Diffraction of a plane wave by a circular cone with an impedance boundary condition

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Abstract

We consider the boundary value problem for the Helmholtz equation in a circular cone with an impedance boundary condition on its face. A new approach for its solution is proposed. The scheme of solution includes applying the Kontorovich-Lebedev transform, derivation of a second-order difference equation in a strip of a complex variable and its reduction to an integral equation of the convolution type with variable coefficients. It is shown that the equation is equivalent to a 2×2 matrix Riemann-Hilbert problem with a discontinuous coefficient. We analyze the behaviour of the solution of the integral equation at the ends of the contour and construct an approximate solution using a collocation method. The diffraction coefficients are computed. For some particular cases of the problem an exact solution is found.

1 Introduction

The problem of diffraction by a circular cone with Dirichlet and Neumann boundary conditions on its face admits an exact solution by applying either the Sommerfeld integral representation or the Kontorovich-Lebedev integral transformation. Both these methods use the fundamental ideas of the separation variables method. For a non-circular cone, it is possible to find an exact solution (in terms of the Lamé functions) only in the case of an elliptic cone [19]. An approximate approach based on the Sommerfeld integral, the theory of potentials and numerical solution of boundary integral equations was developed in [21, 2, 3]. The strength of this method is in its applicability for an arbitrary cone with a smooth cross-section and possibility to find numerical values for the diffraction coefficients, an important characteristic in the geometric theory of diffraction [17]. Diffraction by an arbitrarily shaped cone was analysed in [15] using the Kontorovich-Lebedev transform.

Problems of diffraction by obstacles with the impedance boundary conditions on their surfaces are more complicated than those with the Dirichlet and Neumann boundary conditions. We mention the solution [20] for an impedance wedge and [5] for a semi-infinite plane with different impedance boundary conditions on each face. The exact solutions of the aforementioned papers are based either on the theory of a first-order difference equation in a strip ([20]) or a matrix Wiener-Hopf problem [5].

Recently, Bernard [6] considered the problem of diffraction by an impedance circular cone using the Sommerfeld integral representation, the theory of the Maliuzhinets difference equation and reduced the problem to an integral equation on infinite imaginary axis. He analysed the solvability of the equation in the space L_1. The leading asymptotic term in the case of a narrow impedance cone was found in [7]. The behaviour of the solution of the equation at the ends, a technique of solution of the integral equation, formulae for the diffraction coefficients and numerical analysis were not a subject of consideration in [6]. These aspects among others we pay our attention to.

The structure of the present paper is following. We formulate the problem and establish a class of solutions in Section 2. Section 3 reduces the original boundary value problem for the Helmholtz equation in a cone to a second-order difference equation. For that we use the Kontorovich-Lebedev transform. This approach derives a new basic integral equation of the convolution type with variable coefficients of the structure

$$\psi(\sigma) + \lambda L(\sigma) \int_{-i\infty}^{i\infty} \sec(\tau - \sigma)\psi(\tau)d\tau = f(\sigma), \quad \sigma \in (-i\infty, i\infty) \tag{1.1}$$

with an additional condition of symmetry for the function $\psi(\sigma)$. It is shown that this equation is reducible to a singular integral equation with Dixon's type kernel (see, e.g. [23]) with a variable coefficient

$$\chi_0(x) + \frac{K_0(x)}{2\pi} \int_0^1 \chi_0(y) \left(\frac{1}{y+x} - \frac{y}{yx+1} \right) dy = -h_0(x), \quad 0 < x < 1. \tag{1.2}$$

The kernel possesses an immovable singularity at the point $x = y = 0$, the Hölder condition for the function $K_0(x)$ fails in a neighbourhood of the point $x = 0$, and the function $h_0(x)$ has a power-logarithmic singularity at zero: $h_0(x) = O(x^{\varkappa}\sqrt{\log(-x)})$, $x \to 0$ $(-\frac{1}{2} < \varkappa < 0)$. In Section 5 we consider the axially symmetric case of the problem. We find an explicit expression for the right-hand side of the equation, study the behaviour of the singular integral

$$\Sigma(x) = \int_0^1 y^{\varkappa} \sqrt{\log \frac{1}{y}} \frac{dy}{y+x}, \quad \Re(\varkappa) > -1, \quad \text{as } x \to +0. \tag{1.3}$$

This behaviour is necessary to establish the convergence of the integral representation for the wave field and the diffraction coefficients (Section 7). Section 6 derives the numerical scheme of solution (a version of the collocation method). We find an exact formula (in terms of the solution of the integral equation) for the diffraction coefficients in Section 7.1. Additionally, an approximate formula sufficient for numerical computations is recorded. In particular cases of the Neumann and Dirichlet boundary conditions the classical formulae [8] are re-derived (Section 7.2). Numerical results are discussed in Section 7.3. An exact solution of the problem for the Laplace equation in a cone with a boundary condition of the third kind is constructed in Appendix A. For its solution we use the technique [4, 13, 22] of solution of the Carleman boundary value problem with one shift [9] in a strip. The solution for the Laplace equation may be used for construction of an asymptotic expansion for small wave numbers k. Appendix B shows an alternative approach of analysis of the original problem, namely, formulates the problem as a 2×2 matrix Riemann-Hilbert problem with a discontinuous coefficient. It is reducible to a problem with a continuous matrix coefficient [11] that admits formulation of an existence factorization theorem. Finally, in Appendix C, we give another simple derivation of the integral equation, based on the classical Sokhotski-Plemelj formula.

2 Formulation of the problem

Let us consider the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{2.1}$$

in a cone $C = \{0 < r < \infty, -\pi \le \phi \le \pi, 0 \le \theta < \alpha\}$ $(\frac{\pi}{2} < \alpha < \pi)$ with the non-homogeneous boundary condition of the third kind ("impedance" boundary condition) on its face

$$\left(u + \frac{\gamma}{r} \frac{\partial u}{\partial \theta} \right)_{\theta=\alpha} = g(r, \phi), \quad 0 < r < \infty, -\pi \le \phi \le \pi. \tag{2.2}$$

Here the symbol Δ denotes the Laplacian in spherical coordinates (r, ϕ, θ)

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}, \tag{2.3}$$

k, γ are complex constants and we assume that $-\pi < \arg k \le 0$.

Let $g(r, \phi) \in C[-\pi, \pi]$ with respect to ϕ, and as a function of r satisfy the following conditions

$$\int_0^{1/2} |g(r,\phi)| \log \frac{1}{r} \frac{dr}{\sqrt{r}} < \infty, \quad \int_{1/2}^{B} |g(r,\phi)| dr < \infty \ \left(\frac{1}{2} < B < \infty\right) \quad \text{and} \quad g(r,\phi) = O(1),\ r \to \infty \tag{2.4}$$

uniformly with respect to $\phi \in [-\pi, \pi]$. We are looking for the solution of problem (2.1), (2.2) in the following space of functions

$$u \in C[-\pi,\pi] \quad \text{as a function of} \quad \phi,$$
$$r^{-1/2} u \log \frac{1}{r} \in L_1\left(0, \frac{1}{2}\right), \quad u \in L_1\left(\frac{1}{2}, B_0\right) \quad (0 < B_0 < \infty) \quad \text{with respect to } r,$$
$$\text{and} \quad u = O(1) \quad \text{as} \quad \theta \to 0 \quad u = O(1), \quad \text{as} \quad r \to \infty.$$

We assume that $|k| > 0$. The corresponding problem (2.1), (2.2) for $k = 0$ admits an exact solution. Its derivation is recorded in Appendix A.

3 Carleman boundary value problem with two shifts

To remove the variable ϕ from equation (2.1) and the boundary condition (2.2), we take the finite Fourier transform

$$u_m(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\phi,\theta)e^{im\phi}d\phi,$$

$$u(r,\phi,\theta) = \sum_{m=-\infty}^{\infty} u_m(r,\theta)e^{-im\phi} \tag{3.1}$$

and reduce the original problem to the following one

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_m}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_m}{\partial\theta}\right) + \left(k^2 - \frac{m^2}{r^2\sin^2\theta}\right)u_m = 0,$$

$$0 < r < \infty, \quad 0 \le \theta < \alpha, \tag{3.2}$$

$$\left(u_m + \frac{\gamma}{r}\frac{\partial u_m}{\partial\theta}\right)_{\theta=\alpha} = g_m(r), \quad 0 < r < \infty, \tag{3.3}$$

where

$$g_m(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(r,\phi)e^{im\phi}d\phi. \tag{3.4}$$

Let us introduce the Kontorovich-Lebedev transform [18]

$$u_{ms}(\theta) = \int_0^{\infty} u_m(r,\theta)H_s^{(2)}(kr)\frac{dr}{\sqrt{r}},$$

$$u_m(r,\theta) = -\frac{1}{2\sqrt{r}}\int_{-i\infty}^{i\infty} s u_{ms}(\theta)J_s(kr)ds. \tag{3.5}$$

here $J_s(kr)$ and $H_s^{(2)}(kr)$ are the Bessel functions. The function $u_{ms}(\theta)$ satisfies the following conditions

(i) $u_{ms}(\theta)$ is analytic in the strip $|\Re(s)| < \delta_0 \quad (0 < \delta_0 < 1/2)$;

(ii)

$$u_{m,s}(\theta) = e^{i\pi s}u_{m,-s}(\theta); \tag{3.6}$$

(iii) the integral

$$\int_{-\infty}^{\infty} |(\delta + it)u_{m,\delta+it}(\theta)|\, e^{-t\arg k + \pi|t|/2}dt \tag{3.7}$$

is convergent uniformly with respect to $\delta \in [-\delta_0, \delta_0]$. Applying the Kontorovich-Lebedev transform to equation (3.2) we get the ordinary differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}u_{ms}(\theta)\right) + \left(s^2 - \frac{1}{4} - \frac{m^2}{\sin^2\theta}\right)u_{ms}(\theta) = 0, \tag{3.8}$$

whose general solution, bounded at $\theta = 0$, is

$$u_{ms}(\theta) = A_m(s)P_{s-1/2}^m(\cos\theta). \tag{3.9}$$

Here $P_{s-1/2}^m(\cos\theta)$ is the associated Legendre function of the first kind, $A_m(s)$ is an arbitrary function to be determined from the boundary condition. To apply the Kontorovich-Lebedev transform to the boundary condition (3.3), we have to assume that the integral

$$\frac{d}{d\theta}\int_0^{\infty} u_m(r,\theta)H_s^{(2)}(kr)\frac{dr}{r^{3/2}} \tag{3.10}$$

as $\theta = \alpha$, is an analytic function at least in the strip $-1 < \Re(s) < 1$. Due to the relation for the Bessel function

$$\frac{1}{r}H_s^{(2)}(kr) = \frac{k}{2s}\left[H_{s-1}^{(2)}(kr) + H_{s+1}^{(2)}(kr)\right] \tag{3.11}$$

and condition (i), it means that the function $\frac{d}{d\theta}u_{ms}(\alpha)$ is analytically continuable into the strip $-1 < \Re(s) < 1$. Thus, the boundary condition (3.3) in its transformed form can be written as follows

$$u_{m,s}(\alpha) + \frac{1}{\gamma_0 s}\frac{d}{d\theta}[u_{m,s-1}(\alpha) + u_{m,s+1}(\alpha)] = \frac{h(s)}{k^{1/2}\gamma_0 s}, \tag{3.12}$$

where

$$h(s) = k^{1/2}\gamma_0 s\int\limits_0^\infty g_m(r)H_s^{(2)}(kr)\frac{dr}{\sqrt{r}}, \quad \gamma_0 = \frac{2}{k\gamma}. \tag{3.13}$$

Now we introduce a new function

$$\Phi(s) = k^{1/2}\frac{d}{d\theta}u_{m,s+1}(\alpha). \tag{3.14}$$

Due to formula (3.9) and the analyticity of the $\frac{d}{d\theta}u_{ms}(\alpha)$ in the strip $-1 < \Re(s) < 1$, we obtain that the function

$$\Phi(s) = k^{1/2}\frac{d}{d\theta}P_{s+1/2}^m(\cos\alpha)A_m(s+1) \tag{3.15}$$

is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$. If we compare expressions (3.9) and (3.15), we can find the transform $u_{ms}(\theta)$ in terms of the function $\Phi(s)$

$$u_{ms}(\theta) = \frac{P_{s-1/2}^m(\cos\theta)}{k^{1/2}\frac{d}{d\theta}P_{s-1/2}^m(\cos\alpha)}\Phi(s-1). \tag{3.16}$$

Now we substitute formulae (3.14), (3.16) into equation (3.12) and obtain the Carleman type boundary value problem of the theory of analytic functions for a strip:

Find the function $\Phi(s)$, that is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$, satisfies the boundary condition

$$\Phi(\sigma) + K(\sigma)\Phi(\sigma-1) + \Phi(\sigma-2) = h(\sigma), \quad \sigma \in \Omega = \{-i\infty, i\infty\}, \tag{3.17}$$

where

$$K(s) = \frac{\gamma_0 s P_{s-1/2}^m(\cos\alpha)}{\frac{d}{d\theta}P_{s-1/2}^m(\cos\alpha)} \tag{3.18}$$

and the following auxiliary conditions:

$$\int\limits_{-\infty}^{\infty}|\Phi(x+it)|^2 dt < \infty \tag{3.19}$$

uniformly with respect to $x \in [-2, 0]$ and

$$\Phi(\sigma-1) = e^{i\pi\sigma}\Phi(-\sigma-1), \quad \sigma \in \Omega. \tag{3.20}$$

The formulated problem can be considered as a difference equation of the second order in the strip Π with the two additional conditions (3.19) and (3.20).

4 Integral equation

The exact solution of problem (3.17), (3.19), (3.20) can be found only for particular cases of the coefficient $K(\sigma)$. In general, it could be reduced to a 2×2 matrix Riemann-Hilbert boundary value problem (see Appendix B). To find a constructive solution of the problem, we reformulate the original problem as an integral equation. First, we rewrite equation (3.17) in such a way

$$\Phi(\sigma) + \Phi(\sigma-2) = h^*(\sigma), \quad \sigma \in \Omega, \tag{4.1}$$

where

$$h^*(\sigma) = h(\sigma) - K(\sigma)\Phi(\sigma-1). \tag{4.2}$$

The function $\Phi(\sigma)$ is analytic in the strip $\Pi = \{-2 < \Re(s) < 0\}$, satisfies the boundary condition (4.1) and the auxiliary condition (3.19). Therefore, due to the Sokhotski-Plemelj formula modified for a strip, we write

$$\Phi(s) = \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin\frac{\pi}{2}(\tau - s)}, \quad s \in \Pi, \tag{4.3}$$

$$\Phi(\sigma) = \frac{1}{2}h^*(\sigma) + \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin\frac{\pi}{2}(\tau - \sigma)}, \quad \sigma \in \Omega, \tag{4.4}$$

$$\Phi(\sigma - 2) = \frac{1}{2}h^*(\sigma) - \frac{1}{4i} \int_{\Omega} \frac{h^*(\tau)d\tau}{\sin\frac{\pi}{2}(\tau - \sigma)}, \quad \sigma \in \Omega. \tag{4.5}$$

The first formula (4.3) defines the function $\Phi(s)$ in the interior of the strip Π in terms of its values on the contour $\{-1 - i\infty, -1 + i\infty\}$. The last two formulae (4.4) and (4.5) relate to the boundary values of the function $\Phi(s)$ on the contour Ω and on the contour $\{-2 - i\infty, -2 + i\infty\}$. Now letting $s = \sigma - 1$ in equality (4.3) and take into account (4.2) we get a convolution integral equation with respect to the function $\Phi(\sigma - 1)$

$$\Phi(\sigma - 1) + \frac{1}{4i} \int_{-i\infty}^{i\infty} \frac{K(\tau)\Phi(\tau - 1) - h(\tau)}{\cos\frac{\pi}{2}(\tau - \sigma)} d\tau = 0, \quad \sigma \in \Omega. \tag{4.6}$$

Let us transform this equation into a new one on the segment (0,1). We take into account that

$$\Phi(-\sigma - 1) = e^{-i\pi\sigma}\Phi(\sigma - 1),$$

$$K(-\tau) = -K(\tau), \quad h(-\tau) = -e^{-i\pi\tau}h(\tau), \tag{4.7}$$

and write equation (4.6) in the form

$$\Phi(i\xi - 1) + \frac{1}{4} \int_{-\infty}^{0} [K(i\eta)\Phi(i\eta - 1) - h(i\eta)] \left(\frac{1}{\cosh\frac{\pi}{2}(\eta - \xi)} - \frac{e^{\pi\eta}}{\cosh\frac{\pi}{2}(\eta + \xi)} \right) d\eta = 0,$$

$$-\infty < \xi < 0. \tag{4.8}$$

Then, introducing a new function

$$\chi(\xi) = K(i\xi)\Phi(i\xi - 1) - h(i\xi) \tag{4.9}$$

we get

$$\chi(\xi) + \frac{K(i\xi)}{4} \int_{-\infty}^{0} \chi(\eta) \left(\frac{1}{\cosh\frac{\pi}{2}(\eta - \xi)} - \frac{e^{\pi\eta}}{\cosh\frac{\pi}{2}(\eta + \xi)} \right) d\eta = -h(i\xi), \quad -\infty < \xi < 0. \tag{4.10}$$

Finally, setting in this equation

$$e^{\pi\eta} = y, \; e^{\pi\xi} = x, \tag{4.11}$$

and using the following relations

$$\frac{1}{\cosh\frac{\pi}{2}(\eta - \xi)} = \frac{2\sqrt{yx}}{y + x}, \qquad \frac{e^{\pi\eta}}{\cosh\frac{\pi}{2}(\eta + \xi)} = \frac{2y\sqrt{yx}}{yx + 1}, \tag{4.12}$$

we arrive at an integral equation that is, obviously, equivalent to the original equation (4.6)

$$\chi_0(x) + \frac{K_0(x)}{2\pi} \int_{0}^{1} \chi_0(y) \left(\frac{1}{y + x} - \frac{y}{yx + 1} \right) dy = -h_0(x), \quad 0 < x < 1, \tag{4.13}$$

where

$$\chi_0(x) = \frac{1}{\sqrt{x}}\chi\left(\frac{1}{\pi}\log x\right), \quad K_0(x) = K\left(\frac{i}{\pi}\log x\right),$$

$$h_0(x) = \frac{1}{\sqrt{x}}h\left(\frac{i}{\pi}\log x\right). \tag{4.14}$$

Equation (4.13) is a singular integral equation with an immovable singularity at the point $x = 0$. The function $K_0(x)$ does not belong to the Hölder space. Indeed, in a neighbourhood of the point $x = 0$, one can write (see Appendix A)

$$K_0(x) = -\gamma_0 i + \frac{r_0}{\log x} + O\left(\frac{1}{\log^2 x}\right), \quad x \to 0, \quad r_0 = \text{const} \tag{4.15}$$

and the Hölder condition for the function $K_0(x)$ fails near the point $x = 0$. Obviously, the behaviour of the function $\chi_0(x)$ at the singular point $x = 0$ is defined by a singularity of the function $h_0(x)$ at this point.

Remark. One can see that the key step in the derivation of equation (4.13) is formula (4.3). We note that the same equation can be obtained in another way using the classical Sokhotski-Plemelj formula (see Appendix C).

5 Axially symmetric incident wave

5.1 Formulation

Let us consider a wave field Usatisfying the Helmholtz equation

$$\Delta U + k^2 U = 0 \tag{5.1}$$

in the exterior of a conical obstacle, namely in the domain $C = \{0 < r < \infty, -\pi \le \phi \le \pi, 0 \le \theta < \alpha\}$. We stipulate an "impedance" boundary condition on the face of the obstacle

$$\left(\frac{1}{r}\frac{\partial U}{\partial \theta} + ik\beta U\right)_{\theta=\alpha} = 0, \quad 0 < r < \infty, \ \ -\pi \le \phi \le \pi, \tag{5.2}$$

where β is the complex parameter that specifies admittance of the surface of the obstacle. In the case of an absorbing boundary $\Re(\beta) > 0$. We study the axially symmetric problem of scattering of an incident plane wave

$$u^i = e^{ikr\cos\theta}. \tag{5.3}$$

by a circular impedance cone. The wave field U is represented in the form $U = u^i + u$, where u is the solution of the problem

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + k^2 u = 0, \quad 0 < r < \infty, \ 0 \le \theta < \alpha, \tag{5.4}$$

with the boundary condition

$$\left(u + \frac{\gamma}{r}\frac{\partial u}{\partial\theta}\right)_{\theta=\alpha} = g_0(r), \quad 0 < r < \infty. \tag{5.5}$$

The parameter γ is defined by

$$\gamma = \frac{1}{ik\beta}, \tag{5.6}$$

The formulated problem is equivalent to the symmetric case of the boundary value problem (2.1), (2.2) for the function u. The function $g(r,\phi)$ is independent of ϕ and for $m = 0$ we get

$$g(r,\phi) = g_0(r) = -k_0 e^{-ikr\cos\alpha_0}, \tag{5.7}$$

where

$$k_0 = 1 - \frac{\sin\alpha}{\beta}, \quad \alpha_0 = \pi - \alpha, \quad 0 < \alpha_0 < \frac{\pi}{2}. \tag{5.8}$$

We specify the parameter γ_0 and the function $h(s)$ in (3.13) as follows

$$\gamma_0 = 2i\beta, \quad h(s) = -k^{1/2}k_0\gamma_0 s\int_0^\infty e^{-ikr\cos\alpha_0}H_s^{(2)}(kr)\frac{dr}{\sqrt{r}}. \tag{5.9}$$

5.2 Analysis of the function h(s)

Using the integral representation of the Bessel function ([14], formula 8.421(2))

$$H_s^{(2)}(kr) = -\frac{e^{i\pi s/2}}{\pi i} \int\limits_{-\infty}^{\infty} e^{-ikr\cosh\eta - s\eta} d\eta \tag{5.10}$$

we calculate the interior integral with respect to r and instead of (5.9) we get

$$h(s) = \frac{k_0\gamma_0 s}{i\sqrt{\pi}} e^{i\pi(s/2-1/4)} \int\limits_{-\infty}^{\infty} \frac{e^{-s\eta}d\eta}{\sqrt{\cos\alpha_0 + \cosh\eta}}, \quad -\frac{1}{2} < \Re(s) < \frac{1}{2}. \tag{5.11}$$

The last integral is expressible in terms of the conical function $P_{it-1/2}(\cos\alpha_0)$ ([14], formula 8.842(1)). Therefore we may find an explicit expression for the function $h(s)$ through the Legendre function

$$h(s) = \frac{k_0\gamma_0 s\sqrt{2\pi}}{\cos\pi s} e^{i\pi(s/2-3/4)} P_{s-1/2}(\cos\alpha_0). \tag{5.12}$$

Now we take into account the asymptotic expansion for large $|s|$ for the Legendre function (see Appendix A, formula (A.21)) and obtain the asymptotics of the function $h(s)$ as $\Im(s) \to \pm\infty$

$$h(s) \sim \frac{2k_0\gamma_0 s^{1/2}}{i\sqrt{\sin\alpha_0}} e^{i(3\pi/2-\alpha_0)s}, \quad s \to +i\infty,$$

$$h(s) \sim -\frac{2k_0\gamma_0 s^{1/2}}{\sqrt{\sin\alpha_0}} e^{i(-\pi/2+\alpha_0)s}, \quad s \to -i\infty, \tag{5.13}$$

and therefore from (4.14) the behaviour of the function $h_0(x)$ at the point $x = 0$ is defined by

$$h_0(x) = C_0 x^{-\alpha_0/\pi}\sqrt{\log\frac{1}{x}} + O\left(x^{-\alpha_0/\pi}\frac{1}{\sqrt{-\log x}}\right), \quad x \to +0, \tag{5.14}$$

where

$$C_0 = \frac{\sqrt{2}k_0\gamma_0(i-1)}{\sqrt{\pi}\sin\alpha_0}. \tag{5.15}$$

5.3 Behaviour of the solution of the integral equation at the singular point

To analyze the function $\chi(x)$ in a neighbourhood of the point $x = 0$, we study the following singular integral with the density that has a power-logarithmic singularity[1]

$$\Sigma(x) = \int\limits_0^1 y^{\varkappa}\sqrt{\log\frac{1}{y}}\frac{dy}{y+x}, \quad \Re(\varkappa) > -1, \quad \text{as } x \to +0. \tag{5.16}$$

First, we note that the function $\Sigma(x)$ is an integral of the Mellin convolution type. Indeed, if we introduce the functions

$$v_-(y) = \begin{cases} y^{\varkappa}\sqrt{\log\frac{1}{y}}, & 0 < y < 1 \\ 0, & y > 1 \end{cases}, \quad w(y) = \frac{1}{1+y} \tag{5.17}$$

due to the convolution theorem, we can represent integral (5.16) as follows

$$\Sigma(x) = \int\limits_0^{\infty} v_-(y) w\left(\frac{x}{y}\right)\frac{dy}{y} = \frac{1}{2\pi i}\int\limits_{\Gamma} V_-(p)W(p)x^{-p}dp, \tag{5.18}$$

[1] The Mel'nik formula [12] describes the behaviour of the Cauchy integral

$$\frac{1}{2\pi i}\int\limits_{ab} \frac{\phi(\tau)\log^p(\tau-a)}{(\tau-a)^{\gamma}(\tau-z)}d\tau$$

for positive integer values of p and is not suitable in our case p=1/2.

where $\Gamma = \{p \in \mathbb{C} : \Re(p) = p_0 \in (\varkappa_0, 1)\}$, $\varkappa_0 = \max\{0, -\varkappa\}$ and

$$V_-(p) = \int_0^\infty v_-(y) y^{p-1} dy = \frac{\sqrt{\pi}}{2}(\varkappa + p)^{-3/2}, \tag{5.19}$$

$$W(p) = \int_0^\infty w(y) y^{p-1} dy = \frac{\pi}{\sin \pi p}. \tag{5.20}$$

Here we used formulae 4.269(3) and 3.241(2) [14] and assumed that $\varkappa_0 < \Re(p) < 1$, $\arg(\varkappa + p) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as $p \in \Gamma$. Thus, we have reduced the original integral to another one

$$\Sigma(x) = \frac{1}{2}\pi^{3/2}\sigma\left(x, \frac{3}{2}\right), \tag{5.21}$$

where

$$\sigma(x, q) = \frac{1}{2\pi i}\int_\Gamma \frac{x^{-p} dp}{(\varkappa + p)^q \sin \pi p}, \quad 0 < x < 1. \tag{5.22}$$

At first, we assume that $\Re(q) < 1$. The function $(\varkappa + p)^q$ has two branch points in the half-plane $\Re(p) < p_0$. They are $p = -\varkappa$ and the point at infinity. We cut the p-plane by a straight line from the point $p = -\varkappa$ to infinity in the domain $\Re(p) < p_0$ and fix a single-valued branch of the function $(\varkappa + p)^q$ in such a way: $-\frac{5\pi}{4} \leq \arg(\varkappa + p) \leq \frac{3\pi}{4}$. The Cauchy theorem and Jordan's lemma lead to the following relation

$$\sigma(x, q) + I^+(x, q) + I^-(x, q) = \sigma^{(0)}(x, q), \tag{5.23}$$

where

$$I^\pm(x, q) = \mp\frac{x^\varkappa}{2\pi i}e^{-i\lambda^\pm(q-1)}\int_0^\infty \frac{x^{-r\exp(i\lambda^\pm)} dr}{r^q \sin\{\pi[r\exp(i\lambda^\pm) - \varkappa]\}},$$

$$\lambda^+ = \frac{3\pi}{4}, \quad \lambda^- = -\frac{5\pi}{4}, \tag{5.24}$$

and

$$\sigma^{(0)}(x, q) = \frac{1}{\pi}\sum_{m=0}^\infty \frac{(-1)^m x^m}{(\varkappa - k)^q}, \quad -\frac{5\pi}{4} \leq \arg(\varkappa - k) \leq \frac{3\pi}{4}. \tag{5.25}$$

Using the asymptotic expansion

$$\frac{1}{\sin\{\pi[r\exp(i\lambda^\pm) - \varkappa]\}} = -\frac{1}{\sin \pi\varkappa} - \frac{\pi e^{i\lambda^\pm}\cot \pi\varkappa}{\sin \pi\varkappa} r + O(r^2), \quad r \to 0, \tag{5.26}$$

we obtain

$$I^\pm(x, q) = \mp\frac{x^\varkappa e^{\mp\pi i q}}{2\pi i \sin \pi\varkappa}\left[\Gamma(1-q)\log^{q-1}\frac{1}{x} - \pi\cot\pi\varkappa\Gamma(2-q)\log^{q-2}\frac{1}{x}\right]$$
$$+O\left(x^\varkappa \log^{q-3}\frac{1}{x}\right), \quad x \to +0. \tag{5.27}$$

Then, combining (5.27) and (5.24) we have

$$\sigma(x, q) = \sigma^{(0)}(x, q) + \sigma^{(1)}(x, q), \tag{5.28}$$

where the $\sigma^{(0)}(x, q)$ is defined in (5.25) and

$$\sigma_q^{(1)}(x) = \frac{x^\varkappa \log^{q-1}\frac{1}{x}}{\Gamma(q)\sin \pi\varkappa}\left[-1 + \frac{\pi(1-q)\cot \pi\varkappa}{\log\frac{1}{x}} + O\left(\frac{1}{\log^2\frac{1}{x}}\right)\right], \quad x \to +0, \quad \Re(q) < 1. \tag{5.29}$$

Next, we continue analytically formula (5.28) into the domain $\Re(q) > 1$, $q \neq 2, 3, , \ldots$ and find the representation for the original function $\Sigma(x)$

$$\Sigma(x) = \frac{1}{2}\pi^{3/2}[\sigma^{(0)}(x, \frac{3}{2}) + \sigma^{(1)}(x, \frac{3}{2})]. \tag{5.30}$$

In particularly, when $-1 < \Re(\varkappa) < 0$, formula (5.30) yields

$$\Sigma(x) = -\frac{\pi x^{\varkappa}}{\sin\pi\varkappa}\sqrt{\log\frac{1}{x}}\left[1+\frac{\pi}{2}\frac{\cot\pi\varkappa}{\log\frac{1}{x}}+O\left(\frac{1}{\log^2\frac{1}{x}}\right)\right], \quad x\to+0. \tag{5.31}$$

Now one can answer for the question: what is the singularity of the solution of equation (4.13) at the point $x=0$? Explicitly, if we compare representations (5.14) and (5.31) we may conclude that there is only one possibility: the solution of equation (4.13) behaves at the end $x=0$ as

$$\chi_0(x) = A_0 x^{-\alpha_0/\pi}\sqrt{\log\frac{1}{x}}+O\left(x^{-\alpha_0/\pi}\frac{1}{\sqrt{-\log x}}\right), \quad x\to+0, \tag{5.32}$$

In any other cases the left hand-side of equation (4.13) will not be consistent with the right hand-side as $x\to 0$. We substitute formulae (5.14), (5.32) and (4.15) into (4.13) and get

$$A_0\left(1+\frac{\gamma_0}{2i\sin\alpha}\right)x^{-\alpha_0/\pi}\sqrt{\log\frac{1}{x}}$$

$$= -C_0 x^{-\alpha_0/\pi}\sqrt{\log\frac{1}{x}}+O\left(x^{-\alpha_0/\pi}\frac{1}{\sqrt{-\log x}}\right), \quad x\to+0. \tag{5.33}$$

From here we immediately find the constant A_0

$$A_0 = -\frac{C_0}{1+\frac{\beta}{\sin\alpha}}. \tag{5.34}$$

Moreover, there are no obstacles on the way of construction of full asymptotic expansions for the functions $h_0(x)$ and $\chi_0(x)$ as $x\to 0$ with the coefficients defined in explicit form.

Now, it is easily to restore the behaviour of the function $\chi(\xi)$ as $\xi\to-\infty$ and of the basic function $\Phi(\sigma-1)$ as $\sigma\to\pm i\infty$. Indeed, relations (4.14), (4.11), (4.15) and (5.14), (5.32) yield

$$\chi(\xi) = \sqrt{\pi}A_0 e^{(\pi/2-\alpha_0)\xi}\left\{\sqrt{-\xi}+O\left(\frac{1}{\sqrt{-\xi}}\right)\right\}, \quad \xi\to-\infty,$$

$$h(\xi) = \sqrt{\pi}C_0 e^{(\pi/2-\alpha_0)\xi}\left\{\sqrt{-\xi}+O\left(\frac{1}{\sqrt{-\xi}}\right)\right\}, \quad \xi\to-\infty \quad \left(0<\alpha_0<\frac{\pi}{2}\right) \tag{5.35}$$

and therefore the function $\Phi(\sigma-1)$ decay exponentially as $\sigma\to\pm i\infty$

$$\Phi(\sigma-1) = \frac{\sqrt{\pi}i}{\gamma_0}(A_0+C_0)e^{(\pi/2-\alpha_0)\xi}\left\{\sqrt{-\xi}+O\left(\frac{1}{\sqrt{-\xi}}\right)\right\}, \quad \sigma=i\xi\to-i\infty,$$

$$\Phi(\sigma-1) = \frac{\sqrt{\pi}i}{\gamma_0}(A_0+C_0)e^{(-3\pi/2+\alpha_0)\xi}\left\{\sqrt{\xi}+O\left(\frac{1}{\sqrt{\xi}}\right)\right\}, \quad \sigma=i\xi\to+i\infty, \tag{5.36}$$

6 Numerical solution of equation (4.10)

Despite the singularity of the integral equation (4.10), [2]it can be solved numerically rather sufficiently. We have established the exponential decay of the functions $\chi(\xi)$ and $h(\xi)$ as $\xi\to-\infty$ and therefore even the following simple interpolation should provide a good rate of convergence of an approximate solution to the exact one. Let us consider a set of points

$$0=x_0>x_1>x_2>\ldots>x_N\to-\infty, \quad N\to\infty,$$
$$x_n\sim-\nu n^{\delta}, \quad n\to\infty, \quad \delta>0, \quad \nu>0, \tag{6.1}$$

introduce notations

$$\chi(\xi_m)=X_m \quad \text{as} \quad \xi_m=\frac{x_{m-1}+x_m}{2}, \quad m=1,2,\ldots,N \tag{6.2}$$

[2]Equation (4.10) is defined on the unbounded set $\xi\in(-\infty,0)$), and the kernel of the corresponding integral equation (4.13) on the segment (0,1) has an immovable singularity at the point $x=y=0$.

and approximate the function $\chi(\xi)$ by a piecewise constant function: $\chi(\xi) = X_m,\ \xi \in (x_m, x_{m-1})$. Then we discretise equation(4.10) as follows

$$X_n + \frac{K(i\xi_n)}{4} \sum_{m=1}^{N} X_m \int\limits_{x_m}^{x_{m-1}} \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi_n)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi_n)} \right) d\eta = -h(i\xi_n),$$

$$n = 1, 2, \ldots, N. \tag{6.3}$$

Thus, for the coefficients X_k we get the following system of algebraic equations

$$X_n + \sum_{m=1}^{N} c_{nm} X_m = b_n, \quad n = 1, 2, \ldots, N, \tag{6.4}$$

where

$$c_{nm} = \frac{K(i\xi_n)}{4} \int\limits_{x_m}^{x_{m-1}} \left(\frac{1}{\cosh \frac{\pi}{2}(\eta - \xi_n)} - \frac{e^{\pi\eta}}{\cosh \frac{\pi}{2}(\eta + \xi_n)} \right) d\eta,$$

$$b_n = -h(i\xi_n). \tag{6.5}$$

The integrals in the representation of the coefficients c_{nk} can be estimated explicitly

$$c_{nm} = \frac{K(i\xi_n)}{\pi} (t_{nm}^{(1)} + t_{nm}^{(2)}), \tag{6.6}$$

where

$$t_{nm}^{(1)} = \tan^{-1} \frac{\lambda_{m-1}}{\mu_n} - \tan^{-1} \frac{\lambda_m}{\mu_n}, \tag{6.7}$$

$$t_{nm}^{(2)} = \frac{\lambda_m - \lambda_{m-1}}{\mu_n} + \frac{1}{\mu_n^2} [\tan^{-1}(\lambda_{m-1}\mu_n) - \tan^{-1}(\lambda_m \mu_n)], \tag{6.8}$$

$$\lambda_m = e^{\pi x_m/2} \quad (m = 0, 1, \ldots, N); \quad \mu_n = \sqrt{\lambda_n \lambda_{n-1}} \quad (n = 1, 2, \ldots, N),$$

$$0 < \lambda_m \leq 1, \quad 0 < \mu_n < 1. \tag{6.9}$$

We note that for small μ_n instead of formula (6.8), it is more convenient to use the series representation of the function $\tan^{-1} x$, namely

$$t_{nm}^{(2)} = -\frac{1}{3} \mu_n (\lambda_{m-1}^3 - \lambda_m^3) + \frac{1}{5} \mu_n^3 (\lambda_{m-1}^5 - \lambda_m^5) - \ldots. \tag{6.10}$$

One may conclude that $t_{nm}^{(2)} \to 0$ as $n \to \infty$ (m is fixed) and $m \to \infty$ (n is fixed). The rate of the decay is exponential. For calculation of the coefficients $t_{nm}^{(1)}$ the following two series representations are useful

$$t_{nm}^{(1)} = \frac{\lambda_{m-1}}{\mu_n} - \frac{\lambda_m}{\mu_n} - \frac{1}{3} \left(\frac{\lambda_{m-1}}{\mu_n} \right)^3 + \frac{1}{3} \left(\frac{\lambda_m}{\mu_n} \right)^3 - \ldots, \quad \frac{\lambda_m}{\mu_n} \to 0,$$

$$t_{nm}^{(1)} = -\frac{\mu_n}{\lambda_{m-1}} + \frac{\mu_n}{\lambda_m} + \frac{1}{3} \left(\frac{\mu_n}{\lambda_{m-1}} \right)^3 - \frac{1}{3} \left(\frac{\mu_n}{\lambda_m} \right)^3 + \ldots, \quad \frac{\lambda_m}{\mu_n} \to \infty. \tag{6.11}$$

Thus, the coefficients c_{nm}, b_n decay exponentially as $n \to \infty$ (m is fixed) or in the case of the c_{nm} if $m \to \infty$ (n is fixed), namely

$$c_{nm} = O\left(e^{\pi x_n/2}\right), \quad n \to \infty \ \ (x_n \to -\infty), \quad m \text{ is fixed},$$

$$c_{nm} = O\left(e^{\pi x_m/2}\right), \quad m \to \infty, \quad n \text{ is fixed},$$

$$b_n = O\left(e^{(\pi/2 - \alpha_0)x_n} \sqrt{-x_n}\right), \quad n \to \infty. \tag{6.12}$$

Let us denote by

$$\mathbf{X}^* = (X_1^*, X_2^*, \ldots, X_N^*, \ldots) \tag{6.13}$$

the exact solution of an infinite system of algebraic equations corresponding to the limit case $N=\infty$ of system (6.4)

$$X_n^* + \sum_{m=1}^{\infty} c_{nm} X_m^* = b_n, \quad n = 1, 2, \dots \tag{6.14}$$

$(\mu_1 \to 1, N \to \infty)$. We introduce an operator S defined in the space l_2 of numerical sequences

$$\mathbf{Y} = (Y_1, Y_2, \dots) \in l_2 \ \text{ iff } \ \sum_{n=1}^{\infty} |Y_n|^2 < \infty$$

by the matrix $\{\delta_{nm} + c_{nm}\}$ $(n, m = 1, 2, \dots, N)$ and assume that there exists S^{-1}. Then an approximate solution $\mathbf{X}^{(N)} = (X_1, X_2, \dots, X_N, 0, 0, \dots)$ converges to the exact one $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_N^*, X_{N+1}^*, \dots)$ and the rate of the convergence is determined by

$$||\mathbf{X}^{(N)} - \mathbf{X}^*||_{l_2} = O\left(N^{\delta/2} e^{(\alpha_0 - \pi/2)\nu N^{\delta}}\right), \tag{6.15}$$

where $0 < \alpha_0 < \frac{\pi}{2}$, $\eta > 0$, $\delta > 0$. Estimation (6.15) is derived from the Kantorovich-Krylov theorem [16]. The total error of approximation is defined by (6.15) and by the discrepancy of the piecewise constant approximation (6.2).

7 Diffraction coefficients

7.1 An impedance cone

We now turn to the wave field u, the solution of problem (5.4), (5.5). Applying the inverse Kontorovich-Lebedev transform (3.5) for $m = 0$ we get

$$u(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^{i\infty} s u_s(\theta) J_s(kr) ds. \tag{7.1}$$

Since the function $u_s(\theta)$ satisfies the equality $u_s(\theta) = e^{i\pi s} u_{-s}(\theta)$ (see (3.6)) then

$$u(r, \theta) = -\frac{1}{2\sqrt{r}} \int_{-i\infty}^{0} s u_s(\theta) [J_s(kr) - e^{-i\pi s} J_{-s}(kr)] ds, \tag{7.2}$$

where the function $u_s(\theta)$ is defined in terms of the solution of the integral equation (4.10)

$$u_s(\theta) = \frac{1}{k^{1/2}} \frac{P_{s-1/2}(\cos\theta)\Phi(s-1)}{\frac{d}{d\theta} P_{s-1/2}(\cos\alpha)} = \frac{P_{s-1/2}(\cos\theta)[\chi(-is) + h(s)]}{k^{1/2}\gamma_0 s P_{s-1/2}(\cos\alpha)}. \tag{7.3}$$

Let us find the diffraction coefficients $D(\theta)$, an important characteristic in the geometric theory of diffraction [17]. They are introduced in such a way

$$u(r, \theta) = \frac{D(\theta)}{kr} e^{-ikr} + O\left(\frac{1}{(kr)^2}\right), \quad kr \to \infty \tag{7.4}$$

and describe the high-frequency asymptotics of the solution. To find the diffraction coefficients, we use the behaviour of the Bessel function at infinity

$$J_{\pm s}(kr) = \sqrt{\frac{2}{\pi kr}} \cos\left(kr \mp \frac{\pi s}{2} - \frac{\pi}{4}\right) + O\{(kr)^{-3/2}\}, \quad kr \to \infty, \tag{7.5}$$

and from formula (7.2) we obtain

$$u(r, \theta) = \frac{(1-i)e^{-ikr}}{2\sqrt{\pi k r}} \int_{-i\infty}^{0} s u_s(\theta) e^{-i\pi s/2} \sin \pi s ds + O\left(\frac{1}{(kr)^2}\right), \quad kr \to \infty. \tag{7.6}$$

One needs to study the convergence of this integral. The behaviour of the function $\Phi(s-1)$ as $s \to -i\infty$ has been analysed in Section 5.3:

$$\Phi(s-1) = O\{e^{-(\pi/2-\alpha_0)is}(is)^{1/2}\}, \quad s \to -i\infty. \tag{7.7}$$

From formulae (7.3), (A.19) and (A.24) we easily deduce the asymptotics

$$su_s(\theta)e^{-i\pi s/2}\sin\pi s = O(s^{1/2}e^{is(\pi+\theta-2\alpha)}), \quad s \to -i\infty \tag{7.8}$$

and therefore integral (7.6) converges if

$$\theta < 2\alpha - \pi. \tag{7.9}$$

Now we substitute expression (7.3) for the function $u_s(\theta)$ into the asymptotic formula (7.6) and compare the new formula with the classical representation (7.4). It yields the expression for the diffraction coefficients

$$D(\theta) = \frac{i-1}{2\sqrt{\pi}\gamma_0}\int_{-\infty}^{0}\frac{P_{i\xi-1/2}(\cos\theta)}{P_{i\xi-1/2}(\cos\alpha)}[\chi(\xi)+h(i\xi)]e^{\pi\xi/2}\sinh\pi\xi d\xi. \tag{7.10}$$

The integral is improper, it converges uniformly in the domain $0 \le \theta < 2\alpha - \pi$ and the rate of the convergence is exponential. The function $\chi(\xi)$ is the solution of equation (4.10).

Let us find an approximate formula for the diffraction coefficients, based on an approximate solution of equation (4.10) constructed in Section 6. We split the interval $(-\infty, 0)$ as in Section 6 and obtain from (7.10)

$$D^{(N)}(\theta) = \frac{i-1}{2\sqrt{\pi}\gamma_0}\left[\sum_{m=1}^{N} X_m \int_{x_m}^{x_{m-1}} \Lambda(\xi,\theta)d\xi + \int_{-\infty}^{0} h(i\xi)\Lambda(\xi,\theta)d\xi\right], \tag{7.11}$$

where $D^{(N)}(\theta)$ is an approximate value of the diffraction coefficients $D(\theta)$ and

$$\Lambda(\xi,\theta) = \frac{P_{i\xi-1/2}(\cos\theta)}{P_{i\xi-1/2}(\cos\alpha)}e^{\pi\xi/2}\sinh\pi\xi. \tag{7.12}$$

One can simplify this formula further using an approximate formula for integrals (7.11)

$$D^{(N)}(\theta) = \frac{i-1}{2\sqrt{\pi}\gamma_0}\sum_{m=1}^{N}(X_m - b_m)\Lambda(\xi_m,\theta)(x_{m-1}-x_m). \tag{7.13}$$

7.2 The Dirichlet and Neumann conditions

Let us consider the particular case of the problem (5.4), (5.5) when $\gamma = 0$, ($\beta = \infty$) i.e. the Dirichlet problem for the Helmholtz equation in a cone (axially symmetric case). Then from (7.6) we have

$$u(r,\theta) = -\frac{(1+i)e^{-ikr}}{2\sqrt{\pi kr}}\int_{-\infty}^{0}\xi u_{i\xi}(\theta)e^{\pi\xi/2}\sinh\pi\xi, \quad kr \to \infty, \tag{7.14}$$

where

$$u_s(\theta) = A(s)P_{s-1/2}(\cos\theta) \tag{7.15}$$

with the function $A(s)$ defined from the condition

$$A(s)P_{s-1/2}(\cos\alpha) = -\int_0^\infty e^{-ikr\cos\alpha_0}H_s^{(2)}(kr)\frac{dr}{\sqrt{r}} \tag{7.16}$$

by the equality

$$A(s) = \frac{\sqrt{2\pi}}{k^{1/2}}e^{i\pi s/2-3\pi i/4}\frac{P_{s-1/2}(\cos\alpha_0)}{\cos\pi s P_{s-1/2}(\cos\alpha)}. \tag{7.17}$$

If we compare formulae (7.4), (7.14) and take into account relations (7.15), (7.17), we finally obtain the expression for the diffraction coefficients

$$D_{\mathcal{D}}(\theta)=i\int\limits_0^\infty \frac{\xi\tanh\pi\xi P_{i\xi-1/2}(-\cos\alpha)}{P_{i\xi-1/2}(\cos\alpha)}P_{i\xi-1/2}(\cos\theta)d\xi \tag{7.18}$$

that coincides with the classical formula [8].

In the case of the Neumann condition on the surface of the cone we get $\gamma=\infty$ ($\beta=0$)

$$\frac{1}{r}\frac{\partial u}{\partial\theta}=-\frac{1}{r}\frac{\partial u^i}{\partial\theta}\quad\text{as }\ \theta=\alpha, \tag{7.19}$$

and for the function $A(s)$ in relation (7.15) instead of (7.16) we have

$$A(s)\frac{d}{d\theta}P_{s-1/2}(\cos\theta)=-\frac{d}{d\theta}\int\limits_0^\infty e^{ikr\cos\theta}H_s^{(2)}(kr)\frac{dr}{\sqrt{r}}\quad\text{as }\ \theta=\alpha. \tag{7.20}$$

Therefore

$$A(s)=\frac{\sqrt{2\pi}}{k^{1/2}}e^{i\pi s/2-3\pi i/4}\frac{\frac{d}{d\theta}P_{s-1/2}(-\cos\theta)}{\cos\pi s\frac{d}{d\theta}P_{s-1/2}(\cos\theta)}\quad\text{as }\ \theta=\alpha. \tag{7.21}$$

The diffraction coefficients are given by

$$D_{\mathcal{N}}(\theta)=i\int\limits_0^\infty \frac{\xi\tanh\pi\xi\frac{d}{d\theta}P_{i\xi-1/2}(-\cos\alpha)}{\frac{d}{d\theta}P_{i\xi-1/2}(\cos\alpha)}P_{i\xi-1/2}(\cos\theta)d\xi, \tag{7.22}$$

where

$$\frac{d}{d\theta}P_{s-1/2}(\pm\cos\alpha)=-\frac{s+1/2}{\sin\alpha}[\mp P_{s+1/2}(\pm\cos\alpha)+\cos\alpha P_{s-1/2}(\pm\cos\alpha)]. \tag{7.23}$$

Finally, we note that if we compare formulae (7.18) and (7.10), (5.12) then diffraction coefficients in the case of the impedance boundary condition can be expressed in terms of the corresponding coefficients of the Dirichlet problem as follows

$$D(\theta)=k_0D_{\mathcal{D}}(\theta)+D_0(\theta), \tag{7.24}$$

where $D_{\mathcal{D}}$ are coefficients (7.18), the parameter k_0 is defined in (5.8) and $D_0(\theta)$ is an additional term

$$D_0(\theta)=\frac{1-i}{2\sqrt{\pi}\gamma_0}\int\limits_0^\infty \frac{P_{i\xi-1/2}(\cos\theta)}{P_{i\xi-1/2}(\cos\alpha)}\chi(-\xi)e^{-\pi\xi/2}\sinh\pi\xi d\xi. \tag{7.25}$$

7.3 Numerical results

We confine ourself in considering the case of the axially symmetric incident wave (5.3). In the table below, we write down the values of the diffraction coefficients for $\alpha=\frac{5\pi}{6}$ for the Dirichlet problem ($\beta=\infty$), the Neumann problem ($\beta=0$) and the two cases of the impedance condition: $\beta=1$ and $\beta=1+i$.

	$\beta=\infty$ [2]	$\beta=\infty$	$\beta=1$	$\beta=1+i$		$\beta=0$
θ	$2\pi i f$	$-iD$	$-iD$	$\Re(D)$	$Im(D)$	$-iD$
0	0.2399	0.2399	0.03486	-0.05287	0.05152	-0.05217
$3\pi/24$	0.2576	0.2577	0.03839	-0.05776	0.05697	-0.05824
$6\pi/24$	0.3231	0.3232	0.05196	-0.07620	0.07812	-0.08117
$9\pi/24$	0.4966	0.4967	0.09110	-0.1269	0.1399	-0.1520
$12\pi/24$	1.0860	1.0862	0.2423	-0.3063	0.3843	-0.4590
$15\pi/24$	8.3073	8.2856	2.4441	-2.5373	3.9465	-6.0754

The coefficients $\Im(f)$ were calculated in [2] for the same angle $\alpha=\frac{5\pi}{6}$ and $\beta=\infty$, and we adduce them to compare with our own results (the third column of the table). In Fig.1, we present the graph of the diffraction coefficients $-iD$ versus θ for $\alpha=\frac{5\pi}{6}$ for some real values of the parameter β.

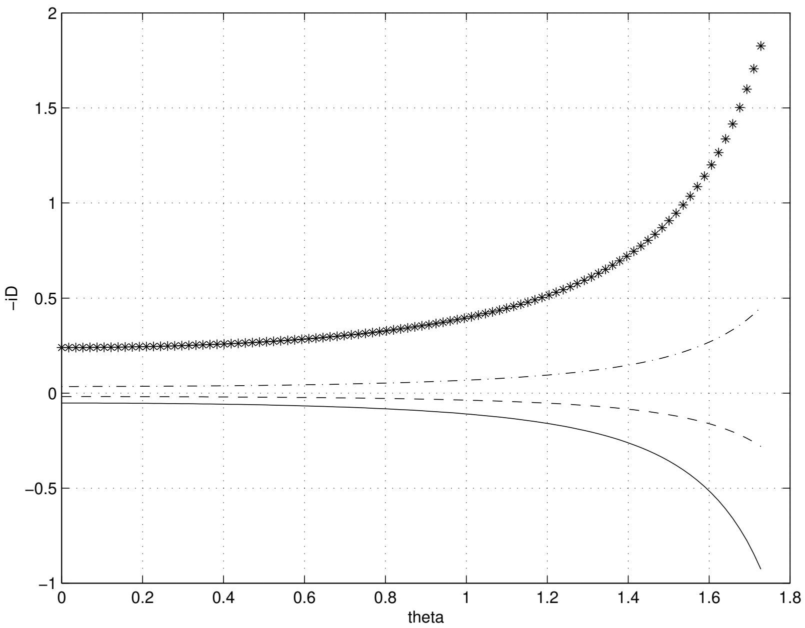

Figure 1: Diffraction coefficients $-iD(\theta)$ versus θ for $\alpha = \frac{5\pi}{6}$, and $\beta = 10^{-4}(-),\ 0.3(--),\ 1(-\cdot-\cdot),\ 10^4(***)$.

It is clear from (5.8) that the value $\beta = \sin\alpha$ is critical for the coefficients $D(\theta)$: $D(\theta) = 0$ for all θ. If $\beta < \sin\alpha$ then $-iD(\theta) < 0$ and for $\beta > \sin\alpha$ the coefficients $-iD(\theta)$ are positive and $-iD(\theta) \to \infty$ as $\theta \to 2\alpha - \pi - 0$. The situation becomes evident from a study of Fig.2 and Fig.3.

The graph represents the coefficients $-iD(\theta)$ as a function of θ $(0 < \theta < 2\alpha - \pi)$ for $\alpha = \frac{7\pi}{12}$, $\alpha = \frac{3\pi}{4}$ and $\alpha = \frac{11\pi}{12}$ when $\beta = 1$. It is seen that the diffraction coefficients for "blunt" cones are greater then those for narrow cones.

The case of complex β is illustrated by the graph in Fig.4, where $\Re(D)$ and $\Im(D)$ are parametrically defined as functions of $\beta \in (1 - 20i, 1 + 20i)$ for $\theta = 0$ and $\theta = \frac{2\pi}{3}$ $(\alpha = \frac{5\pi}{6})$. The point traverses along the orbit from the start point $\beta = 1 - i\infty$ to the terminal point $\beta = 1 + i\infty$ (these two point are not shown in the graph) and the direction of the motion is clock-wise. The points $\beta = 1 - i\infty$ and $\beta = 1 + i\infty$ provide the same values of the diffraction coefficients corresponding to those for the Dirichlet problem.

Conclusion

The boundary value problem for the Helmholtz equation in a circular cone with an impedance boundary condition on its face was examined using the Kontorovich-Lebedev integral transform. The problem was reduced to a difference equation of the second order in a strip of a complex variable and then to a convolution type integral equation with variable coefficients on an imaginary axis subject to an additional condition of symmetry. It was shown that the equation was equivalent to a 2×2 matrix Riemann-Hilbert problem with the discontinuous coefficient. To analyse the singularities of the solution at infinity, the original equation was reduced to a singular integral equation with Dixon's kernel. Its coefficients are variable and the right-hand side has a power-logarithmic singularity. The behaviour of the Dixon's integral at the ends in the case of a power-logarithmic singularity in the density was described. An approximate solution of the integral equation was constructed. For the diffraction coefficients an exact formula (in terms of the solution of the integral equation) and an approximate expression convenient for numerical computations, were derived. The limit cases of the Neumann and Dirichlet boundary conditions were considered. Additionally, an exact solution of the problem for the Laplace equation in a cone with a condition of the third kind was found.

The approach of this paper is applicable for 3-D-problems of thermal conductivity and the theory of elasticity in a wedge and in a cone with boundary condition of the third kind.

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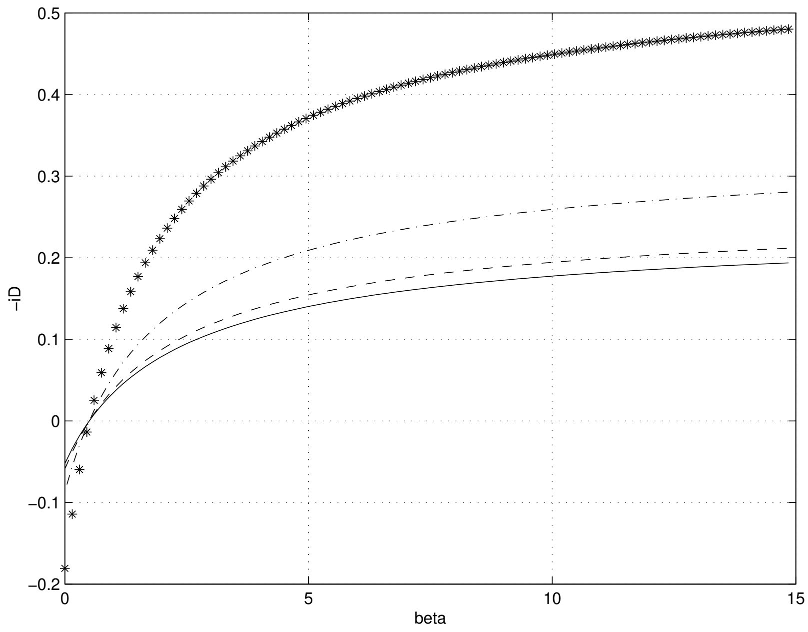

Figure 2: Diffraction coefficients $-iD(\theta)$ versus β for $\alpha = \frac{5\pi}{6}$, and $\theta = 0(-)$, $\varphi/5(--)$, $2\varphi/5(-\cdot-\cdot)$, $3\varphi/5(***)$ $(\varphi = 2\alpha - \pi)$.

Appendix A. Laplace equation in a cone with a boundary condition of the third kind: an exact solution

We construct an exact solution of the boundary value problem for the Laplace equation in a cone

$$\Delta u(r,\phi,\theta) = 0, \quad 0 < r < \infty, \ \ -\pi \le \phi \le \pi, \ \ 0 \le \theta < \alpha \tag{A.1}$$

with the boundary condition

$$\left(u + \frac{\gamma}{r}\frac{\partial u}{\partial \theta} \right)_{\theta=\alpha} = g(r,\phi), \quad 0 < r < \infty, \ \ -\pi \le \phi \le \pi. \tag{A.2}$$

where γ =const, $g(r,\phi)$ is a given function. Let us apply the finite Fourier transform (3.1) and consider the following boundary value problem

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_m}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u_m}{\partial \theta} \right) - \frac{m^2}{r^2 \sin^2\theta} u_m = 0, \quad 0 < r < \infty, \quad 0 \le \theta < \alpha, \tag{A.3}$$

$$\left(u_m + \frac{\gamma}{r}\frac{\partial u_m}{\partial \theta} \right)_{\theta=\alpha} = g_m(r), \quad 0 < r < \infty, \tag{A.4}$$

where $u_m(r,\theta)$ and $g_m(r)$ were defined in (3.1) and (3.4). We assume the following behaviour of the function $u_m(r,\theta)$ as $r \to 0$ and $r \to \infty$

$$|u_m(r,\theta)| \le C r^{\varepsilon_1}, \quad r \to 0, \quad C = const, \quad -1 < \varepsilon_1 < 0,$$
$$u_m(r,\theta) = O(r^{\varepsilon_2}), \quad r \to \infty, \quad -1 \le \varepsilon_2 < 0, \quad (\varepsilon_1 > \varepsilon_2). \tag{A.5}$$

By taking the Mellin transform

$$u_{ms}(\theta) = \int_0^\infty u_m(r,\theta) r^s dr, \quad -1 - \varepsilon_1 < \Re(s) < -1 - \varepsilon_2,$$

$$u_m(\theta) = \frac{1}{2\pi i}\int_\Omega u_{ms}(\theta) r^{-s-1} ds,$$

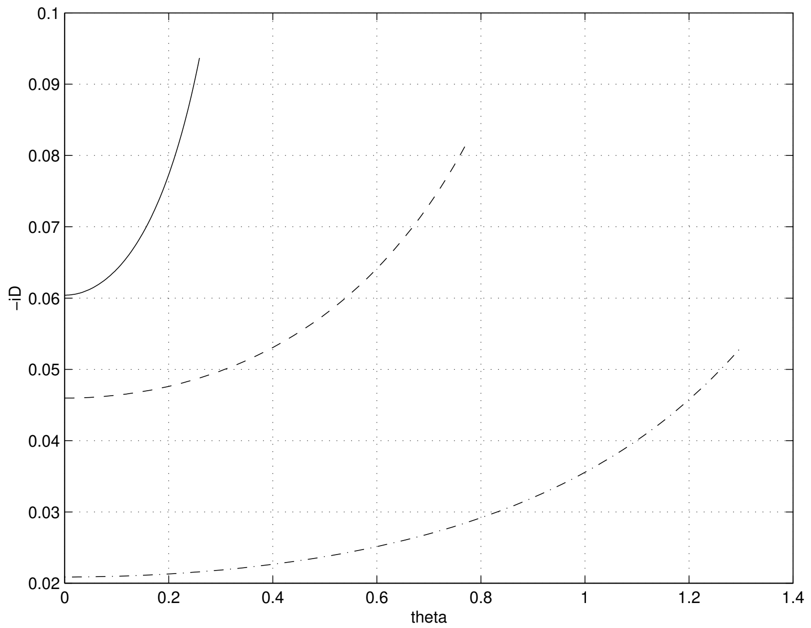

Figure 3: Diffraction coefficients $-iD(\theta)$ versus θ for $\beta = 1$, and $\alpha = \frac{7\pi}{12}(-)$, $\frac{3\pi}{4}(--)$, $\frac{11\pi}{12}(-\cdot-\cdot)$.

$$\Omega = \{s : \Re(s) = c \in (-1-\varepsilon_1, -1-\varepsilon_2) \subset (-1, 0)\} \tag{A.6}$$

of equation (A.3) and condition (A.4) we get the boundary value problem for the ordinary differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}u_{ms}(\theta)\right) + \left[s(s+1) - \frac{m^2}{\sin^2\theta}\right]u_{ms}(\theta) = 0, \quad 0 \le \theta < \alpha, \tag{A.7}$$

$$\left(u_{m,s} + \gamma\frac{d}{d\theta}u_{m,s-1}\right)_{\theta=\alpha} = G(s), \tag{A.8}$$

where

$$G(s) = g_{ms} = \int_0^\infty g_m(r) r^s dr. \tag{A.9}$$

Remark. The function $g_m(r)$ is supposed to satisfy the following conditions

$$r^{c+1}g_m(r) \in L_{2,\frac{1}{r}}(0,\infty), \quad r^{c+2}g'_m(r) \in L_{2,\frac{1}{r}}(0,\infty), \tag{A.10}$$

where the weight space $L_{2,\frac{1}{r}}(0,\infty)$ consists of such functions $g(r)$ that

$$\int_0^\infty |g(r)|^2\frac{dr}{r} < \infty.$$

As far as the function g_{ms} is concerned, we assume that $G(s) \in H(\Omega)$. Here H is the Hölder space. The function $u_{ms}(\theta)$ is bounded at the point $\theta = 0$. Therefore the solution of equation (A.7) has the form

$$u_{ms}(\theta) = A_m(s)P_s^m(\cos\theta). \tag{A.11}$$

Let us show that the function $A_m(s)$ can be determined from the boundary condition (A.8) explicitly. We apply the inverse Mellin transform (A.6) and get

$$\frac{1}{2\pi i}\int_\Omega [A_m(s)P_s^m(\cos\alpha) - G(s)]r^{-s-1}ds = -\frac{\gamma}{2\pi i}\int_\Omega A_m(s)\frac{d}{d\theta}P_s^m(\cos\alpha)r^{-s-2}ds. \tag{A.12}$$

Now we introduce a new function

$$\Phi(s) = A_m(s)\frac{d}{d\theta}P_s^m(\cos\theta)|_{\theta=\alpha} \tag{A.13}$$

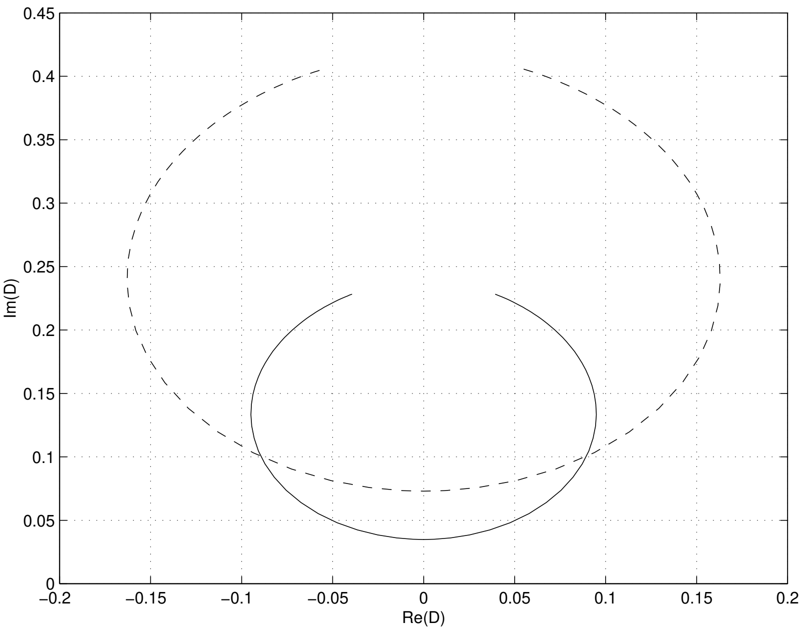

Figure 4: Parametrically defined complex-valued diffraction coefficients ($\Im(D)$ versus $\Re(D)$), with $\beta = 1 + iB$, $-20 < B < 20$, for $\alpha = \frac{5\pi}{6}$ and $\theta = 0(-)$, $\frac{2\pi}{3}(--)$.

and assume that

(i) the function $\Phi(s)$ is analytically continuable into the strip $\Pi = \{c - 1 < \Re(s) < c\}$;
(ii) it is bounded and continuous up to the boundary in the domain $\{c - 1 \leq \Re(s) \leq c\}$;
(iii) there exists a constant C such that

$$\int_{-\infty}^{\infty} |\Phi(x + it)|^2 dt < \text{const} \tag{A.14}$$

uniformly with respect to $x \in [-1 + c, c]$.

Due to the Cauchy theorem, we may move the contour Ω in the integral

$$\frac{1}{2\pi i} \int_{\Omega} \Phi(s) r^{-s-2} ds \tag{A.15}$$

from the position $\Re(s) = c$ to the new one $\Omega_{-1} = \{\Re(s) = c - 1\}$. Finally, we get

$$\frac{1}{2\pi i} \int_{\Omega} \Phi(s) r^{-s-2} ds = \frac{1}{2\pi i} \int_{\Omega_{-1}} \Phi(s) r^{-s-2} ds = \frac{1}{2\pi i} \int_{\Omega} \Phi(s-1) r^{-s-1} ds. \tag{A.16}$$

It means that equation (A.12) can be rewritten in such a way

$$\frac{1}{2\pi i} \int_{\Omega} \left[\gamma \Phi(s-1) + \frac{P_s^m(\cos\alpha)}{\frac{d}{d\theta} P_s^m(\cos\alpha)} \Phi(s) - G(s) \right] r^{-s-1} ds = 0 \tag{A.17}$$

for any $r \in (0, \infty)$. The last equation leads to the boundary condition of the following Carleman boundary value problem for a strip:

Find the function $\Phi(s)$ that is analytic in the strip $\Pi = \{c - 1 < \Re(s) < c\}$, satisfies conditions (i)-(iii) and the boundary condition

$$\Phi(\sigma - 1) + K(\sigma)\Phi(\sigma) = \frac{1}{\gamma} G(\sigma), \quad \sigma \in \Omega, \tag{A.18}$$

where

$$K(s) = \frac{P_m^s(\cos\alpha)}{\gamma \frac{d}{d\theta} P_s^m(\cos\alpha)}. \tag{A.19}$$

Let us study the behaviour of the coefficient $K(s)$ at infinity $s \to c \pm i\infty$. We rewrite the expression for the function $K(s)$ as follows

$$K(s) = \frac{1}{\gamma}\left\{-(s+1)\cot\alpha + \frac{s-m+1}{\sin\alpha}\frac{P^m_{s+1}(\cos\alpha}{P^m_s(\cos\alpha)}\right\}^{-1} \tag{A.20}$$

and use the asymptotic expansion for the Legendre function ([14], formula 8.721(1))

$$P^m_s(\cos\alpha) = \frac{2}{\sqrt{\pi}}\Gamma(s+m+1)\sum_{l=0}^{\infty}\frac{\Gamma\left(m+l+\frac{1}{2}\right)}{\Gamma\left(m-l+\frac{1}{2}\right)}\frac{\cos\left[\left(s+l+\frac{1}{2}\right)\alpha - \frac{\pi}{4}(2l+1)+\frac{m\pi}{2}\right]}{l!\Gamma\left(s-l+\frac{3}{2}\right)(2\sin\alpha)^{l+1/2}},$$

$$0 < \delta \le \alpha \le \pi - \delta < \pi, \tag{A.21}$$

and for the Γ-function

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O\left(\frac{1}{z^2}\right)\right], \quad z \to \infty. \tag{A.22}$$

Now we can obtain

$$\frac{P^m_{c+i\tau+1}(\cos\alpha)}{P^m_{c+i\tau}(\cos\alpha)} = e^{\mp i\alpha}\left[1 + O\left(\frac{1}{\tau}\right)\right], \quad \tau \to \pm\infty. \tag{A.23}$$

Thus, the function $K(s)$ is discontinuous on the contour Ω:

$$K(s) = \frac{1}{\gamma(s+1)}\left[\pm i + O\left(\frac{1}{\tau}\right)\right], \quad s = c + i\tau, \quad \tau \to \pm\infty. \tag{A.24}$$

Let us split the function $K(s)$ as follows

$$K(s) = -\frac{\cot\frac{\pi s}{2}}{\gamma(s+1)}K_1(s). \tag{A.25}$$

The function $K_1(s)$ is continuous at infinity and the increment of its argument as the point s traverses the contour Ω in the positive direction, is equal to zero. Next, we factorise the constant γ, the function $s+1$ and the $\cot\frac{\pi s}{2}$:

$$\frac{1}{\gamma} = \frac{\gamma^{s-1}}{\gamma^s}, \quad \frac{1}{s+1} = \frac{\Gamma(s+1)}{\Gamma(s+2)}, \quad -\cot\frac{\pi s}{2} = \frac{\sin\frac{\pi}{2}(s-1)}{\sin\frac{\pi}{2}s}. \tag{A.26}$$

Let us consider the problem of factorization of the function $K_1(s)$

$$K_1(\sigma) = \frac{X(\sigma)}{X(\sigma-n)}, \quad \sigma \in \Omega \tag{A.27}$$

possessing the following properties

$$K_1(s) = 1 + O\left(\frac{1}{s}\right), \quad s = \sigma + i\tau, \quad \tau \to \pm\infty,$$

$$K_1(\sigma) \in H(\Omega), \quad [\arg K_1(\sigma)]_\Omega = 0, \tag{A.28}$$

where $H(\Omega)$ is the Hölder class and $[f(\sigma)]_\Omega$ is the increment of the function $f(\sigma)$ when the point σ passes round the contour Ω in the positive direction. The function $X(s)$ is analytic in the strip $\Pi = \{c - n < \Re(s) < c\}$. In this case problem (A.27) admits an exact solution by quadratures ([4, 22, 13])

$$X(\sigma) = \exp\left\{\frac{1}{2}\log K_1(\sigma) + \frac{1}{2ni}\int_\Omega \cot\frac{\pi}{n}(\tau-\sigma)\log K_1(\tau)d\tau + C\right\}, \quad \sigma \in \Omega,$$

$$X(\sigma-n) = \exp\left\{-\frac{1}{2}\log K_1(\sigma) + \frac{1}{2ni}\int_\Omega \cot\frac{\pi}{n}(\tau-\sigma)\log K_1(\tau)d\tau + C\right\}, \quad \sigma \in \Omega,$$

$$X(s) = \exp\left\{\frac{1}{2ni}\int_\Omega \cot\frac{\pi}{n}(\tau-s)\log K_1(\tau)d\tau + C\right\}, \quad s \in \Pi, \tag{A.29}$$

where C is an arbitrary constant.

Substituting formulae (A.26), (A.27) and (A.29) into the boundary condition (A.18) and canceling the C yield a problem with a given jump for a strip

$$\Phi_0(\sigma) + \Phi_0(\sigma - 1) = G_0(\sigma), \quad \sigma \in \Omega \tag{A.30}$$

with respect to the new function

$$\Phi_0(s) = \frac{\Phi(s)X(s)}{\gamma^s \Gamma(s+2)\sin\frac{\pi s}{2}}. \tag{A.31}$$

The jump in (A.30) is defined by

$$G_0(s) = -\frac{G(s)X(s-1)}{\gamma^s \Gamma(s+1)\cos\frac{\pi s}{2}}. \tag{A.32}$$

We note that the function $\Phi_0(s)$ is analytic everywhere in the strip Π and the point $s = -2$ is a removable point. Let $G_0(s)$ be an $L_2(\Omega)$-function. Then the solution of the problem with a given jump

$$\Phi_0(\sigma) + \Phi_0(\sigma - n) = G_0(\sigma), \quad \sigma \in \Omega \tag{A.33}$$

is determined by the following integrals

$$\Phi_0(\sigma) = \frac{1}{2}G_0(\sigma) + \frac{1}{2ni}\int_\Omega \frac{G_0(\tau)d\tau}{\sin\frac{\pi}{n}(\tau - \sigma)}, \quad \sigma \in \Omega,$$

$$\Phi_0(\sigma - n) = \frac{1}{2}G_0(\sigma) - \frac{1}{2ni}\int_\Omega \frac{G_0(\tau)d\tau}{\sin\frac{\pi}{n}(\tau - \sigma)}, \quad \sigma \in \Omega,$$

$$\Phi_0(s) = \frac{1}{2ni}\int_\Omega \frac{G_0(\tau)d\tau}{\sin\frac{\pi}{n}(\tau - s)}, \quad s \in \Pi. \tag{A.34}$$

Now we put $n = 1$ in formulae (A.29), (A.34), take into account expression (A.31) and obtain the solution of the boundary value problem of the theory of analytic functions in the strip (A.18), (A.19)

$$\Phi(s) = \gamma^s \Gamma(s+2)\sin\frac{\pi s}{2}\frac{T(s)}{X(s)}, \quad s \in \Pi,$$

$$\Phi(\sigma) = \frac{G(\sigma)}{2\gamma K(\sigma)} + \gamma^\sigma \Gamma(\sigma+2)\sin\frac{\pi\sigma}{2}\frac{T(\sigma)}{X(\sigma)}, \quad \sigma \in \Omega,$$

$$\Phi(\sigma - 1) = \frac{G(\sigma)}{2\gamma} + \gamma^{\sigma-1}\Gamma(\sigma+1)\cos\frac{\pi\sigma}{2}\frac{T(\sigma)}{X(\sigma-1)}, \quad \sigma \in \Omega, \tag{A.35}$$

where

$$T(s) = -\frac{1}{2i}\int_\Omega \frac{X(\tau-1)G(\tau)d\tau}{\gamma^\tau \Gamma(\tau+1)\cos\frac{\pi\tau}{2}\sin\pi(\tau - s)}. \tag{A.36}$$

All the functions in (A.35) and (A.36) are defined in closed form. If we apply the inverse Mellin and Fourier transforms we can find the expression for the solution of the original boundary value problem for the Laplace equation in the cone (A.1) with the boundary condition of the third kind (A.2)

$$u(r,\phi,\theta) = \sum_{n=-\infty}^{\infty} e^{-in\phi}\frac{1}{2\pi i}\int_\Omega \frac{P_s^m(\cos\theta)}{\frac{d}{d\theta}P_s^m(\cos\alpha)}\Phi(s)r^{-s-1}ds. \tag{A.37}$$

We note that the above method of exact solution of problem (2.1), (2.2) for the case $k = 0$ can be used for construction of an asymptotic expansion for the solution of the corresponding boundary value problem for the Helmholtz equation for small k.

Appendix B. A 2×2 matrix Riemann-Hilbert problem with a discontinuous coefficient

To reduce the second-order difference equation (3.17) in the strip Π to a Riemann-Hilbert problem, we develop Čerskiĭ's idea ([10], see also [1]) proposed for Carleman's problem with a shift. Let us map the strip $\Pi = \{-2 < \Re(s) < 0\}$ on a complex plane with the cut along the positive semi-axis

$$\Gamma = \{\zeta \in \mathbb{C} : \ \zeta = \xi, \quad \arg\zeta = 0, \quad 0 < \xi < \infty\}$$

using the "gluing" function $\zeta = e^{-\pi i s}$. The upper side ($\zeta = \xi + i0$, $\arg\zeta = 0$) of the positive semi-axis of the ζ-plane corresponds to the contour Ω of the s-plane. The contour $\{s \in \mathbb{C} : \ \Re(s) = -1\}$ is mapped on to the negative semi-axis ($-\infty < \xi < 0$) of the ζ-plane, and the contour $\{\Re(s) = -2\}$ of the s-plane is mapped on to the lower side ($\zeta = \xi - i0$, $\arg\zeta = 2\pi$) of the contour Γ. The points $\sigma = -i\infty$, $\sigma = +i\infty$ are mapped on to the points $\xi = 0$, $\xi = \infty$, respectively. Therefore, when the point s moves along Ω from the point $-i\infty$ to $+i\infty$, the corresponding point ζ moves along the contour Γ in the positive direction from 0 to ∞. We introduce the function

$$\omega_1(\zeta) = \frac{1}{\zeta^{1/2}}\Phi\left(\frac{i}{\pi}\log\zeta\right), \qquad (B.1)$$

where $s = \frac{i}{\pi}\log\zeta$. The functions $\log\zeta$ and $\zeta^{1/2}$ are defined and analytic in the ζ-plane with the cut Γ. Moreover, on the upper side of the contour Γ, the logarithmic function is real and the $\zeta^{1/2}$ is positive. Thus, the limit values of the function $\omega_1(\zeta)$ are

$$\omega_1^+(\xi) = \omega_1(\xi + i0) = \frac{1}{\sqrt{\xi}}\Phi\left(\frac{i}{\pi}\log\xi\right) = e^{\pi i\sigma/2}\Phi(\sigma),$$

$$\omega_1^-(\xi) = \omega_1(\xi - i0) = -\frac{1}{\sqrt{\xi}}\Phi\left(-2 + \frac{i}{\pi}\log\xi\right) = -e^{\pi i\sigma/2}\Phi(\sigma - 2),$$

$$\omega_1(-\xi) = \frac{1}{i\sqrt{\xi}}\Phi\left(-1 + \frac{i}{\pi}\log\xi\right) = -ie^{\pi i\sigma/2}\Phi(\sigma - 1), \quad 0 < \xi < \infty. \qquad (B.2)$$

Due to equation (3.17), the new function $\omega_1(\xi)$ satisfies the following boundary condition

$$\omega_1^+(\xi) + iK_0(\xi)\omega_1(-\xi) - \omega_1^-(\xi) = h_0(\xi), \quad 0 < \xi < \infty, \qquad (B.3)$$

where

$$K_0(\xi) = K\left(\frac{i}{\pi}\log\xi\right), \quad h_0(\xi) = \frac{1}{\sqrt{\xi}}h\left(\frac{i}{\pi}\log\xi\right). \qquad (B.4)$$

Now putting $\omega_2(\zeta) = \omega_1(-\zeta)$, we get the function $\omega_2(\zeta)$ that is analytic in the $\zeta-$plane with the cut along the negative semi-axis ($\xi < 0$). Then for negative ξ we have the following relations

$$\omega_2^+(\xi) = \omega_2(\xi + i0) = \omega_1(-\xi - i0) = \omega_1^-(-\xi),$$

$$\omega_2^-(\xi) = \omega_2(\xi - i0) = \omega_1(-\xi + i0) = \omega_1^+(-\xi). \qquad (B.5)$$

In virtue of (B.3) and (B.5), the boundary condition for the function $\omega_2(\zeta)$ is

$$\omega_2^-(\xi) + iK_0(-\xi)\omega_1(\xi) - \omega_2^+(\xi) = h_0(-\xi), \quad -\infty < \xi < 0. \qquad (B.6)$$

We note that the functions $\omega_1(\zeta), \omega_2(\zeta)$ satisfy the two additional "natural" boundary conditions

$$\omega_1^+(\xi) = \omega_1^-(\xi), \quad -\infty < \xi < 0,$$

$$\omega_2^+(\xi) = \omega_2^-(\xi), \quad 0 < \xi < \infty. \qquad (B.7)$$

Inequality (3.19), formulated for the limit values of the functions $\omega_1(\zeta), \omega_2(\zeta)$, yields

$$\int\limits_{-\infty}^{\infty} |\omega_j^{\pm}(\xi)|^2 d\xi < \infty, \quad j = 1, 2. \qquad (B.8)$$

i.e. $\omega_j^{\pm}(\xi) \in L_2(-\infty,\infty)$. Due to the Čerskiĭ theorem [10], the function $\Phi(s)$ is analytically continuable into the strip Π and satisfies condition (3.19) if and only if the functions $\omega_j(\zeta)$ can be represented as Cauchy integrals

$$\omega_j(\zeta) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\Sigma_j(\xi)}{\xi-\zeta}d\xi, \qquad (B.9)$$

with $\Sigma_j(\xi) \in L_2(-\infty,\infty)$.

Now let us combine the boundary conditions (B.3), (B.6) and (B.7) and introduce the vector

$$\boldsymbol{\omega}(\zeta) = \begin{pmatrix} \omega_1(\zeta) \\ \omega_2(\zeta) \end{pmatrix}. \qquad (B.10)$$

We have reduced the difference equation of the second order in the strip Π (3.17) to the 2×2 matrix Riemann-Hilbert problem:

It is required to determine the vector-function $\boldsymbol{\omega}(\zeta)$ analytic in the entire ζ-plane except for the real axis, representable by a Cauchy integral with the $L_2(\mathbb{R}_1)$-density; its boundary values $\boldsymbol{\omega}^+(\xi)$, $\boldsymbol{\omega}^-(\xi)$ are L_2-integrable on the real axis $\mathbb{R}_1$ and satisfy the boundary condition

$$\boldsymbol{\omega}^+(\xi) = \mathbf{G}(\xi)\boldsymbol{\omega}^-(\xi) + \mathbf{f}(\xi), \quad \xi \in \mathbb{R}_1 = \{-\infty,\infty\}, \qquad (B.11)$$

where

$$\mathbf{G}(\xi) = \begin{pmatrix} 1 & (\operatorname{sgn}\xi+1)a(\xi) \\ (\operatorname{sgn}\xi-1)a(\xi) & 1 \end{pmatrix}, \quad a(\xi) = -\frac{i}{2}K_0(|\xi|), \qquad (B.12)$$

$$\mathbf{f}(\xi) = \frac{1}{2}h_0(|\xi|)\begin{pmatrix} \operatorname{sgn}\xi+1 \\ \operatorname{sgn}\xi-1 \end{pmatrix}. \qquad (B.13)$$

We note that the matrix $\mathbf{G}(\xi)$ is discontinuous at the point $\xi = 0$:

$$\mathbf{G}(+0) = \begin{pmatrix} 1 & -\gamma_0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G}(-0) = \begin{pmatrix} 1 & 0 \\ \gamma_0 & 1 \end{pmatrix}. \qquad (B.14)$$

Using Gakhov's algorithm [11] one can remove the discontinuity at the point $\xi = 0$ and reduce the matrix problem (B.11) to a 2×2 matrix Riemann-Hilbert problem with a continuous coefficient.

Appendix C. An alternative derivation of the integral equation

We note that the functional equation (B.3) with condition (B.8) and therefore the matrix problem (B.11) are equivalent to an integral equation. Indeed, from (B.3) by the Sokhotski-Plemelj formulae we may write

$$\omega_1(\zeta) = \frac{1}{2\pi i}\int\limits_0^{\infty} \frac{h_0(y) - iK_0(y)\omega_1(-y)}{y-\zeta}dy, \quad \zeta \in \mathbb{C}\setminus[0,\infty). \qquad (C.1)$$

Putting in the last formula $\zeta = -x$ $(0 < x < \infty)$ we get the integral equation with respect to the function $\omega_1(-x)$

$$\omega_1(-x) + \frac{1}{2\pi}\int\limits_0^{\infty} \frac{K_0(y)\omega_1(-y) + ih_0(y)}{y+x}dy, \quad 0 < x < \infty, \qquad (C.2)$$

which is reducible to both equations (4.6) and (4.13). To show that, we put in (C.2) $y = e^{-i\pi\eta}$, $x = e^{-i\pi\xi}$, take into account the following relations (see (B.2), (B.4) and (B.5))

$$\omega_1(-e^{-i\pi\eta}) = -ie^{i\pi\eta/2}\Phi(\eta-1), \quad K_0(e^{-i\pi\eta}) = K(\eta),$$

$$h_0(e^{-i\pi\eta}) = e^{i\pi\eta/2}h(\eta), \quad \frac{1}{y+x} = \frac{e^{i\pi(\eta+\xi)/2}}{2\cos\frac{\pi}{2}(\eta-\xi)},$$

and obtain the integral equation (4.6) that is equivalent to (4.13).

References

[1] Y.A.Antipov and H.Gao, *Exact solution of integro-differential equations of diffusion along a grain boundary*, submitted to Q. Jl Mech. appl. Math., May 1999 (preprint 99/15, University of Bath).

[2] V.Babich, D.Dement'ev, B.Samokish, *On the diffraction of high-frequency waves by a cone of arbitrary shape*, Wave Motion, 21 (1995), pp.203-207.

[3] V.M.Babich, D.B.Dement'ev, B.A.Samokish, V.P.Smyshlyaev, *On evaluation of the diffraction coefficients for arbitrary "non-singular" directions of a smooth convex cone*, to appear in SIAM J. Appl. Math., 1999.

[4] R.D.Bantsuri, *Contact problem for a wedge with an elastic support*, Sov. Phys. Dokl., 18, No 8 (1974), pp. 561-562.

[5] P.G.Barton and A.D.Rawlins, *Acoustic diffraction by a semi-infinite plane with different face impedances*, Q. Jl Mech. appl. Math., 52 (1999), pp. 469-487.

[6] J.M.L.Bernard, *Méthode analytique et transformées functionnelles pour la diffraction d'ondes par une singularité conique*, Rapport CEA-R-5764 Editions Dist/Saslay (1997).

[7] J.M.L.Bernard and M.A.Lyalinov, *The leading asymptotic term for the scattering diagram in the problem of diffraction by a narrow circular impedance cone*, J. Phys. A: Math. Gen., 32 (1999), L43-L48.

[8] J.J. Bowman, T.B.A. Senior and P.L.E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, New-York (1987).

[9] T. Carleman, *Sur la théorie des équations intégrales et ses applications*, Verhandl. des Internat. Mathem. Kongr., Zurich (1932), pp. 138-151.

[10] Ju.I. Čerskiĭ, *A normally solvable equation of smooth transition*, Soviet Math. Dokl., 11, No 1 (1970), pp. 55-59.

[11] F.D. Gakhov, *Riemann boundary value problem for a system of n pairs of functions*, Russian Mathematical Surveys (1952) 7, No 4 (50), pp. 3-54.

[12] F.D. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford (1963).

[13] F.D. Gakhov and Ju.I. Čerskiĭ, *Equations of the Convolution Type*, Nauka, Moscow (1978).

[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, Orlando, et al (1980).

[15] D.S. Jones, *Scattering by a cone* , Q. Jl Mech. appl. Math., 50 (1997), pp. 499-523.

[16] L.V. Kantorovich and G.P.Akilov, *Functional Analysis*, Pergamon Press, Oxford (1982).

[17] J.B. Keller, *The geometrical theory of diffraction*, J. Opt. Soc. Amer., 52 (1962), pp. 116-130.

[18] M.I. Kontorovich and N.N. Lebedev, *On a method of solution of some problems of diffraction theory and relative problems*, J. Experim. and Theor. Physics., 8 (10-11) (1938), pp. 1192-1206.

[19] L. Kraus and L.M. Levine, *Diffraction by an elliptic cone*, Comm. Pure Appl. Math. 14(1) (1961) pp. 49-68.

[20] G.D. Maliuzhinets, *Excitation, reflection and emission of surface waves from a wedge with given wave impedances*, Sov. Phys.Dokl., 3 (1958) pp. 752-755.

[21] V.P. Smyshlyaev, *The high-frequency diffraction of electromagnetic waves by cones of arbitrary cross-sections*, SIAM J. Appl. Math., 53 (1993) pp. 670-688.

[22] L.Ia. Tikhonenko, *A plane mixed problem of thermal conductivity for a wedge*, Differencial'nye Uravnenija, 9 (1973), 1915-1918.

[23] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press (1948).