}{dz}(0) = 0$ will be used as the boundary data for the differential equation (1.5) at the upper end of the fibre.

Remark. In our asymptotic model the function v_3 represents the leading-order approximation of the displacement jump, and the requirement for $v_3(0)$ and $\frac{dv_3}{dz}(0)$ to vanish is consistent with the analysis of Section 4 dealing with a semi-infinite frictional fibre.

To make sure that the error produced by the expansion (1.4) in the boundary condition (1.1) is small, we set

$$v_3(-l_s) = \phi. \tag{1.9}$$

1.4.4 Solution of the one-dimensional model

The leading term of the longitudinal displacement jump can be specified explicitly. Equation (1.5) and the boundary conditions (1.7), (1.8) and (1.9) give the function v_3 and the length l_s of the friction region, i.e.

$$v_3(z) = -\frac{2(1-\nu^2)\tilde{\sigma}_r}{E\nu}\left[z - \frac{a}{\mu\nu}(e^{\frac{\mu\nu z}{a}} - 1)\right],$$

and l_s is a unit positive root of the transcendental equation

$$e^{-\frac{\mu\nu l_s}{a}} = 1 - \frac{\mu\nu l_s}{a} + \frac{E\mu\nu^2}{2a(1-\nu^2)\tilde{\sigma}_r}\phi.$$

Fig. 1 shows the length l_s of the slipping region as a function of the non-dimensional parameter $E\phi/\tilde{\sigma}a$, for different values of the Poisson ratio ν and the friction coefficient $\varepsilon\mu$. One can see that the bigger value of l_s corresponds to the bigger Poisson ratio ν and the smaller friction coefficient $\varepsilon\mu$.

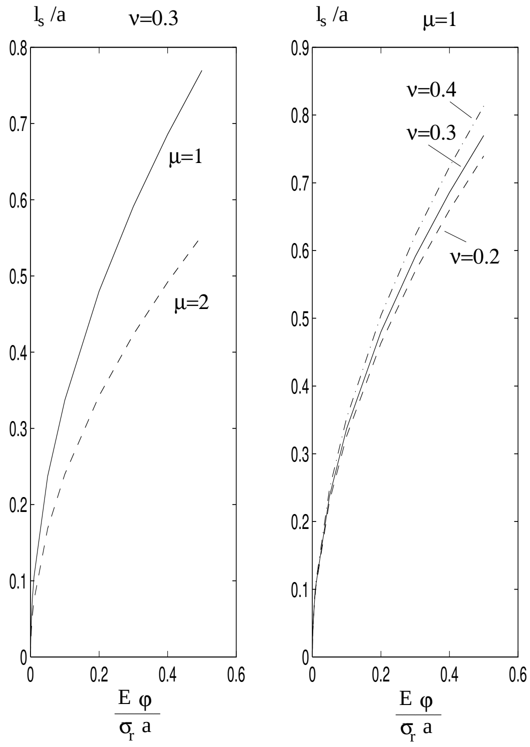

Figure 1: Pull-out problem for a thin frictional fibre: evaluation of the length of the frictional region as a function of normalized pull-out displacement. The diagram on the left presents the graphs of the frictional length function for different values of the friction coefficient $\varepsilon\mu$ and fixed value of the Poisson ratio ν; the diagram on the right includes the graphs of the frictional length function for different values of the Poisson ration and a fixed value of the friction coefficient

1.4.5 The challenge still exists

The asymptotic scheme described above is sufficiently simple and accurate to predict the length of the slippage region on the fibre-matrix interface. However, in order to know the distribution of stress and strain in the immediate neighbourhood of the end of the slippage region, one needs the field of the boundary layer type (we used the notation $\mathfrak{W}$ in the text above). The analysis of the model describing the boundary layer in the end region of the frictional fibre was an open problem. Here, we present an analytical solution of this problem and study its asymptotic behaviour at the end of the slippage zone and at infinity.

2 Formulation of the model problem

The problem considered is shown schematically in Fig. 2. An infinite circular cylinder Π of radius a,

$$\Pi = \{(r, \theta, z) : \ 0 < r < a, \ 0 < \theta \leq 2\pi, \ z \in \mathbb{R}_1\},$$

is embedded into an infinite matrix $\mathbb{R}_3 \setminus \Pi$; here (r, θ, z) are the cylindrical coordinates. Both the cylinder and matrix materials are isotropic and have the same Young's modulus E and Poisson's ratio ν.

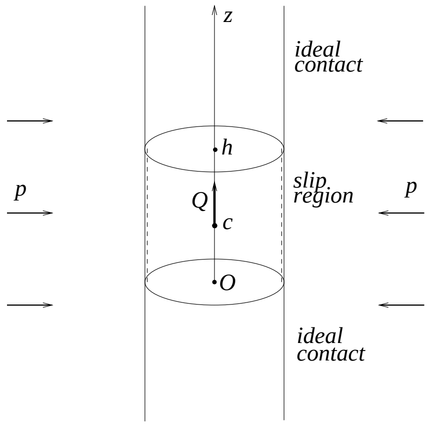

Figure 2: Finite region of frictional contact on the fibre-matrix interface.

We assume that a point force $Q\mathbf{e}^{(3)}$ acts at $(0,0,c)$ along the fibre axis, and a uniform lateral compression $\sigma_r^\infty = -p,\ p \geq 0$, is applied at infinity. The interface $\partial\Pi$ consists of two parts: $\partial\Pi_f = \{(r,\theta,z) \in \partial\Pi :\ \ 0 < z < h\}$ where frictional sliding occurs, and $\partial\Pi_c = \{(r,\theta,z) \in \partial\Pi :\ z \in (-\infty, 0) \cup (h,\infty)\}$ where the fibre and the surrounding matrix are perfectly bonded; $\partial\Pi = \partial\Pi_f \bigcup \partial\Pi_c$. The transfer of load across $\partial\Pi_f$ is governed by a Coulomb friction law, so the conditions on $\partial\Pi_f$ are

$$\tau_{rz}^{(total)} + \mu\sigma_r^{(total)} = 0, \quad \sigma_r^{(total)} < 0, \quad [u_r^{(total)}] = 0, \quad [\sigma_r^{(total)}] = [\tau_{rz}^{(total)}] = 0 \ \text{ on } \ \partial\Pi_f.$$

Outside the slipping region

$$|\tau_{rz}^{(total)}| < |\mu\sigma_r^{(total)}|,\ \sigma_r^{(total)} < 0,\ [u_r^{(total)}] = [u_z^{(total)}] = 0,\ [\sigma_r^{(total)}] = [\tau_{rz}^{(total)}] = 0 \ \text{ on } \ \partial\Pi_s.$$

Here, $[\cdot]$ denotes the jump of a function across the interface: $[f] := f\,|_{r=a-0} - f|_{r=a+0}$; $u_r^{(total)}$, $u_z^{(total)}$ are the radial and axial displacements respectively, $\tau_{rz}^{(total)}$ $\sigma_r^{(total)}$ are the stress components, and $|\mu|$ is the friction coefficient; the sign of μ is verified a posteriori from the condition $\mu[u_z] < 0$ (see, for example, Dundurs and Comninou (1979)). The problem is axi-symmetric, i.e. the stress and displacement fields are independent of θ, and $\tau_{r\theta}^{(total)} = \tau_{z\theta}^{(total)} = 0, \quad u_\theta^{(total)} = 0$. The quantities c and h are so far unknown, they will be chosen in such a way that on $\partial\Pi_c$ the stress components are bounded as $z \to 0$ and as $z \to h$.

3 An integral equation and components of tractions

3.1 Representation for the displacement and stress

The resulting solution $\boldsymbol{\sigma}^{(total)}$ is represented as a sum of three model fields: $\boldsymbol{\sigma}^{(total)} = \boldsymbol{\sigma}^{(\infty)} + \boldsymbol{\sigma}^{(K)} + \boldsymbol{\sigma}$, where

i) $\boldsymbol{\sigma}^\infty$ is the uniform stress field due to the lateral compression,

$$\sigma_\theta^{(\infty)} = \sigma_r^{(\infty)} = -p \quad (p \geq 0), \quad \sigma_z^{(\infty)} = -2\nu p, \quad \tau_{rz}^{(\infty)} = 0;$$

ii) $\boldsymbol{\sigma}^{(K)}$ is the solution of Kelvin's problem on the point force $Q\mathbf{e}^{(3)}$ in a unbounded three-dimensional elastic space (see, e.g. Westergaard, 1952)

$$\sigma_r^{(K)} = B\left(\kappa_-\frac{z_c}{R^3} - \frac{3r^2 z_c}{R^5}\right), \quad \sigma_\theta^{(K)} = B\kappa_-\frac{z_c}{R^3}, \quad \sigma_z^{(K)} = -B\left(\kappa_-\frac{z_c}{R^3} + \frac{3z_c^3}{R^5}\right),$$

$$\tau_{rz}^{(K)} = -B\left(\kappa_-\frac{r}{R^3} + \frac{3rz_c^2}{R^5}\right), \quad \tau_{r\theta}^{(K)} = \tau_{z\theta}^{(K)} = 0,$$

with $z_c = z - c$, $R^2 = r^2 + z_c^2$, $B = [4\kappa_+\pi]^{-1}Q$, $\kappa_\pm = \frac{1}{2}(\kappa \pm 1)$, $\kappa = 3 - 4\nu$; and

iii) $\boldsymbol{\sigma}$ is the field which compensates for the discrepancies left by $\boldsymbol{\sigma}^{(\infty)}, \boldsymbol{\sigma}^{(K)}$ in the conditions of the frictional contact on $\partial\Pi_f$:

$$\tau_{rz} + \mu\sigma_r = f(z), \quad [u_r] = 0 \text{ on } \partial\Pi_f, \tag{3.1}$$

where

$$f(z) = \mu p + \frac{Q}{4\pi\kappa_+ R_a^3}\left[\kappa_-(a - \mu z_c) + \frac{3a}{R_a^2}z_c(z_c + \mu a)\right], \quad R_a = \sqrt{a^2 + z_c^2}.$$

The displacement and stress components are represented via the Love function $\phi(r, z)$ as follows:

$$2Gu_r = -\frac{\partial^2\phi}{\partial r\partial z}, \quad 2Gu_z = \left(\kappa_+\Delta - \frac{\partial^2}{\partial z^2}\right)\phi + const,$$

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\Delta - \frac{\partial^2}{\partial r^2}\right)\phi, \quad \tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\phi,$$

where $G = [2(1+\nu)]^{-1}E$ is the shear modulus, and Δ is the Laplacian operator; for the axi-symmetric case

$$\Delta = \Delta_r + \frac{\partial^2}{\partial z^2}, \quad \Delta_r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

3.2 Boundary value problem for the Love function

The function ϕ satisfies the following boundary value problem

$$\Delta^2\phi(r, z) = 0, \quad r \in (0, \infty) \setminus \{a\}, \ -\infty < z < \infty \tag{3.2}$$

$$\frac{\partial}{\partial z}\left(\kappa_+\Delta_r + \kappa_-\frac{\partial^2}{\partial z^2}\right)\phi(a-0, z) - \frac{\partial}{\partial z}\left(\kappa_+\Delta_r + \kappa_-\frac{\partial^2}{\partial z^2}\right)\phi(a+0, z) = \chi(z), \tag{3.3}$$

$$\frac{\partial^2\phi}{\partial r\partial z}(a-0, z) = \frac{\partial^2\phi}{\partial r\partial z}(a+0, z), \tag{3.4}$$

$$\frac{\partial}{\partial z}\left(\nu\Delta - \frac{\partial^2}{\partial r^2}\right)\phi(a-0, z) = \frac{\partial}{\partial z}\left(\nu\Delta - \frac{\partial^2}{\partial r^2}\right)\phi(a+0, z), \tag{3.5}$$

$$\frac{\partial}{\partial r}\left[(1-\nu)\Delta_r - \nu\frac{\partial^2}{\partial z^2}\right]\phi(a-0, z) = \frac{\partial}{\partial r}\left[(1-\nu)\Delta_r - \nu\frac{\partial^2}{\partial z^2}\right]\phi(a+0, z), \quad z \in \mathbb{R}_1. \tag{3.6}$$

The right-hand side $\chi(z)$ in (3.3) represents the unknown jump of the function $\frac{\partial}{\partial z}(2Gu_z)$ across the interface. The solution of problem (3.2)-(3.6), the function $\phi(r,z)$ is sought in the class $\mathcal{P}^4\{a\}$ with respect to r and $\mathcal{A}_1^4(\mathbb{R}_1)$ with respect to z. Here, $\mathcal{P}^m\{a\}$ denotes a class of functions whose first m derivatives are continuous everywhere on the interval $(0,\infty)$ except at the point a. A function $f(x)$ is said to belong to the class $\mathcal{A}_1^n(\mathbb{R}_1)$ if its k-th derivative $f^{(k)}(x)$ $(k=0,1,\dots,n)$ is absolutely integrable in $\mathbb{R}_1$. Note that the function χ is equal to zero outside the interval $[0,h]$, and the components of the stress tensor vanish as $z^2+r^2\to\infty$.

Remark. In the formulation of the problem (3.2)-(3.6) we did not use the first condition (3.1) of the frictional contact on $\partial\Pi_f$. It will be employed on the later stage to define the function χ.

3.3 Fourier transforms and an integral equation

Applying the Fourier transform

$$\phi_\alpha(r)=\int_{-\infty}^{\infty}\phi(r,z)e^{i\alpha z}dz,\quad \phi(r,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_\alpha(r)e^{-i\alpha z}d\alpha$$

to the boundary value problem (3.2)-(3.6), we obtain

$$(\Delta_r^2-2\alpha^2\Delta_r+\alpha^4)\phi_\alpha(r)=0,\quad r\in(0,a)\cup(a,\infty), \tag{3.7}$$

$$-i\alpha[(\kappa_+\Delta_r-\kappa_-\alpha^2)\phi_\alpha(a-0)-(\kappa_+\Delta_r-\kappa_-\alpha^2)\phi_\alpha(a+0)]=\chi_\alpha, \tag{3.8}$$

$$\frac{d}{dr}\phi_\alpha(a-0)-\frac{d}{dr}\phi_\alpha(a+0)=0, \tag{3.9}$$

$$\left(\nu\Delta_r-\frac{d^2}{dr^2}-\nu\alpha^2\right)\phi_\alpha(a-0)-\left(\nu\Delta_r-\frac{d^2}{dr^2}-\nu\alpha^2\right)\phi_\alpha(a+0)=0, \tag{3.10}$$

$$[(1-\nu)\Delta_r+\nu\alpha^2]\phi_\alpha(a-0)-[(1-\nu)\Delta_r+\nu\alpha^2]\phi_\alpha(a+0)=0, \tag{3.11}$$

where

$$\chi_\alpha=\int_0^h\chi(z)e^{i\alpha z}dz.$$

The bounded solution of the boundary value problem (3.7)-(3.11) has the form

$$\phi_\alpha(r)=\begin{cases} C_0(\alpha)I_0(|\alpha|r)+C_1(\alpha)|\alpha|rI_1(|\alpha|r),\ r<a\\ D_0(\alpha)K_0(|\alpha|r)+D_1(\alpha)|\alpha|rK_1(|\alpha|r),\ r>a\end{cases},$$

with the coefficients

$$C_0=\frac{\kappa_+K_1(|\alpha|a)+|\alpha|aK_0(|\alpha|a)}{i\kappa_+\alpha^2d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\ \chi_\alpha,\quad C_1=-\frac{K_1(|\alpha|a)}{i\kappa_+\alpha^2d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\ \chi_\alpha,$$

$$d(\alpha)=\alpha[I_0(\alpha)K_1(\alpha)+I_1(\alpha)K_0(\alpha)],$$

and

$$D_0=\frac{-\kappa_+I_1(|\alpha|a)+|\alpha|aI_0(|\alpha|a)}{i\kappa_+\alpha^2d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\ \chi_\alpha,\quad D_1=-\frac{I_1(|\alpha|a)}{i\kappa_+\alpha^2d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\ \chi_\alpha,$$

$I_j(x), K_j(x)$ $(j = 0, 1)$ are Bessel functions of imaginary argument.

Formal analysis (technical derivation has been moved to Appendix B; see (B.6) and (B.4)) of interface contact conditions and the inverse Fourier transform give the following singular integral equation for the function χ with the Cauchy kernel

$$\frac{1}{\pi}\int_0^{\lambda_0} \chi(a\zeta)\left[\frac{1}{\zeta - z} + \mu_0 \log\frac{1}{|\zeta - z|} + K(\zeta - z)\right] d\zeta = F(z;\lambda_1), \quad 0 < z < \lambda_0, \tag{3.12}$$

where $\mu_0 = \nu_0\mu,\ \lambda_1 = c/a,\ K(t)$ is a continuous function (its explicit form is given by (B.9)), and

$$F(z;\lambda_1) = -2\kappa_+\mu p - \frac{\kappa_-[1-\mu(z-\lambda_1)] + 3(z-\lambda_1)(z-\lambda_1+\mu)[(z-\lambda_1)^2+1]^{-1}}{2\pi a^2[(z-\lambda_1)^2+1]^{3/2}} Q. \tag{3.13}$$

3.4 Components of traction

Due to formulae (B.6), expressions (B.2) for the stresses can be rewritten as follows

$$\sigma_r(a, az) = \frac{1}{\pi\kappa_+}\int_0^{\lambda_0} \chi(a\zeta)\left[\frac{\nu_0}{2}\log|\zeta - z| + s_0^{(0)}(\zeta - z)\right] d\zeta, \tag{3.14}$$

$$\tau_{rz}(a, az) = -\frac{1}{\pi\kappa_+}\int_0^{\lambda_0} \chi(a\zeta)\left[\frac{1}{2(\zeta - z)} + s_1^{(0)}(\zeta - z)\right] d\zeta \tag{3.15}$$

Finally, we note that the stresses are bounded as $z \to -0$ and $z \to \lambda_0 + 0$. This fact is verified by the analysis of integrals with the logarithmic and Cauchy kernels and the density that is bounded and vanishes at the ends, i.e. that is of the form (A.2). The position of the unknown a priori points h and c is defined from the system of two non-linear algebraic equations (A.6) and (A.7) with respect to λ_0 and λ_1. The auxiliary function, $\chi(z)$, is represented by expansion (A.2) with the coefficients X_m defined approximately from the truncated system (A.8).

Generalization to the case of debonding with zones of opening and frictional sliding is recorded in Appendix B.

4 Semi-infinite slipping region

4.1 Reduction to a Wiener-Hopf problem

Now, consider the limit case when $h \to \infty$ (the region of the frictional contact is semi-infinite). In this case we assume that $p = 0$,[1] and the only external load is a point force applied at the point $(r, z) = (0, c)$. The contact conditions on $\partial\Pi$ have the form

$$(\tau_{rz} + \mu\sigma_r)|_{r=a} = f(z), \quad [u_r] = 0, \quad z > 0,$$
$$[\tau_{rz}] = [\sigma_r] = 0, \quad [u_r] = [u_z] = 0, \quad z < 0. \tag{4.1}$$

[1]If $p \neq 0$ and the region of frictional contact is semi-infinite, the displacement field has an inadmissible growth at infinity required for the balance of forces applied to the infinite fibre.

To apply the Fourier transform to the boundary conditions (4.1), we introduce the two functions

$$f_+(z) = \left\{ \begin{array}{cc} f(z), & z > 0 \\ 0, & z < 0 \end{array} \right., \quad f_-(z) = \left\{ \begin{array}{cc} 0, & z > 0 \\ (\tau_{rz} + \mu\sigma_r)|_{r=a}, & z < 0 \end{array} \right.,$$

and re-write the first condition (4.1) of the frictional contact in the form

$$\tau_{rz} + \mu\sigma_r = f_+(z) + f_-(z), \quad |z| < \infty, \quad r = a. \tag{4.2}$$

We also introduce the notations

$$\Phi^+(\alpha) = \int_0^\infty \chi(a\xi)e^{i\alpha\xi}d\xi, \quad H^+(\alpha;\lambda) = \int_0^\infty h(\xi;\lambda)e^{i\alpha\xi}d\xi, \quad \lambda = \frac{c}{a},$$

$$h(\xi;\lambda) = \frac{\kappa_-[1 - \mu(\xi - \lambda)] + 3(\xi - \lambda)(\xi - \lambda + \mu)[(\xi - \lambda)^2 + 1]^{-1}}{4\pi\kappa_+[(\xi - \lambda)^2 + 1]^{3/2}},$$

$$\Phi^-(\alpha) = \int_{-\infty}^0 [\tau_{rz}(a, a\xi) + \mu\sigma_r(a, a\xi)]e^{i\alpha\xi}d\xi.$$

Here, Φ^+ and Φ^- are unknown functions, analytic in the upper ($\mathbb{C}^+ : Im\alpha > 0$) and the lower ($\mathbb{C}^- : Im\alpha < 0$) half-planes respectively. The function H^+ is known and analytic in the upper half-plane.

Applying the Fourier transform to equation (4.2), we arrive at the following Wiener-Hopf problem

$$\Xi(\alpha)\Phi^+(\alpha) = \Phi^-(\alpha) + \frac{Q}{a^2}H^+(\alpha;\lambda), \quad \alpha \in \mathbb{R}_1 = (-\infty, +\infty), \tag{4.3}$$

where

$$\Xi(\alpha) = \frac{i\,\mathrm{sgn}\,\alpha\ \Sigma_1(|\alpha|) + \mu\Sigma_0(|\alpha|)}{\kappa_+ d(|\alpha|)}.$$

4.2 Factorization and solution of the Wiener-Hopf problem

To factorize the function $\Xi(\alpha)$, we study its properties. Using the asymptotic formulae (B.5), we establish the behaviour of the function $\Xi(\alpha)$ at $\alpha = 0$ and at infinity

$$\Xi(\alpha) = \frac{\mu\nu}{\kappa_+} + \frac{i\alpha}{\kappa_+} + O(\alpha^2\log\alpha), \quad \alpha \to 0, \quad \Xi(\alpha) = \frac{i\,\mathrm{sgn}\,\alpha}{2\kappa_+} - \frac{\mu\nu_0}{2\kappa_+\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \alpha \to \pm\infty.$$

Thus, the function $\Xi(\alpha)$ has a discontinuity at infinity. Let us split $\Xi(\alpha)$ into two parts

$$\Xi(\alpha) = \frac{i}{2\kappa_+}\tanh\pi(\alpha - i\rho)\Xi_0(\alpha)$$

where ρ is a parameter such that $\rho = \frac{1}{\pi}\tan^{-1}2\mu\nu \in \left(0, \frac{1}{2}\right)$. Then the function $\Xi_0(\alpha)$ is continuous at infinity and tends to 1 as $\alpha \to \pm\infty$ and $\alpha \to 0$, i.e.

$$\Xi_0(\alpha) = 1 + O\left(\frac{1}{\alpha}\right), \quad \alpha \to \pm\infty; \quad \Xi_0(\alpha) = 1 + O(\alpha), \quad \alpha \to 0.$$

The values of the real and imaginary parts of the function $\Xi_0(\alpha)$ are presented in Fig. 3.

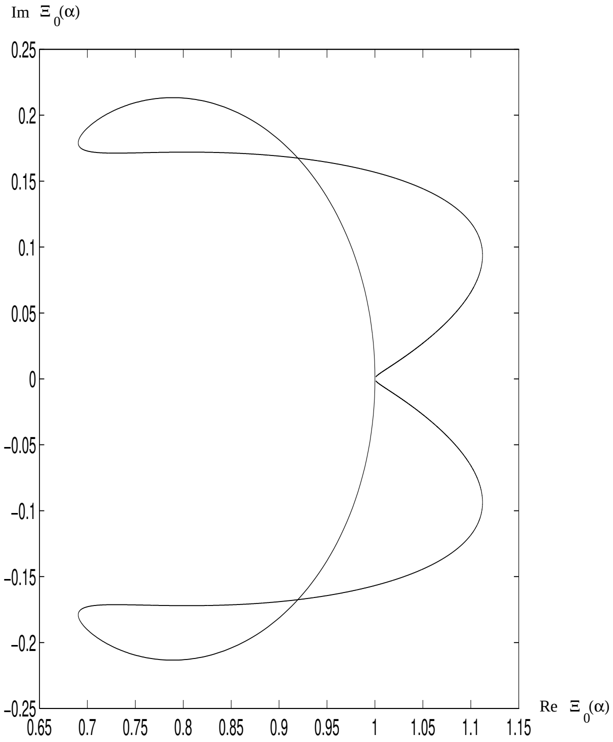

Figure 3: Parametrically defined complex-valued function $\Xi_0(\alpha)$, with α travelling along the contour $\mathbb{R}_1$.

The point α traverses the contour $\mathbb{R}_1$ in the positive direction. The values of ν and μ are 0.3 and 0.1, respectively. The similar behaviour of Re $\Xi_0(\alpha)$ and Im $\Xi_0(\alpha)$ is observed for other admissible values of the parameters ν and μ and $0 < \text{Re}\ \Xi_0(\alpha) < \infty$. Thus,

$$\operatorname{ind}\Xi_0(\alpha) = \frac{1}{2\pi}[\arg \Xi_0(\alpha)]_{\mathbb{R}_1} = 0,$$

and the factorization of the function $\Xi_0(\alpha)$ is given by

$$\Xi_0(\alpha) = \frac{X^+(\alpha)}{X^-(\alpha)}, \quad \alpha \in \mathbb{R}_1, \quad X^\pm(\alpha) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \Xi_0(\beta)}{\beta - \alpha} d\beta \right\}, \quad \alpha \in \mathbb{C}^\pm. \tag{4.4}$$

Using the splitting of the hyperbolic tangent

$$\frac{i}{2\kappa_+} \tanh \pi(\alpha - i\rho) = \frac{K^+(\alpha)}{K^-(\alpha)},$$

$$K^+(\alpha) = \frac{\Gamma(\frac{1}{2} - \rho - i\alpha)}{2\kappa_+ \Gamma(1 - \rho - i\alpha)}, \quad K^-(\alpha) = \frac{\Gamma(\rho + i\alpha)}{\Gamma(\frac{1}{2} + \rho + i\alpha)}, \tag{4.5}$$

we arrive at the desired factorization formula

$$\Xi(\alpha) = \frac{K^+(\alpha)X^+(\alpha)}{K^-(\alpha)X^-(\alpha)}, \quad \alpha \in \mathbb{R}_1.$$

We substitute the last formula into (4.3) and represent the function $K^-(\alpha)X^-(\alpha)H^+(\alpha;\lambda)$ as follows

$$K^-(\alpha)X^-(\alpha)H^+(\alpha;\lambda) = \Psi^+(\alpha) - \Psi^-(\alpha), \quad \alpha \in \mathbb{R}_1,$$

$$\Psi^\pm(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K^-(\beta)X^-(\beta)H^+(\beta;\lambda)}{\beta-\alpha}d\beta, \quad \alpha \in \mathbb{C}^\pm. \tag{4.6}$$

Hence, (4.3) can be written in the form

$$K^+(\alpha)X^+(\alpha)\Phi^+(\alpha) - \frac{Q}{a^2}\Psi^+(\alpha) = K^-(\alpha)X^-(\alpha)\Phi^-(\alpha) - \frac{Q}{a^2}\Psi^-(\alpha), \quad \alpha \in \mathbb{R}_1. \tag{4.7}$$

Then, we take into account the behaviour at infinity

$$|\Phi^\pm(\alpha)| \le C|\alpha|^{-1/2}, \quad \Psi^\pm(\alpha) = O\left(\frac{1}{\alpha}\right), \quad K^\pm(\alpha) = O(\alpha^{-1/2}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm$$

and use Liouville's theorem. Finally, we obtain

$$\Phi^\pm(\alpha) = \frac{Q}{a^2}\frac{\Psi^\pm(\alpha)}{K^\pm(\alpha)X^\pm(\alpha)}, \quad \alpha \in \mathbb{C}^\pm. \tag{4.8}$$

4.3 Evaluation of the stress-intensity factor

Now, let us find the stress intensity factor

$$K_{II} = \lim_{\zeta\to -0}\sqrt{-2\pi a\zeta}\,\tau_{rz}(a, a\zeta). \tag{4.9}$$

The function $\Phi^+(\alpha)$ defined in (4.8) specifies the behaviour of the jump χ as $\zeta \to +0$, i.e. $\chi(a\zeta) \sim X\zeta^{-1/2}, \quad \zeta \to +0, \quad X = const$. According to the relation (3.15), we have

$$\tau_{rz}(a, az) \sim -\frac{X}{2\pi\kappa_+}\int_0^B \frac{\zeta^{-1/2}}{\zeta - z}d\zeta \sim -\frac{X(-z)^{-1/2}}{2\kappa_+} \quad \text{as} \quad z \to -0, \quad B > 0. \tag{4.10}$$

From (4.9) and (4.10), we obtain $K_{II} = -\sqrt{\frac{\pi a}{2}}\frac{X}{\kappa_+}$. Now, we take into account the behaviour at infinity of the function $\Phi^+(\alpha)$ (see Noble, 1988)

$$\Phi^+(\alpha) \sim \frac{X\sqrt{\pi}}{(-i\alpha)^{1/2}}, \quad \alpha \to \infty,\ \alpha \in \mathbb{C}^+$$

and the asymptotics of the same function obtained directly from (4.8)

$$\Phi^+(\alpha) \sim \frac{\kappa_+ Q}{\pi(-i\alpha)^{1/2}a^2}\Psi_0(\lambda), \quad \alpha \to \infty,\ \alpha \in \mathbb{C}^+,$$

$$\Psi_0(\lambda) = \int_{-\infty}^{\infty} K^-(\alpha)X^-(\alpha)H^+(\alpha;\lambda)d\alpha.$$

Thus, $K_{II} = -\frac{1}{\pi}\frac{Q}{\sqrt{2}a^{3/2}}\Psi_0(\lambda)$. The stress component τ_{rz} is bounded at the point $(r, z) = (a, 0)$ if and only if

$$\Psi_0(\lambda) = 0. \tag{4.11}$$

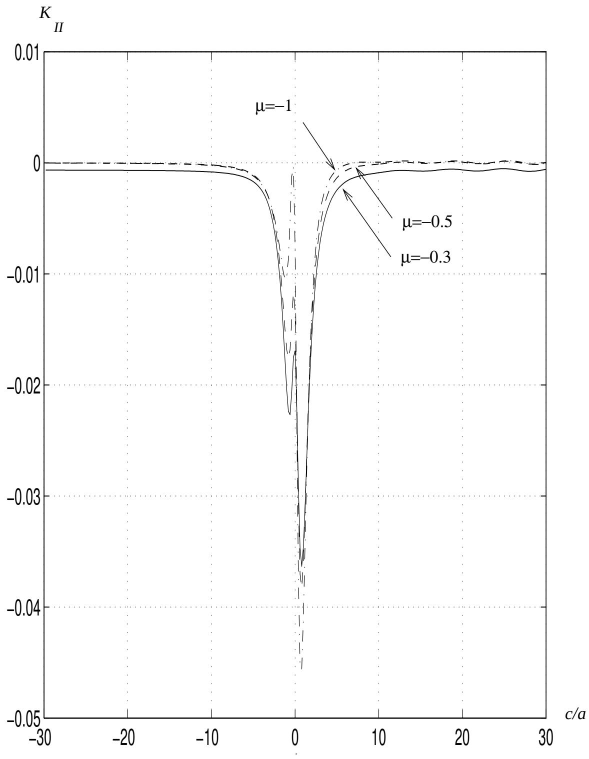

Figure 4: Stress-intensity factor K_{II} versus c/a with $Q = 1$.

This is the transcendental equation for the parameter $\lambda = c/a$.

Note that the stress component σ_r is bounded for any λ, not only for the roots of equation (4.11). This follows from the relation (3.14). Equation (4.11) yields $K_{II} = 0$ which is equivalent to either of the following[2]

$$\left[\frac{\partial u_z}{\partial z}\right]_{r=a} = O(z^{1/2}), \quad (\tau_{rz} + \mu\sigma_r)_{r=a} = O(z^{1/2}) \quad \text{as} \quad z \to 0.$$

In Figs 4 and 5 we present the graphs of the stress-intensity factor K_{II} versus c/a, for

(i) $\mu = -0.3,\ -0.5,\ -1.0$ and $Q = 1$ (Fig. 4).

(ii) $\mu = 0.3,\ 0.5,\ 1.0$ and $Q = -1$ (Fig. 5).

There exists a countable set of roots of equation (4.11) for $c > 0$. The scaled graph of K_{II} versus c/a is shown in Fig. 6 (with $Q = 1, \mu = -0.4$), where the first several roots are observed clearly.

In all computations, the Poisson ratio ν was set to be 0.3. As $|\mu|$ increases, the first root approaches the origin, and the amplitude of oscillations of K_{II} becomes larger. In the case (i) the first root is much greater compared to the first root corresponding to the case (ii). In the earlier papers by Spence (1973), Atkinson (1982), Simonov (1984) and Antipov (1995)

[2]The stress components τ_{rz} and σ_r are bounded and continuous as $z \to 0$, i.e.

$$\sigma_r = A + O(z^{1/2}), \quad \tau_{rz} = -\mu A + O(z^{1/2}), \quad z \to 0, \quad A = const.$$

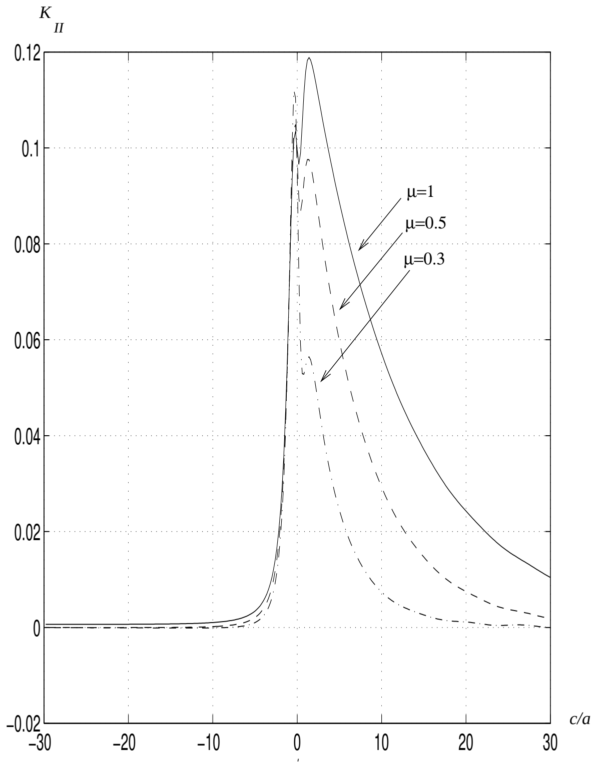

Figure 5: Stress-intensity factor K_{II} versus c/a with $Q = -1$.

similar situation with a countable set of roots was observed in contact problems and models of interfacial cracks.

4.4 Behaviour of the displacement field at infinity

To find the relations for the displacements, we write the expressions for their Fourier transforms

$$2G\kappa_+ \hat{u}_z(r;\alpha) = \frac{a}{i\alpha}\frac{\Phi^+(\alpha a)}{d(|\alpha|a)}\mathfrak{F}(|\alpha|a, r), \tag{4.12}$$

$$2G\kappa_+ \hat{u}_r(r;\alpha) = \frac{a^2\Phi^+(\alpha a)}{d(|\alpha|a)}\mathfrak{G}(|\alpha|a, r), \tag{4.13}$$

where

$$\mathfrak{F}(\alpha, r) = \begin{cases} \alpha\left\{[-\kappa_+ K_1(\alpha) + \alpha K_0(\alpha)]I_0\left(\frac{\alpha r}{a}\right) - \frac{r\alpha}{a}K_1(\alpha)I_1\left(\frac{\alpha r}{a}\right)\right\}, \; r < a \\ \alpha\left\{[\kappa_+ I_1(\alpha) + \alpha I_0(\alpha)]K_0\left(\frac{\alpha r}{a}\right) - \frac{r\alpha}{a}I_1(\alpha)K_1\left(\frac{\alpha r}{a}\right)\right\}, \; r > a \end{cases}, \tag{4.14}$$

$$\mathfrak{G}(\alpha, r) = \begin{cases} [\kappa_+ K_1(\alpha) + \alpha K_0(\alpha)]I_1\left(\frac{\alpha r}{a}\right) - \frac{r\alpha}{a}K_1(\alpha)I_0\left(\frac{\alpha r}{a}\right), \; r < a \\ [\kappa_+ I_1(\alpha) - \alpha I_0(\alpha)]K_1\left(\frac{\alpha r}{a}\right) + \frac{r\alpha}{a}I_1(\alpha)K_0\left(\frac{\alpha r}{a}\right), \; r > a \end{cases}.$$

For technical detail of analysis we refer to Appendix C. The asymptotic behaviour of the function $\frac{\partial u_z}{\partial z}$ as $z \to \infty$ inside of the cylindrical surface differs from the behaviour of this

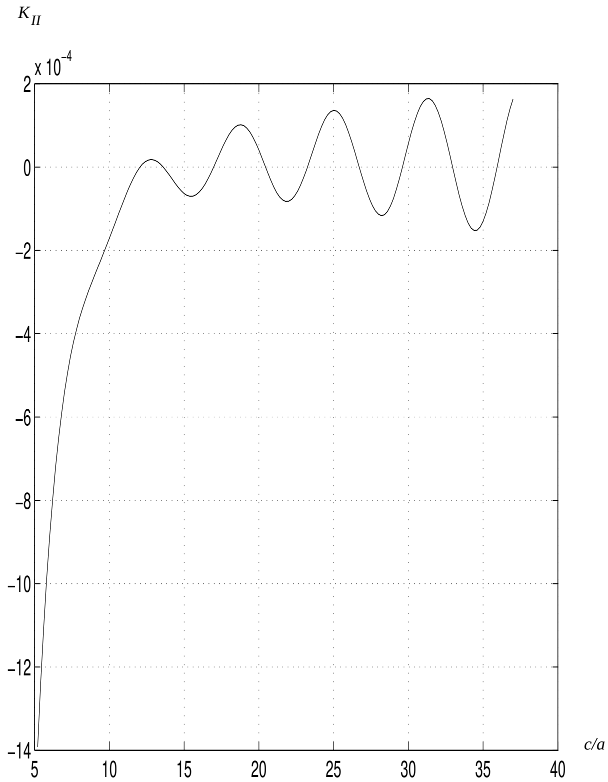

Figure 6: Position of the first several roots for the case $\mu = -0.4, Q = 1$.

function as $|z| \to \infty$ outside the surface, i.e.

$$2G\kappa_+ \frac{\partial u_z}{\partial z}(r,z) = \frac{a^3\Phi^+(0)}{z^3} \left\{ \begin{array}{c} \frac{1}{\nu} \\ \nu - 2 \end{array} \right\} + o\left(\frac{1}{z^3}\right), \quad |z| \to \infty, \quad \left\{ \begin{array}{l} (r,\theta,z) \in \Pi_+ \\ (r,\theta,z) \notin \Pi_+ \end{array} \right., \tag{4.15}$$

where $\Pi_+ = \{(r,\theta,z) : \ 0 \le r < a, \ 0 \le \theta \le 2\pi, \ 0 \le z < \infty\}$.

Omitting the intermediate formulae, let us write the final asymptotic relations for the radial displacement

$$2G\kappa_+ u_r(r,z) = \frac{a^4\Phi^+(0)}{2z^3}\left[(\nu+1)\frac{r}{a} + (\nu-1)\frac{a}{r}\right] + o\left(\frac{1}{z^3}\right), \quad z \to +\infty \quad r > a, \tag{4.16}$$

$$2G\kappa_+ u_r(r,z) = \frac{a^3 r\nu\Phi^+(0)}{z^3} + o\left(\frac{1}{z^3}\right), \quad z \to +\infty \quad r < a, \tag{4.17}$$

$$2G\kappa_+ u_r(r,z) = \frac{a^3 r(1+\nu)\Phi^+(0)}{2z^3} + o\left(\frac{1}{z^3}\right), \quad z \to -\infty \quad 0 < r < \infty. \tag{4.18}$$

We see that the formulae (4.16) and (4.17) coincide as $r \to a$. On the other hand, as $|z| \to \infty$ and $r \to \infty$ the asymptotic formulae (4.16) and (4.18) also come to an agreement with each other.

4.5 Coefficients in the asymptotics at infinity in the exterior of the fibre

Formulae (4.15)-(4.18) give the asymptotics for u_r and $\frac{\partial u_z}{\partial z}$, with r fixed and $|z|$ going to infinity. These relations can be used to derive the asymptotic expressions for u_r and u_z that are uniformly valid on a sphere of a large radius $R = \sqrt{r^2 + z^2}$. Namely, the displacement field $\mathbf{u} = (u_r, 0, u_z)$ decays like a dipole, i.e. $||\mathbf{u}|| = O(R^{-2})$ as $R \to \infty$, or, more specifically, outside the cylinder Π the field $\mathbf{u}$ has the form

$$\mathbf{u} = A\left(\frac{\partial}{\partial x_1}\mathbf{T}^{(1)} + \frac{\partial}{\partial x_2}\mathbf{T}^{(2)}\right) + B\frac{\partial}{\partial z}\mathbf{T}^{(3)} + O(R^{-3}) \quad \text{as} \quad R \to \infty,$$

where A and B are constant coefficients, and $\mathbf{T} = \{\mathbf{T}^{(1)}, \mathbf{T}^{(2)}, \mathbf{T}^{(3)}\}$ is proportional to the Green's tensor, i.e.

$$\mathbf{T}^{(1)} = \begin{pmatrix} \frac{\kappa}{R} + \frac{x_1^2}{R^3} \\ \frac{x_1 x_2}{R^3} \\ \frac{x_1 z}{R^3} \end{pmatrix}, \quad \mathbf{T}^{(2)} = \begin{pmatrix} \frac{x_1 x_2}{R^3} \\ \frac{\kappa}{R} + \frac{x_2^2}{R^3} \\ \frac{x_2 z}{R^3} \end{pmatrix}, \quad \mathbf{T}^{(3)} = \begin{pmatrix} \frac{x_1 z}{R^3} \\ \frac{x_2 z}{R^3} \\ \frac{\kappa}{R} + \frac{z^2}{R^3} \end{pmatrix}.$$

In cylindrical coordinates

$$\frac{\partial}{\partial x_1}\mathbf{T}^{(1)} + \frac{\partial}{\partial x_2}\mathbf{T}^{(2)} = -\frac{r[\kappa r^2 + (\kappa - 3)z^2]}{R^5}\mathbf{e}^{(r)} - \frac{z(r^2 - 2z^2)}{R^5}\mathbf{e}^{(z)},$$

$$\frac{\partial}{\partial z}\mathbf{T}^{(3)} = \frac{r(r^2 - 2z^2)}{R^5}\mathbf{e}^{(r)} - \frac{z[(\kappa - 2)r^2 + (\kappa + 1)z^2]}{R^5}\mathbf{e}^{(z)},$$

where $\mathbf{e}^{(r)}$ and $\mathbf{e}^{(z)}$ are the basis vectors. Thus, the components u_r and u_θ have the following asymptotics as $R \to \infty$ and $(r, \theta, z) \notin \Pi_+$

$$u_r \sim -A\frac{r[\kappa r^2 + (\kappa - 3)z^2]}{R^5} + B\frac{r(r^2 - 2z^2)}{R^5}$$

$$u_z \sim -A\frac{z(r^2 - 2z^2)}{R^5} - B\frac{z[(\kappa - 2)r^2 + (\kappa + 1)z^2]}{R^5}. \tag{4.19}$$

The constant A and B can be determined by comparison of (4.19) with (4.15)-(4.18). Namely, the constants A and B satisfy the following equations

$$2\nu A - B = \frac{a^3(1 + \nu)}{8G\kappa_+}\Phi^+(0), \quad A - \kappa_+ B = \frac{a^3(2 - \nu)}{8G\kappa_+}\Phi^+(0),$$

and hence

$$A = -\frac{\nu a^3}{8(1 - 2\nu)G\kappa_+}\Phi^+(0) \quad \text{and} \quad B = -\frac{(1 - \nu)a^3}{8(1 - 2\nu)G\kappa_+}\Phi^+(0). \tag{4.20}$$

Remark. Note that the asymptotic behaviour of the displacement components in the cylindrical fibre differs from the asymptotics (4.19) obtained for a sphere of a large radius R when $(r, \theta, z) \in \mathbb{R}_3 \setminus \Pi_+$. The problem discussed above can be regarded as a junction problem for a semi-infinite elastic fibre and an elastic matrix. The power decay of displacement components is due to the presence of the frictional contact with the surrounding matrix; in the case of homogeneous or self-balanced traction conditions on the fibre surface the decay would be exponential.

4.6 Frictionless contact

Let us consider the case of a semi-infinite frictionless ($\mu = 0$) region, i.e. let the following conditions be held

$$\tau_{rz}(a,z) = f(z), \quad [u_r] = 0, \quad z > 0,$$
$$[\tau_{rz}] = [\sigma_r] = 0, \quad [u_r] = [u_z] = 0, \quad z < 0.$$

As before, we have the functional equation (4.3), with

$$\Xi(\alpha) = \frac{i \operatorname{sgn} \alpha \Sigma_1(|\alpha|)}{\kappa_+ d(|\alpha|)}.$$

In this case the behaviour of the function $\Xi(\alpha)$ as $\alpha \to 0$ and $\alpha \to \infty$ is described by the formulae

$$\Xi(\alpha) = \frac{i\alpha}{\kappa_+} + O(\alpha^3 \log \alpha), \quad \alpha \to 0,$$
$$\Xi(\alpha) = \frac{i \operatorname{sgn} \alpha}{2\kappa_+} + \frac{3i \operatorname{sgn} \alpha}{16\kappa_+ \alpha^2} + o\left(\frac{1}{\alpha^2}\right), \quad \alpha \to \pm\infty.$$

Next, we construct the factorization of $\Xi(\alpha)$ in the form

$$\Xi(\alpha) = \frac{i\alpha K_0^+(\alpha) X^+(\alpha)}{K_0^-(\alpha) X^-(\alpha)}, \quad \alpha \in \mathbb{R}_1, \tag{4.21}$$

where

$$K_0^+(\alpha) = \frac{\Gamma(\frac{1}{2} - i\alpha)}{\Gamma(1 - i\alpha)}, \quad K_0^-(\alpha) = \frac{\Gamma(1 + i\alpha)}{\Gamma(\frac{1}{2} + i\alpha)},$$

and $X^\pm(\alpha)$ are determined by (4.4), with

$$\Xi_0(\alpha) = \frac{2\Sigma_1(|\alpha|)}{\tanh \pi|\alpha| d(|\alpha|)},$$

$$\Xi_0(\alpha) = 1 + O\left(\frac{1}{\alpha^2}\right), \quad \alpha \to \pm\infty; \quad \Xi_0(\alpha) = \frac{2}{\pi} + O(\alpha^2 \log \alpha), \quad \alpha \to 0.$$

Since $\Xi_0(\alpha)$ is an even function, $\operatorname{ind} \Xi_0(\alpha) = 0$. Substituting (4.19) into the boundary condition (4.3) and introducing the function (4.6,) we arrive at the condition

$$\mathbb{F}(\alpha) := i\alpha K_0^+(\alpha) X^+(\alpha) \Phi^+(\alpha) - \frac{Q}{a^2}\Psi^+(\alpha) = K_0^-(\alpha) X^-(\alpha) \Phi^-(\alpha) - \frac{Q}{a^2}\Psi^-(\alpha), \ \alpha \in \mathbb{R}_1.$$

Due to Liouville's theorem, $\mathbb{F}(\alpha) = C, \quad \alpha \in \mathbb{C}$, where C is an arbitrary constant. Thus, the solution of the problem is given by

$$\Phi^-(\alpha) = \frac{C + Qa^{-2}\Psi^-(\alpha)}{K_0^-(\alpha) X^-(\alpha)}, \quad \Phi^+(\alpha) = 2\kappa_+ \frac{C + Qa^{-2}\Psi^+(\alpha)}{i\alpha K_0^+(\alpha) X^+(\alpha)}.$$

The function $\Phi^+(\alpha)$ is analytic at $\alpha = 0$ if and only if $C = -Qa^{-2}\Psi^+(0)$.

Calculating the stress intensity factor K_{II} as in the case $\mu \neq 0$, we obtain $K_{II} = \sqrt{2a}C$.

Conclusion

We have analyzed a model problem of elasticity for an infinite frictional fibre in an elastic space. The problem was reduced to an integral equation, and a numerical algorithm based on the method of orthogonal polynomials has been developed to solve this equation. For the case of a semi-infinite frictional crack, the Wiener-Hopf technique was applied to obtain the exact solution and explicit asymptotic formulae for the stress and displacement fields at infinity. It has been shown that the displacement field has a power decay at infinity with the coefficients which are different for the elastic fibre and surrounding matrix. These coefficients have been evaluated explicitly. The shear-mode stress intensity factor has been calculated for the case when a point force was applied along the fibre axis. The position of the friction contact region at the interface has been found by solving the equation $K_{II} = 0$. Generalizations are considered in Appendix for the model formulation including an infinite fibre with a finite slippage and opening at the interface subject to nonzero lateral compression. We reduce the problem to a system of integral equations with respect to the displacement jumps, and outline the algorithm of solution.

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Appendix A: Discrepancy terms associated with derivatives of the cut-off function

Here, we give the structure of the discrepancy terms associated with derivatives of the cut-off function employed in the asymptotic example of Section 1.4.

The discrepancy terms in the Lamé system are defined by

$$\mathbf{L}\{[\chi_f \mathfrak{U}_f(\mathbf{x},\varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x},\varepsilon)]\chi\left(\frac{R}{\varepsilon}\right)\}$$

$$= \frac{G}{\varepsilon(1-2\nu)}\frac{\partial\chi}{\partial z}\begin{pmatrix} 2(1-2\nu)\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 2(1-2\nu)\frac{\partial}{\partial z} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 4(1-\nu)\frac{\partial}{\partial z}\end{pmatrix}[\chi_f \mathfrak{U}_f(\mathbf{x},\varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x},\varepsilon)]$$

$$+\frac{G}{\varepsilon^2(1-2\nu)}\begin{pmatrix} 0 & 0 & \frac{\partial^2}{\partial z\partial\zeta_1}\chi \\ 0 & 0 & \frac{\partial^2}{\partial z\partial\zeta_2}\chi \\ \frac{\partial^2}{\partial z\partial\zeta_1}\chi & \frac{\partial^2}{\partial z\partial\zeta_2}\chi & 0\end{pmatrix}[\chi_f \mathfrak{U}_f(\mathbf{x},\varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x},\varepsilon)]$$

$$+\frac{G}{\varepsilon^2(1-2\nu)}\frac{\partial^2\chi}{\partial z^2}\begin{pmatrix} 1-2\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & 2(1-\nu)\end{pmatrix}[\chi_f \mathfrak{U}_f(\mathbf{x},\varepsilon) + \chi_m \mathfrak{U}_m(\mathbf{x},\varepsilon)]. \tag{A.1}$$

Similarly, the operator $\boldsymbol{\sigma}^{(n)}$ of tractions on the lateral surface of the fibre applied to the product (1.6) gives

$$\boldsymbol{\sigma}^{(n)}[(\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m)\chi\left(\frac{R}{\varepsilon}\right)]$$

$$= \chi\left(\frac{R}{\varepsilon}\right)\boldsymbol{\sigma}^{(n)}(\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m) + \varepsilon^{-1}[\boldsymbol{\sigma}^{(n)},\ \chi]_{\zeta z}(\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m), \tag{A.2}$$

where $[\boldsymbol{\sigma}^{(n)},\ \chi]_{\zeta z}$ is the commutator of $\boldsymbol{\sigma}^{(n)}$ and χ represented in the scaled coordinates (ζ_1, ζ_2, Z). The boundary layer term $\mathfrak{W}$ is introduced to compensate for the leading parts of the error in the right-hand sides of (A.1) and (A.2). Introducing the scaled variable $Z = \frac{z-l_s}{\varepsilon}$ in the vicinity of the junction, we represent the sum $\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m$ in the form

$$\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m = \chi_f v_3(0)\mathbf{e}^{(3)}$$

$$+\varepsilon \left\{ \left(\chi_f + \chi_m \frac{a^2}{||\zeta||^2} \right) \frac{dv_3}{dz}(0)(\zeta_1, \zeta_2, 0)^T + \chi_f Z \frac{dv_3}{dz}(0)\mathbf{e}^{(3)} + \chi_f v_3^{(1)} \mathbf{e}^{(3)} \right\} + O(\varepsilon^2). \qquad \text{(A.3)}$$

Next, we consider the terms of order $O(\varepsilon)$ that include $\frac{dv_3}{dz}(0)$. The multiplication by the cut-off function imposes the error terms in the equilibrium equations and the conditions of the frictional contact on the lateral surface of the fibre

$$\varepsilon \mathbf{L} \left\{ \left(\chi_f + \frac{a^2 \chi_m}{||\zeta||^2} \right) \frac{dv_3}{dz}(0)(\zeta_1, \zeta_2, 0)^T + \chi_f Z \frac{dv_3}{dz}(0)\mathbf{e}^{(3)} \right\} \chi \left(\frac{R}{\varepsilon} \right)$$

$$:= \varepsilon^{-1} \frac{G}{1-2\nu} \frac{dv_3}{dz}(0)(\mathfrak{F}_f + \mathfrak{F}_m), \qquad \text{(A.4)}$$

where

$$\mathfrak{F}_f := \chi_f \left\{ 2(1-\nu) \left(2\frac{\partial \chi}{\partial Z} + Z \frac{\partial^2 \chi}{\partial Z} \right) \mathbf{e}^{(3)} \right.$$

$$\left. +2\frac{\partial \chi}{\partial Z}\mathbf{e}^{(3)} + \left(\frac{\partial^2 \chi}{\partial Z \partial \zeta_1}\zeta_1 + \frac{\partial^2 \chi}{\partial Z \partial \zeta_2}\zeta_2 \right) + (1-2\nu)\frac{\partial^2 \chi}{\partial Z^2}(\zeta_1, \zeta_2, 0)^T \right\},$$

$$\mathfrak{F}_m := \chi_m \left\{ \frac{\partial \chi}{\partial Z} \left(\frac{\partial}{\partial \zeta_1} \frac{a^2 \zeta_1}{||\zeta||^2} + \frac{\partial}{\partial \zeta_2} \frac{a^2 \zeta_2}{||\zeta||^2} \right) \mathbf{e}^{(3)} \right.$$

$$\left. + \frac{\partial^2 \chi}{\partial Z \partial \zeta_1}\mathbf{e}^{(3)} \frac{a^2}{||\zeta||^2}\zeta_1 + \frac{\partial^2 \chi}{\partial Z \partial \zeta_2}\mathbf{e}^{(3)} \frac{a^2}{||\zeta||^2}\zeta_2 + (1-2\nu)\frac{\partial^2 \chi}{\partial Z^2}\frac{a^2}{||\zeta||^2}(\zeta_1, \zeta_2, 0)^T \right\}.$$

It is verified directly that the principal force vector associated with the right-hand side of (A.4) is not equal to zero. The discrepancy in the friction contact condition (1.2) is given in the form

$$\tau_{rz} \left[(\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m)\chi \left(\frac{R}{\varepsilon} \right) \right] + \varepsilon \mu \sigma_r \left[(\chi_f \mathfrak{U}_f + \chi_m \mathfrak{U}_m)\chi \left(\frac{R}{\varepsilon} \right) \right]$$

$$\sim \varepsilon \frac{dv_3}{dz}(0)[\tau_{rz} + \varepsilon\mu\sigma_r, \chi]_{\zeta z} \left[\left(\chi_f + \chi_m \frac{a^2}{||\zeta||^2} \right) (\zeta_1, \zeta_2, 0)^T + \chi_f Z \mathbf{e}^{(3)} \right]. \qquad \text{(A.5)}$$

Appendix B: Integral equation and its solution

Derivation of the integral equation

This Appendix includes derivation of an integral equation for the jump function discussed in Section 3.3.

The Fourier transforms of tractions, written in terms of the function ϕ_α and its derivatives, are given by

$$\hat{\sigma}_r(r;\alpha) = \int_{-\infty}^{\infty} \sigma_r(r,z)e^{i\alpha z}dz = -i\alpha \left(\nu \Delta_r - \frac{d^2}{dr^2} - \nu\alpha^2 \right) \phi_\alpha(r),$$

$$\hat{\tau}_{rz}(r;\alpha) = \int_{-\infty}^{\infty} \tau_{rz}(r,z)e^{i\alpha z}dz = \frac{d}{dr}[(1-\nu)\Delta_r + \nu\alpha^2]\phi_\alpha(r).$$

Thus, at $r = a$,

$$\hat{\sigma}_r(a;\alpha) = \frac{\Sigma_0(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\chi_\alpha, \quad \hat{\tau}_{rz}(a;\alpha) = \frac{i\,\mathrm{sgn}\,\alpha\,\Sigma_1(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\chi_\alpha, \tag{B.1}$$

where

$$\Sigma_0(\alpha) = \alpha^2 I_0(\alpha)K_0(\alpha) + \alpha[I_0(\alpha)K_1(\alpha) - I_1(\alpha)K_0(\alpha)] - (\kappa_+ + \alpha^2)I_1(\alpha)K_1(\alpha),$$
$$\Sigma_1(\alpha) = \alpha^2[I_0(\alpha)K_1(\alpha) - I_1(\alpha)K_0(\alpha)].$$

Applying the inverse Fourier transform and using the integral representation for χ_α, after interchanging the order of integration, we obtain that

$$\sigma_r(a,az) = \frac{1}{\pi\kappa_+}\int_0^{\lambda_0} \chi(a\zeta)s_0(\zeta - z)d\zeta, \quad \tau_{rz}(a,az) = -\frac{1}{\pi\kappa_+}\int_0^{\lambda_0} \chi(a\zeta)s_1(\zeta - z)d\zeta, \tag{B.2}$$

where $\lambda_0 = h/a$, and

$$s_0(t) = \int_0^{\infty} \frac{\Sigma_0(\alpha)}{d(\alpha)}\cos(\alpha t)d\alpha, \quad s_1(t) = \int_0^{\infty} \frac{\Sigma_1(\alpha)}{d(\alpha)}\sin(\alpha t)d\alpha. \tag{B.3}$$

Using the first condition (3.1) of the frictional contact on $\partial\Pi_f$, we obtain the following integral equation with respect to the jump χ

$$\frac{1}{\pi\kappa_+}\int_0^{\lambda_0} \chi(a\zeta)[-s_1(\zeta - z) + \mu s_0(\zeta - z)]d\zeta = f(az). \tag{B.4}$$

To study the properties of the kernel of the integral equation (B.4), first, we consider the weakly convergent parts of the integrals (B.3). Taking into account the asymptotic behaviour of the cylindrical functions at $\alpha = 0$ and at infinity (see Gradshteyn and Ryzhik (1980)) we arrive at the following asymptotic formulae

$$\frac{\Sigma_0(\alpha)}{d(\alpha)} = \nu + O(\alpha^2\log\alpha), \quad \frac{\Sigma_1(\alpha)}{d(\alpha)} = \alpha + O(\alpha^3\log\alpha), \quad \alpha \to 0,$$
$$\frac{\Sigma_0(\alpha)}{d(\alpha)} = -\frac{\nu_0}{2\alpha} + O\left(\frac{1}{\alpha^3}\right), \quad \frac{\Sigma_1(\alpha)}{d(\alpha)} = \frac{1}{2} + \frac{3}{16\alpha^2} + O\left(\frac{1}{\alpha^4}\right), \quad \alpha \to \infty, \tag{B.5}$$

where $\nu_0 = \frac{1}{2} - 2\nu$. The integrals in (B.3) can be written in the form

$$s_0(t) = -\frac{\nu_0}{2}\log\frac{1}{|t|} + s_0^{(0)}(t), \quad s_1(t) = \frac{1}{2t} + s_1^{(0)}(t), \tag{B.6}$$

where $s_0^{(0)}(t)$ and $s_1^{(0)}(t)$ are continuous functions,

$$s_0^{(0)}(t) = \int_0^1 \frac{\Sigma_0(\alpha)}{d(\alpha)}\cos\alpha t d\alpha + \int_1^{\infty} \sigma_0(\alpha)\cos\alpha t d\alpha + \frac{\nu_0}{2}\left[\gamma + \sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{2n(2n)!}\right],$$

$$s_1^{(0)}(t) = \int_0^1 \left(\frac{\Sigma_1(\alpha)}{d(\alpha)} - \frac{1}{2}\right) \sin \alpha t d\alpha + \int_1^\infty \sigma_1(\alpha) \sin \alpha t d\alpha + \frac{3}{16} r(t), \tag{B.7}$$

$$\sigma_0(\alpha) = \frac{\Sigma_0(\alpha)}{d(\alpha)} + \frac{\nu_0}{2\alpha}, \quad \sigma_1(\alpha) = \frac{\Sigma_1(\alpha)}{d(\alpha)} - \frac{1}{2} - \frac{3}{16\alpha^2}, \quad r(t) = \sin t - t \operatorname{ci}(|t|).$$

$ci(x)$ is the cosine integral, and γ is Euler's constant. The asymptotic behaviour of the functions $\sigma_0(\alpha)$, $\sigma_1(\alpha)$ at infinity is defined by $\sigma_0(\alpha) = O\left(\alpha^{-3}\right), \quad \sigma_1(\alpha) = O\left(\alpha^{-4}\right)$ as $\alpha \to \infty$. Substitution of (B.6) into equation (B.4) gives the following singular integral equation with the Cauchy kernel

$$\frac{1}{\pi} \int_0^{\lambda_0} \chi(a\zeta) \left[\frac{1}{\zeta - z} + \mu_0 \log \frac{1}{|\zeta - z|} + K(\zeta - z)\right] d\zeta = F(z; \lambda_1), \quad 0 < z < \lambda_0, \tag{B.8}$$

where $\mu_0 = \nu_0 \mu$, $\lambda_1 = c/a$, and

$$K(t) = 2s_1^{(0)}(t) - 2\mu s_0^{(0)}(t) \in H(-\lambda_0, \lambda_0), \tag{B.9}$$

$$F(z; \lambda_1) = -2\kappa_+ \mu p - \frac{\kappa_-[1 - \mu(z - \lambda_1)] + 3(z - \lambda_1)(z - \lambda_1 + \mu)[(z - \lambda_1)^2 + 1]^{-1}}{2\pi a^2 [(z - \lambda_1)^2 + 1]^{3/2}} Q. \tag{B.10}$$

Let us describe the class of solutions of (3.12) we are looking for. The function $\chi(a\zeta)$ will be sought in the class of Hölder functions, bounded at the both ends $\zeta = 0$ and $\zeta = \lambda_0$ and satisfying the complementary condition

$$\int_0^{\lambda_0} \chi(a\zeta) d\zeta = 0. \tag{B.11}$$

This class will be called $H^0[0, \lambda_0]$. The unknown parameters λ_0, λ_1 will be determined, together with the function χ, from the equation (3.12) and the embedding $\chi \in H^0[0, \lambda_0]$.

Solution of the integral equation

To solve equation (3.12), we use the method of orthogonal polynomials (see, for example, Popov, 1969). Taking into account the spectral relationship for the Chebyshev polynomials of the first and the second kind

$$\int_0^{\lambda_0} \frac{\sqrt{\zeta(\lambda_0 - \zeta)}}{\zeta - z} U_m \left(1 - \frac{2\zeta}{\lambda_0}\right) d\zeta = \frac{\pi \lambda_0}{2} T_{m+1} \left(1 - \frac{2z}{\lambda_0}\right), \quad 0 < z < \lambda_0; \ m = 0, 1, \ldots, \tag{B.12}$$

we expand the solution of equation (3.12) in a series

$$\chi(a\zeta) = \sqrt{\zeta(\lambda_0 - \zeta)} \sum_{m=0}^{\infty} X_m U_m \left(1 - \frac{2\zeta}{\lambda_0}\right), \tag{B.13}$$

with unknown coefficients X_m. The function $\chi(a\zeta) \in H^0[0, \lambda_0]$ if and only if it satisfies the condition (B.11), and therefore, $X_0 = 0$. We use the spectral relation for the logarithmic kernel (compare with Reade, 1977)

$$\frac{8}{\pi \lambda_0^2} \int_0^{\lambda_0} \log \frac{1}{|\zeta - z|} \sqrt{\zeta(\lambda_0 - \zeta)} U_m \left(1 - \frac{2\zeta}{\lambda_0}\right) d\zeta$$

$$= \sigma_m T_m \left(1 - \frac{2z}{\lambda_0}\right) - \frac{1}{m+2} T_{m+2} \left(1 - \frac{2z}{\lambda_0}\right), \quad 0 < z < \lambda_0; \; m = 0, 1, \ldots,$$

where

$$\sigma_m = \begin{cases} \frac{1}{m}, \; m > 0 \\ \log \frac{4}{\lambda_0}, \; m = 0 \end{cases},$$

and reduce equation (3.12), with $X_0 = 0$, to the form

$$\frac{\lambda_0}{2} \sum_{m=1}^{\infty} X_m \left\{ T_{m+1} \left(1 - \frac{2z}{\lambda_0}\right) + \frac{\mu_0 \lambda_0}{4} \left[\sigma_m T_m \left(1 - \frac{2z}{\lambda_0}\right) - \frac{1}{m+2} T_{m+2} \left(1 - \frac{2z}{\lambda_0}\right) \right] \right\}$$

$$+ \frac{1}{\pi} \sum_{m=1}^{\infty} X_m \int_0^{\lambda_0} \sqrt{\zeta(\lambda_0 - \zeta)} U_m \left(1 - \frac{2\zeta}{\lambda_0}\right) K(\zeta - z) d\zeta = F(z; \lambda_1), \quad 0 < z < \lambda_0. \tag{B.14}$$

The interchange of order of integration and summation in formula (B.14) is justified by the fact that the series (B.13) converges absolutely and uniformly. The orthogonality condition for the Chebyshev polynomials

$$\int_0^{\lambda_0} \frac{T_m \left(1 - \frac{2z}{\lambda_0}\right) T_n \left(1 - \frac{2z}{\lambda_0}\right)}{\sqrt{z(\lambda_0 - z)}} dz = \begin{cases} \frac{\pi}{2}, \; n = m \neq 0 \\ \pi, \; n = m = 0 \\ 0, \; n \neq m \end{cases} \tag{B.15}$$

leads to the following infinite algebraic system for the coefficients X_n

$$X_{n-1} + \frac{\mu_0 \lambda_0}{4n} (X_n - X_{n-2}) + \sum_{m=1}^{\infty} a_{nm} X_m = b_n, \quad n = 2, 3, \ldots. \tag{B.16}$$

For $n = 0, 1$, we have two additional equations

$$\sum_{m=1}^{\infty} a_{0m} X_m = b_0, \tag{B.17}$$

$$\frac{\mu_0 \lambda_0}{4} X_1 + \sum_{m=1}^{\infty} a_{1m} X_m = b_1, \tag{B.18}$$

where $a_{nm} = a_{nm}(\lambda_0)$ and $b_n = b_n(\lambda_0, \lambda_1)$,

$$a_{nm} = \frac{2}{\pi^2 \lambda_0 \rho_n} \int_0^{\lambda_0} \int_0^{\lambda_0} \sqrt{\frac{\zeta(\lambda_0 - \zeta)}{z(\lambda_0 - z)}} U_m \left(1 - \frac{2\zeta}{\lambda_0}\right) T_n \left(1 - \frac{2z}{\lambda_0}\right) K(\zeta - z) d\zeta dz,$$

$$b_n = \frac{2}{\pi \lambda_0 \rho_n} \int_0^{\lambda_0} \frac{F(z; \lambda_1)}{\sqrt{z(\lambda_0 - z)}} T_n \left(1 - \frac{2z}{\lambda_0}\right) dz, \quad \rho_n = \begin{cases} \frac{1}{2}, \; n > 0 \\ 1, \; n = 0 \end{cases}.$$

For the given values λ_0 and λ_1, the system (A.5) defines all coefficients $X_1, X_2, \ldots,$ as functions of λ_0, λ_1. This system is solved using the reduction method, i.e. by considering a finite algebraic system

$$X_{n-1}^{(N)} + \frac{\mu_0 \lambda_0}{4n} (X_n^{(N)} - X_{n-2}^{(N)}) + \sum_{m=1}^{N} a_{nm} X_m^{(N)} = b_n, \quad n = 2, 3, \ldots, N+1. \tag{B.19}$$

Convergence of the approximate solution $\{X_1^{(N)}, X_2^{(N)}, \ldots, X_N^{(N)}, 0, 0, \ldots\}$ to the exact solution in the space l_2 follows from the completeness of a set of the Chebyshev polynomials and the Parseval formula. Equations (B.17) and (B.18) form a system of two transcendental equations for the parameters λ_0, λ_1.

Appendix C: Asymptotics at infinity

Here, we give the details of analysis of the asymptotics at infinity discussed in Section 4.4 of the main text. We start with the longitudinal displacement. To avoid the pole at 0 for the integrand in the integral representation of u_z, we study the derivative of this function

$$2G\kappa_+ \frac{\partial u_z}{\partial z}(r,z) = -\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\alpha z/a} \frac{\Phi^+(\alpha)}{d(|\alpha|)} \mathfrak{F}(|\alpha|, r) d\alpha.$$

The quantity $|\alpha|$ can be interpreted as $|\alpha| = \lim_{\delta\to+0}(\alpha^2+\delta^2)^{1/2}$, so that

$$2G\kappa_+ \frac{\partial u_z}{\partial z}(r,z) = -\frac{1}{2\pi} \lim_{\delta\to+0} \int\limits_{-\infty}^{\infty} e^{-i\alpha z/a} \frac{\Phi^+(\alpha)}{d(\sqrt{\alpha^2+\delta^2})} \mathfrak{F}(\sqrt{\alpha^2+\delta^2}, r) d\alpha. \tag{C.1}$$

Let us consider the case $r < a$ and $z < 0$. We cut the α-plane along straight lines from the point $\alpha = i\delta$ to $\alpha = -i\delta$ through infinity and define the branch

$$(\alpha^2+\delta^2)^{1/2} = (\alpha+i\delta)^{1/2}(\alpha-i\delta)^{1/2},$$

$$\theta_1 = \arg(\alpha - i\delta) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right), \quad \theta_2 = \arg(\alpha+i\delta) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

The function

$$\frac{\Phi^+(\alpha)}{d(\sqrt{\alpha^2+\delta^2})} \mathfrak{F}(\sqrt{\alpha^2+\delta^2}, r) \tag{C.2}$$

is meromorphic in the plane with a cut and has no singularities on the real axis. Thus, as $z \to -\infty$, we arrive at the formula

$$2G\kappa_+ \frac{\partial u_z}{\partial z}(r,z) = \frac{1}{2\pi i} \int\limits_0^{\infty} e^{\beta z/a} \Phi^+(i\beta) \left[\frac{\mathfrak{F}(i\beta, r)}{d(i\beta)} - \frac{\mathfrak{F}(-i\beta, r)}{d(-i\beta)}\right] d\beta + O(e^{Im\zeta_1 z/a}), \tag{C.3}$$

where ζ_1 is the pole of the function (C.2) (as a function of $\sqrt{\alpha^2+\delta^2}$) in the upper half-plane with the smallest imaginary part. Integration by parts in (C.3) gives

$$2G\kappa_+ \frac{\partial u_z}{\partial z}(r,z)$$
$$= -\frac{a}{2\pi i z} \int\limits_0^{\infty} e^{\beta z/a} \frac{\partial}{\partial \beta} \left\{\Phi^+(i\beta) \left[\frac{\mathfrak{F}(i\beta, r)}{d(i\beta)} - \frac{\mathfrak{F}(-i\beta, r)}{d(-i\beta)}\right]\right\} d\beta + O(e^{Im\zeta_1 z/a}), \quad z \to -\infty.$$

Here, we have taken into account that

$$\lim_{\beta\to 0} \mathfrak{F}(\pm i\beta, r) = -\kappa_+, \quad \lim_{\beta\to 0} d(\pm i\beta) = 1, \quad |\Phi^+(i0)| < \infty, \quad r < a.$$

Then, we integrate by parts twice using the following relations verified by the direct calculations

$$\Phi^+(i\beta) = O(1), \quad \frac{d}{d\beta}\Phi^+(i\beta) = O(\log\beta), \quad \frac{d^2}{d\beta^2}\Phi^+(i\beta) = O(\frac{1}{\beta}), \quad \beta \to 0,$$

$$\frac{\partial^m}{\partial \beta^m}\left[\frac{\mathfrak{F}(i\beta,r)}{d(i\beta)}-\frac{\mathfrak{F}(-i\beta,r)}{d(-i\beta)}\right]=\left\{\begin{array}{cc} O(\beta^{2-m}), & m=0,1 \\ \pi i(2+\kappa_+)+o(1), & m=2 \end{array}\right., \quad \beta\to 0.$$

Finally, we arrive at the asymptotics

$$2G\kappa_+\frac{\partial u_z}{\partial z}(r,z)=-\frac{a^3(2+\kappa_+)}{2z^3}\Phi^+(0)+o\left(\frac{1}{z^3}\right), \quad z\to-\infty \quad (r<a). \tag{C.4}$$

Consider the case $z\to+\infty$ and assume that, as before, $r<a$. Let ε be a given positive number. Then we can choose A_0 to be so large that $|H^+(\alpha;\lambda)-H_A^+(\alpha;\lambda)|<\varepsilon$ as $A>A_0$, where

$$H_A^+(\alpha;\lambda)=\int_0^A h(\xi;\lambda)e^{i\alpha\xi}d\xi.$$

Using the boundary condition (4.3), we replace the function $\Phi^+(\alpha)$ in (C.1) by

$$\frac{\Phi^-(\alpha)+Qa^{-2}H_A^+(\alpha;\lambda)}{\Xi(\alpha)}$$

and study the behaviour of the integral

$$2G\frac{\partial u_z}{\partial z}(r,z)=-\frac{1}{2\pi}\lim_{\delta\to+0}\int_{-\infty}^{\infty}e^{-i\alpha z/a}\frac{\Phi^-(\alpha)+Qa^{-2}H_A^+(\alpha;\lambda)}{E_\delta(\alpha)}\mathfrak{F}(\sqrt{\alpha^2+\delta^2},r)d\alpha,$$

as $z\to+\infty$; here

$$E_\delta(\alpha)=\frac{i\alpha\Sigma_1(\sqrt{\alpha^2+\delta^2})}{\sqrt{\alpha^2+\delta^2}}+\mu\Sigma_0(\sqrt{\alpha^2+\delta^2}).$$

The function $H_A^+(\alpha;\lambda)$ is analytic in each finite region of the lower half-plane and

$$H_A^+(\alpha;\lambda)=O(1), \quad \frac{d}{d\alpha}H_A^+(\alpha;\lambda)=O(\log\alpha), \quad \alpha\to 0.$$

As before, we can write

$$2G\frac{\partial u_z}{\partial z}(r,z)=\frac{1}{2\pi i}\int_0^\infty e^{-\beta z/a}\left[\Phi^-(-i\beta)+\frac{Q}{a^2}H_A^+(-i\beta;\lambda)\right]\left[\frac{\mathfrak{F}(i\beta,r)}{e_-(\beta)}-\frac{\mathfrak{F}(-i\beta,r)}{e_+(\beta)}\right]d\beta$$
$$+\,\Omega(r,z), \quad z\to+\infty,$$

where the function $\Omega(r,z)$ decays exponentially as $z\to+\infty$, and $e_\pm(\beta)=\pm\Sigma_1(\mp i\beta)+\mu\Sigma_0(\mp i\beta)$. Taking into account the asymptotics

$$\frac{\partial^m}{\partial\beta^m}\left[\Phi^-(-i\beta)+\frac{Q}{a^2}H_A^+(-i\beta;\lambda)\right]=\left\{\begin{array}{cc} O(1), & m=0 \\ O(\log\beta), & m=1 \end{array}\right., \quad \beta\to 0,$$

$$\frac{\partial^m}{\partial\beta^m}\left[\frac{\mathfrak{F}(i\beta,r)}{e_-(\beta)}-\frac{\mathfrak{F}(-i\beta,r)}{e_+(\beta)}\right]=O(\beta^{2-m}), \quad m=0,1, \quad \beta\to 0,$$

and integrating by parts, we arrive at the following relations as $z\to+\infty$

$$2G\frac{\partial u_z}{\partial z}(r,z)=\frac{a^3}{2\pi i z^3}\left[\Phi^-(0)+\frac{Q}{a^2}H_A^+(0;\lambda)\right]\frac{\partial^2}{\partial\beta^2}\left[\frac{\mathfrak{F}(i\beta,r)}{e_-(\beta)}-\frac{\mathfrak{F}(-i\beta,r)}{e_+(\beta)}\right]_{\beta=0}+o\left(\frac{1}{z^3}\right).$$

The second derivative with respect to β can be evaluated using the formulae

$$\frac{d^2}{d\beta^2}\left[\frac{1}{e_-(\beta)} - \frac{1}{e_+(\beta)}\right]_{\beta=0} = -\frac{\pi i \kappa_+}{2\mu\nu^2}, \quad \frac{\partial^2}{\partial\beta^2}\left[\mathfrak{F}(i\beta, r) - \mathfrak{F}(-i\beta, r)\right]_{\beta=0} = \pi i(2+\kappa_+).$$

Next, we return to the function $\Phi^+(0)$:

$$\left|\Phi^-(0) + \frac{Q}{a^2}H_A^+(0;\lambda) - \Xi(0)\Phi^+(0)\right| < \varepsilon, \quad A > A_0,$$

where $\Xi(0) = \kappa_+^{-1}\mu\nu$, and obtain the asymptotics

$$2G\kappa_+\frac{\partial u_z}{\partial z}(r,z) = \frac{a^3\Phi^+(0)}{\nu z^3} + o\left(\frac{1}{z^3}\right), \quad z \to +\infty \quad (r < a).$$

In the case $r > a$ we use the formulae (4.12) and (4.14) and, in a similar way, arrive at the same formula (C.4) in both situations $z \to +\infty$ and $z \to -\infty$.

Appendix D: Generalization to the case of debonding with opening and frictional sliding

Let us generalize the formulation of the problem described in Section 2 by introducing a finite zone $\partial\Pi_d = \{(r,\theta,z) \in \partial\Pi:\ b < z < d\}$, where the following boundary conditions hold

$$\tau_{rz}^{(total)} = \sigma_r^{(total)} = 0, \quad [u_r^{(total)}] \neq 0, \quad [u_z^{(total)}] \neq 0. \tag{D.1}$$

There are also two finite friction zones at the interface

$$\partial\Pi_{f_1} = \{(r,\theta,z) \in \partial\Pi:\ 0 < z < b\}, \quad \partial\Pi_{f_2} = \{(r,\theta,z) \in \partial\Pi:\ d < z < h\}.$$

As before, outside the region $\partial\Pi_{f_1} \cup \partial\Pi_d \cup \partial\Pi_{f_2}$ the matrix and fibre are perfectly bonded.

We assume that a point force $Q\mathbf{e}^{(3)}$ acts at the point $(0,0,c)$, and at infinity the uniform compressive load $\sigma_r^{(\infty)} = -p$, $\tau_{rz}^{(\infty)} = 0$ is applied. The quantities b, d, h and c which specify the boundaries of the regions are unknown and form a part of the solution.

In a similar way, instead of the relations (B.1), we obtain the following expressions for the Fourier transforms of the stresses

$$\hat{\sigma}_r(a;\alpha) = \frac{\Sigma_0(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\chi_\alpha + \frac{i\,\mathrm{sgn}\,\alpha\Sigma_2(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\omega_\alpha, \quad \hat{\tau}_{rz}(a;\alpha) = \frac{i\,\mathrm{sgn}\,\alpha\Sigma_1(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\chi_\alpha - \frac{\Sigma_0(|\alpha|a)}{\kappa_+ d(|\alpha|a)}\omega_\alpha, \tag{D.2}$$

where

$$\Sigma_2(\alpha) = 2\alpha I_0(\alpha)K_0(\alpha) + (\alpha^2 + \kappa_+ + 1)[I_0(\alpha)K_1(\alpha) - I_1(\alpha)K_0(\alpha)] - \frac{2}{\alpha}(\kappa_+ + \alpha^2)I_1(\alpha)K_1(\alpha),$$

$$\omega_\alpha = \int_b^d \omega(z)e^{i\alpha z}dz.$$

The function $\omega(z)$ is an unknown jump in the $z-$derivative of the normal displacement $\omega(z) = 2G\frac{\partial}{\partial z}[u_r(a-0,z) - u_r(a+0,z)]$. The other functions in (D.2) are as defined in Section 3.

We invert the Fourier transforms (D.2) and arrive at the expressions for the stresses

$$\pi\kappa_+\tau_{rz}(a,az) = -\int_{\lambda_0}^{\lambda_1}\omega(a\zeta)s_0(\zeta-z)d\zeta - \int_0^{\lambda_2}\chi(a\zeta)s_1(\zeta-z)d\zeta,$$

$$\pi\kappa_+\sigma_r(a,az) = -\int_{\lambda_0}^{\lambda_1}\omega(a\zeta)s_2(\zeta-z)d\zeta + \int_0^{\lambda_2}\chi(a\zeta)s_0(\zeta-z)d\zeta, \tag{D.3}$$

where

$$\lambda_0 = \frac{b}{a}, \quad \lambda_1 = \frac{d}{a}, \quad \lambda_2 = \frac{h}{a}, \quad 0<\lambda_0<\lambda_1<\lambda_2, \quad s_2(t) = \int_0^\infty \frac{\Sigma_2(\alpha)}{d(\alpha)}\sin(\alpha t)d\alpha.$$

The kernels $s_0(t)$ and $s_1(t)$ coincide with (B.3). We substitute the stresses (D.3) into the boundary conditions (3.1) and (D.1) and re-write them in the form

$$\sigma_r(a,z) = f_0(z), \quad \tau_{rz}(a,z) = f_1(z), \quad b<z<d,$$

$$(\tau_{rz}+\mu\sigma_r)|_{r=a} = f(z), \quad 0<z<b \quad \text{and} \quad d<z<h,$$

where

$$f_0(z) = p - \frac{Qz_c}{4\pi\kappa_+R_a^5}[(\kappa_- - 3)a^2 + \kappa_- z_c^2], \quad f_1(z) = \frac{Q}{4\pi\kappa_+R_a^5}[\kappa_- aR_a^2 + 3az_c^2], \quad z_c = z-c.$$

Thus, the formulated problem is reduced to the system of integral equations

$$\int_{\lambda_0}^{\lambda_1} s_0(\zeta-z)\omega(a\zeta)d\zeta + \int_0^{\lambda_2} s_1(\zeta-z)\chi(a\zeta)d\zeta = -\pi\kappa_+ f_1(az), \quad \lambda_0<z<\lambda_1,$$

$$\int_{\lambda_0}^{\lambda_1} s_2(\zeta-z)\omega(a\zeta)d\zeta - \int_0^{\lambda_2} s_0(\zeta-z)\chi(a\zeta)d\zeta = -\pi\kappa_+ f_0(az), \quad \lambda_0<z<\lambda_1,$$

$$\int_{\lambda_0}^{\lambda_1}[s_0(\zeta-z)+\mu s_2(\zeta-z)]\omega(a\zeta)d\zeta + \int_0^{\lambda_2}[s_1(\zeta-z)-\mu s_0(\zeta-z)]\chi(a\zeta)d\zeta$$
$$= -\pi\kappa_+ f(az), \quad 0<z<\lambda_0 \cup \lambda_1<z<\lambda_2.$$

The functions $s_0(t)$, $s_1(t)$ have been studied in Section 3 (see the formulae (B.6) and (B.7)). For the kernel $s_2(t)$ we have

$$s_2(t) = \frac{1}{2t} + s_2^{(0)}(t),$$

$$s_2^{(0)}(t) = \int_0^1\left(\frac{\Sigma_2(\alpha)}{d(\alpha)} - \frac{1}{2}\right)\sin\alpha t d\alpha + \int_1^\infty \sigma_2(\alpha)\sin\alpha t d\alpha + \kappa_* r(t),$$

$$\sigma_2(\alpha) = \frac{\Sigma_2(\alpha)}{d(\alpha)} - \frac{1}{2} - \frac{\kappa_*}{\alpha^2}, \quad \kappa_* = \frac{15-4\kappa}{16}.$$

The function $\sigma_2(\alpha) = O(\alpha^{-4})$ as $\alpha \to \infty$, and the function $s_2^{(0)}(t)$ is continuous. Hence we obtain the system of three singular integral equations

$$\int_0^{\lambda_2} \left[\frac{1}{\zeta - z} + 2s_1^{(0)}(\zeta - z)\right] \chi(a\zeta) d\zeta + \int_{\lambda_0}^{\lambda_1} \left[\nu_0 \log|\zeta - z| + 2s_0^{(0)}(\zeta - z)\right] \omega(a\zeta) d\zeta$$

$$= -2\pi\kappa_+ f_1(az), \quad \lambda_0 < z < \lambda_1,$$

$$-\int_0^{\lambda_2} \left[\nu_0 \log|\zeta - z| + 2s_0^{(0)}(\zeta - z)\right] \chi(a\zeta) d\zeta + \int_{\lambda_0}^{\lambda_1} \left[\frac{1}{\zeta - z} + 2s_2^{(0)}(\zeta - z)\right] \omega(a\zeta) d\zeta$$

$$= -2\pi\kappa_+ f_0(az), \quad \lambda_0 < z < \lambda_1,$$

$$\int_0^{\lambda_2} \left[\frac{1}{\zeta - z} - \mu_0 \log|\zeta - z| + K(\zeta - z)\right] \chi(a\zeta) d\zeta$$

$$+ \int_{\lambda_0}^{\lambda_1} \left[\frac{\mu}{\zeta - z} + \nu_0 \log|\zeta - z| + 2s_0^{(0)}(\zeta - z) + 2\mu s_2^{(0)}(\zeta - z)\right] \omega(a\zeta) d\zeta$$

$$= -2\pi\kappa_+ f(az), \quad 0 < z < \lambda_0 \cup \lambda_1 < z < \lambda_2, \tag{D.4}$$

where the kernel $K(t)$ is defined by (B.9). The functions $\chi(a\zeta)$, $\omega(a\zeta)$ should be Hölder functions on the segments $(0, \lambda_2)$ and (λ_0, λ_1) respectively, satisfy the two additional conditions

$$\int_0^{\lambda_2} \chi(a\zeta) d\zeta = 0, \quad \int_{\lambda_0}^{\lambda_1} \omega(a\zeta) d\zeta = 0,$$

and possess the following asymptotics at the ends

$$\chi(a\zeta) = O(\zeta^{1/2}), \ \zeta \to 0, \quad \chi(a\zeta) = O(1), \ \zeta \to \lambda_0,$$

$$\chi(a\zeta) = O(1), \ \zeta \to \lambda_1, \quad \chi(a\zeta) = O((\lambda_2 - \zeta)^{1/2}), \ \zeta \to \lambda_2,$$

$$\omega(a\zeta) = O((\zeta - \lambda_0)^{1/2}), \ \zeta \to \lambda_0, \quad \omega(a\zeta) = O((\lambda_1 - \zeta)^{1/2}), \ \zeta \to \lambda_1.$$

To construct an approximate solution of the system (D.4), one can use the method of orthogonal polynomials (see Section 3) that leads to an infinite algebraic system of linear equations and gives a system of four transcendental equations for the unknown parameters $\lambda_0, \ \lambda_1, \ \lambda_2$ and c/a. The choice of the parameters provides the boundedness of both stresses $\sigma_r(a, z)$ and $\tau_{rz}(a, z)$ at the points $z = 0, \ z = b, \ z = d$ and $z = h$.

References

Antipov, Y.A., 1995. An interface crack between elastic materials when there is dry friction. J. Appl. Math. Mech. (PMM) 59, 273-287.

Atkinson, C., 1982. The interface crack with a contact zone (an analytical treatment). Int. J. Fracture 18, 161-177.

Atkinson, A., Avila, J., 1982. Special analytical solutions for use in debond stress analysis. IMA J. Appl. Math. 29, 143-171.

Aveston, J., Kelly, A., 1973. Theory of multiple fracture of fibrous composites. J. Mater. Sci. 8, 352-362.
Budiansky, B., Hutchinson, J.W., Evans, A.G., 1986. Matrix fracture in fibre-reinforced ceramics, J. Mech. Phys. Solids 34, No 2, 167-189.
Dollar, A., Steif, P.S., 1988. Load transfer in composites with a Coulomb friction interface. Int. J. Solids Structures 24, 789-803.
Dollar, A., Steif, P.S., 1989. A tension crack impinging upon frictional interfaces. J. Appl. Mech. 56, 291-298.
Dollar, A., Steif, P.S., 1991. Stresses in fibres spanning an infinite matrix crack. Int. J. Solids Structures 27, 1011-1024.
Dundurs, J., Comninou, M., 1979. Some consequences of the inequality conditions in contact and crack problems. J. Elasticity 9, 71-82.
Freund, L.B., 1992. The axial force needed to slide a circular fibre along a hole in an elastic material and implications for fibre pull-out. Eur. J. Mech., A/Solids 11, No 1, 1-19.
Gradshteyn, I.S., Ryzhik, I.M., 1980. Table of integrals, series, and products. Academic Press.
Kelly, A., 1970. Interface effects and the work of fracture of a fibrous composite. Proc. R. Soc. Lond. A319, 95-116.
Meda, G., Steif, P.S., 1994a. A detailed analysis of cracks bridged by fibers - I. Limiting cases of short and long cracks. J. Mech. Phys. Solids 42, 1293-1321.
Meda, G., Steif, P.S., 1994b. A detailed analysis of cracks bridged by fibers - II. Cracks of intermediate size. J. Mech. Phys. Solids 42, 1323-1341.
Movchan, N.V., Willis, J.R., 1997. Asymptotic analysis of reinforcement by frictional fibres. Proc. R. Soc. London. A 453, 757-784.
Muki, R., Sternberg, E., 1970. Elastostatic load-transfer to a half-space from a partially embedded axially loaded rod. Int. J. Solids Structures 6, 69-90.
Noble, B., 1988. *Methods Based on the Wiener-Hopf Technique*, 2nd edn. Chelsea Publishing Company, New York.
Popov, G.Ia., 1969. On the method of orthogonal polynomials in contact problems of the theory of elasticity. J. Appl. Math. Mech. (PMM), 33, 503-517.
Reade, J.B., 1979. Asymptotic behaviour of eigen-values of certain integral equations. Proc. Edinburgh Math. Soc. 22, 137-144.
Selvadurai, A.P.S., Rajapakse, R.K.N.D., 1985. On the load transfer from a rigid cylindrical inclusion into an elastic half-space. Int. J. Solids Structures 21, No 12, 1213-1229.
Simonov, I.V., 1984. The steady motion of a crack with creep and separation sections along the interface between two elastic materials. J. Appl. Math. Mech. (PMM) 48, 347-353.
Spence, D.A., 1973. An eigenvalue problem for elastic contact with finite friction. Proc. Camb. Phil. Soc. 73, 249-268.
Steif, P.S., Hoysan, S.F., 1986. On load transfer between imperfectly bonded constituents. Mech. Materials 5, 375-382.
Westergaard, H.M., 1952. *Theory of Elasticity and Plasticity*, Dover, New York.