Absolute Stability Results for Well-Posed Infinite-Dimensional Systems with Applications to Low-Gain Integral Control *

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Abstract: We derive absolute stability results for well-posed infinite-dimensional systems which, in a sense, extend the well-known circle criterion to the case that the underlying linear system is the series interconnection of an exponentially stable well-posed infinite-dimensional system and an integrator and the nonlinearity ϕ satisfies a sector condition of the form $\langle \phi(u), \phi(u) - au \rangle \leq 0$ for some constant $a > 0$. These results are used to prove convergence and stability properties of low-gain integral feedback control applied to exponentially stable, linear, well-posed systems subject to actuator nonlinearities. The class of actuator nonlinearities under consideration contains standard nonlinearities which are important in control engineering such as saturation and deadzone.

Keywords: Absolute stability; actuator nonlinearities; circle criterion; integral control; positive real; robust tracking; well-posed infinite-dimensional systems.

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* This work was supported by the UK-DUTCH Joint Scientific Research Programme and by the UK EPSRC Council (Grant GR/L78086).

1 Introduction

Absolute stability problems and their relations to positive-real conditions have played a prominent role in finite-dimensional systems and control theory and have led to a number of important stability criteria for closed-loop systems obtained by applying unity feedback controls to linear dynamical systems subject to static input or output nonlinearities, see, for example, Khalil [12], Lefschetz [13], Leonov et al. [14] and Vidyasagar [27]. Although there is some literature on absolute stability problems in infinite dimensions (for example, Bucci [3], Corduneanu [6], Leonov et al. [14], Logemann [16], Wexler [32], [33]), the number of results available in the literature is fairly limited, in particular for systems with unbounded control and observation.

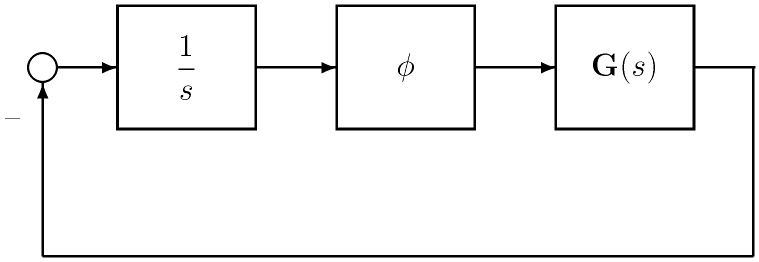

Figure 1

In this paper we study a certain absolute stability problem for the class of well-posed infinite-dimensional systems which are documented in Salamon [23], [24], Staffans [25], [26] and Weiss [28] – [31]. We remark that the class of well-posed, linear, infinite-dimensional systems is rather general: it includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. Consider the system shown in Figure 1, where $\mathbf{G}$ is the transfer function of an exponentially stable, well-posed, infinite-dimensional, linear system and $\phi : U \to U$ is a locally Lipschitz nonlinearity which, for some $a \geq 0$, satisfies the sector condition

$$\langle \phi(u), \phi(u) - au \rangle \leq 0\,, \quad \forall\, u \in U\,, \tag{1.1}$$

where U denotes the input space of the well-posed system which is assumed to be a real Hilbert space. Given $b > 0$, we study the absolute stability problem of finding conditions on $\mathbf{G}$ such that the feedback system in Figure 1 is stable for all locally Lipschitz ϕ satisfying (1.1) for some $a \in [0, b)$. In Section 3 we show that if $\mathbf{G}(0)$ is invertible and the positive real condition

$$I + \frac{b}{2}\left(\frac{1}{s}\mathbf{G}(s) + \frac{1}{\bar{s}}\mathbf{G}^*(s)\right) \geq 0\,, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0 \tag{1.2}$$

holds, then, for all locally Lipschitz ϕ satisfying (1.1) for some $a \in [0, b)$, the equilibrium of the closed-loop system shown in Figure 1 is stable in the large. Moreover, under suitable extra assumptions on ϕ, we prove that the equilibrium is semi-globally exponentially stable. These results extend, in a certain sense, a part of the well-known circle criterion, see Remark 3.2, part (e).

In Section 4 we apply the absolute stability results obtained in Section 3 to the low-gain integral control problem illustrated in Figure 2, where $r \in U$ is the reference vector and $k > 0$ is the integral gain.

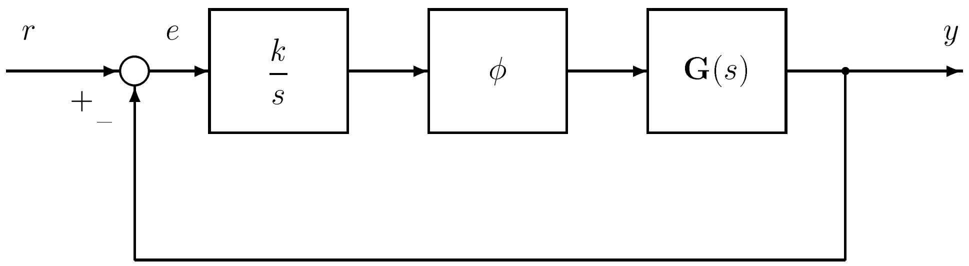

Figure 2

We assume that $U = \mathbb{R}^m$, $\mathbf{G}$ is the transfer function of an exponentially stable well-posed system such that $\mathbf{G}(0)$ is invertible and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a decoupled nonlinearity of the form

$$\phi(u) = [\phi_1(u_1), \phi_2(u_2), \ldots, \phi_m(u_m)]^T, \quad \forall\, u = (u_1, u_2, \ldots, u_m)^T \in \mathbb{R}^m,$$

where the functions $\phi_i : \mathbb{R} \to \mathbb{R}$ are non-decreasing and globally Lipschitz with Lipschitz constants $\lambda_i > 0$. Setting

$$K := \sup\{b > 0 : (1.2) \text{ holds}\}, \quad \lambda := \max \lambda_i,$$

we prove that for all $k \in (0, K/\lambda)$ and for all reference vectors r satisfying $[\mathbf{G}(0)]^{-1} r \in \operatorname{im} \phi$, the error $e(\cdot) = r - y(\cdot)$ is in $L^2(\mathbb{R}_+, \mathbb{R}^m)$. Moreover, we show that if the impulse response of the linear system (i.e. the inverse Laplace transform of the transfer function $\mathbf{G}$) is a (matrix-valued) Borel measure and if the initial condition of the plant is "sufficiently smooth", then $e(t) \to 0$ as $t \to \infty$, i.e., the output $y(t)$ asymptotically tracks the reference vector r. Under mild extra assumptions on r and ϕ, the convergence will be exponentially fast. We remark that these results considerably improve earlier work by Logemann et al. [19] in the sense that (i) our results guarantee better asymptotic and faster convergence properties and (ii) our results are not restricted to single-input single-output regular systems, but apply to the wider class of multivariable well-posed systems.

The paper is organized as follows. Section 2 contains some preliminaries on well-posed infinite-dimensional systems. Section 3 is devoted to a detailed analysis of the absolute stability problem described above. In Section 4 we apply the absolute stability theory developed in Section 3 to derive results on the low-gain integral control problem illustrated in Figure 2. An example of a diffusion process with output delay illustrating our results is given in Section 5. Finally, some technicalities are relegated to the Appendix.

Notation. Let X be a real or complex Hilbert space; for $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the operator of the right-shift by τ on $L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$, where $\mathbb{R}_+ := [0, \infty)$; the truncation operator $\mathbf{P}_\tau : L^p_{\mathrm{loc}}(\mathbb{R}_+, X) \to L^p(\mathbb{R}_+, X)$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ otherwise; for $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, X) |\ f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, X)\}$ and endow it with the norm $\|f\|_{p,\alpha} := \|f(\cdot)\exp(-\alpha\,\cdot)\|_{L^p}$, where $\|\cdot\|_{L^p}$ denotes the usual norm in $L^p(\mathbb{R}_+, X)$; for $\tau > 0$, $W^{1,2}([0, \tau], X)$ denotes the space of all functions $f : [0, \tau] \to X$ for which there exists $g \in L^2([0, \tau], X)$ such that $f(t) = f(0) + \int_0^t g(s)\, ds$ for all $t \in [0, \tau]$; $W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, X)$ denotes the space of all functions $f : \mathbb{R}_+ \to X$ such that for all $\tau > 0$, the restriction of f to $[0, \tau]$ belongs to $W^{1,2}([0, \tau], X)$; $\mathcal{M}$ denotes the space of all $\mathbb{R}^{m\times m}$-valued Borel measures on $\mathbb{R}_+$; for $\alpha \in \mathbb{R}$, we define $\mathcal{M}_\alpha$ to be the space of all $\mathbb{R}^{m\times m}$-valued Borel measures on $\mathbb{R}_+$ with the property that the exponentially weighted measure $E \mapsto \int_E e^{-\alpha t}\mu(dt)$ belongs to $\mathcal{M}$; for $\alpha \in \mathbb{R}$, $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$; $H^2(\mathbb{C}_\alpha, X)$ denotes the Hardy-Lebesgue space of square-integrable functions defined on $\mathbb{C}_\alpha$ with values in X; for a Banach space Z, $H^\infty(\mathbb{C}_\alpha, Z)$ denotes the space of bounded holomorphic functions defined on $\mathbb{C}_\alpha$ with values in Z; $\mathcal{B}(X_1, X_2)$ denotes the space of bounded linear operators from a Hilbert space X_1 to a Hilbert space X_2; we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$; the Laplace transform is denoted by $\mathfrak{L}$.

If X is a real Hilbert space, then its complexification is denoted by X_c. Every vector $z \in X_c$ can be uniquely expressed in the form $z = x + iy$, where $x, y \in X$. In particular, $X \subset X_c$. The inner product $\langle \cdot\,, \cdot\rangle$ on X extends in a natural way to a (complex) inner product on X_c; a similar statement is true for a linear operator S on X (see Halmos [11], pp. 150, for details). We shall use the same symbol $\langle \cdot\,, \cdot\rangle$ (respectively, S) for the original inner product (respectively, operator) and the associated extensions. A linear operator $S : \mathrm{dom}(S) \subset X_c \to Y_c$, where Y is a real Hilbert space, is called *real*, if $Sx \in Y$ for all $x \in \mathrm{dom}(S) \cap X$.

2 Preliminaries on well-posed systems

We assemble some fundamental facts pertaining to well-posed linear systems and regular linear systems and tailored to later requirements: the reader is referred to Salamon [23], [24], Staffans [25], [26] and Weiss [28] – [31] for full details.

Well-posed systems. The concept of a well-posed linear system which will be used in this paper was introduced in [31]; an equivalent definition can be found in [23]. Let U, X and Y be real Hilbert spaces and let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a *well-posed* linear system with *state space* X, *input space* U and *output space* Y, i.e.

$\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X;

$\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that, for all $\tau, t \geq 0$,

$$\mathbf{\Phi}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v\,, \quad \forall\ u, v \in L^2(\mathbb{R}_+, U)\,;$$

$\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{\Psi}_0 = 0$ and, for all $\tau, t \geq 0$,

$$\mathbf{\Psi}_{\tau+t} x^0 = \mathbf{P}_\tau \mathbf{\Psi}_\tau x^0 + \mathbf{R}_\tau \mathbf{\Psi}_t \mathbf{T}_\tau x^0\,, \quad \forall\ x^0 \in X\,;$$

$\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that $\mathbf{F}_0 = 0$ and, for all $\tau, t \geq 0$,

$$\mathbf{F}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{P}_\tau \mathbf{F}_\tau u + \mathbf{R}_\tau(\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,, \quad \forall\ u, v \in L^2(\mathbb{R}_+, U).$$

For an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and initial state $x^0 \in X$, the associated state function $x \in C(\mathbb{R}_+, X)$ and output function $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ of $\mathbf{\Sigma}$ are given by

$$x(t) = \mathbf{T}_t x^0 + \mathbf{\Phi}_t \mathbf{P}_t u\,, \tag{2.1a}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t x^0 + \mathbf{F}_t \mathbf{P}_t u\,. \tag{2.1b}$$

$\mathbf{\Sigma}$ is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable:

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\| < 0\,.$$

$\mathbf{\Psi}_\infty$ and $\mathbf{F}_\infty$ will denote the unique operators $X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and $L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, respectively, satisfying

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty\,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty\,; \quad \forall\ \tau \geq 0\,.$$

For any $\alpha > \omega(\mathbf{T})$, $\mathbf{\Psi}_\infty$ is a bounded operator from X into $L^2_\alpha(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty$ maps $L^2_\alpha(\mathbb{R}_+, U)$ boundedly into $L^2_\alpha(\mathbb{R}_+, Y)$. If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$, $\mathbf{\Psi}_t$ and $\mathbf{F}_t$ are uniformly bounded. Since $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$, $\mathbf{F}_\infty$ is a *causal* operator, called the *input-output operator* of $\mathbf{\Sigma}$.

Transfer functions. Weiss [28] has established that if $\alpha > \omega(\mathbf{T})$, then there exists a unique holomorphic function $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{B}(U_c, Y_c)$ such that

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s)\,, \quad \forall\ s \in \mathbb{C}_\alpha\,,\ \forall\ u \in L^2_\alpha(\mathbb{R}_+, U)\,,$$

where $\mathfrak{L}$ denotes Laplace transform. In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. Moreover, for all $s \in (\omega(\mathbf{T}), \infty)$, $\mathbf{G}(s)$ is a real operator. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if there exists a linear operator D such that

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)u = Du\,, \quad \forall\ u \in U,$$

in which case, by the principle of uniform boundedness, it follows that $D \in \mathcal{B}(U, Y)$ (in particular, D is real). The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$.

Control and observation operators. The generator of $\mathbf{T}$ is denoted by A with domain $\mathrm{dom}(A)$. Let X_1 be the space $\mathrm{dom}(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$, where $\lambda \in \varrho(A)$ is any fixed element of the resolvent set $\varrho(A)$ of A. Then $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathcal{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [23] (see also Weiss [29, 30]), there exist unique operators $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that, for all $t \geq 0$, $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $x^0 \in X_1$,

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x^0)(t) = C\mathbf{T}_t x^0 \ .$$

The so-called *Lebesgue extension* of C, denoted by C_L, is defined by

$$C_L x^0 = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x^0 \, d\tau \,,$$

where $\mathrm{dom}(C_L)$ is the set of all those $x^0 \in X$ for which the above limit exists (see [30]). Clearly $X_1 \subset \mathrm{dom}(C_L) \subset X$. Furthermore, for any $x^0 \in X$, we have that $\mathbf{T}_t x^0 \in \mathrm{dom}(C_L)$ for almost every $t \geq 0$ and

$$(\mathbf{\Psi}_\infty x^0)(t) = C_L \mathbf{T}_t x^0 \,, \quad \text{a.e. } t \geq 0\,. \tag{2.2}$$

B is said to be *bounded* if it is so as a map from the input space U to the state space X, otherwise, B is said to be *unbounded.* C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded.* If $\mathbf{T}$ is exponentially stable, then there exist constants β_1, $\beta_2 > 0$ such that, for all $t \geq 0$, $u \in L^2(\mathbb{R}_+, U)$ and $x^0 \in X$,

$$\|\mathbf{\Phi}_t \mathbf{P}_t u\| = \left\| \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \right\| \leq \beta_1 \|u\|_{L^2(0,t;U)}, \tag{2.3}$$

$$\|\mathbf{\Psi}_\infty x^0\|_{L^2(0,t;Y)} = \left(\int_0^t \|C\mathbf{T}_\tau x^0\|^2 d\tau \right)^{1/2} \leq \beta_2 \|x^0\|\,. \tag{2.4}$$

For any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the state trajectory $x(\cdot)$ defined by (2.1a) satisfies the equation

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x^0\,, \quad \text{a.e. } t \geq 0\,. \tag{2.5}$$

The derivative on the left-hand side of (2.5) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (2.5) in the space X_{-1}, then for any $x^0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$, (2.5) has a unique strong solution (in the sense of Pazy [21], p. 109) given by the variation of parameters formula

$$t \mapsto x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\,. \tag{2.6}$$

It has been shown in [23] that for any $x^0 \in X$, $u \in W^{1,2}_{\text{loc}}(\mathbb{R}_+, U)$ with $u(0) = 0$ and $s_0 \in \varrho(A)$, the output function $y(\cdot)$ defined by (2.1b) can be expressed as

$$y(t) = C_L \mathbf{T}_t x^0 + C\left[\int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau - (s_0 I - A)^{-1} B u(t)\right] + \mathbf{G}(s_0) u(t)\,, \ \text{ a.e. } t \geq 0\,. \tag{2.7}$$

If $\mathbf{\Sigma}$ is regular (with feedthrough D), then the state trajectory $x(\cdot)$ defined by (2.1a) satisfies $x(t) \in \text{dom}(C_L)$ for almost every $t \geq 0$ and the output $y(t)$ given by (2.1b) can be written in the familiar form

$$y(t) = C_L x(t) + D u(t)\,, \quad \text{a.e. } t \geq 0\,. \tag{2.8}$$

We know from [9] and [23] that for all $s, s_0 \in \varrho(A)$, $s \neq s_0$

$$\frac{1}{s - s_0}\left(\mathbf{G}(s) - \mathbf{G}(s_0)\right) = -C(sI - A)^{-1}(s_0 I - A)^{-1} B \tag{2.9}$$

Moreover, it has been demonstrated in [28] that, if $\mathbf{\Sigma}$ is regular (with feedthrough D), then $(sI - A)^{-1} BU \subset \text{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed as

$$\mathbf{G}(s) = C_L (sI - A)^{-1} B + D\,, \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}\,, \tag{2.10}$$

which is familiar from finite-dimensional systems theory. The operators A, B, C (and D, in the regular case) are called the *generating operators* of $\mathbf{\Sigma}$. Note that in the non-regular case, the generating operators do not completely determine the system $\mathbf{\Sigma}$, since A, B and C determine the input-output operator $\mathbf{F}_\infty$ only up to an additive constant, namely $\mathbf{G}(s_0)$ (see (2.7)).

Four technical lemmas. In the following let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a well-posed linear system with state space X, input space U, output space Y, generating operators A, B and C, input-output operator $\mathbf{F}_\infty$ and transfer function $\mathbf{G}$. We state four lemmas on the asymptotic behaviour and the regularity of the solutions to (2.1) and on the existence and uniqueness of solutions for a certain nonlineear feedback system with (2.1) in the forward loop.

Lemma 2.1 *Let $x^0 \in X$. If $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$ and $u^\infty \in U$ are such that $u - u^\infty \in L^2_\alpha(\mathbb{R}_+, U)$ for some $\alpha > \omega(\mathbf{T})$, then the output y of $\mathbf{\Sigma}$ (given by (2.1b)) satisfies*

$$y - \mathbf{G}(0) u^\infty \in L^2_\alpha(\mathbb{R}_+, Y)\,.$$

Proof: Since $\alpha > \omega(\mathbf{T})$, the function $t \mapsto C_L \mathbf{T}_t x^0$ is in $L^2_\alpha(\mathbb{R}_+, Y)$. Combining this with the identity

$$y(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty u)(t)\,, \quad \text{a.e.}\, t \geq 0\,,$$

it follows that it is sufficient to show that

$$\mathbf{F}_\infty u - y^\infty \in L^2_\alpha(\mathbb{R}_+, Y)\,,$$

where $y^\infty := \mathbf{G}(0)u^\infty$. Trivially, we have

$$\mathbf{F}_\infty u - y^\infty = \mathbf{F}_\infty(u - u^\infty) + \mathbf{F}_\infty(u^\infty) - y^\infty\,. \tag{2.11}$$

Since $u - u^\infty \in L^2_\alpha(\mathbb{R}_+, U)$ for some $\alpha > \omega(\mathbf{T})$, we may conclude that $\mathbf{F}_\infty(u - u^\infty) \in L^2_\alpha(\mathbb{R}_+, Y)$. Therefore, by (2.11), the claim follows if we can show that

$$\mathbf{F}_\infty(u^\infty) - y^\infty \in L^2_\alpha(\mathbb{R}_+, Y)\,. \tag{2.12}$$

To this end take the Laplace transform of $\mathbf{F}_\infty(u^\infty) - y^\infty$ to obtain

$$(\mathfrak{L}(\mathbf{F}_\infty(u^\infty) - y^\infty))(s) = \frac{1}{s}\mathbf{G}(s)u^\infty - \frac{1}{s}y^\infty = \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))u^\infty\,. \tag{2.13}$$

It is clear that the function $s \mapsto (\mathbf{G}(s) - \mathbf{G}(0))u^\infty/s$ is in $H^2(\mathbb{C}_\alpha, Y)$ and therefore, appealing to a well-known theorem due to Paley and Wiener, it follows from (2.13) that $\mathbf{F}_\infty(u^\infty) - y^\infty \in L^2_\alpha(\mathbb{R}_+, Y)$, which is (2.12). $\square$

Lemma 2.2 *Suppose that* $\mathbf{T}$ *is exponentially stable. Then, for all* $x^0 \in X$ *and* $u \in L^2(\mathbb{R}_+, U)$*, the solution* $x(\cdot)$ *of the initial-value problem (2.5) satisfies*

$$\lim_{t\to\infty} \|x(t)\| = 0\,, \quad x \in L^2(\mathbb{R}_+, X)\,.$$

Proof: Let $x_0 \in X$ and $u \in L^2(\mathbb{R}_+, U)$ and assume that $\mathbf{T}$ is exponentially stable. It has been shown in [20], Lemma 2.2, that under these assumptions $x \in L^2(\mathbb{R}_+, X)$. It remains to show that $\lim_{t\to\infty} \|x(t)\| = 0$. To this end note that by the exponential stability of $\mathbf{T}$ and (2.3), there exists $\beta > 0$ such that

$$\| \int_s^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \| \leq \beta \left(\int_s^t \|u(\tau)\|^2\, d\tau \right)^{1/2}\,, \quad t \geq s \geq 0\,. \tag{2.14}$$

Since $u \in L^2(\mathbb{R}_+, U)$, there exists $s_1 \geq 0$ such that

$$\int_s^t \|u(\tau)\|^2\, d\tau \leq \varepsilon^2/4\beta^2\,, \quad t \geq s \geq s_1\,. \tag{2.15}$$

Let $\varepsilon > 0$. By the exponential stability of $\mathbf{T}$ there exists $s_2 \geq 0$ such that

$$\|\mathbf{T}_t x(s_1)\| \leq \varepsilon/2\,, \quad \forall\, t \geq s_2\,. \tag{2.16}$$

Now

$$x(t) = \mathbf{T}_{t-s_1}x(s_1) + \int_{s_1}^{t} \mathbf{T}_{t-\tau}Bu(\tau)\,d\tau\,,$$

and hence, by combining (2.14)-(2.16), we obtain for all $t \geq s_1 + s_2$

$$\|x(t)\| \leq \|\mathbf{T}_{t-s_1}x(s_1)\| + \beta\left(\int_{s_1}^{t}\|u(\tau)\|^2\,d\tau\right)^{1/2} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon\,,$$

showing that $\lim_{t\to\infty}\|x(t)\| = 0$. □

The following lemma can be found in Salamon [24] (more precisely, it is part of Lemma 2.5 in [24]).

Lemma 2.3 *If $x^0 \in X$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, U)$ are such that $Ax^0 + Bu(0) \in X$, then the solution $x(\cdot)$ of the initial-value problem (2.5) is continuously differentiable in X.*

Suppose that $Y = U$ and consider the following nonlinear system

$$\dot{x}(t) = Ax(t) + B\phi(u(t))\,,\quad x(0) = x^0 \in X \tag{2.17a}$$

$$\dot{u}(t) = k\{r - C_L\mathbf{T}_t x^0 - [\mathbf{F}_\infty(\phi(u))](t)\}\,,\quad u(0) = u^0 \in U\,, \tag{2.17b}$$

where the reference vector $r \in U$, the gain parameter $k \in \mathbb{R}$ and the nonlinearity $\phi : U \to U$ satisfies a *local Lipschitz condition*, that is, for every bounded set $W \subset U$ there exists a constant $l \geq 0$ such that

$$\|\phi(u) - \phi(v)\| \leq l\|u - v\|\,,\quad \forall\, u, v \in W\,.$$

Of course, $y(t) := C_L\mathbf{T}_t x^0 + [\mathbf{F}_\infty(\phi(u))](t)$ is the output of the well-posed system $\mathbf{\Sigma}$ corresponding to the initial condition $x(0) = x^0$ and the input $\phi \circ u$, and so (2.17b) may be written in the compact form $\dot{u}(t) = k(r - y(t))$.

For $T \in (0, \infty]$, a continuous function

$$[0, T) \to X \times U\,,\quad t \mapsto (x(t), u(t))$$

is a *solution* of (2.17) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times U)$-valued function, $(x(0), u(0)) = (x^0, u^0)$ and the differential equations in (2.17) are satisfied almost everywhere on $[0, a)$, where the derivative in (2.17a) should be interpreted in the space X_{-1}.† An application of a well-known result on abstract Cauchy problems (see Pazy [21], Theorem 2.4, p. 107) shows that a continuous $(X \times U)$-valued function $(x(\cdot), u(\cdot))$ is a solution of (2.17) if, and only if, it satisfies the following integrated version of (2.17)

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau}B\phi(u(\tau))\,d\tau\,, \tag{2.18a}$$

$$u(t) = u^0 - k\int_0^t [r - C_L\mathbf{T}_\tau x^0 - (\mathbf{F}_\infty\phi(u))(\tau)]\,d\tau\,. \tag{2.18b}$$

† Being a Hilbert space $X_{-1} \times U$ is reflexive, and hence any absolutely continuous $(X_{-1} \times U)$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [2], Theorem 3.1, p. 10.

The next result asserts that (2.17) has a unique solution: the proof is given in [19] under the assumption that $\mathbf{\Sigma}$ is regular. An inspection of the proof shows that it carries over to well-posed systems without any changes.

Lemma 2.4 *Let $\phi : U \to U$ be locally Lipschitz. For each $(x^0, u^0) \in X \times U$, there exists a unique solution $(x(\cdot), u(\cdot))$ of (2.17) defined on a maximal interval $[0, T)$. If $T < \infty$, then*

$$\limsup_{t \to T} (\|x(t)\| + \|u(t)\|) = \infty\,.$$

3 Absolute stability results

In the following let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a well-posed linear system with state space X, input space U, output space $Y = U$, generating operators A, B and C, input-output operator $\mathbf{F}_\infty$ and transfer function $\mathbf{G}$. In this section we consider the nonlinear system (2.17) with $r = 0$ and $k = 1$, i.e.,

$$\dot{x}(t) = Ax(t) + B\phi(u(t))\,, \quad x(0) = x^0 \in X \tag{3.1a}$$
$$\dot{u}(t) = -C_L \mathbf{T}_t x^0 - [\mathbf{F}_\infty(\phi(u))](t)\,, \quad u(0) = u^0 \in U\,. \tag{3.1b}$$

We assume that ϕ is locally Lipschitz (in the sense of Section 2) and sector bounded, i.e., there exist numbers $a \le b$ such that

$$\langle \phi(v) - av, \phi(v) - bv \rangle \le 0\,, \quad \forall\, v \in U\,. \tag{3.2}$$

Let $\mathcal{S}[a,b]$ denote the set of all functions $\phi : U \to U$ such that (3.2) holds. It is easy to show that (3.2) holds if and only if

$$\left\| \phi(v) - \frac{a+b}{2}\, v \right\| \le \frac{b-a}{2}\, \|v\|\,, \quad \forall\, v \in U\,.$$

This shows in particular, that if $\phi \in \mathcal{S}[a,b]$, then the graph of ϕ is contained in a (non-convex) cone with vertex at the origin. For $a < b$ we define

$$\mathcal{S}[a,b) := \bigcup_{a \le c < b} \mathcal{S}[a,c]\,.$$

Stability in the large

The zero solution of (3.1) is called *stable in the large* if: (i) for all $(x^0, u^0) \in X \times U$ there exists a solution of (3.1) on $\mathbb{R}_+$ (i.e. $T = \infty$ in Lemma 2.4); and (ii) there exists a continuous, strictly increasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ with $p(0) = 0$ and such that for any $l > 0$ the solution of $(x(\cdot), u(\cdot))$ of the initial value (3.1) satisfies

$$\|x^0\| + \|u^0\| \le l \quad \Longrightarrow \quad \|x(t)\| + \|u(t)\| \le p(l)\,, \ \forall\, t \ge 0\,.$$

Let $R, S \in \mathcal{B}(U_c)$. If

$$\langle Rv, v \rangle \geq 0\,, \quad \forall\, v \in U_c\,,$$

we write $R \geq 0$; if the above inequality is strict for all $u \in U_c \setminus \{0\}$, we write $R > 0$. Since U_c is a complex Hilbert space, $R \geq 0$ implies that $R = R^*$. We write $S \geq R$ if $S - R \geq 0$.

The following theorem shows that a suitable positive real condition in terms of the transfer function $\mathbf{G}(s)/s$ will ensure that the zero solution of (3.1) is stable in the large if $\mathbf{\Sigma}$ is exponentially stable and $\phi \in \mathcal{S}[0, b)$.

Theorem 3.1 *Suppose that $\mathbf{\Sigma}$ is exponentially stable, $\mathbf{G}(0)$ is invertible in $\mathcal{B}(U)$ and $\phi : U \to U$ is locally Lipschitz. Let $(x^0, u^0) \in X \times U$. If there exists $b > 0$ such that*

$$I + \frac{b}{2}\left(\frac{1}{s}\mathbf{G}(s) + \frac{1}{\bar{s}}\mathbf{G}^*(s)\right) \geq 0\,, \quad \forall\, s \in \mathbb{C}_0\,, \tag{PR}$$

and if $\phi \in \mathcal{S}[0, b)$, then the following statements hold:

(a) the zero solution of (3.1) is stable in the large;

(b) the solution $(x(\cdot), u(\cdot))$ of (3.1) satisfies

$$\lim_{t\to\infty} \|x(t)\| = 0\,, \quad x \in L^2(\mathbb{R}_+, X)\,, \quad \lim_{t\to\infty} \|\phi(u(t))\| = 0\,, \quad \phi \circ u \in L^2(\mathbb{R}_+, U)\,;$$

(c) under the extra assumption that $\dim U = 1$, $u^\infty := \lim_{t\to\infty} u(t)$ exists, is finite and satisfies $\phi(u^\infty) = 0$.

Remark 3.2 (a) Since $\mathbf{\Sigma}$ is exponentially stable, it follows that $\mathbf{G}(s)/s$ is holomorphic in $\mathbb{C}_0$. Combining this with (PR) shows that the function $I + b\,\mathbf{G}(s)/s$ is positive real. As in the finite-dimensional case (see Anderson and Vongpanitlerd [1], pp. 53), it can be shown that (PR) holds if and only if $\mathbf{G}(0) = \mathbf{G}^*(0) \geq 0$ and

$$I + \frac{b}{2}\left(\frac{1}{i\omega}\mathbf{G}(i\omega) - \frac{1}{i\omega}\mathbf{G}^*(i\omega)\right) \geq 0\,, \quad \forall\, \omega \in \mathbb{R} \setminus \{0\}\,.$$

In this context it is interesting to note that there exists $b > 0$ such that (PR) holds if and only if $\mathbf{G}(0) = \mathbf{G}^*(0) \geq 0$ (see Logemann and Townley [20]).

(b) Note that assertion (a) implies the boundeness of the solution $(x(\cdot), u(\cdot))$ of (3.1) for all $(x^0, u^0) \in X \times U$.

(c) If $\phi^{-1}(\{0\}) = \{0\}$ and $\dim U = 1$, then it follows from a combination of assertions (a)-(c) that the zero solution of (3.1) is globally asymptotically stable.

(d) Some of the statements in Theorem 3.1 remain true for time-varying sector bounded nonlinearities. More precisely, let $\phi : \mathbb{R}_+ \times U \to U$, $(t, v) \mapsto \phi(t, v)$ be continuous in t and locally Lipschitz in v, uniformly in t on bounded intervals. An inspection of the proof of Theorem 3.1 shows that statement (a) remains true for all such ϕ satisfying

$$\langle \phi(t,v), \phi(t,v) - cv \rangle \leq 0\,, \quad \forall\,(t,v) \in \mathbb{R}_+ \times U$$

for some $c \in [0,b)$; furthermore, apart from the convergence of $\phi(t,u(t))$ to 0 as $t \to \infty$, statement (b) remains true also.

(e) Consider the feedback system shown in Figure 3, where $\mathbf{H}(s)$ is the (rational) transfer function of a m-input, m-output, exponentially stable, finite-dimensional system and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity. The circle criterion says that if

$$I + \frac{b}{2}\left(\mathbf{H}(i\omega) + \mathbf{H}^*(i\omega)\right) \geq 0\,, \quad \forall\,\omega \in \mathbb{R}\,,$$

then for all locally Lipschitz $\phi \in \mathcal{S}[0,b)$, the closed-loop system shown in Figure 3 is globally exponentially stable (see [12], p. 409 and [27], p. 227). Hence Theorem 3.1 can be considered as an extension of the circle criterion to the case where $\mathbf{H}(s)$ is of the form $\mathbf{H}(s) = \mathbf{G}(s)/s$ with $\mathbf{G}$ being the transfer function of an exponentially stable well-posed infinite-dimensional system, i.e. the plant is the series interconnection of an exponentially stable well-posed infinite-dimensional system and an integrator.

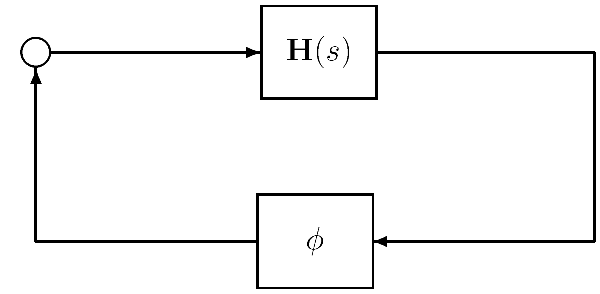

Figure 3

(f) For exponentially stable single-input single-output systems with bounded control and bounded observation, the stability properties of the zero solution of (3.1) have been investigated in [32] under the assumption that ϕ is locally Lipschitz, $\phi(v)v > 0$ for $v \neq 0$ and $\lim_{|v|\to\infty} \int_0^v \phi(w)\,dw = \infty$. It is shown in [32] that for all such ϕ, the zero solution of (3.1) is uniformly asymptotically stable in the large, provided that

$$\operatorname{Re} \mathbf{G}(i\omega) \geq \varepsilon > 0\,, \quad \forall\,\omega \in \mathbb{R}\,, \tag{3.3}$$

i.e., $\mathbf{G}$ is positive-real in a strict sense.[†] Trivially, there are many examples where (3.3) is not satisfied, whilst (PR) holds for some $b > 0$. $\diamond$

[†] If the underlying semigroup is analytic, then this result remains true for unbounded control action, see [3].

Proof of Theorem 3.1: Let $(x^0, u^0) \in X \times U$. By Lemma 2.4 the corresponding solution of the initial-value problem (3.1), denoted by $(x(\cdot), u(\cdot))$, exists on a maximal interval $[0, T)$ (where $0 < T \le \infty$) and is unique. Set $Q := \mathbf{G}(0)$ and define for $t \in [0, T)$

$$z(t) := A^{-1}x(t)\,, \quad v(t) := Q^{-1}(Cz(t) + u(t))\,. \tag{3.4}$$

We proceed in several steps.

Step 1: A differential equation for (z, v).
We claim that $(z(t), v(t))$ is continuously differentiable in $X \times U$ for all $t \in (0, T)$ and satisfies

$$\dot z(t) = Az(t) + A^{-1}B\phi(u(t))\,, \quad z(0) = z^0 \in X_1\,, \ \ \forall\, t \in (0, T)\,, \tag{3.5a}$$
$$\dot v(t) = -\phi(u(t))\,, \quad v(0) = v^0 \in U\,, \ \ \forall\, t \in (0, T)\,, \tag{3.5b}$$

where $z^0 := A^{-1}x^0$ and $v^0 := Q^{-1}(CA^{-1}x^0 + u^0)$.

Using the continuity of the functions $t \mapsto \phi(u(t))$ and

$$[0, T) \to X\,, \ \ t \mapsto \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau\,,$$

it follows from a well-known result on the existence of classical solutions to abstract Cauchy problems (see Pazy [21], p. 107) that for all $t \in (0, T)$, $x(t)$ is continuously differentiable in X_{-1} and (3.1a) holds. As a consequence, $z(t)$ is continuously differentiable in X for all $t \in (0, T)$ and (3.5a) holds. The proof of the claim for $v(t)$ is more difficult and requires an approximation argument. To this end let $T_0 \in (0, T)$ be arbitrary. Choose $x_n^0 \in X_1$ and $w_n \in W^{1,2}([0, T_0], U)$ with $w_n(0) = 0$ such that

$$\lim_{n\to\infty} \|x^0 - x_n^0\| = 0\,, \quad \lim_{n\to\infty} \|\phi(u) - w_n\|_{L^2(0,T;U)} = 0\,. \tag{3.6}$$

Consider the initial-value problem

$$\dot\zeta(t) = A\zeta(t) + Bw_n(t)\,, \quad \zeta(0) = x_n^0\,, \tag{3.7a}$$
$$\dot\eta(t) = -C\mathbf{T}_t x_n^0 - (\mathbf{F}_\infty w_n)(t)\,, \quad \eta(0) = u^0\,, \tag{3.7b}$$

and denote its solution by $(x_n(\cdot), u_n(\cdot))$. Denoting the output function of $\mathbf{\Sigma}$ corresponding to the initial value x_n^0 and the control $w_n(\cdot)$ by $y_n(\cdot)$, it is clear that the right-hand side of (3.7b) is equal to $-y_n(t)$. By Lemma 2.3, x_n is continuously differentiable in X and hence the function defined by

$$v_n(t) := Q^{-1}[CA^{-1}x_n(t) + u_n(t)] \tag{3.8}$$

is absolutely continuous with derivative

$$\begin{aligned}\dot v_n(t) &= Q^{-1}\left(C[x_n(t) + A^{-1}Bw_n(t)] - y_n(t)\right)\\ &= Q^{-1}\left(C[\mathbf{T}_t x_n^0 + \int_0^t \mathbf{T}_{t-\tau}Bw_n(\tau)\, d\tau + A^{-1}Bw_n(t)] - y_n(t)\right)\,, \quad \text{a.e. } t \in [0, T_0]\end{aligned}$$

Invoking (2.7), we obtain

$$\dot{v}_n(t) = -Q^{-1}\mathbf{G}(0)w_n(t) = -w_n(t)\,, \quad \text{a.e. } t \in [0, T_0]\,,$$

and thus

$$v_n(t) = v_n(0) - \int_0^t w_n(\tau)\, d\tau\,, \quad \forall\, t \in [0, T_0]\,. \tag{3.9}$$

From (3.6) it follows via standard properties of well-posed systems that for all $t \in [0, T_0]$

$$\lim_{n\to\infty} \|x_n(t) - x(t)\| = 0\,, \quad \lim_{n\to\infty} u_n(t) = u(t)\,,$$

and therefore, by (3.4) and (3.8)

$$\lim_{n\to\infty} v_n(t) = v(t)\,, \quad \forall\, t \in [0, T_0]\,.$$

On the other hand, by (3.6) and (3.9)

$$\lim_{n\to\infty} v_n(t) = v^0 - \int_0^t \phi(u(\tau))\, d\tau\,, \quad \forall\, t \in [0, T_0]\,,$$

showing that

$$v(t) = v^0 - \int_0^t \phi(u(\tau))\, d\tau\,, \quad \forall\, t \in [0, T_0]\,,$$

which upon differentiation yields (3.5b).

Step 2: Exploiting the positive-real condition (PR).
By assumption, $\phi \in \mathcal{S}[0, b)$, and hence there exists $\tilde{b} \in (0, b)$ such that

$$\phi \in \mathcal{S}[0, \tilde{b}]\,. \tag{3.10}$$

Choose $c \in (\tilde{b}, b)$. We consider the quadruple $\Xi = (A, A^{-1}B, C, c^{-1}I)$ of operators, which are the generating operators of an exponentially stable Pritchard-Salamon system[†] on the spaces $X_1 \hookrightarrow X$ (this implies that Ξ defines an exponentially stable regular system). The function

$$\mathbf{H}(s) = C(sI - A)^{-1}A^{-1}B + \frac{1}{c}I$$

is the transfer function of Ξ. By (2.9) we have

$$\mathbf{H}(s) = \frac{1}{s}\,(\mathbf{G}(s) - \mathbf{G}(0)) + \frac{1}{c}I = \frac{1}{s}\,(\mathbf{G}(s) - Q) + \frac{1}{c}I\,, \quad \forall\, s \in \varrho(A),\ s \neq 0\,.$$

It is not difficult to show that (PR) implies that $Q = \mathbf{G}(0) \geq 0$, and hence, in particular, $Q = Q^*$ (this can be proved exactly as in the finite-dimensional case, see [1], pp. 53; cf.

[†] See [8, 15] for the concept of a Pritchard-Salamon system.

also Remark 3.2, part (a)). Therefore, since $0 < c < b$, (PR) guarantees the existence of a constant $\varepsilon > 0$ such that

$$\mathbf{H}(i\omega) + \mathbf{H}^*(i\omega) \geq \varepsilon I\,, \quad \forall\, w \in \mathbb{R}\,.$$

Consequently, by a result in van Keulen [15] (see Theorem 3.10 and Remark 3.14 in [15]), there exists $\tilde{P} \in \mathcal{B}(X)$, $\tilde{P} = \tilde{P}^*$, such that

$$\langle Ax_1, \tilde{P}x_2\rangle + \langle \tilde{P}x_1, Ax_2\rangle = \frac{c}{2}\langle[(A^{-1}B)^*\tilde{P} + C]x_1, [(A^{-1}B)^*\tilde{P} + C]x_2\rangle\,, \quad \forall\, x_1, x_2 \in X_1\,. \tag{3.11}$$

Setting

$$P := -\tilde{P} \in \mathcal{B}(X)\,, \quad L := \sqrt{\frac{c}{2}}\,[C - (A^{-1}B)^*P] \in \mathcal{B}(X_1, U)\,,$$

we obtain using (3.11)

$$\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle = -\langle Lx_1, Lx_2\rangle\,, \quad \forall\, x_1, x_2 \in X_1 \tag{3.12a}$$

$$(A^{-1}B)^*Px_1 = Cx_1 - \sqrt{2/c}\,Lx_1\,, \quad \forall\, x_1 \in X_1\,. \tag{3.12b}$$

Moreover, by a routine argument it follows from (3.12a) and the exponential stability of the semigroup $\mathbf{T}_t$ that $P \geq 0$.

Step 3: A Lyapunov-type argument.
For $t \in [0, T)$ define

$$V(t) := \langle z(t), Pz(t)\rangle + \langle v(t), Qv(t)\rangle \geq 0\,. \tag{3.13}$$

By Step 1, V is continuously differentiable. Differentiating V and using (3.5), (3.12) and the definition of v (see (3.4)) yields for all $t \in [0, T)$

$$\begin{aligned}\dot{V}(t) &= -\|Lz(t)\|^2 + 2\langle\phi(u(t)), Cz(t) - \sqrt{2/c}\,Lz(t)\rangle - 2\langle\phi(u(t)), Qv(t)\rangle \\ &= -\|Lz(t)\|^2 - 2\sqrt{2/c}\,\langle\phi(u(t)), Lz(t)\rangle - 2\langle\phi(u(t)), u(t)\rangle\,.\end{aligned}$$

Completing the square gives for all $t \in [0, T)$

$$\dot{V}(t) = -\|Lz(t) + \sqrt{2/c}\,\phi(u(t))\|^2 + \frac{2}{c}\left(\|\phi(u(t))\|^2 - c\langle\phi(u(t)), u(t)\rangle\right)\,. \tag{3.14}$$

Since $c > \tilde{b}$ there exists $\delta > 0$ such that $c = \tilde{b} + \delta$ and therefore, using (3.10), we obtain for all $t \in [0, T)$

$$\begin{aligned}\|\phi(u(t))\|^2 - c\langle\phi(u(t)), u(t)\rangle &= \langle\phi(u(t)), \phi(u(t)) - \tilde{b}u(t)\rangle - \delta\langle\phi(u(t)), u(t)\rangle \\ &\leq -\delta\langle\phi(u(t)), u(t)\rangle \\ &\leq -\frac{\delta}{\tilde{b}}\|\phi(u(t))\|^2\,.\end{aligned}$$

Using this inequality in (3.14), it follows that

$$\dot{V}(t) \leq -\frac{2\delta}{\tilde{b}c}\|\phi(u(t))\|^2\,, \quad \forall\, t \in [0,T)\,.$$

Hence, since $V(t) \geq 0$, integration leads to

$$\int_0^t \|\phi(u(\tau))\|^2\, d\tau \leq \frac{\tilde{b}c}{2\delta}V(0)\,, \quad \forall\, t \in [0,T)\,. \tag{3.15}$$

The inequality (3.15) shows that $\phi \circ u \in L^2(0,T;U)$. Combining this with the well-posedness and the exponential stability of $\mathbf{\Sigma}$, we may conclude that $x(t)$ is bounded on the interval $[0,T)$ and $\dot{u} \in L^2(0,T;U)$. This implies the boundedness of $(x(t),u(t))$ on $[0,T)$ if $T < \infty$, and hence, by the maximality of T, it follows via Lemma 2.4 that $T = \infty$. Therefore the solution $(x(t),u(t))$ exists for all $t \geq 0$ and by (3.15)

$$\int_0^\infty \|\phi(u(\tau))\|^2\, d\tau \leq \frac{\tilde{b}c}{2\delta}V(0)\,. \tag{3.16}$$

Step 4: Proof of statement (a) – stability in the large.
As an immediate consequence of the exponential stability of $\mathbf{\Sigma}$, (2.3) (3.4), (3.13) and (3.16) there exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ not depending on x^0 and u^0 and such that

$$\begin{aligned}\|x(t)\|^2 &\leq \alpha_1(\|x^0\|^2 + V(0)) \leq \alpha_2(\|x^0\|^2 + \|A^{-1}x^0\|^2 + \|CA^{-1}x^0 + u^0\|^2)\\ &\leq \alpha_3(\|x^0\|^2 + \|u^0\|^2)\,.\end{aligned} \tag{3.17}$$

Moreover, as was pointed out in Step 2, (PR) implies that $Q = Q^* \geq 0$. Combining this with the invertibility of Q, it follows from standard results on positive operators (see e.g. Rudin [22], p. 314) that the map $w \mapsto \langle w, Qw\rangle^{1/2}$ defines a new norm on U which is equivalent to the original norm on U. Therefore, using (3.13), the fact that $V(t)$ is non-increasing and $P \geq 0$, we may conclude that

$$\|v(t)\|^2 \leq \beta_1 V(0) \leq \beta_2(\|x^0\|^2 + \|u^0\|^2)\,,$$

for some constants $\beta_1, \beta_2 > 0$ which do not depend on x^0 and u^0. From the definition of $v(t)$ (see (3.4)) it follows that

$$\|Q^{-1}u(t)\|^2 \leq 2\beta_2(\|x^0\|^2 + \|u^0\|^2) + 2\|Q^{-1}CA^{-1}x(t)\|^2\,.$$

Combining this with (3.17) yields the existence of a constant $\beta_3 > 0$ (not depending on x^0 and u^0) such that

$$\|u(t)\|^2 \leq \beta_3(\|x^0\|^2 + \|u^0\|^2)\,. \tag{3.18}$$

Statement (a) now follows from (3.17) and (3.18).

STEP 5: Proof of statement (b) – asymptotic behaviour of x and $\phi \circ u$.
Note that by (3.16),

$$\phi \circ u \in L^2(\mathbb{R}_+, U), \tag{3.19}$$

and hence, by Lemma 2.2, $x \in L^2(\mathbb{R}_+, X)$ and $\lim_{t\to\infty} \|x(t)\| = 0$. It only remains to show that $\lim_{t\to\infty} \phi(u(t)) = 0$. To this end note, that that by statement (a) proved in Step 4, $u(\cdot)$ is bounded, i.e. there exists a number $\gamma > 0$ such that

$$\|u(t)\| \leq \gamma, \quad \forall\, t \geq 0. \tag{3.20}$$

Moreover, by the exponential stability of $\mathbf{\Sigma}$ we obtain from (3.1b) and (3.19) that $\dot{u} \in L^2(\mathbb{R}_+, U)$, which implies in particular that $u(\cdot)$ is uniformly continuous. By (3.20) and the local Lipschitz property of ϕ, the restricted map $\phi|_{\operatorname{im} u}$ is globally Lipschitz, and therefore, using the uniform continuity of $u(\cdot)$, we may conclude that $\phi \circ u$ is uniformly continuous. Combining this with (3.19) and applying Barbălat's lemma (see [12], p. 192) shows that $\lim_{t\to\infty} \phi(u(t)) = 0$.

STEP 6: Proof of statement (c) – asymptotic behaviour of u.
Assume that $\dim U = 1$, i.e. $U = \mathbb{R}$. Note, that by Step 3, $V_\infty := \lim_{t\to\infty} V(t)$ exists and is finite (since $V(\cdot)$ is non-negative and non-increasing). Combining this with the fact that $\lim_{t\to\infty} \|x(t)\| = 0$, we may conclude from (3.4) and (3.13) that $\lim_{t\to\infty} u(t) = u^\infty$ exists and is finite. By continuity of ϕ and statement (b), it follows that $\phi(u^\infty) = \lim_{t\to\infty} \phi(u(t)) = 0$. □

In the following we introduce extra assumptions on the nonlinearity ϕ which will guarantee that $u(t)$ converges as $t \to \infty$ in the case that $\dim U > 1$.

Corollary 3.3 *Suppose that the assumptions of Theorem 3.1 hold. If there exists $b > 0$ such that (PR) is satisfied, if $\phi \in \mathcal{S}[0, b)$ and if the extra assumptions*

(A1) $\phi^{-1}(\{0\}) \cap W$ *is finite for any bounded set* $W \subset U$*,*

(A2) $\inf_{w\in W} \|\phi(w)\| > 0$ *for any bounded, closed and nonempty set* $W \subset U$ *with* $\phi^{-1}(\{0\}) \cap W = \emptyset$*,*

are satisfied, then statements (a) and (b) of Theorem 3.1 hold. Moreover, the limit $u^\infty := \lim_{t\to\infty} u(t)$ *exists, is finite and* $\phi(u^\infty) = 0$*.*

Remark 3.4 Of course, assumption (A2) is automatically satisfied if $\dim U < \infty$. Corollary 3.3 shows in particular that if $\phi^{-1}(\{0\}) = \{0\}$ and (A2) holds, then the zero solution of (3.1) is globally asymptotically stable. ◇

Proof of Corollary 3.3: Let $(x(\cdot), u(\cdot))$ denote the solution of (3.1). Clearly, since the assumptions of Theorem 3.1 are satisfied, statements (a) and (b) of Theorem 3.1 hold. Therefore, in particular, $\operatorname{im} u$ is bounded and

$$\lim_{t\to\infty} \phi(u(t)) = 0\,. \tag{3.21}$$

By assumption (A1), the set $Z := \phi^{-1}(\{0\}) \cap \operatorname{clos}(\operatorname{im} u)$ is finite. Moreover, by (A2) and (3.21), $Z \neq \emptyset$. It is sufficient to show that

$$\lim_{t\to\infty} \operatorname{dist}(u(t), Z) = 0\,, \tag{3.22}$$

since by the finiteness of Z, the continuity of u and the continuity of ϕ, (3.22) implies that $u^\infty := \lim_{t\to\infty} u(t)$ exists and $u^\infty \in Z \subset \phi^{-1}(\{0\})$.

Seeking a contradiction, suppose that (3.22) is not true. Then there exist a sequence $(t_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$ and a number $\varepsilon > 0$ such that

$$u(t_n) \notin \{v \in U : \operatorname{dist}(v, Z) < \varepsilon\} =: Z_\varepsilon\,.$$

Defining $W := \operatorname{clos}(\operatorname{im} u \setminus Z_\varepsilon)$, it follows that W is closed, bounded and $\phi^{-1}(\{0\}) \cap W = \emptyset$. Consequently, by assumption (A2), there exists $\delta > 0$ such that $\|\phi(w)\| \geq \delta$ for all $w \in W$. Since $u(t_n) \in W$ for all $n \in \mathbb{N}$, we obtain that $\|\phi(u(t_n))\| \geq \delta$ for all $n \in \mathbb{N}$, contradicting (3.21). □

Exponential stability

The following theorem shows that under the assumptions of Theorem 3.1 the zero solution of (3.1) is globally exponentially stable, provided that $\phi \in \mathcal{S}[a, b)$ for some $a \in (0, b)$ (i.e., the nonlinearity ϕ is assumed to satisfy a more restrictive sector condition than in Theorem 3.1).

Theorem 3.5 *Suppose that $\mathbf{\Sigma}$ is exponentially stable, $\mathbf{G}(0)$ is invertible and $\phi : U \to U$ is locally Lipschitz. If there exists $b > 0$ such that (PR) holds and if $\phi \in \mathcal{S}[a, b)$ for some $a \in (0, b)$, then the zero solution of (3.1) is globally exponentially stable, that is, there exist $N \geq 1$ and $\nu \in (\omega(\mathbf{T}), 0)$ such that for all $(x^0, u^0) \in X \times U$ the solution $(x(\cdot), u(\cdot))$ of (3.1) satisfies*

$$\|x(t)\| + \|u(t)\| \leq N e^{\nu t}(\|x^0\| + \|u^0\|)\,, \quad \forall\, t \geq 0\,.$$

Proof: Let K be the supremum of all numbers $k > 0$ such that

$$I + \frac{k}{2}\left(\frac{1}{s}\mathbf{G}(s) + \frac{1}{\bar{s}}\mathbf{G}^*(s)\right) \geq 0\,, \quad \forall\, s \in \mathbb{C}_0\,.$$

By assumption, $K \geq b > 0$. Since $\phi \in \mathcal{S}[a,b)$, there exists $c \in (a,b)$ such that $\phi \in \mathcal{S}[a,c]$. Let $\delta \in (0,a)$ such that $c+\delta < K$. We define

$$\kappa := \frac{c+\delta}{2} < \frac{K}{2}, \quad \mathbf{H}(s) := \frac{1}{s}\mathbf{G}(s)\left(I + \frac{\kappa}{s}\mathbf{G}(s)\right)^{-1}, \quad \mathbf{L}(s) := \frac{1}{s}\left(I + \frac{\kappa}{s}\mathbf{G}(s)\right)^{-1}.$$

We know from [20]† that

$$\mathbf{H},\, \mathbf{L} \in H^\infty(\mathbb{C}_0, \mathcal{B}(U))\,. \tag{3.23}$$

Moreover, Lemma 3.10 in [20] † yields

$$\|\mathbf{H}\|_\infty = \sup_{s \in \mathbb{C}_0} \|\mathbf{H}(s)\| = \frac{1}{\kappa}\,. \tag{3.24}$$

Setting

$$\psi(v) := \phi(v) - \kappa v\,, \quad \gamma := \frac{c-\delta}{2}$$

and using the fact that $\phi \in \mathcal{S}[\delta, c] \subset \mathcal{S}[a,c]$, we obtain

$$\|\psi(v)\| \leq \gamma \|v\|\,, \quad \forall\, v \in U\,. \tag{3.25}$$

Clearly, $\kappa > \gamma$, and hence by (3.24)

$$\gamma \|\mathbf{H}\|_\infty < 1\,. \tag{3.26}$$

Let $\varepsilon \in (0,\delta)$ be sufficiently small such that the semigroup $e^{\varepsilon t}\mathbf{T}_t$ is exponentially stable,

$$\mathbf{H},\, \mathbf{L} \in H^\infty(\mathbb{C}_{-\varepsilon}, \mathcal{B}(U))\,, \tag{3.27}$$

and

$$\gamma \sup_{s \in \mathbb{C}_{-\varepsilon}} \|\mathbf{H}(s)\| < 1\,. \tag{3.28}$$

For all sufficiently small $\varepsilon > 0$, (3.27) follows via a routine argument from (3.23) and the fact that $\mathbf{G} \in H^\infty(\mathbb{C}_{-\varepsilon}, \mathcal{B}(U))$, whilst (3.28) is a consequence of (3.26) and (3.27) combined with the fact that a holomorphic function which is bounded in an open vertical strip in the complex plane is uniformly continuous in any closed vertical substrip (see [7], p. 72).

Let $(x^0, u^0) \in X \times U$ and let $(x(\cdot), u(\cdot))$ denote the solution of (3.1) which satisfies the initial conditions $x(0) = x^0$ and $u(0) = u^0$. Rewrite (3.1b) in the form

$$\dot{u}(t) = -(\mathbf{\Psi}_\infty x^0)(t) - (\mathbf{F}_\infty(\psi \circ u + \kappa u))(t)\,, \quad \text{a.e. } t \geq 0\,. \tag{3.29}$$

† The results in [20] are proved for matrix-valued $\mathbf{G}$, i.e. $\dim U < \infty$. However, it is easy to see that the relevant results in [20] extend to infinite-dimensional input spaces U.

By Theorem 3.1, u is bounded and $\dot{u} \in L^2(\mathbb{R}_+, U)$. Thus the Laplace transforms $(\mathfrak{L}(u))(s)$, $(\mathfrak{L}(\psi \circ u))(s)$ and $(\mathfrak{L}(\dot{u}))(s)$ exist for all $s \in \mathbb{C}_0$ and an application of the Laplace transform to (3.29) combined with a straightforward calculation yields

$$(\mathfrak{L}(u))(s) = \mathbf{L}(s)u^0 - \mathbf{L}(s)C(sI - A)^{-1}x^0 - \mathbf{H}(s)(\mathfrak{L}(\psi \circ u))(s)\,, \quad \forall\, s \in \mathbb{C}_0\,. \tag{3.30}$$

Define bounded operators H, L from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, U)$ by setting

$$Hv = \mathfrak{L}^{-1}(\mathbf{H}\mathfrak{L}(v))\,, \quad Lv = \mathfrak{L}^{-1}(\mathbf{L}\mathfrak{L}(v))\,; \quad \forall\, v \in L^2(\mathbb{R}_+, U)\,.$$

By (3.27), H and L restrict to bounded operators from $L^2_{-\varepsilon}(\mathbb{R}_+, U)$ to $L^2_{-\varepsilon}(\mathbb{R}_+, U)$. The $L^2_{-\varepsilon}(\mathbb{R}_+, U)$-induced operator norms of H and L are given by

$$\sup_{s \in \mathbb{C}_{-\varepsilon}} \|\mathbf{H}(s)\| =: h \quad \text{and} \quad \sup_{s \in \mathbb{C}_{-\varepsilon}} \|\mathbf{L}(s)\| =: l\,, \tag{3.31}$$

respectively. Taking inverse Laplace transforms in (3.30), we obtain

$$u = \mathfrak{L}^{-1}(\mathbf{L}u^0) - L(\mathbf{\Psi}_\infty x^0) - H(\psi \circ u)\,.$$

Taking the $L^2_{-\varepsilon}$-norm of $\mathbf{P}_t u$ (where $t \geq 0$), using the causality of H and estimating gives

$$\left(\int_0^t \|e^{\varepsilon\tau} u(\tau)\|^2\, d\tau\right)^{1/2} \leq \left(\int_0^\infty \|e^{\varepsilon\tau}(\mathfrak{L}^{-1}(\mathbf{L}u^0))(\tau)\|^2\, d\tau\right)^{1/2} + l\left(\int_0^\infty \|e^{\varepsilon\tau} C_L \mathbf{T}_\tau x^0\|^2\, d\tau\right)^{1/2}$$
$$+\, h\left(\int_0^t \|e^{\varepsilon\tau}\psi(u(\tau))\|^2\, d\tau\right)^{1/2}, \quad \forall\, t \geq 0\,. \tag{3.32}$$

It is clear that the function $s \mapsto \mathbf{L}(s)u^0$ is in $H^2(\mathbb{C}_{-\varepsilon}, U)$, and therefore by a well-known theorem due to Paley and Wiener

$$\int_0^\infty \|e^{\varepsilon\tau}(\mathfrak{L}^{-1}(\mathbf{L}u^0))(\tau)\|^2\, d\tau = \frac{1}{2\pi}\sup_{\alpha > -\varepsilon}\int_{-\infty}^\infty \|\mathbf{L}(\alpha + i\omega)u^0\|^2\, d\omega \leq N_1^2\|u^0\|^2\,, \tag{3.33}$$

where

$$N_1 := \left(\frac{1}{2\pi}\sup_{\alpha > -\varepsilon}\int_{-\infty}^\infty \|\mathbf{L}(\alpha + i\omega)\|^2\, d\omega\right)^{1/2} < \infty\,.$$

Combining (3.25), (3.32), (3.33) and using the exponential stability of $e^{\varepsilon t}\mathbf{T}_t$, we may conclude that there exists $N_2 > 0$ (not depending on x^0 and u^0) such that

$$\left(\int_0^t \|e^{\varepsilon\tau} u(\tau)\|^2\, d\tau\right)^{1/2} \leq N_1\|u^0\| + N_2\|x^0\| + \gamma h\left(\int_0^t \|e^{\varepsilon\tau} u(\tau)\|^2\, d\tau\right)^{1/2}, \quad \forall\, t \geq 0\,.$$

By (3.28) and (3.31), $\gamma h < 1$, and therefore, $u \in L^2_{-\varepsilon}(\mathbb{R}_+, U)$, and, moreover,

$$\left(\int_0^\infty \|e^{\varepsilon\tau} u(\tau)\|^2\, d\tau\right)^{1/2} \leq N_3(\|x^0\| + \|u^0\|)\,, \tag{3.34}$$

where

$$N_3 := \frac{\max(N_1, N_2)}{1 - \gamma h}.$$

Furthermore, since $\mathbf{F}_\infty$ is bounded from $L^2_{-\varepsilon}(\mathbb{R}_+, U)$ to $L^2_{-\varepsilon}(\mathbb{R}_+, U)$ with $L^2_{-\varepsilon}$-induced operator norm equal to

$$f_\infty := \sup_{s \in \mathbb{C}_{-\varepsilon}} \|\mathbf{G}(s)\|$$

and $\phi \in \mathcal{S}[a, c]$, it follows from (3.34) that

$$\left(\int_0^\infty \|e^{\varepsilon\tau} (\mathbf{F}_\infty \phi(u))(\tau)\|^2 \, d\tau \right)^{1/2} \leq N_3 f_\infty c(\|x^0\| + \|u^0\|), \tag{3.35}$$

In order to obtain an exponential estimate for $x(t)$, we multiply the integrated version (3.1a) by $e^{\varepsilon t}$ to obtain

$$e^{\varepsilon t} x(t) = e^{\varepsilon t} \mathbf{T}_t x^0 + \int_0^t e^{\varepsilon(t-\tau)} \mathbf{T}_{t-\tau} B e^{\varepsilon\tau} \phi(u(\tau)) \, d\tau, \quad \forall\, t \geq 0.$$

Using the exponential stability of the semigroup $e^{\varepsilon t} \mathbf{T}_t$ and the fact that $\phi \in \mathcal{S}[a, b)$, it follows that there exists a constant $N_4 \geq 1$ (not depending on x^0 and u^0) such that

$$e^{\varepsilon t} \|x(t)\| \leq N_4 \left[\|x^0\| + \left(\int_0^t \|e^{\varepsilon\tau} u(\tau)\|^2 \, d\tau \right)^{1/2} \right], \quad \forall\, t \geq 0.$$

Combining this with (3.34) shows that

$$e^{\varepsilon t} \|x(t)\| \leq N_5 (\|x^0\| + \|u^0\|), \quad \forall\, t \geq 0, \tag{3.36}$$

for some $N_5 \geq 1$ (not depending on x^0 and u^0).

Finally, to derive an exponential estimate for $u(t)$, let $\eta \in (0, \varepsilon)$. Using (3.1b), a straightforward calculation gives

$$\frac{d}{dt}(e^{\eta t} u(t)) = -e^{(\eta-\varepsilon)t} C_L e^{\varepsilon t} \mathbf{T}_t x^0 - e^{(\eta-\varepsilon)t} [e^{\varepsilon t} (\mathbf{F}_\infty \phi(u))(t)] + \eta e^{(\eta-\varepsilon)t} [e^{\varepsilon t} u(t)], \quad \text{a.e. } t \geq 0. \tag{3.37}$$

Clearly, the functions

$$t \mapsto e^{(\eta-\varepsilon)t}, \; t \mapsto C_L e^{\varepsilon t} \mathbf{T}_t x^0$$

are in $L^2(\mathbb{R}_+, \mathbb{R})$ and $L^2(\mathbb{R}_+, U)$, respectively. Moreover, by (3.34) and (3.35) the functions

$$t \mapsto e^{\varepsilon t} u(t), \; t \mapsto e^{\varepsilon t} (\mathbf{F}_\infty \phi(u))(t)$$

are in $L^2(\mathbb{R}_+, U)$. Integrating (3.37), using the Cauchy-Schwarz inequality and invoking (3.34) and (3.35) shows that there exists a constant $N_6 \geq 1$ (not depending on x^0 and u^0) such that

$$e^{\eta t} \|u(t)\| \leq N_6 (\|x^0\| + \|u^0\|), \quad \forall\, t \geq 0. \tag{3.38}$$

The claim now follows from (3.36) and (3.38) with $N = 2\max(N_5, N_6)$ and $\nu = -\eta$. $\square$

For $W \subset U$ and $a \le b$, let $\mathcal{S}_W[a,b]$ denote the set of all functions $\phi : U \to U$ such that

$$\langle \phi(w) - aw, \phi(w) - bw \rangle \le 0\,, \quad \forall\, w \in W\,.$$

For $a < b$ we define

$$\mathcal{S}_W[a,b) := \bigcup_{a \le c < b} \mathcal{S}_W[a,c]\,.$$

Of course, $\mathcal{S}_U[a,b] = \mathcal{S}[a,b]$ and $\mathcal{S}_U[a,b) = \mathcal{S}[a,b)$.

Theorem 3.1 and Theorem 3.5 can be used to derive the following semi-global exponential stability result. The assumptions on the nonlinearity ϕ are more restrictive than in Theorem 3.1, but less restrictive than in Theorem 3.5.

Theorem 3.6 *Suppose that $\mathbf{\Sigma}$ is exponentially stable, $\mathbf{G}(0)$ is invertible, $\phi : U \to U$ is locally Lipschitz and there exists $b > 0$ such that (PR) holds. If $\phi \in \mathcal{S}[0,b)$ and ϕ satisfies the two extra assumptions*

(A3) $\phi \in \mathcal{S}_V[a,b)$ for some open set $V \subset U$ with $0 \in V$ and some $a \in (0,b)$,

(A4) $\inf_{w\in W} \|\phi(w)\| > 0$ for any bounded, closed, nonempty set $W \subset U$ with $0 \notin W$,

then the zero solution of (3.1) is semi-globally exponentially stable, that is, for every $M > 0$, there exists $N \ge 1$ and $\nu \in (\omega(\mathbf{T}), 0)$ such that for all $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \le M$, the solution $(x(\cdot), u(\cdot))$ of (3.1) satisfies

$$\|x(t)\| + \|u(t)\| \le N e^{\nu t}(\|x^0\| + \|u^0\|)\,, \quad \forall\, t \ge 0\,.$$

Of course, for finite-dimensional U, (A4) holds if $\phi^{-1}(\{0\}) = \{0\}$ (by the continuity of ϕ).

Proof of Theorem 3.6: Let $M > 0$. By statement (a) of Theorem 3.1, there exists $R > 0$ such that for all $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \le M$, the solution $(x(\cdot), u(\cdot))$ of (3.1) satisfies

$$\|x(t)\| + \|u(t)\| \le R\,, \quad \forall\, t \ge 0\,. \tag{3.39}$$

Since $\phi \in \mathcal{S}[0,b)$ and by assumption (A3), there exists $c \in (a,b)$ such that

$$\phi \in \mathcal{S}[0,c] \cap \mathcal{S}_V[a,c]\,.$$

Define $\psi : U \to U$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if} \quad \|v\| \le R \\ c\left(1 - \dfrac{R}{\|v\|}\right) v + \phi(\dfrac{R}{\|v\|} v) & \text{if} \quad \|v\| > R\,. \end{cases} \tag{3.40}$$

Clearly, ψ is locally Lipschitz. Moreover, by Corollary A.2 in Part 1 of the Appendix, there exists $\varepsilon > 0$ such that $\psi \in \mathcal{S}[\varepsilon, b)$ (this is evident in the case $\dim U = 1$). Consequently, we may apply Theorem 3.5 to the system

$$\dot{\zeta}(t) = A\zeta(t) + B\psi(\eta(t)) \,, \quad \zeta(0) = \zeta^0 \,, \tag{3.41a}$$
$$\dot{\eta}(t) = -C_L \mathbf{T}_t x^0 - [\mathbf{F}_\infty(\psi(\eta))](t) \,, \quad \eta(0) = \eta^0 \,, \tag{3.41b}$$

to conclude that the zero solution of (3.41) is globally exponentially stable. By (3.39) and (3.40) it is clear that for all initial conditions $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \leq M$, the solution $(x(\cdot), u(\cdot))$ of (3.1) is also a solution of (3.41). Therefore, by Theorem 3.5, there exists $N \geq 1$ and $\nu \in (\omega(\mathbf{T}), 0)$ such that for all initial conditions $(x^0, u^0) \in X \times U$ with $\|x^0\| + \|u^0\| \leq M$,

$$\|x(t)\| + \|u(t)\| \leq N e^{\nu t}(\|x^0\| + \|u^0\|) \,, \quad \forall\, t \geq 0 \,.$$

$\square$

4 Applications to low-gain integral control

In this section we apply the results in Section 3 to the so-called low-gain tracking problem described in the Introduction (see also Figure 2). As in Section 3, let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a well-posed linear system with state space X, input space U, output space $Y = U$, generating operators A, B and C, input-output operator $\mathbf{F}_\infty$ and transfer function $\mathbf{G}$. In this section we assume that $U = Y = \mathbb{R}^m$.

For $\lambda > 0$, let $\mathcal{N}_1(\lambda)$ denote the set of all nondecreasing globally Lipschitz nonlinearities $f : \mathbb{R} \to \mathbb{R}$ such that λ is a Lipschitz constant for f. Moreover, for $\lambda > 0$, $\mathcal{N}_m(\lambda)$ denotes the set of all nonlinearities $f : \mathbb{R}^m \to \mathbb{R}^m$ which are of the form

$$f(v) = [f_1(v_1), f_2(v_2), \ldots , f_m(v_m)]^T \,, \quad \forall\, v = (v_1, v_2, \ldots , v_m)^T \in \mathbb{R}^m \,, \tag{4.1}$$

where $f_i \in \mathcal{N}_1(\lambda)$ for all $i = 1, \ldots , m$. Sometimes it will be convenient to write (4.1) in the form $f = \operatorname{diag}(f_i)$.

Consider the following nonlinear system

$$\dot{x}(t) = Ax(t) + B\phi(u(t)) \,, \quad x(0) = x^0 \in X \tag{4.2a}$$
$$\dot{u}(t) = k\{r - C_L \mathbf{T}_t x^0 - [\mathbf{F}_\infty(\phi(u))](t)\} \,, \quad u(0) = u^0 \in \mathbb{R}^m \,, \tag{4.2b}$$

where $r \in \mathbb{R}^m$ is the reference vector, $k \in \mathbb{R}$ is a gain parameter and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinearity in $\mathcal{N}_m(\lambda)$. The aim is to show that under suitable conditions on $\mathbf{\Sigma}$, the error $e(t) = r - y(t)$, where $y(t) = C_L \mathbf{T}_t x^0 + [\mathbf{F}_\infty(\phi(u))](t)$, becomes small in some sense as $t \to \infty$.

If $\mathbf{G}$ is holomorphic and bounded on $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$ (which for example is the case if $\boldsymbol{\Sigma}$ is exponentially stable) and $\mathbf{G}(0) = \mathbf{G}^*(0) > 0$, then it is not difficult to show that the following positive-real condition

$$I + \frac{k}{2}\left(\frac{1}{s}\mathbf{G}(s) + \frac{1}{\bar{s}}\mathbf{G}^*(s)\right) \geq 0\,, \quad \forall\, s \in \mathbb{C}_0 \tag{4.3}$$

holds for all sufficiently small $k > 0$, see Lemma 3.10 in [20]. We define

$$K := \sup\{k > 0 : (4.3) \text{ holds}\}\,.$$

Recall that $\mathcal{M}$ denotes the space of all $\mathbb{R}^{m\times m}$-valued Borel measures on $\mathbb{R}_+$.

Theorem 4.1 *Assume that $\boldsymbol{\Sigma}$ is exponentially stable and $\mathbf{G}(0) = \mathbf{G}^*(0) > 0$. Let $\lambda > 0$, $\phi \in \mathcal{N}_m(\lambda)$, $k \in (0, K/\lambda)$ and let $r \in \mathbb{R}^m$ be such that*

$$\phi^r := [\mathbf{G}(0)]^{-1} r \in \operatorname{im}\phi\,. \tag{4.4}$$

Then the solution $(x(\cdot), u(\cdot))$ of (4.2) is unique and exists on $\mathbb{R}_+$, and for each $u^r \in \phi^{-1}(\{\phi^r\})$, there exists $N > 0$ such that for all $(x^0, u^0) \in X \times \mathbb{R}^m$

$$\|x(t) - x^r\| + \|u(t) - u^r\| \leq N(\|x^0 - x^r\| + \|u^0 - u^r\|)\,, \quad \forall\, t \geq 0\,, \tag{4.5}$$

where $x^r := -A^{-1}B\phi^r$. Moreover, the following statements hold:

(a) $\lim_{t\to\infty}\phi(u(t)) = \phi^r\,, \quad \phi\circ u - \phi^r \in L^2(\mathbb{R}_+, \mathbb{R}^m)\,;$

(b) $\lim_{t\to\infty}\|x(t) - x^r\| = 0\,, \quad x - x^r \in L^2(\mathbb{R}_+, X)$*;*

(c) $e := r - y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$*, where* $y(t) = C_L\mathbf{T}_t x^0 + [\mathbf{F}_\infty(\phi(u))](t)$*;*

(d) under the additional assumption that $m = 1$, the limit $\lim_{t\to\infty} u(t) =: u^\infty$ exists, is finite and satisfies $\phi(u^\infty) = \phi^r$;

(e) under the additional assumption that $\phi^{-1}(\{\phi^r\}) \cap W$ is finite for any bounded set $W \subset \mathbb{R}^m$, the limit $\lim_{t\to\infty} u(t) =: u^\infty$ exists, is finite and satisfies $\phi(u^\infty) = \phi^r$;

(f) under the additional assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, the error e satisfies $e = e_1 + e_2$, where e_1 is a bounded function with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)$ for any $\alpha > \omega(\mathbf{T})$; if additionally $\mathbf{T}_{t^0}x^0 \in X_1$ for some $t^0 \geq 0$, then $\lim_{t\to\infty} e_2(t) = 0$.

Remark 4.2 (a) Whilst statement (c) of Theorem 4.1 need not imply asymptotic tracking, it does imply that the error is small for large t in the sense that for all $\delta, \varepsilon > 0$, there exists $\tau > 0$ such that

$$\mu_L(\{t \geq \tau : \|e(t)\| \geq \delta\}) \leq \varepsilon\,,$$

where μ_L denotes the Lebesgue measure on $\mathbb{R}_+$.

(b) The assumption in statement (f) of Theorem 4.1 that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ implies the regularity of $\mathbf{\Sigma}$. However, this assumption is not very restrictive in the sense that it seems to be satisfied in all practical examples of exponentially stable well-posed systems. In particular, it is satisfied if B or C is bounded (see Lemma 2.3 in [17]). Statement (f) implies that $\lim_{t\to\infty} e(t) = 0$, provided that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ and $\mathbf{T}_{t^0}x^0 \in X_1$ for some $t^0 \geq 0$.

(c) In applying Theorem 4.1 it is important to know the constant K or at least a lower bound for K. In the single-input, single-output case it has been shown in [18] how K can be obtained from frequency-response experiments performed on the linear part of the plant.

(d) In the multivariable case, the applicability of Theorem 4.1 is severely limited by the assumption $\mathbf{G}(0) = \mathbf{G}^*(0) > 0$. If we relax this hypothesis on $\mathbf{G}(0)$ and only assume that $\det \mathbf{G}(0) \neq 0$, then there exists $\Gamma \in \mathbb{R}^{m\times m}$ such that $\Gamma\mathbf{G}(0) = \mathbf{G}^*(0)\Gamma^* > 0$ (choose, for example, $\Gamma = [\mathbf{G}(0)]^{-1}$). If the control law (4.2b) is replaced by

$$\dot{u}(t) = k\Gamma\{r - C_L\mathbf{T}_t x^0 - [\mathbf{F}_\infty(\phi(u))](t)\}$$

and $\mathbf{G}(s)$ is replaced by $\Gamma\mathbf{G}(s)$ in (4.3), then the conclusions of Theorem 4.1 remain true. However, finding a suitable "matrix gain" Γ will usually require exact knowledge of $\mathbf{G}(0)$.

(e) We remark that Theorem 4.1 considerably improves the main result of [19] (see Theorem 3.3 in [19]) in the sense that Theorem 4.1 (i) guarantees better asymptotic and faster convergence properties and (ii) is not restricted to single-input single-output regular systems, but applies to the wider class of multivariable well-posed systems. In particular, the following parts of Theorem 4.1 are new: the stability property (4.5), the fact that $\phi \circ u - \phi^r \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $x - x^r \in L^2(\mathbb{R}_+, X)$ and statements (c)-(e). $\diamond$

Proof of Theorem 4.1: Let $(x^0, u^0) \in X \times U$. By Lemma 2.4, the corresponding solution of the initial-value problem (4.2), denoted by $(x(\cdot), u(\cdot))$, is unique and exists on $\mathbb{R}_+$. Let $u^r \in \phi^{-1}(\{\phi^r\})$ and introduce the nonlinearity

$$\psi : \mathbb{R}^m \to \mathbb{R}^m\,, \quad w \mapsto \phi(w + u^r) - \phi^r\,. \tag{4.6}$$

Since $\phi \in \mathcal{N}_m(\lambda)$, it is straightforward to show that $\psi \in \mathcal{S}[0, \lambda]$. Set

$$z(\cdot) := x(\cdot) - x^r\,, \quad v(\cdot) := u(\cdot) - u^r\,. \tag{4.7}$$

We proceed in two steps.

Step 1. We claim that

$$\dot{z}(t) = Az(t) + B\psi(v(t))\,, \quad z(0) = z^0 := x^0 - x^r \in X\,, \tag{4.8a}$$
$$\dot{v}(t) = -k\{C_L\mathbf{T}_t z^0 + [\mathbf{F}_\infty(\psi(v))](t)\}\,, \quad v(0) = v^0 := u^0 - u^r \in U\,, \tag{4.8b}$$

where, as usual, the derivative on the left-hand side of (4.8a) has to be interpreted in X_{-1}. Equation (4.8a) follows easily from (4.2a). Moreover, setting

$$\tilde{v}(t) := r - C_L\mathbf{T}_t x^r - [\mathbf{F}_\infty(\phi^r)](t)\,,$$

we obtain from (4.2b) that

$$\dot{v}(t) = k\{\tilde{v}(t) - C_L \mathbf{T}_t z^0 - [\mathbf{F}_\infty(\psi(v))](t)\}\,.$$

It remains to show that $\tilde{v}(t) = 0$ for a.e. $t \geq 0$. Using Laplace transforms and (2.9), we obtain

$$\begin{aligned}(\mathfrak{L}(\tilde{v}))(s) &= \frac{r}{s} - C(sI - A)^{-1}x^r - \frac{1}{s}\mathbf{G}(s)\phi^r \\ &= -\frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))\phi^r + C(sI - A)^{-1}A^{-1}B\phi^r \\ &= 0\,,\end{aligned}$$

showing that $\tilde{v}(t) = 0$ for a.e. $t \geq 0$.

STEP 2. For $k \in (0, K/\lambda)$, we may apply Theorem 3.1 to (4.8) in order to derive (4.5) and

$$\lim_{t\to\infty} \|z(t)\| = 0\,, \quad z \in L^2(\mathbb{R}_+, X)\,, \quad \lim_{t\to\infty} \psi(v(t)) = 0\,, \quad \psi \circ v \in L^2(\mathbb{R}_+, \mathbb{R}^m)\,. \tag{4.9}$$

Statements (a) and (b) now follow from (4.9). Furthermore, using that $\phi^{-1}(\{\phi^r\}) = \psi^{-1}(\{0\}) + u^r$, statements (d) and (e) are easy consequences of Theorem 3.1, Corollary 3.3 and Remark 3.4. Statement (c) follows from Lemma 2.1, the exponential stability of $\mathbf{\Sigma}$ and statement (a). To prove statement (f), decompose $e = e_1 + e_2$, where

$$e_1(t) := r - [\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u)](t)\,, \quad e_2(t) := -C_L \mathbf{T}_t x^0\,. \tag{4.10}$$

Clearly, using the exponential stability of $\mathbf{T}$, $e_2 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)$ for any $\alpha > \omega(\mathbf{T})$ and $\lim_{t\to\infty} e_2(t) = 0$ if $\mathbf{T}_{t^0}x^0 \in X_1$ for some $t^0 \geq 0$. Finally, since $\lim_{t\to\infty} \phi(u(t)) = \phi^r$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, it follows from Gripenberg et al. [10] (Theorem 6.1, part (ii), p. 96) that

$$\lim_{t\to\infty} e_1(t) = r - \mathbf{G}(0)\phi^r = r - r = 0\,.$$

$\square$

Remark 4.3 Suppose that $\phi = \mathrm{diag}\,(\phi_i)$, where the functions $\phi_i : \mathbb{R} \to \mathbb{R}$ are non-decreasing. If we replace the global Lipschitz assumption on ϕ_i by the weaker assumption that ϕ_i is locally Lipschitz and $\phi_i(\cdot) - \phi_i(0) \in \mathcal{S}[0, b]$ for some $b \geq 0$, then it is not difficult to prove that the function ψ defined in (4.6) is in $\mathcal{S}[0, \tilde{b}]$ for some $\tilde{b} \geq 0$. An inspection of the proof of Theorem 4.1 then shows that the conclusions of Theorem 4.1 remain true for all sufficiently small $k > 0$. $\diamond$

Under certain additional assumptions on ϕ and r, it can be shown that the variables x, u and y converge exponentially fast. This will be addressed in the next result, Theorem 4.5. In order to state and prove this theorem, we need some preparation.

Let $f \in \mathcal{N}_1(\lambda)$. The Clarke [4],[5] *generalized directional derivative* $f^\circ(v;w)$ of f at v in direction w is given by

$$f^\circ(v;w) = \limsup_{\substack{\xi \to v \\ h \downarrow 0}} \frac{f(\xi + hw) - f(\xi)}{h} .$$

Define $f^-(\cdot) := -f^\circ(\cdot;-1)$ (if f is C^1 with derivative f', then $f^- \equiv f'$). A point $v \in \mathbb{R}$ is said to be a *critical point* (and $f(v)$ is said to be a *critical value*) of f if $f^-(v) = 0$. A point $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$ is called a *critical point* (and $f(v)$ is said to be a *critical value*) of $f = \operatorname{diag}(f_i) \in \mathcal{N}_m(\lambda)$ if there exists $j \in \{1, \ldots, m\}$ such that v_j is a critical point of f_j. Note that if $f \in \mathcal{N}_m(\lambda)$ and $w \in \operatorname{im} f$ is not a critical value of f, then $f^{-1}(\{w\})$ is a singleton.

We record the following technicality for later use. The proof can be found in the Appendix, Part 2.

Lemma 4.4 *Let $\lambda > 0$ and $f \in \mathcal{N}_m(\lambda)$. If $f(0) = 0$ and 0 is not a critical point of f, then there exist $a \in (0, \lambda)$ and an open set $V \subset \mathbb{R}^m$ with $0 \in V$ such that $f \in \mathcal{S}_V[a, \lambda]$.*

Recall that by definition a $\mathbb{R}^{m\times m}$-valued Borel measure μ on $\mathbb{R}_+$ is in $\mathcal{M}_\alpha$ (where $\alpha \in \mathbb{R}$) if the exponentially weighted measure $E \mapsto \int_E e^{-\alpha t}\mu(dt)$ belongs to $\mathcal{M}$. Equivalently, $\mathcal{M}_\alpha$ is the space of all $\mathbb{R}^{m\times m}$-valued Borel measures μ on $\mathbb{R}_+$ such that $\int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$, where $|\mu|$ denotes the total variation of μ.

Theorem 4.5 *Assume that $\mathbf{\Sigma}$ is exponentially stable and $\mathbf{G}(0) = \mathbf{G}^*(0) > 0$. Let $\lambda > 0$, $\phi \in \mathcal{N}_m(\lambda)$ and $k \in (0, K/\lambda)$. Let $r \in \mathbb{R}^m$ be such that*

$$\phi^r := [\mathbf{G}(0)]^{-1} r \in \operatorname{im} \phi$$

and ϕ^r is not a critical value of ϕ. Let u^r be the unique element in $\mathbb{R}^m$ such that $\phi(u^r) = \phi^r$ and set $x^r := -A^{-1}B\phi^r$. Then, for given $M > 0$, there exist $N \geq 1$ and $\nu \in (\omega(\mathbf{T}), 0)$ such that for all $(x^0, u^0) \in X \times \mathbb{R}^m$ with $\|x^0 - x^r\| + \|u^0 - u^r\| \leq M$, the solution $(x(\cdot), u(\cdot))$ of (4.2) satisfies

$$\|x(t) - x^r\| + \|u(t) - u^r\| \leq N e^{\nu t}(\|x^0 - x^r\| + \|u^0 - u^r\|) , \quad \forall\, t \geq 0 , \tag{4.11}$$

and

$$\|\phi(u(t)) - \phi^r\| \leq \lambda N e^{\nu t}(\|x^0 - x^r\| + \|u^0 - u^r\|) , \quad \forall\, t \geq 0 . \tag{4.12}$$

Moreover, for all $(x^0, u^0) \in X \times \mathbb{R}^m$ with $\|x^0 - x^r\| + \|u^0 - u^r\| \leq M$, the following statements hold:

(a) for any $\alpha > \nu$, $e := r - y \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)$, where $y(t) = C_L \mathbf{T}_t x^0 + [\mathbf{F}_\infty(\phi(u))](t)$;

(b) under the additional assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$, the error e satisfies $e = e_1 + e_2$, where $e_1 \in L^\infty_\beta(\mathbb{R}_+, \mathbb{R}^m)$ for any $\beta \geq \max(\alpha, \nu)$, and $e_2 \in L^2_\beta(\mathbb{R}_+, \mathbb{R}^m)$ for any $\beta > \omega(\mathbf{T})$; if additionally $\mathbf{T}_{t^0}x^0 \in X_1$ for some $t^0 \geq 0$, then $e_2 \in L^\infty_\beta(\mathbb{R}_+, \mathbb{R}^m)$ for any $\beta > \omega(\mathbf{T})$.

Remark 4.6 (a) Statement (b) shows that exponentially fast asymptotic tracking is guaranteed if $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$ and $\mathbf{T}_{t^0}x^0 \in X_1$ for some $t^0 \geq 0$. Again, the assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_\alpha$ for some $\alpha < 0$ is not very restrictive and seems to be satisfied in all practical examples of exponentially stable well-posed systems. In particular, this assumption is satisfied if B or C is bounded (see Lemma 2.3 in [17]).

(b) We mention that parts (c) and (d) of Remark 4.2 and Remark 4.3 remain relevant in the context of Theorem 4.5.

(c) As compared to the main result in [19] (see Theorem 3.3 in [19]), Theorem 4.5 is entirely new: the issue of exponential decay is not addressed in [19]. $\diamond$

Proof of Theorem 4.5: Let $(x^0, u^0) \in X \times U$. By Lemma 2.4, the corresponding solution of the initial-value problem (4.2), denoted by $(x(\cdot), u(\cdot))$, is unique and exists on $\mathbb{R}_+$. Let ψ, z and v be defined by (4.6) and (4.7). Clearly, by the assumptions on ϕ, $\psi \in \mathcal{N}_m(\lambda)$, $\psi^{-1}(\{0\}) = \{0\}$ and 0 is not a critical value of ψ. Invoking Lemma 4.4, we see that there exists $a \in (0, \lambda)$ and an open set $V \subset \mathbb{R}^m$ with $0 \in V$ such that $\psi \in \mathcal{S}_V[a, \lambda]$. For $k \in (0, K/\lambda)$ we may apply Theorem 3.6 to (4.8) in order to derive (4.11) and (4.12). Statement (a) follows from Lemma 2.1 and (4.12). In order to prove statement (b), we write $e = e_1 + e_2$, with e_1 and e_2 defined by (4.10). The claim for e_2 follows immediately from standard results on admissible observation operators. Finally, to prove exponential convergence of e_1, set $\mu := \mathfrak{L}^{-1}(\mathbf{G})$. By assumption $\mu \in \mathcal{M}_\alpha$ for some $\alpha < 0$. The function $t \mapsto \|e_1(t)\|$ can be estimated as follows

$$\|e_1(t)\| \leq \|[\mu \star (\phi(u) - \phi^r\theta)(t)]\| + \|(\mu \star \phi^r\theta)(t) - \mathbf{G}(0)\phi^r\|\,, \quad \forall\, t \geq 0\,, \tag{4.13}$$

where θ denotes the unit-step (Heaviside) function. Let $\beta \geq \max(\alpha, \nu)$. Then, by (4.12), the function $t \mapsto e^{-\beta t}\|\phi(u(t)) - \phi^r\|$ remains bounded as $t \to \infty$. Since $\mu \in \mathcal{M}_\alpha$, the measure $E \mapsto \int_E e^{-\alpha t}\mu(dt)$ belongs to $\mathcal{M}$. Hence, by [10] (Theorem 3.5, part (i), p. 119), we may conclude that there exists $M_1 > 0$ such that

$$e^{-\beta t}\|[\mu \star (\phi(u) - \phi^r\theta)](t)\| \leq M_1\,, \quad \forall\, t \geq 0\,. \tag{4.14}$$

Moreover, $M_2 := \int_0^\infty e^{-\alpha t}|\mu|(dt) < \infty$, and thus

$$e^{-\beta t}\|(\mu \star \phi^r\theta)(t) - \mathbf{G}(0)\phi^r\| \leq \|\phi^r\|e^{-\alpha t}\int_t^\infty |\mu|(d\tau) \leq \|\phi^r\|\int_0^\infty e^{-\alpha\tau}|\mu|(d\tau) = M_2\|\phi^r\|\,. \tag{4.15}$$

Consequently, appealing to (4.13)–(4.15), we deduce that the function $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto e^{-\beta t}\|e_1(t)\|$ is bounded. $\square$

5 Example: diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = (0, 1)$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity ϕ with Lipschitz constant $\lambda > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (at point $x_c \in I$, $x_c \geq x_b$.). We formally write this diffusion process as

$$\begin{aligned} &z_t(t,x) = \kappa z_{xx}(t,x) + \delta(x - x_b)\phi(u(t)), \quad y(t) = z(t-h, x_c) \\ &z(t,0) = 0 = z(t,1)\,, \qquad \text{for all } t > 0\,. \end{aligned}$$

For simplicity, we assume zero initial conditions:

$$z(t,x) = 0\,, \qquad \text{for all } (t,x) \in [-h, 0] \times [0,1]\,.$$

This system was analyzed in the context of low-gain integral control in [18], [19]. With input $\phi(u(\cdot))$ and output $y(\cdot)$, this example qualifies as a well-posed linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh}\sinh\left(x_b\sqrt{(s/\kappa)}\right)\sinh\left((1-x_c)\sqrt{(s/\kappa)}\right)}{\kappa\sqrt{(s/\kappa)}\sinh\sqrt{(s/\kappa)}}\,.$$

It is not difficult to show that $\mathfrak{L}^{-1}(\mathbf{G}) \in L^1_\alpha(\mathbb{R}_+, \mathbb{R}) \subset \mathcal{M}_\alpha$ for any $\alpha > -\kappa\pi^2$.

From [18] we know that

$$\begin{aligned} K &= \sup\{k > 0 \mid (4.3) \text{ holds}\} \\ &= \frac{1}{|\mathbf{G}'(0)|} = \frac{6\kappa^2}{x_b(1-x_c)(6h\kappa + 1 - x_b^2 - (1-x_c)^2)}. \end{aligned}$$

Note the dependence of K on the time-delay h: the larger h, the smaller K. By Theorem 4.5, part (b), for each $k \in (0, K/\lambda)$, the integral control

$$u(t) = k\int_0^t [r - y(t)]\,dt$$

guarantees exponentially fast asymptotic tracking of every constant reference value r such that

$$\phi^r = \frac{r}{\mathbf{G}(0)} = \frac{\kappa r}{x_b(1-x_c)} \in \operatorname{im}\phi$$

and ϕ^r is not a critical value of ϕ. Note that ϕ^r does not depend on h. It is easy to see that if $0 \in \operatorname{im}\phi$, then the range of values of r which can be tracked is maximal if $x_b = x_c = 1/2$.

For purposes of illustration, we adopt the following values

$$\kappa = 0.1, \quad x_b = \frac{1}{3}, \quad x_c = \frac{2}{3}, \quad h = 1$$

and we consider a nonlinearity ϕ of saturation type, defined as follows

$$u \mapsto \phi(u) := \begin{cases} 1, & u \geq 1 \\ u, & u \in (0,1) \\ 0, & u \leq 0\,. \end{cases}$$

In this case, $K = 243/620 \approx 0.3919$ and $\lambda = 1$. The critical values of ϕ are 0 and 1. For $r = 1$, we have

$$\phi^r = \frac{r}{\mathbf{G}(0)} = \frac{\kappa}{x_b(1-x_c)} = 0.9 \in [0,1] = \operatorname{im}\phi\,.$$

In particular, ϕ^r is not a critical value of ϕ. In each of the following three cases of controller gains

$$\text{(i) } k = 0.39, \quad \text{(ii) } k = 0.26, \quad \text{(iii) } k = 0.13,$$

Figure 4 depicts the output behaviour of the system under integral control, while Figures 5 and 6 depict the corresponding control input and integrator state, respectively. Figure 7 illustrates the evolution of the temperature profile $z(t,\cdot)$ in case (i). These figures were generated using SIMULINK Simulation Software within MATLAB, wherein a truncated eigenfunction expansion, of order 20, was adopted to model the diffusion process.

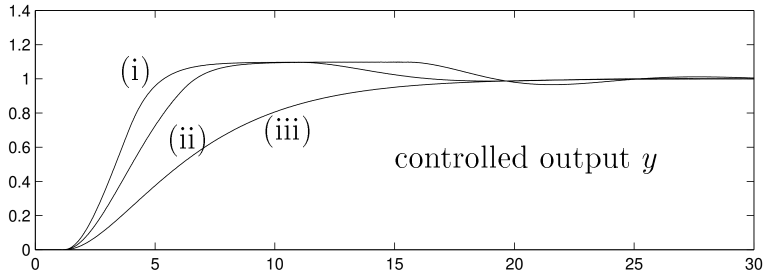

Figure 4: Controlled output

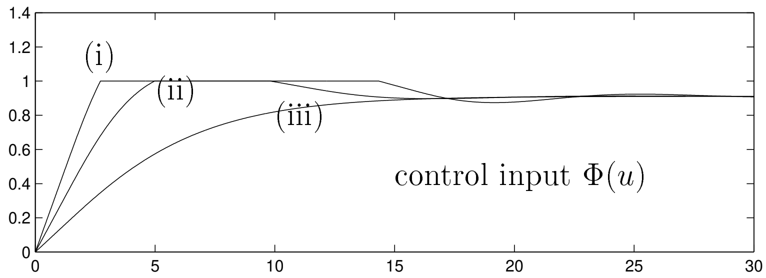

Figure 5: Control input

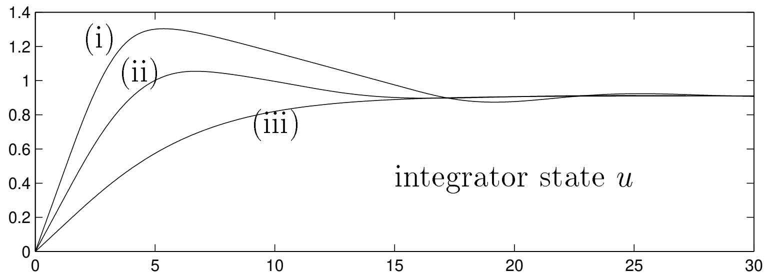

Figure 6: Integrator state

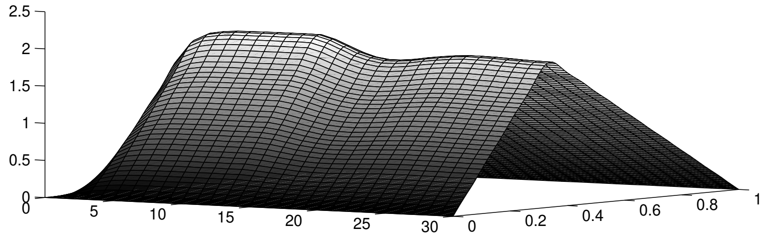

Figure 7: Temperature profile in case (i) ($k = 0.39$)

Appendix

Part 1

In this part of the Appendix we prove that there exists $\varepsilon > 0$ such that the function $\psi : U \to U$ as defined in (3.40) is in $\mathcal{S}[\varepsilon, b)$, provided that $\phi \in \mathcal{S}[0, b)$ and ϕ satisfies assumptions (A3) and (A4) of Theorem 3.6. To this end we recall some properties of sector bounded nonlinearities which will be used freely in the following.

Let $W \subset U$ and $a \leq b$. For any function $f : U \to U$ we have

$$f \in \mathcal{S}_W[a, b] \quad \Longleftrightarrow \quad \left\| f(w) - \frac{a+b}{2}\, w \right\| \leq \frac{b-a}{2}\, \|w\|\,, \ \ \forall\, w \in W\,. \tag{A.1}$$

From this it follows that if $f \in \mathcal{S}_W[a, b]$, then $f \in \mathcal{S}_W[c, d]$ for all numbers c and d with $c \leq a \leq b \leq d$.

Lemma A.1 *Let $f : U \to U$ be a function. Suppose that $f \in \mathcal{S}[0, b)$ for some $b > 0$ and that*

(a) there exists a neighbourhood $V \subset U$ of 0 and a number $a \in (0, b)$ such that $f \in \mathcal{S}_V[a, b)$;

(b) $\inf_{w\in W}\|f(w)\| > 0$ *for any bounded closed nonempty set* $W \subset U$ *with* $0 \notin W$.

Under these conditions, for any bounded nonempty set $Y \subset U$*, there exists* $\varepsilon > 0$ *such that* $f \in \mathcal{S}_Y[\varepsilon, b)$.

Of course, Lemma A.1 is not surprising and, if $\dim U = 1$, the result is obvious and trivial. However, the case $\dim U > 1$ requires a proof.

Proof of Lemma A.1: Let $Y \subset U$ be bounded and nonempty. Since $f \in \mathcal{S}[0, b)$ and by assumption (a), there exists $c \in (a, b)$ such that $f \in \mathcal{S}[0, c] \cap \mathcal{S}_V[a, c]$. Let $\eta > 0$ be such that for all $v \in U$

$$\|v\| < \eta \quad \Rightarrow \quad v \in V\,. \tag{A.2}$$

Let $d \in (c, b)$. We claim that there exists $\varepsilon > 0$ such that $f \in \mathcal{S}_Y[\varepsilon, d] \subset \mathcal{S}_Y[\varepsilon, b)$. Seeking a contradiction, suppose that the claim is not true. Then there exist sequences $(\varepsilon_n) \subset (0, a)$ with $\lim_{n\to\infty}\varepsilon_n = 0$ and $(v_n) \subset Y$ such that

$$\langle f(v_n) - \varepsilon_n v_n, f(v_n) - d v_n\rangle > 0\,, \quad \forall\, n \in \mathbb{N}. \tag{A.3}$$

Since $f \in \mathcal{S}_V[a, c] \subset \mathcal{S}_V[\varepsilon_n, d]$, we obtain from (A.2) and (A.3) that

$$\|v_n\| \geq \eta > 0\,, \quad \forall\, n \in \mathbb{N}. \tag{A.4}$$

By (A.3),

$$\|f(v_n)\|^2 \geq (d + \varepsilon_n)\langle f(v_n), v_n\rangle - d\varepsilon_n\|v_n\|^2\,, \quad \forall\, n \in \mathbb{N}. \tag{A.5}$$

Moreover, since $f \in \mathcal{S}[0, c]$

$$\|f(v_n)\|^2 \leq c\langle f(v_n), v_n\rangle\,, \quad \forall\, n \in \mathbb{N}. \tag{A.6}$$

The set

$$W := \mathrm{clos}\{v_n : n \in \mathbb{N}\}$$

is a bounded closed set, and, by (A.4), $0 \notin W$. By assumption (b), there exists $\nu_1 > 0$ such that

$$\|f(v_n)\|^2 \geq \nu_1\,, \quad \forall\, n \in \mathbb{N}. \tag{A.7}$$

Setting $\nu_2 := \sup_{n\in\mathbb{N}}\|v_n\|^2 < \infty$, it follows from (A.5)–(A.7) that

$$c \geq d + \varepsilon_n - \varepsilon_n c d\frac{\nu_2}{\nu_1}\,, \quad \forall\, n \in \mathbb{N}.$$

Since $\lim_{n\to\infty}\varepsilon_n = 0$, this implies $c \geq d$, in contradiction to $d \in (c, b)$. □

Corollary A.2 *Suppose that* $\phi \in \mathcal{S}[0, b)$ *satisfies the assumptions (A3) and (A4) in Theorem 3.6 and let* $R > 0$. *Then there exists an open neighbourhood* $V \subset U$ *of* 0, *a* *number $c \in (a,b)$ and $\varepsilon > 0$ such that $\phi \in \mathcal{S}[0,c] \cap \mathcal{S}_V[a,c]$ and the function $\psi : U \to U$ defined by*

$$\psi(v) = \begin{cases} \phi(v) & \text{if} \quad \|v\| \le R \\ c\left(1 - \dfrac{R}{\|v\|}\right)v + \phi(\dfrac{R}{\|v\|}v) & \text{if} \quad \|v\| > R\,. \end{cases}$$

is in $\mathcal{S}[\varepsilon, b)$.

Proof: Since $\phi \in \mathcal{S}[0,b)$ and since ϕ satisfies assumption (A3), it is clear that there exist an open neighbourhood $V \subset U$ of 0 and a number $c \in (a,b)$ such that $\phi \in \mathcal{S}[0,c] \cap \mathcal{S}_V[a,c]$. Let $d \in (c,b)$ such that $2c - d > 0$ and let $\delta \in (0, 2c-d)$. Setting

$$b(v) := \phi(\frac{R}{\|v\|}v) - \frac{cR}{\|v\|}v\,,$$

we have for $v \in U$ with $\|v\| > R$,

$$\left\|\psi(v) - \frac{d+\delta}{2}v\right\| \le \frac{2c-d-\delta}{2}\|v\| + \|b(v)\| = \frac{d-\delta}{2}\|v\| + (c-d)\|v\| + \|b(v)\|\,. \quad \text{(A.8)}$$

Now $b : U \setminus \{0\} \to U$ is a bounded function, and so there exists $\tilde{R} > R$ such that

$$\|b(v)\| \le (d-c)\|v\|\,, \quad \forall\, v \in E_{\tilde{R}}\,,$$

where $E_{\tilde{R}} := \{v \in U : \|v\| \ge \tilde{R}\}$. Combining this with (A.8) and applying (A.1) yields

$$\psi \in \mathcal{S}_{E_{\tilde{R}}}[\delta, d] \subset \mathcal{S}_{E_{\tilde{R}}}[\delta, b)\,.$$

To prove the claim, it remains to show that there exists $\varepsilon \in (0,\delta)$ such that

$$\psi \in \mathcal{S}_{B_{\tilde{R}}}[\varepsilon, b)\,,$$

where $B_{\tilde{R}} := \{v \in U : \|v\| < \tilde{R}\}$. This in turn will follow from Lemma A.1 if we can prove that $\psi \in \mathcal{S}[0,b)$ and ψ satisfies assumptions (a) and (b) of Lemma A.1. To this end we proceed in three steps.

Step 1. We show that $\psi \in \mathcal{S}[0,b)$. Using the fact that $\phi \in \mathcal{S}[0,c]$, we obtain that for all $v \in U$ with $\|v\| > R$

$$\begin{aligned} \left\|\psi(v) - \frac{c}{2}v\right\| &= \left\|\frac{c}{2}(1 - \frac{R}{\|v\|})v + \phi(\frac{R}{\|v\|}v) - \frac{cR}{2\|v\|}v\right\| \\ &\le \frac{c}{2}\left(1 - \frac{R}{\|v\|}\right)\|v\| + \frac{c}{2}R \\ &= \frac{c}{2}\|v\|\,. \end{aligned} \quad \text{(A.9)}$$

Since $\psi(v) = \phi(v)$ for all $v \in U$ with $\|v\| \le R$ and $\phi \in \mathcal{S}[0,c]$, it follows from (A.1) and (A.9) that $\psi \in \mathcal{S}[0,c] \subset \mathcal{S}[0,b)$.

Step 2. Clearly, the set $V_R = \{v \in V : \|v\| < R\}$ is an open neighbourhood of 0, and by construction

$$\psi \in \mathcal{S}_{V_R}[a, b)\,,$$

showing that ψ satisfies assumption (a) of Lemma A.1.

Step 3. To prove that ψ satisfies assumption (b) of Lemma A.1, let $W \subset U$ be nonempty, bounded and closed with $0 \notin W$. Define $W_R := \{w \in W : \|w\| \leq R\}$. Then W_R is bounded and closed, and, if $W_R \neq \emptyset$, we obtain using assumption (A4) that

$$\inf_{w \in W_R} \|\psi(w)\| = \inf_{w \in W_R} \|\phi(w)\| > 0\,. \tag{A.10}$$

Furthermore, for all $w \in U$ with $\|w\| \geq R$

$$\begin{aligned}
\|\psi(w)\| &\geq c\left(1 - \frac{R}{2\|w\|}\right)\|w\| - \left\|\phi(\frac{R}{\|w\|}w) - \frac{cR}{2\|w\|}w\right\| \\
&\geq c\left(1 - \frac{R}{2\|w\|}\right)\|w\| - \frac{c}{2}R \\
&= c(\|w\| - R)\,,
\end{aligned}$$

where in the second inequality we have used that $\phi \in \mathcal{S}[0, c]$. We see that for any $\eta > 0$

$$\inf_{\|w\| \geq R+\eta} \|\psi(w)\| > 0\,. \tag{A.11}$$

Finally, for $\eta > 0$, define

$$B(R, \eta) := \{w \in U : R \leq \|w\| \leq R + \eta\}\,.$$

By assumption (A4), $\inf_{\|v\|=R} \|\phi(v)\| > 0$, and hence it follows that there exist $\gamma > 0$ and $\eta^* > 0$ such that for all $\eta \in (0, \eta*)$

$$\inf_{w \in B(R,\eta)} \|\phi(\frac{R}{\|w\|}w)\| \geq \sup_{w \in B(R,\eta)} \left(c(1 - \frac{R}{\|w\|})\|w\|\right) + \gamma\,. \tag{A.12}$$

Therefore, since for all $w \in U$ with $\|w\| \geq R$

$$\psi(w) = \phi(\frac{R}{\|w\|}w) + c\left(1 - \frac{R}{\|w\|}\right)w\,,$$

we may conclude using (A.12) that

$$\inf_{w \in B(R,\eta)} \|\psi(w)\| \geq \gamma > 0\,, \quad \forall\, \eta \in (0, \eta^*)\,.$$

Together with (A.10) and (A.11) this leads to

$$\inf_{w \in W} \|\psi(w)\| > 0\,,$$

showing that ψ satisfies assumption (b) of Lemma A.1. $\square$

Part 2

This part of the Appendix contains a proof of Lemma 4.4. We prove the following result of which Lemma 4.4 is an easy consequence.

Lemma A.3 *Let $b > 0$ and $f \in \mathcal{N}_1(b)$. If $f(0) = 0$ and $f^-(0) > 0$, then there exist constants $\varepsilon > 0$ and $a \in (0, b)$ such that*

$$av^2 \le f(v)v \le bv^2 , \quad \forall\, v \in \ (-\varepsilon, \varepsilon) .$$

Proof: Since $f \in \mathcal{N}_1(b)$ and $f(0) = 0$ it follows easily that

$$f(v)v \le bv^2 , \quad \forall\, v \in \mathbb{R} .$$

It remains to show that there exists $a \in (0, b)$ and $\varepsilon > 0$ such that

$$av^2 \le f(v)v , \quad \forall\, v \in (-\varepsilon, \varepsilon) . \tag{A.13}$$

Seeking a contradiction suppose that (A.13) is not true. Then there exist sequences $(v_n) \subset \mathbb{R} \setminus \{0\}$ and $(a_n) \subset (0, b)$ with $\lim_{n\to\infty} v_n = 0$ and $\lim_{n\to\infty} a_n = 0$ and such that

$$a_n v_n^2 > f(v_n)v_n .$$

Clearly, (v_n) must contain a subsequence (v_{n_j}) with either $v_{n_j} > 0$ for all $j \in \mathbb{N}$ (case 1) or $v_{n_j} < 0$ for all $j \in \mathbb{N}$ (case 2).

CASE 1: Setting $\xi_j = h_j = v_{n_j}$, it follows that

$$\frac{f(\xi_j - h_j) - f(\xi_j)}{h_j} = \frac{f(0) - f(\xi_j)}{\xi_j} = -\frac{f(\xi_j)}{\xi_j} > -a_{n_j} .$$

This yields $f^\circ(0; -1) \ge 0$, and hence, $f^-(0) \le 0$, contradicting the hypothesis that $f^-(0) > 0$.

CASE 2: Setting $\xi_j = 0$ and $h_j = -v_{n_j}$, we have

$$\frac{f(\xi_j - h_j) - f(\xi_j)}{h_j} = \frac{f(v_{n_j})}{|v_{n_j}|} > a_{n_j} \frac{v_{n_j}}{|v_{n_j}|} = -a_{n_j} .$$

Again, this yields $f^\circ(0; -1) \ge 0$, and hence, $f^-(0) \le 0$, contradicting the hypothesis that $f^-(0) > 0$. □

Acknowledgement: We would like to thank Adam D. Mawby (Bath) for performing the simulations in Section 5.

References

[1] B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis: A Modern Systems Theory Approach*, Prentice Hall, Englewood-Cliffs, NJ, 1973.

[2] V. Barbu. *Analysis and Control of Nonlinear Infinite-Dimensional Systems*, Academic Press, Boston, 1993.

[3] F. Bucci. Frequency-domain stability of nonlinear feedback systems with unbounded input operator, Preprint, Dipartimento de Matematica Applicata "G. Sansone", Università degli Studi di Firenze, 1997.

[4] F.H. Clarke. *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[5] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern and P.R. Wolenski. *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, 1998.

[6] C. Corduneanu. *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[7] C. Corduneanu. *Almost Periodic Functions*, Wiley, New York, 1968.

[8] R.F. Curtain, H. Logemann, S. Townley and H. Zwart. Well-posedness, stabilizability and admissibility for Pritchard-Salamon systems, *Mathematical Systems, Estimation and Control* **7** (1997), 439–476.

[9] R.F. Curtain and G. Weiss. Well-posedness of triples of operators in the sense of linear systems theory, pp. 41–59 in *Control and Estimation of Distributed Parameter System* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Basel, 1989.

[10] G. Gripenberg, S.-O. Londen and O. Staffans. *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

[11] P.R. Halmos. *Finite-Dimensional Vector Spaces*, Springer-Verlag, New York, 1987.

[12] H.K. Khalil. *Nonlinear Systems*, 2nd edition, Prentice-Hall, Upper Saddle River, NJ, 1996.

[13] S. Lefschetz. *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.

[14] G.A. Leonov, D.V. Ponomarenko, V.B. Smirnova. *Frequency-Domain Methods for Nonlinear Analysis*, World Scientific, Singapore, 1996.

[15] B.A.M. van Keulen. *H^∞-Control for Infinite-Dimensional Systems: a State-Space Approach*, Birkhäuser Verlag, Boston, 1993.

[16] H. Logemann. Circle criteria, small-gain conditions and internal stability for infinite-dimensional systems, *Automatica* **27** (1991), 677–690.

[17] H. Logemann and E.P. Ryan. Time-varying and adaptive integral control of infinite-dimensional regular linear systems with input nonlinearities, Mathematics Preprint 98/12, University of Bath, 1998 † (to appear in *SIAM J. Control & Optim.*).

[18] H. Logemann, E.P. Ryan and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain, *IEEE Trans. Auto. Control* **44** (1999), 1315–1319.

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[19] H. Logemann, E.P. Ryan and S. Townley. Integral control of infinite-dimensional linear systems subject to input saturation, *SIAM J. Control & Optim.* **36** (1998), 1940–1961.

[20] H. Logemann and S. Townley. Low-gain control of uncertain regular linear systems, *SIAM J. Control & Optim.* **35** (1997), 78–116.

[21] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[22] W. Rudin. *Functional Analysis*, McGraw-Hill, New York, 1973.

[23] D. Salamon. Realization theory in Hilbert space, *Math. Systems Theory* **21** (1989), 147–164.

[24] D. Salamon. Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.* **300** (1987), 383–431.

[25] O.J. Staffans. *Well-Posed Linear Systems*, monograph in preparation (preprint available at http://www.abo.fi/˜staffans/).

[26] O.J. Staffans. Quadratic optimal control of stable well-posed linear systems, *Trans. Amer. Math. Soc.* **349** (1997), 3679–3715.

[27] M. Vidyasagar. *Nonlinear Systems Analysis*, 2nd edition, Prentice Hall, Englewood Cliffs, NJ, 1993.

[28] G. Weiss. Transfer functions of regular linear systems, part I: Characterization of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827–854.

[29] G. Weiss. Admissibility of unbounded control operators, *SIAM J. Control & Optim.* **27** (1989), 527–545.

[30] G. Weiss. Admissible observation operators for linear semigroups, *Israel J. Math.* **65** (1989), 17–43.

[31] G. Weiss. The representation of regular linear systems on Hilbert spaces, pp. 401–416 in *Control and Estimation of Distributed Parameter System* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Basel, 1989.

[32] D. Wexler. On frequency domain stability for evolution equations in Hilbert spaces via the algebraic Riccati equation, *SIAM J. Math. Analysis* **11** (1980), 969–983.

[33] D. Wexler. Frequency domain stability for a class of equations arising in reactor dynamics, *SIAM J. Math. Analysis* **10** (1979), 118-138.