Low-Gain Integral Control of Continuous-Time Linear Systems Subject to Input and Output Nonlinearities *

T. Fliegner, H. Logemann & E.P. Ryan

Department of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, United Kingdom,
Email: T.Fliegner@maths.bath.ac.uk, hl@maths.bath.ac.uk,
epr@maths.bath.ac.uk

Abstract: Continuous-time low-gain integral control strategies are presented for tracking of constant reference signals for finite-dimensional, continuous-time, asymptotically stable, single-input single-output, linear systems subject to a globally Lipschitz and non-decreasing input nonlinearity and a locally Lipschitz, non-decreasing and affinely sector-bounded output nonlinearity (the conditions on the output nonlinearities may be relaxed if the input nonlinearity is bounded). Both non-adaptive (but possibly time-varying) and adaptive integrator gains are considered. In particular, it is shown that applying error feedback using an integral controller ensures asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is ultimately sufficiently small and (c) the reference value is feasible in a very natural sense. The classes of actuator and sensor nonlinearities under consideration contain standard nonlinearities important in control engineering such as saturation and deadzone.

Keywords: Adaptive control; continuous-time systems; input/output nonlinearities; integral control; robust control; tracking.

AMS subject classifications: 93B52, 93C10, 93C60, 93D09, 93D10, 93D21.

1 Introduction

The synthesis of low-gain integral and proportional-plus-integral controllers for (uncertain) stable plants has received considerable attention in the last 20 years. The following principle is well known (see Davison [5], Lunze [13] and Morari [16]): closing the loop around an exponentially stable, finite-dimensional, continuous-time, single-input, single-output plant Σ, with transfer function G, compensated by a pure integral controller (see Fig. 1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $kG(0) > 0$.

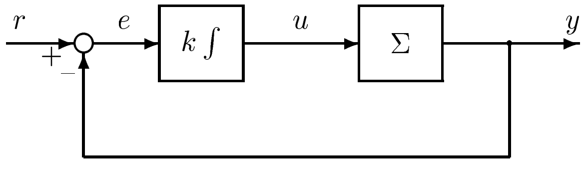

Fig. 1: Low-gain control system

* This work was supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/L78086).

Therefore, if a plant is known to be exponentially stable and if the sign of $G(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design approach ("tuning regulator theory" [5]) has been successfully applied in process control, see, for example, Coppus *et al.* [4] and Lunze [12].

An analogous result holds for finite-dimensional multivariable systems, under suitable assumptions on $G(0)$, see [5], [13] and [16]. Furthermore, the problem of tuning the integrator gain adaptively has been addressed in a number of papers, see Cook [3] and Miller and Davison [14, 15] (with input constraints treated in [14]). Recently, Logemann *et al.* have developed tuning regulator results for infinite-dimensional systems with input nonlinearities, [7, 8, 9].

In this paper (which extends the investigations in [7] to encompass output nonlinearities in a finite-dimensional context) we present results which show that the above principle remains true if the plant to be controlled is a stable, finite-dimensional single-input, single-output, linear system subject to an input and/or output nonlinearity (see Fig. 2). Precisely, we prove that, if $G(0) > 0$ and if the constant reference signal r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all non-decreasing, globally Lipschitz input nonlinearities φ and all non-decreasing, locally Lipschitz and affinely sector-bounded output nonlinearities ψ (the sector-bound assumption on ψ can be removed if φ is bounded) the following holds: for all positive, bounded and continuous integrator gains $k(\cdot)$ (thus in particular for positive constant gains), the output $y(t)$ of the closed-loop system converges to r as $t \to \infty$ provided that $\limsup_{t\to\infty} k(t) < k^*$ and k is not of class L^1 (under some additional assumptions on the nonlinearities, results concerning the rate of convergence are derived).

Moreover, one consequence of the above principle is that the following simple adaptation law (introduced in [7])

$$k(t) = 1/l(t), \qquad \dot{l}(t) = |r - y(t)|, \quad l(0) = l^0 > 0\,,$$

produces an integrator gain k so that the output $y(t)$ of the closed-loop system converges to r as $t \to \infty$.

The number k^* is closely related to the supremum κ^* of the set of all numbers $\kappa > 0$ such that the function

$$s \mapsto 1 + \kappa\frac{G(s)}{s}$$

is positive-real: for example, if both φ and ψ are globally Lipschitz with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then $k^* = \kappa^*/(\lambda_1\lambda_2)$. Formulae for the computation of κ^* have appeared in [10] for various classes of systems.

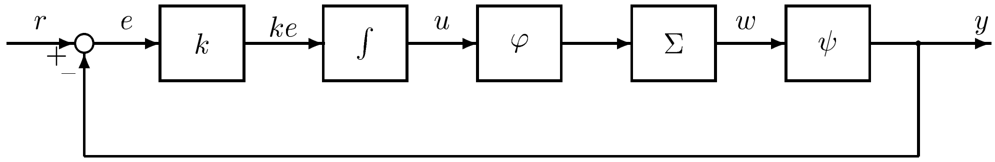

Fig. 2: Low-gain control with input and output nonlinearities

The essence of our approach is to invoke a particular coordinate transformation and perform a Lyapunov-type analysis on the transformed system in which a matrix inequality, connected to a variant of the Kalman-Yakubovich-Popov Lemma, plays a central rôle.

2 Problem formulation

The problem of tracking constant reference signals $r \in \mathbb{R}$ will be addressed in the context of a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$), continuous-time (time domain $\mathbb{R}_+ := [0, \infty)$), real linear systems $\Sigma = (A, B, C, D)$ having a nonlinearity in the input and output channel:

$$\dot{x} = Ax + B\varphi(u)\,, \quad x(0) = x^0 \in \mathbb{R}^N\,, \tag{1a}$$

$$w = Cx + D\varphi(u)\,, \tag{1b}$$

$$y = \psi(w) = \psi(Cx + D\varphi(u))\,. \tag{1c}$$

2.1 The class $\mathcal{S}$ of linear systems

In (1a), A is assumed to be Hurwitz, *i.e.*, each eigenvalue of A has negative real part. Furthermore, the transfer function G, given by

$$G(s) = C(sI - A)^{-1}B + D\,,$$

is assumed to satisfy $G(0) > 0$. Thus, the underlying class of real linear systems $\Sigma = (A, B, C, D)$ is

$$\mathcal{S} := \{\Sigma = (A, B, C, D) \mid A \text{ Hurwitz},\ G(0) = D - CA^{-1}B > 0\}.$$

We denote the L^∞-gain of $\Sigma = (A, B, C, D) \in \mathcal{S}$ by Γ_Σ, and so

$$0 < G(0) \leq \Gamma_\Sigma = \int_0^\infty |C \exp(At)B| dt + |D| < \infty\,. \tag{2}$$

2.1.1 The positive-real condition

If G is the transfer function of a system $\Sigma = (A, B, C, D) \in \mathcal{S}$, then it is readily shown that

$$1 + \kappa \operatorname{Re} \frac{G(s)}{s} \geq 0 \qquad \forall s \in \mathbb{C} \quad \text{with } \operatorname{Re} s > 0 \tag{3}$$

for all sufficiently small $\kappa > 0$, see [11, Lemma 3.10]. We refer to (3) as the positive-real condition. Define

$$\kappa^* := \sup\{\kappa > 0 \,|\, (3) \text{ holds}\}\,. \tag{4}$$

Lower bounds and formulae for κ^* may be found in [10]. The following lemma will be invoked in a later stability analysis.

Lemma 2.1 *Assume that $\Sigma = (A, B, C, D) \in \mathcal{S}$ and let $\Delta > 1/\kappa^*$. Then there exists $P \in \mathbb{R}^{N \times N}$ such that $P = P^T \geq 0$ and*

$$\begin{bmatrix} PA + A^T P & PA^{-1}B - C^T \\ (A^{-1}B)^T P - C & -2\Delta \end{bmatrix} < 0\,.$$

Proof. Define

$$H(s) := C(sI - A)^{-1}A^{-1}B = \begin{cases} \dfrac{G(s) - G(0)}{s} \;, & s \neq 0 \\ G'(0) \;, & s = 0 \end{cases}$$

Let $\tilde{\Delta} \in (1/\kappa^*, \Delta)$. Clearly, H is holomorphic in $\{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$ for some $\alpha < 0$ and the positive-real condition (3) holds for $\kappa = 1/\tilde{\Delta}$. Therefore, $\tilde{\Delta} + \operatorname{Re} H(i\omega) \geq 0$ for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$ and so

$$\Delta + \operatorname{Re} H(i\omega) > 0 \qquad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}\,.$$

Writing

$$M := \begin{bmatrix} 0 & -C^T \\ -C & -2\Delta \end{bmatrix},$$

we may conclude

$$\begin{bmatrix} (i\omega I - A)^{-1}A^{-1}B \\ 1 \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}A^{-1}B \\ 1 \end{bmatrix} = -2(\operatorname{Re} H(i\omega) + \Delta) < 0 \quad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}\,,$$

and the assertion of the lemma immediately follows from a variant of the Kalman-Yakubovich-Popov Lemma as given by Rantzer ([17, Theorem 1]). ∎

2.2 The class $\mathcal{N}$ of input/output nonlinearities

The following sets of monotone, non-decreasing nonlinearities are first introduced:

$$\mathcal{M} := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ locally Lipschitz and non-decreasing}\}\,,$$

$$\mathcal{M}(\lambda) := \{f \in \mathcal{M} \mid 0 \leq (f(\xi) - f(0))\xi \leq \lambda\xi^2 \quad \forall \xi \in \mathbb{R}\}\,,$$

$$\mathcal{M}_L(\lambda) := \{f \in \mathcal{M} \mid f \text{ is globally Lipschitz with Lipschitz constant } \lambda\}\,.$$

Clearly, $\mathcal{M}_L(\lambda) \subset \mathcal{M}(\lambda) \subset \mathcal{M}$.

Remarks 2.2
(i) Let $f \in \mathcal{M}(\lambda)$. Then $f(\xi) = f(0) + \gamma(\xi)\xi$ for all ξ, where

$$\gamma : \xi \mapsto \begin{cases} (f(\xi) - f(0))/\xi \;, & \xi \neq 0 \\ \lambda \;, & \xi = 0 \end{cases}$$

is upper semicontinuous and $\gamma(\xi) \in [0, \lambda]$ for all $\xi \in \mathbb{R}$.

(ii) If $f \in \mathcal{M}(\lambda)$, then for each $\nu \in \mathbb{R}$, there exists $\tilde{\lambda} \geq 0$ such that the function $\xi \mapsto f(\xi+\nu) - f(\nu)$ is of class $\mathcal{M}(\tilde{\lambda})$.

(iii) If $f \in \mathcal{M}_L(\lambda)$, then for each $\nu \in \mathbb{R}$, $\xi \mapsto f(\xi + \nu) - f(\nu)$ is also of class $\mathcal{M}_L(\lambda)$.

As an example of a function f which belongs to $\mathcal{M}_L(\lambda)$ (and hence to $\mathcal{M}(\lambda)$ and $\mathcal{M}$), consider the nonlinearity in Figure 3.

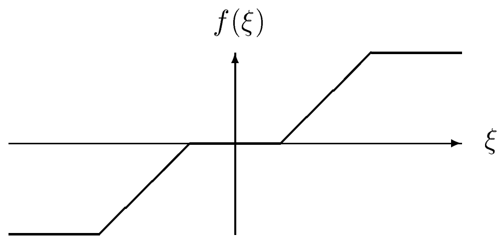

Fig. 3: Nonlinearity with saturation and deadzone

2.2.1 Critical points and critical values

Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. For $\xi, \nu \in \mathbb{R}$, the Clarke [2] directional derivative $f^\circ(\xi;\nu)$ of f at ξ in direction ν is given by

$$f^\circ(\xi;\nu) := \limsup_{\substack{\theta\to\xi \\ h\downarrow 0}} \frac{f(\theta+h\nu)-f(\theta)}{h}.$$

Define $f^-(\cdot) := -f^\circ(\cdot;-1)$ (if f is C^1 with derivative f', then $f^- \equiv f'$). By upper semicontinuity of f°, f^- is lower semicontinuous. Let $f \in \mathcal{M}$: a point $\xi \in \mathbb{R}$ is said to be a *critical point* (and $f(\xi)$ is said to be a *critical value*) of f if $f^-(\xi) = 0$.[†] We denote, by $\mathcal{C}(f)$, the set of critical values of f.

2.2.2 Two technical lemmas

First, we record a lemma given originally in [9]. To render the present paper self-contained, a proof is included in the Appendix.

Lemma 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and (ε_n) be any sequence with $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Define the function $f^\diamond : \mathbb{R} \to \mathbb{R}$ by*

$$f^\diamond(\xi) = \limsup_{n\to\infty} \frac{f(\xi+\varepsilon_n)-f(\xi)}{\varepsilon_n}.$$

Then $f^\diamond$ is Borel measurable and $f^\diamond \in L^\infty_{loc}(\mathbb{R},\mathbb{R})$ ($f^\diamond \in L^\infty(\mathbb{R},\mathbb{R})$ if f is globally Lipschitz). If $g : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, so is $f \circ g$ and

$$\frac{d}{dt}(f\circ g)(t) = f^\diamond(g(t))\dot g(t) \qquad \text{for a.a. } t \in \mathbb{R}_+ .$$

The next lemma collects properties of f^- which will be used later. The proof is relegated to the Appendix.

Lemma 2.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Then*

$$f^-(\xi) \le \liminf_{h\to 0} \frac{f(\xi+h)-f(\xi)}{h} \le f^\diamond(\xi) \qquad \forall \xi \in \mathbb{R}. \tag{5}$$

If $f \in \mathcal{M}_L(\lambda)$ for some $\lambda \ge 0$, then

$$0 \le f^-(\xi) \le f^\diamond(\xi) \le \lambda \qquad \forall \xi \in \mathbb{R}. \tag{6}$$

[†] If f is merely locally Lipschitz, but not in $\mathcal{M}$, then it would be natural to deem $\xi \in \mathbb{R}$ a critical point of f if 0 belongs to the (Clarke [2]) generalized gradient $\partial f(\xi)$ of f at ξ. We remark that, in the case of $f \in \mathcal{M}$, $0 \le \min\{\partial f(\xi)\} = f^-(\xi)$. Therefore, for functions $f \in \mathcal{M}$, the latter concept of a critical point coincides with that given above.

2.2.3 Input/output nonlinearities

We are now in a position to make precise the class $\mathcal{N}$ of input/output nonlinearities: a pair (φ, ψ) is in $\mathcal{N}$ if $\varphi \in \mathcal{M}_L(\lambda_1)$ for some $\lambda_1 > 0$, $\psi \in \mathcal{M}$ and at least one of the following holds:

$$\text{(i) } \varphi \text{ is bounded, or (ii) } \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\,.$$

Equivalently,

$$\mathcal{N} := \{(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M} \mid \lambda_1 > 0,\ \varphi \text{ unbounded } \Rightarrow\ \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\}\,.$$

2.3 The tracking objective and feasibility

Given $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $(\varphi, \psi) \in \mathcal{N}$, the tracking objective is to determine, by feedback, an input u such that, for given $r \in \mathbb{R}$, the output y of (1) has the property $y(t) \to r$ as $t \to \infty$. Clearly, if this objective is achievable, then r is necessarily in the closure of $\operatorname{im} \psi$. We will impose a stronger condition, namely,

$$\Psi^r \cap \overline{\Phi} \neq \emptyset\,, \quad \text{where } \Psi^r := \{v \in \mathbb{R} \mid \psi(G(0)v) = r\}, \quad \Phi := \operatorname{im} \varphi, \quad \overline{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \overline{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values.* The next proposition shows that $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as, if tracking of r is achievable whilst maintaining continuity and boundedness of $\varphi \circ u$, then $r \in \mathcal{R}$.

Proposition 2.5 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ be continuous and let $u : \mathbb{R}_+ \to \mathbb{R}$ be such that $\varphi \circ u$ is continuous and bounded. For $x^0 \in \mathbb{R}^N$, let $x : \mathbb{R}_+ \to \mathbb{R}^N$ denote the bounded solution of the initial-value problem* (1a)*. If*

$$\lim_{t \to \infty} [\psi(Cx(t) + D\varphi(u(t)))] = r\,, \tag{7}$$

then $r \in \mathcal{R}$.

Proof. The function $w = Cx + D\varphi(u)$ is continuous and bounded and so has non-empty, compact, connected ω-limit set Ω. We claim that $\Omega \cap G(0)\overline{\Phi} \neq \emptyset$. Seeking a contradiction, suppose $\Omega \cap G(0)\overline{\Phi} = \emptyset$. Then Ω and $G(0)\overline{\Phi}$ are closed disjoint intervals and so there exist $\varepsilon > 0$ and $\beta = \pm 1$ such that

$$\inf\{\beta\omega \mid \omega \in \Omega\} - \sup\{\beta G(0)v \mid v \in \overline{\Phi}\} = 2\varepsilon\,.$$

Moreover, since $w(t)$ approaches its compact ω-limit set Ω as $t \to \infty$, there exists $T > 0$ such that

$$\inf \Omega - \varepsilon < w(t) < \sup \Omega + \varepsilon \quad \forall t > T\,,$$

and so $\beta w(t) > \inf\{\beta\omega \mid \omega \in \Omega\} - \varepsilon$ for all $t > T$. Hence,

$$\begin{aligned} \beta CA^{-1}\dot{x}(t) &= \beta w(t) - \beta G(0)\varphi(u(t)) \geq \beta w(t) - \sup\{\beta G(0)v \mid v \in \overline{\Phi}\} \\ &> \inf\{\beta\omega \mid \omega \in \Omega\} - \varepsilon - \sup\{\beta G(0)v \mid v \in \overline{\Phi}\} = \varepsilon \qquad \forall t > T\,, \end{aligned}$$

which contradicts the boundedness of x. Therefore, $\Omega \cap G(0)\overline{\Phi} \neq \emptyset$.

Let $w^* \in \Omega \cap G(0)\overline{\Phi}$. Then there exist $v \in \overline{\Phi}$ and $(t_n) \subset \mathbb{R}_+$ such that $t_n \to \infty$ and $w(t_n) \to w^* = G(0)v$ as $n \to \infty$. Therefore, by (7) and continuity of ψ,

$$\psi(w(t_n)) \to \psi(G(0)v) = r \qquad \text{as } n \to \infty$$

and so $v \in \Psi^r \cap \overline{\Phi}$. ∎

3 Integral control

Let $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $(\varphi, \psi) \in \mathcal{N}$. To achieve the objective of tracking feasible reference values $r \in \mathcal{R}$, we will investigate integral control action

$$u(t) = u^0 + \int_0^t k(\tau)[r - \psi(Cx(\tau) + D\varphi(u(\tau))]\, d\tau = u^0 + \int_0^t k(\tau)[r - y(\tau)]\, d\tau\,, \tag{8}$$

with control gain k (possibly constant) which is either prescribed or determined adaptively. In both cases (non-adaptive and adaptive), an application of the integrator (8) will lead to a system of nonlinear non-autonomous differential equations of the general form:

$$\dot{x}(t) = Ax(t) + B\varphi(u(t))\,, \quad x(0) = x^0 \in \mathbb{R}^N\,, \tag{9a}$$

$$\dot{u}(t) = \sigma(t)\theta(t)[r - \psi(Cx(t) + D\varphi(u(t)))]\,, \quad u(0) = u^0 \in \mathbb{R}\,, \tag{9b}$$

$$\dot{\theta}(t) = h(\theta(t))\chi(|r - \psi(Cx(t) + D\varphi(u(t)))|)\,, \quad \theta(0) = \theta^0 \in (0, \infty)\,. \tag{9c}$$

The following existence result subsumes both cases.

Lemma 3.1 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{N}$ and $r \in \mathbb{R}$. Let $\sigma : \mathbb{R}_+ \to \mathbb{R}$ be bounded and continuous, let $h : \mathbb{R} \to (-\infty, 0]$ be locally Lipschitz with $h(0) = 0$, and let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be locally Lipschitz.*
For each $(x^0, u^0, \theta^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the initial-value problem (9) has a unique solution $(x, u, \theta) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$.

Proof. Let $(x^0, u^0, \theta^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$. By standard existence and uniqueness results for ordinary differential equations, there exists a unique maximal solution $(x, u, \theta) : [0, t_{\max}) \to \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$. To conclude that $t_{\max} = \infty$, we show that the solution does not have a finite escape time. First observe that, by properties of h and χ and positivity of θ^0, the solution component θ satisfies $\theta(t) \in (0, \theta^0]$ for all $t \in [0, t_{\max})$ and so the product $\sigma\theta$ is bounded. These observations, together with the properties of φ, ψ and the Hurwitz property of A, imply the existence of some constant $c > 0$, such that

$$\|(x(t), u(t))\| \leq \|(x^0, u^0)\| + c \int_0^t [1 + \|(x(s), u(s))\|] ds \qquad \forall t \in [0, t_{\max})\,,$$

and so, by a standard application of Gronwall's Lemma, the solution components x and u are bounded on every bounded subinterval of $[0, t_{\max})$. Therefore, $t_{\max} = \infty$. ∎

3.1 Prescribed gain

Henceforth, we assume that the gain function $k : \mathbb{R}_+ \to (0, \infty)$ satisfies

$$k \in \mathcal{G} := \{g : \mathbb{R}_+ \to (0, \infty) \mid g \text{ is continuous and bounded}\}.$$

An application of the integrator (8) leads to the following system of nonlinear differential equations

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \quad x(0) = x^0 \in \mathbb{R}^N, \tag{10a}$$
$$\dot{u}(t) = k(t)[r - \psi(Cx(t) + D\varphi(u(t)))], \quad u(0) = u^0 \in \mathbb{R}. \tag{10b}$$

Setting $h \equiv 0$, $\theta^0 = 1$ and $\sigma = k$ in (9), an application of Lemma 3.1 yields the next proposition.

Proposition 3.2 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $(\varphi, \psi) \in \mathcal{N}$, $k \in \mathcal{G}$ and $r \in \mathbb{R}$. For each $(x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}$, the initial-value problem* (10) *has a unique solution $(x, u) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$.*

Before presenting the main results, we describe a convenient family of projection operators. Specifically, with each $p \in [0, \infty]$, we associate an operator $\Pi_p : \mathcal{M} \to \mathcal{M}$, with the property $\Pi_p \circ \Pi_p = \Pi_p$ (whence the terminology projection operator), defined as follows:

$$\text{if} \quad p < \infty, \text{ then} \quad \Pi_p f : \xi \mapsto \begin{cases} f(-p) \ , & \xi < -p \\ f(\xi) \ , & |\xi| \le p \\ f(p) \ , & \xi > p \end{cases} ; \qquad \text{if} \quad p = \infty, \text{ then} \quad \Pi_p f = \Pi_\infty f := f.$$

The next theorem forms the core of the paper and is formulated in such a manner that the tracking results (both non-adaptive and adaptive) follow readily as corollaries (Corollary 3.4 and Corollary 3.8 below).

Theorem 3.3 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{N}$, $r \in \mathcal{R}$ and $k \in \mathcal{G}$. Define*

$$p^* := \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \in (0, \infty].$$

For $(x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}$, let $(x, u) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ be the unique solution of the initial-value problem (10).

A. *Assume that $K(t) := \int_0^t k \to \infty$ as $t \to \infty$ and that $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Then*

   (a1) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

   (a2) $\lim_{t\to\infty} x(t) = -A^{-1}B\varphi^r$,

   (a3) $\lim_{t\to\infty} y(t) = r$, *where* $y(t) = \psi(Cx(t) + D\varphi(u(t)))$,

   (a4) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

   (a5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then $u(\cdot)$ is bounded.*

B. *There exists $k^* > 0$, independent of (x^0, u^0) and k, such that the following hold.*

   (b1) *If* $\limsup_{t\to\infty} k(t) < k^*$, *then* $\lim_{t\to\infty} \varphi(u(t))$ *exists and is finite.*

(b2) *If* $\limsup_{t\to\infty} k(t) < k^*$ *and* $K(t) := \int_0^t k \to \infty$ *as* $t \to \infty$, *then statements* (a1) *to* (a5) *hold. If, in addition,* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ *and* $r \notin \mathcal{C}(\psi)$, *then the convergence in* (a1) *to* (a3) *is of order* $\exp(-\rho K(t))$ *for some* $\rho > 0$.

C. *Let* $\lambda_1 > 0$ *be a Lipschitz constant for* φ. *If* $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ *for some* $\delta > 0$ *and some* $\lambda_2 > 0$, *then* (b1) *and* (b2) *are valid with* $k^* = \kappa^*/(\lambda_1\lambda_2)$, *where* κ^* *is given by* (4).

We are now in a position to state the main tracking result in the non-adaptive situation as an immediate corollary of parts B and C of Theorem 3.3.

Corollary 3.4 *Let* $\Sigma = (A, B, C, D) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{N}$ *and* $r \in \mathcal{R}$. *Define*

$$p^* := \Gamma_\Sigma \sup_{\xi\in\Phi} |\xi| \in (0, \infty]\,.$$

A. *There exists* $k^* > 0$ *such that the following statement holds.*

For each $k \in \mathcal{G}$ *with the properties* $\limsup_{t\to\infty} k(t) < k^*$ *and* $K(t) := \int_0^t k \to \infty$ *as* $t \to \infty$, *and for every* $(x^0, u^0) \in \mathbb{R}^N \times \mathbb{R}$, *assertions* (a1) *to* (a5) *of Theorem* 3.3 *hold for the unique solution of the initial-value problem* (10); *moreover, if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ *and* $r \notin \mathcal{C}(\psi)$, *then the convergence in* (a1) *to* (a3) *is of order* $\exp(-\rho K(t))$ *for some* $\rho > 0$.

B. *Let* $\lambda_1 > 0$ *be a Lipschitz constant for* φ. *If* $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ *for some* $\delta > 0$ *and some* $\lambda_2 > 0$, *then the statement in part* A *holds with* $k^* = \kappa^*/(\lambda_1\lambda_2)$, *where* κ^* *is given by* (4).

Remarks 3.5
(i) The essence of part A of Theorem 3.3 is an assertion, via (a1) to (a3), that the tracking objective is achieved by the integral control (8), provided that $\varphi \circ u$ converges and that the gain $k \in \mathcal{G}$ is *not* of class L^1; moreover, under additional hypotheses on the nonlinearities, (a4) and (a5) capture aspects of the qualitative behaviour of the function u (for example, (a4) implies that u converges if the set $\varphi^{-1}(\varphi^r)$ is a singleton, which, in turn, is true if φ^r is not a critical value of φ). Part A of Theorem 3.3 is of importance in the analysis of the adaptive control proposed in Section 3.2 below.
(ii) Part B of Theorem 3.3 asserts the existence of a positive constant $k^* > 0$ (independent of $k \in \mathcal{G}$ and the initial data (x^0, u^0)) such that, if $k \in \mathcal{G}$ is ultimately strictly bounded above by k^*, then $\varphi \circ u$ converges; if, in addition, k is *not* of class L^1, then statements (a1) to (a5) hold; furthermore, if the sets $\Psi^r \cap \overline{\Phi}$ and $\Psi^r \cap \Phi$ coincide, if Ψ^r contains no critical values of φ and if r is not a critical value of ψ, then (b2) provides an estimate of the rate of convergence in (a1) to (a3).
(iii) Under the assumption that $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta, \lambda_2 > 0$ (which, since $(\varphi, \psi) \in \mathcal{N}$, is equivalent to requiring that $\psi \in \mathcal{M}_L(\lambda_2)$ for some $\lambda_2 > 0$ if φ is unbounded), part C of Theorem 3.3 provides a formula for k^*. If $p^* < \infty$ (or, equivalently, if φ is bounded) and if ψ is continuously differentiable in $(p^* - \varepsilon, p^* + \varepsilon)$ and in $(-p^* - \varepsilon, -p^* + \varepsilon)$ for some $\varepsilon > 0$, then part C of Theorem 3.3 remains true with $\delta = 0$.
(iv) Corollary 3.4 contains a distillation of the results in Theorem 3.3 in the context of integral control with prescribed gain $k \in \mathcal{G}$. An immediate consequence of part A of Corollary 3.4 is the following: if $k \in \mathcal{G}$ is chosen such that, as $t \to \infty$, $k(t)$ tends to zero *sufficiently slowly* (in the sense that $k \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$), then the tracking objective is achieved. Under the hypotheses of part B of Corollary 3.4, we may infer that the tracking objective is achievable by *constant* gain $k \in (0, \kappa^*/(\lambda_1\lambda_2))$ and, moreover, under additional conditions on the nonlinearities, the convergence is of exponential order.

Before proceeding to the proof of Theorem 3.3, we illustrate some of the above results by means of a simple example.

Example 3.6 Consider the second-order example

$$\begin{aligned} \dot{x}_1 &= x_2\,, \quad \dot{x}_2 = -ax_2 - bx_1 + \varphi(u)\,, \\ w &= x_1\,, \quad y = \psi(w)\,, \end{aligned}$$

where $a, b > 0$, the output nonlinearity $\psi \in \mathcal{M}$ is the cubic $\psi : w \mapsto w^3$, and the input nonlinearity $\varphi \in \mathcal{M}_L(1)$ is of saturation type, defined as follows

$$u \mapsto \varphi(u) := \begin{cases} -1\,, & u < -1 \\ u\,, & u \in [-1, 1] \\ 1\,, & u > 1\,, \end{cases}$$

and so $\overline{\Phi} = [-1, 1]$. The transfer function G of the associated linear system Σ is given by

$$G(s) = \frac{1}{s^2 + as + b}\,, \qquad \text{with } G(0) = 1/b > 0\,.$$

For this system

$$p^* = \Gamma_\Sigma = \begin{cases} \frac{1}{b}\coth(a\pi/(2\sqrt{4b - a^2}))\,, & 4b > a^2 \\ \frac{1}{b}\,, & 4b \le a^2 \end{cases}$$

where the first expression on the righthand side is derived by direct computation using (2), while the second follows from the monotonicity of the step response of Σ in the case $4b \le a^2$ which implies that $\Gamma_\Sigma = G(0)$. Again by direct computation we obtain

$$\kappa^* = \begin{cases} a(4b - a^2)/4\,, & 2b > a^2 \\ b^2/a\,, & 2b \le a^2\,. \end{cases}$$

Since φ is bounded and ψ is continuously differentiable and noting that $\Pi_{p^*}(\psi) \in \mathcal{M}_L(\lambda)$ with $\lambda := \psi'(p^*) = 3(p^*)^2$, part C of Theorem 3.3 and Remark 3.5(iii) may be invoked to obtain $k^* = \kappa^*/(3(p^*)^2)$. Finally, $\Psi^r = \{b(r)^{1/3}\}$ and so $\Psi^r \cap \overline{\Phi} \neq \emptyset$ if and only if $-b^{-3} \le r \le b^{-3}$. Thus, the set $\mathcal{R}$ of feasible reference values is given by $\mathcal{R} = [-b^{-3}, b^{-3}]$.

Proof of Theorem 3.3. A. By hypothesis, there exists $\varphi^r \in \mathbb{R}$ such that $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$ which, together with the Hurwitz property of A, implies $\lim_{t\to\infty} w(t) = G(0)\varphi^r$. Evidently, $\varphi^r \in \overline{\Phi}$ and so, to establish (a1), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that

$$\theta := \frac{r - \psi(G(0)\varphi^r)}{2} \neq 0\,.$$

Using continuity of ψ, we obtain for sufficiently large $s > 0$

$$|y(t) - \psi(G(0)\varphi^r)| \le |\theta| \qquad \forall t \ge s\,.$$

As a consequence, and noticing that $\dot{u}(t) = k(t)[r - y(t)] = k(t)[2\theta - y(t) + \psi(G(0)\varphi^r)]$, we have

$$-|\theta|k(t) \le \dot{u}(t) - 2\theta k(t) \le |\theta|k(t) \qquad \forall t \ge s\,.$$

Since $\theta \neq 0$, either $\theta > 0$ or $\theta < 0$. Assume $\theta > 0$. Then

$$\dot{u}(t) \ge \theta k(t) \qquad \forall t \ge s\,,$$

which, on integration, yields

$$u(t) - u(s) \ge \theta(K(t) - K(s)) \qquad \forall t \ge s\,.$$

Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, hence

$$\sup \overline{\Phi} = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r\,. \tag{11}$$

Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Since $\theta > 0$, it follows $\varphi^* > \varphi^r$ and, consequently, by (11), $\sup \overline{\Phi} < \varphi^*$, a contradiction. A similar argument shows that the assumption $\theta < 0$ also leads to a contradiction. Therefore, we may conclude $\varphi^r \in \Psi^r \cap \overline{\Phi}$ which is statement (a1). Statement (a2) follows from (a1) and the Hurwitz property of A. Statement (a3) is a consequence of (a1), (a2) and continuity of ψ. Next, we establish statement (a4). Assume $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ which, together with (a1), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\mathrm{dist}(u(t), \varphi^{-1}(\varphi^r)) \not\to 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $(t_n) \in \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\mathrm{dist}\,(u(t_n), \varphi^{-1}(\varphi^r)) \ge \varepsilon\,. \tag{12}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (12). Therefore, we may assume that $(u(t_n))$ is unbounded. Extracting a subsequence if necessary, we may then assume that either $u(t_n) \to \infty$ or $u(t_n) \to -\infty$ as $n \to \infty$: if the former holds, then $u(t_n) > \xi^*$ for all n sufficiently large; if the latter holds, then $u(t_n) < \xi^*$ for all n sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u(t_n)) = \varphi(\xi^*) = \varphi^r$ for all n sufficiently large. Clearly, this contradicts (12) and so statement (a4) must hold.

Now, assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \mathrm{int}(\Phi)$ and, for contradiction, suppose that u is unbounded. Then there exists a sequence $(t_n) \subset (0,\infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. By monotonicity of φ and (a1), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \mathrm{int}(\Phi) \subset \mathrm{int}(\Phi)$. Therefore, u is bounded. This completes the proof of part A.

B. Let $\varepsilon > 0$. Then the Hurwitz property of A, together with elementary properties of the convolution product, implies that

$$|w(t)| = |Cx(t) + D\varphi(u(t))| \le |C\exp(At)x^0| + \Gamma_\Sigma \sup_{\xi\in\Phi} |\xi| \le p^* + \varepsilon =: q \in (0,\infty]$$

for all t sufficiently large. Hence, there exists $T_0 \ge 0$ such that

$$\psi(w(t)) = \Pi_q \psi(w(t)) \qquad \forall t \ge T_0\,. \tag{13}$$

Define $\hat{\Psi}^r := \{\xi \in \mathbb{R} \,|\, \Pi_q \psi(G(0)\xi) = r\}$. We claim that

$$\hat{\Psi}^r \cap \overline{\Phi} = \Psi^r \cap \overline{\Phi}\,. \tag{14}$$

To see this, note that, using (2), we have $G(0)|\xi| \leq p^*$ for all $\xi \in \overline{\Phi}$: therefore, $\Pi_q\psi(G(0)\xi) = \psi(G(0)\xi)$ for all $\xi \in \overline{\Phi}$ and so $\Psi^r \cap \overline{\Phi} = \hat{\Psi}^r \cap \overline{\Phi}$. This establishes (14).

Since $r \in \mathcal{R}$, we may conclude from (14) that $\hat{\Psi}^r \neq \emptyset$. Let $\varphi^* \in \hat{\Psi}^r$; since $\Pi_q\psi \in \mathcal{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_q\psi(\xi + G(0)\varphi^*) - r \tag{15}$$

is in $\mathcal{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ (cf. Remark 2.2(ii)). Therefore, by Remark 2.2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi \qquad \forall \xi \in \mathbb{R}, \tag{16}$$

where

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi & , \quad \xi \neq 0 \\ \tilde{\lambda}_2 & , \quad \xi = 0\,. \end{cases} \tag{17}$$

Recall that $0 \leq \gamma(\xi) \leq \tilde{\lambda}_2$ for all $\xi \in \mathbb{R}$. Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\lambda := \lambda_1\tilde{\lambda}_2\,, \qquad k^* := \kappa^*/\lambda\,,$$

where κ^* is given by (4).

Let $k \in \mathcal{G}$ be such that $\limsup_{t\to\infty} k(t) < k^*$. Choose $k_* > 0$ such that

$$\limsup_{t\to\infty} k(t) < k_* < k^*\,. \tag{18}$$

By Lemma 2.1 there exists a $P \in \mathbb{R}^{N\times N}$, with $P = P^T \geq 0$, such that

$$\Lambda := \begin{bmatrix} PA + A^TP & PA^{-1}B - C^T \\ (A^{-1}B)^TP - C & -2/(k_*\lambda) \end{bmatrix} < 0\,.$$

Introduce new variables

$$z(t) := x(t) + A^{-1}B\varphi(u(t))\,, \quad v(t) := \varphi(u(t)) - \varphi^* \qquad \forall t \geq 0\,,$$

and notice that, in terms of the new variables, w is given by

$$w = Cx + D\varphi(u) = Cz + G(0)(v + \varphi^*) = \tilde{w} + G(0)\varphi^*\,, \quad \text{where } \ \tilde{w} := Cz + G(0)v\,.$$

By Lemma 2.3, $\dot{v}(t) = \varphi^\diamond(u(t))\dot{u}(t)$ for almost all $t \in [0,\infty)$. Using (13) and (15), an easy calculation yields

$$\begin{aligned} \dot{z}(t) &= \ Az(t) - k(t)\varphi^\diamond(u(t))A^{-1}B\tilde{\psi}(\tilde{w}(t)) && \text{for a.a. } t \geq T_0\,, \\ \dot{v}(t) &= \ -k(t)\varphi^\diamond(u(t))\,\tilde{\psi}(\tilde{w}(t)) && \text{for a.a. } t \geq T_0\,. \end{aligned}$$

Invoking (16),

$$\dot{z}(t) = Az(t) + A^{-1}B\zeta(t) \qquad \text{for a.a. } t \geq T_0\,, \tag{19a}$$

$$\dot{v}(t) = \zeta(t) \qquad \text{for a.a. } t \geq T_0\,, \tag{19b}$$

where

$$\zeta(t) := -k(t)\eta(t)\tilde{w}(t)\,, \quad \text{with} \;\; \eta(t) := \varphi^{\diamond}(u(t))\gamma(\tilde{w}(t))\,. \tag{20}$$

Note that

$$0 \le \eta(t) \le \lambda_1\tilde{\lambda}_2 = \lambda \qquad \forall t \ge 0\,. \tag{21}$$

We will investigate asymptotic properties of (z,v) using a Lyapunov approach. Define the absolutely continuous function

$$V : \mathbb{R}_+ \to \mathbb{R}_+\,, \quad t \mapsto \langle z(t), Pz(t)\rangle + G(0)v^2(t)\,.$$

Then, using (19) and (20)

$$\begin{aligned}
\dot{V}(t) &= \langle z(t), (PA + A^TP)z(t)\rangle + 2(A^{-1}B)^TPz(t)\zeta(t) + 2G(0)v(t)\zeta(t) \\
&= \langle z(t), (PA + A^TP)z(t)\rangle + 2[(A^{-1}B)^TP - C]z(t)\zeta(t) + 2\tilde{w}(t)\zeta(t) \\
&= \big\langle [z^T(t),\zeta(t)]^T, \Lambda[z^T(t),\zeta(t)]^T\big\rangle + (2/(k_*\lambda))\zeta^2(t) - 2k(t)\eta(t)\tilde{w}^2(t) \\
&\le -\alpha\left[\|z(t)\|^2 + \zeta^2(t)\right] - 2k(t)\eta(t)\left[1 - \frac{k(t)\eta(t)}{k_*\lambda}\right]\tilde{w}^2(t) \quad \text{for a.a. } t \ge T_0\,,
\end{aligned}$$

where $\alpha = 1/\|\Lambda^{-1}\|$. Invoking (18) and (21), we see that there exists $T_1 \ge T_0$ such that

$$\sup_{t\ge T_1} k(t)\eta(t) < k_*\lambda\,.$$

Consequently, there exists $\beta > 0$ such that

$$\dot{V}(t) \le -\alpha\|z(t)\|^2 - \alpha\zeta^2(t) - \beta k(t)\eta(t)\tilde{w}^2(t) \qquad \text{for a.a. } t \ge T_1. \tag{22}$$

It follows that $z \in L^2(\mathbb{R}_+,\mathbb{R}^N)$ and $\zeta \in L^2(\mathbb{R}_+,\mathbb{R})$. Since $k\eta v = -(G(0))^{-1}[k\eta Cz + \zeta]$, we may conclude that $k\eta v \in L^2(\mathbb{R}_+,\mathbb{R})$ and thus

$$k\eta vCz \in L^1(\mathbb{R}_+,\mathbb{R})\,. \tag{23}$$

Moreover, using (22) and the definition of $\tilde{w}$, we obtain

$$\dot{V}(t) \le -2\beta G(0)k(t)\eta(t)v(t)Cz(t) - \beta G^2(0)k(t)\eta(t)v^2(t) \qquad \text{for a.a. } t \ge T_1\,,$$

and so we have

$$0 \le V(t) \le V(T_1) - 2\beta G(0)\int_{T_1}^t k(s)\eta(s)v(s)Cz(s)ds - \beta G^2(0)\int_{T_1}^t k(s)\eta(s)v^2(s)ds \quad \forall t \ge T_1\,,$$

which, together with (23), implies that

$$k\eta v^2 \in L^1(\mathbb{R}_+,\mathbb{R})\,. \tag{24}$$

From (23) and (24), we may infer that $v\zeta = -k\eta v[Cz + G(0)v] \in L^1(\mathbb{R}_+,\mathbb{R})$. Multiplying (19b) by $v(t)$ and integrating yields

$$\lim_{t\to\infty} v^2(t) = v^2(T_1) + 2\lim_{t\to\infty}\int_{T_1}^t v\zeta = c$$

for some constant $c \in [0, \infty)$. By continuity of v it follows that

$$\lim_{t\to\infty} v(t) = \sqrt{c} \qquad \text{or} \qquad \lim_{t\to\infty} v(t) = -\sqrt{c}\,.$$

This establishes statement (b1).

To prove (b2), assume that $\limsup_{t\to\infty} k(t) < k^*$ and $K(t) \to \infty$ as $t \to \infty$. Then it immediately follows from A and (b1) that (a1) to (a5) hold; in particular, there exists $\varphi^r \in \Psi^r \cap \overline{\Phi}$ such that $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$.

To show that the convergence is of order $\exp(-\rho K(t))$ for some $\rho > 0$, notice that, by hypothesis, $\varphi^r \in \Phi$ and $\varphi^r \notin \mathcal{C}(\varphi)$. Thus, by monotonicity, the preimage $\varphi^{-1}(\varphi^r)$ is a singleton $\{u^r\}$ and, by Lemma 2.4,

$$\varphi^\diamond(u^r) \geq \varphi^-(u^r) > 0\,.$$

By (a4), $u(t) \to u^r$ as $t \to \infty$. By lower semicontinuity of φ^-, there exists $T_2 \geq T_1$ such that

$$\varphi^\diamond(u(t)) \geq \varphi^-(u(t)) \geq \varphi^-(u^r)/2 > 0 \qquad \forall t \geq T_2\,. \tag{25}$$

Let $\rho > 0$ and introduce weighted variables by defining, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} z_\rho(t) &:= \exp(\rho K(t))[x(t) + A^{-1}B\varphi(u(t))] = \exp(\rho K(t))z(t)\,, \\ v_\rho(t) &:= \exp(\rho K(t))[\varphi(u(t)) - \varphi^r] = \exp(\rho K(t))v(t)\,. \end{aligned}$$

Note that (z_ρ, v_ρ) is absolutely continuous. Replacing φ^* in (15) by φ^r, it follows from Lemma 2.3, (13), (15) and a straightforward calculation that

$$\begin{aligned} \dot z_\rho(t) &= (A + \rho k(t) I) z_\rho(t) - k(t)\varphi^\diamond(u(t)) \exp(\rho K(t)) A^{-1}B\tilde\psi(\tilde w(t)) && \text{for a.a. } t \geq T_2\,, \\ \dot v_\rho(t) &= \rho k(t) v_\rho(t) - k(t)\varphi^\diamond(u(t)) \exp(\rho K(t))\tilde\psi(\tilde w(t)) && \text{for a.a. } t \geq T_2\,. \end{aligned}$$

Define

$$\tilde w_\rho(t) := \exp(\rho K(t))\tilde w(t) = C z_\rho(t) + G(0) v_\rho(t)$$

and

$$\zeta_\rho(t) := \exp(\rho K(t))\zeta(t) = -k(t)\eta(t)\tilde w_\rho(t)$$

with η defined as in (20). Invoking (16) yields

$$\dot z_\rho(t) = (A + \rho k(t) I) z_\rho(t) + A^{-1}B\zeta_\rho(t) \qquad \text{for a.a. } t \geq T_2\,, \tag{26a}$$

$$\dot v_\rho(t) = \rho k(t) v_\rho + \zeta_\rho(t) \qquad \text{for a.a. } t \geq T_2\,. \tag{26b}$$

Since by assumption $r \notin \mathcal{C}(\psi)$, we have that

$$\tilde\psi^-(0) = (\Pi_q\psi)^-(G(0)\varphi^r) = \psi^-(G(0)\varphi^r) > 0\,.$$

We claim that there exists $T_3 \geq T_2$ and $0 < \mu < \lambda$ such that $\mu \leq \eta(t)$ for all $t \geq T_3$. Seeking a contradiction, suppose the claim is not true. Then, by (25), there is a sequence $t_n \to \infty$ such that

$$0 \leq \gamma(\tilde w(t_n)) \leq 1/n\,.$$

Define $h_n := \tilde{w}(t_n) = Cz(t_n) + G(0)v(t_n)$. By (a1) and (a2) we know that $h_n \to 0$ for $n \to \infty$. Now, using (17) and Lemma 2.4,

$$0 = \lim_{n\to\infty} \gamma(h_n) \geq \liminf_{h\to 0} \frac{\tilde{\psi}(h)}{h} \geq \tilde{\psi}^-(0) > 0\,,$$

a contradiction. Thus, there exists $\mu \in (0,\lambda)$ and $T_3 \geq T_2$ such that $\mu \leq \eta(t)$ for all $t \geq T_3$. We will use the absolutely continuous function

$$V_\rho : \mathbb{R}_+ \to \mathbb{R}_+\,, \quad t \mapsto \langle z_\rho(t), P z_\rho(t)\rangle + G(0)v_\rho^2(t)$$

to investigate asymptotic properties of (26). By calculations similar to those leading to inequality (22) (with φ^* replaced by φ^r), we conclude that

$$\begin{aligned}\dot{V}_\rho(t) &\leq -\alpha\|z_\rho(t)\|^2 - \alpha\zeta_\rho^2(t)\\ &\quad - \beta\mu k(t)\tilde{w}_\rho^2(t) + 2\rho k(t)G(0)v_\rho^2(t) + 2\rho k(t)\langle z_\rho(t), P z_\rho(t)\rangle \quad \text{for a.a. } t \geq T_3\,, \end{aligned} \tag{27}$$

for some $\alpha, \beta > 0$. Let $\rho^* > 0$ be such that for all $\rho \in (0,\rho^*)$

$$-\alpha\|z_\rho(t)\|^2 + 2\rho k(t)\langle z_\rho(t), P z_\rho(t)\rangle \leq -\frac{\alpha}{2}\|z_\rho(t)\|^2 \qquad \forall t \geq T_3\,, \tag{28}$$

let $\nu \in (0,1)$ be such that

$$\beta\mu k(t)\left(\nu^{-1} - 1\right)\|C\|^2 \leq \frac{\alpha}{4} \qquad \forall\, t \geq T_3\,, \tag{29}$$

and let $\rho \in (0,\rho^*)$ be such that

$$(1-\nu)\beta\mu G(0) - 2\rho =: \alpha_\rho > 0\,. \tag{30}$$

Observing that

$$\begin{aligned}\tilde{w}_\rho^2(t) &= [Cz_\rho(t) + G(0)v_\rho(t)]^2\\ &= (1-\nu^{-1})|Cz_\rho(t)|^2 + (1-\nu)G^2(0)v_\rho^2(t) + \nu^{-1}[Cz_\rho(t) + \nu G(0)v_\rho(t)]^2\\ &\geq (1-\nu^{-1})\|C\|^2\|z_\rho(t)\|^2 + (1-\nu)G^2(0)v_\rho^2(t)\,,\end{aligned}$$

it now follows from (27), (28), (29) and (30) that

$$\begin{aligned}\dot{V}_\rho(t) &\leq -\frac{\alpha}{2}\|z_\rho(t)\|^2 - \alpha\zeta_\rho^2(t) + \beta\mu k(t)(\nu^{-1}-1)\|C\|^2\|z_\rho(t)\|^2\\ &\qquad - \beta\mu(1-\nu)k(t)G^2(0)v_\rho^2(t) + 2\rho k(t)G(0)v_\rho^2(t)\\ &\leq -\frac{\alpha}{4}\|z_\rho(t)\|^2 - \alpha\zeta_\rho^2(t) - \alpha_\rho k(t)G(0)v_\rho^2(t) \qquad \text{for a.a. } t \geq T_3\,.\end{aligned}$$

Integration yields

$$0 \leq V_\rho(t) \leq V_\rho(T_3) - \frac{\alpha}{4}\int_{T_3}^t \|z_\rho(s)\|^2 ds - \alpha\int_{T_3}^t \zeta_\rho^2(s)ds - \alpha_\rho G(0)\int_{T_3}^t k(s)v_\rho^2(s)ds \quad \forall t \geq T_3\,,$$

which implies $z_\rho \in L^2(\mathbb{R}_+, \mathbb{R}^N)$ and $\zeta_\rho \in L^2(\mathbb{R}_+, \mathbb{R})$. Thus, (26a) may be considered as an asymptotically stable system with a square-integrable forcing term $\rho k z_\rho + A^{-1}B\zeta_\rho$, implying that $\lim_{t\to\infty} z_\rho(t) = 0$ and, hence, boundedness of z_ρ. Since, moreover, $v_\rho^2(t) \leq V_\rho(T_3)/G^2(0)$

for all $t \geq T_3$, we also have boundedness of v_ρ, proving convergence of order $\exp(-\rho K(t))$ in (a1) and (a2). To establish the claim for (a3), notice that $r \notin \mathcal{C}(\psi)$ implies that $\psi^{-1}(r)$ is a singleton: $\psi^{-1}(r) = \{w^r\}$. Convergence of order $\exp(-\rho K(t))$ in (a1) and (a2) then implies the existence of $L > 0$ such that

$$|w^r - w(t)| \leq L \exp(-\rho K(t)) \quad \forall t \in \mathbb{R}_+ .$$

Let λ be a Lipschitz constant for ψ restricted to the compact set $\operatorname{clos}(w(\mathbb{R}_+))$. Then

$$|r - \psi(w(t))| \leq \lambda |w^r - w(t)| \leq \lambda L \exp(-\rho K(t)) \quad \forall t \in \mathbb{R}_+ ,$$

implying convergence of order $\exp(-\rho K(t))$ in (a3).

C. Let $\lambda_1 > 0$ be a Lipschitz constant for φ. By hypothesis, $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$. By Remark 2.2(iii) it follows that, for any fixed $r \in \mathcal{R}$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_{(p^*+\delta)}\psi(\xi + G(0)\varphi^r) - r$$

is also in $\mathcal{M}_L(\lambda_2)$. The argument used in part B above applies *mutatis mutandis* (specifically, on replacing ε by δ, φ^* by φ^r and $\tilde{\lambda}_2$ by λ_2) to conclude that (b1) and (b2) hold with $k^* = \kappa^*/(\lambda_1\lambda_2)$. $\square$

3.2 Adaptive gain

Whilst Corollary 3.4 identifies conditions under which the tracking objective is achieved through the use of a prescribed gain function, the resulting control strategy is somewhat unsatisfactory insofar as the gain function is selected *a priori:* no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain function. Let $\mathcal{L}$ denote the class of locally Lipschitz functions $\mathbb{R}_+ \to \mathbb{R}_+$ with value zero only at zero and with linear growth near zero, specifically:

$$\mathcal{L} := \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ locally Lipschitz},\ f^{-1}(0) = \{0\},\ \liminf_{\xi \downarrow 0} \xi^{-1} f(\xi) > 0\}.$$

Let $\chi \in \mathcal{L}$ and let the gain $k(\cdot)$ be generated by the following adaptation law:

$$k(t) = \frac{1}{l(t)}, \quad \dot{l}(t) = \chi(|r - y(t)|), \quad l(0) = l^0 > 0 . \tag{31}$$

This leads to the feedback system

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \quad x(0) = x^0 \in \mathbb{R}^N , \tag{32a}$$
$$\dot{u}(t) = k(t)[r - \psi(Cx(t) + D\varphi(u(t)))], \quad u(0) = u^0 \in \mathbb{R}, \tag{32b}$$
$$\dot{k}(t) = -k^2(t)\chi(|r - \psi(Cx(t) + D\varphi(u(t)))|), \quad k(0) = k^0 \in (0,\infty) . \tag{32c}$$

Setting $h(\theta) = -\theta^2$ and $\sigma \equiv 1$ in (9), an application of Lemma 3.1 yields the following result.

Proposition 3.7 *Let $\Sigma = (A,B,C,D) \in \mathcal{S}$, $(\varphi,\psi) \in \mathcal{N}$, $\chi \in \mathcal{L}$ and $r \in \mathbb{R}$.*
For each $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0,\infty)$, the initial-value problem (32) has a unique solution $(x,u,k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0,\infty)$.

We now arrive at the main adaptive tracking result.

Corollary 3.8 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{N}$, $\chi \in \mathcal{L}$ and $r \in \mathcal{R}$. Assume further that, if φ is unbounded, then there exists $\delta > 0$ such that $\chi(\xi) \geq \delta\xi$ for all $\xi \in \mathbb{R}_+$.*
For each $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the unique solution $(x, u, k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ of the initial-value problem (32) *is such that statements* (a1) *to* (a5) *of Theorem* 3.3 *hold. Moreover, if $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$, then the monotone gain k converges to a positive value.*

Proof. Let $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ and let $(x, u, k) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ be the unique solution of the initial-value problem (32). For all $t \in \mathbb{R}_+$, let $K(t) = \int_0^t k$, $l(t) = 1/k(t)$ and $e(t) = r - \psi(Cx(t) + D\varphi(u(t)))$. Since l is a continuous non-decreasing function, either $l(t) \to \infty$ as $t \to \infty$ (*Case* 1), or $l(t) \to \hat{l} \in (0, \infty)$ as $t \to \infty$ (*Case* 2). We consider these two cases separately.

Case 1: In this case, $k(t) \downarrow 0$ as $t \to \infty$ and by Theorem 3.3(b1), $\varphi(u(t))$ and consequently $x(t)$ converge as $t \to \infty$, and so, in particular, $\varphi \circ u$ and x are bounded functions. Therefore, there exists $\beta > 0$ such that

$$\dot{l}(t) = \chi(|e(t)|) \leq \beta \quad \forall t \geq 0$$

whence

$$k(t) = \frac{1}{l(t)} \geq \frac{1}{l^0 + \beta t} \qquad \forall t \geq 0\,, \tag{33}$$

and so $K(t) \to \infty$ as $t \to \infty$. Therefore, by Theorem 3.3(b2), statements (a1) to (a5) of Theorem 3.3 hold.

Case 2: In this case, $k(t) \to \hat{k} := 1/\hat{l} > 0$ as $t \to \infty$ and so $K(t) \to \infty$ as $t \to \infty$. In order to conclude that statements (a1) to (a5) of Theorem 3.3 hold, it suffices to show (by part A of Theorem 3.3) that $u(t)$ converges to a finite limit as $t \to \infty$: in view of (32b), we will establish the latter convergence by showing that e is of class $L^1(\mathbb{R}_+, \mathbb{R})$. By boundedness of l and (31),

$$\int_0^\infty \chi(|e(t)|)dt < \infty\,. \tag{34}$$

First assume φ is unbounded. Then, by hypothesis, $\chi(|e(t)|) \geq \delta|e(t)|$ for all $t \geq 0$ which, together with (34), implies that $e \in L^1(\mathbb{R}_+, \mathbb{R})$. Next, assume φ is bounded. Then, by the Hurwitz property of A and (32a,b), it follows that e is uniformly continuous. Moreover, (34) holds and so, by Barbălat's Lemma [1, 6], $\chi(|e(t)|) \to 0$ as $t \to \infty$ which, recalling that $\chi^{-1}(0) = \{0\}$, implies that $e(t) \to 0$ as $t \to \infty$. Since $\liminf_{\xi \downarrow 0} \xi^{-1}\chi(\xi) > 0$, we may infer the existence of $T \in \mathbb{R}_+$ and $\delta > 0$ such that $\chi(|e(t)|) \geq \delta|e(t)|$ for all $t \geq T$ which, together with (34), implies that $e \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, $u(t)$ converges to a finite limit as $t \to \infty$. Hence, the hypotheses of part A of Theorem 3.3 are satisfied and so statements (a1) to (a5) of Theorem 3.3 hold.

Finally, assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$. We will show that the monotone gain k converges to a positive value. Seeking a contradiction, suppose that the monotone function l is unbounded (equivalently, $k(t) \downarrow 0$ as $t \to \infty$). Then the hypothesis of Theorem 3.3(b1) is satisfied: therefore, $\varphi \circ u$ and x are bounded and so (33) holds for some $\beta > 0$. Therefore, $K(t) \to \infty$ as $t \to \infty$ and so, by Theorem 3.3(b2), $e(t) \to 0$ as $t \to \infty$, and the convergence is of order $\exp(-\rho K(t))$ for some $\rho > 0$. This, together with the fact that χ is locally Lipschitz with $\chi(0) = 0$, implies the existence of a constant $L > 0$ such that

$$\chi(|e(t)|) \leq L \exp(-\rho K(t)) \qquad \forall t \in \mathbb{R}_+ . \tag{35}$$

Choose $\alpha \geq \beta$ such that $\rho/\alpha < 1$. By (33), $k(t) = 1/l(t) \geq (l^0 + \alpha t)^{-1}$ for all $t \in \mathbb{R}_+$. Therefore,

$$K(t) = \int_0^t k \ \geq \ \frac{1}{\alpha} \ln\left(\frac{l^0 + \alpha t}{l^0}\right) \qquad \forall t \geq 0 .$$

Consequently

$$\dot{l}(t) = \chi(|e(t)|) \leq L \exp(-\rho K(t)) \leq \left(\frac{l^0 + \alpha t}{l^0}\right)^{-\nu} \qquad \forall t \geq 0 ,$$

where $\nu = \rho/\alpha \in (0, 1)$. Integration gives

$$l(t) \leq M(l^0 + \alpha t)^{1-\nu} \qquad \forall t \geq 0$$

for some suitable constant $M > 0$. It follows that

$$-K(t) = -\int_0^t k \leq -(M\alpha\nu)^{-1} \left[(l^0 + \alpha t)^\nu - (l^0)^\nu\right] \qquad \forall t \geq 0 .$$

Therefore, $\exp(-\rho K)$ is of class $L^1(\mathbb{R}_+, \mathbb{R})$. By (35), it follows that $\chi(|e|) \in L^1(\mathbb{R}_+, \mathbb{R})$ which, recalling that $\dot{l} = \chi(|e|)$, contradicts the supposition of unboundedness of l. Therefore, l is bounded. $\square$

Example 3.9 Consider again the second-order system given in Example 3.6. Let $\chi \in \mathcal{L}$. By Corollary 3.4, it follows that the adaptive control (31) achieves the tracking objective for each feasible reference value $r \in \mathcal{R} = [-b^{-3}, b^{-3}]$. Moreover, if r is both non-zero (so that $r \notin \mathcal{C}(\psi) = \{0\}$) and in the interior of $\mathcal{R}$ (so that $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$), then the adapting gain $k(\cdot)$ converges to a positive value. For purposes of illustration, let $a = 2$ and $b = 1$, in which case $\mathcal{R} = [-1, 1]$, and let $\chi = \varphi|_{\mathbb{R}_+}$ (the restriction of the saturation function to $\mathbb{R}_+$). For initial data $(x_1(0), x_2(0), u(0), k(0)) = (0, 0, 0, 1)$ and the feasible reference value $r = 0.75$, Figure 4 (generated using SIMULINK Simulation Software under MATLAB) depicts the system performance under adaptive control.

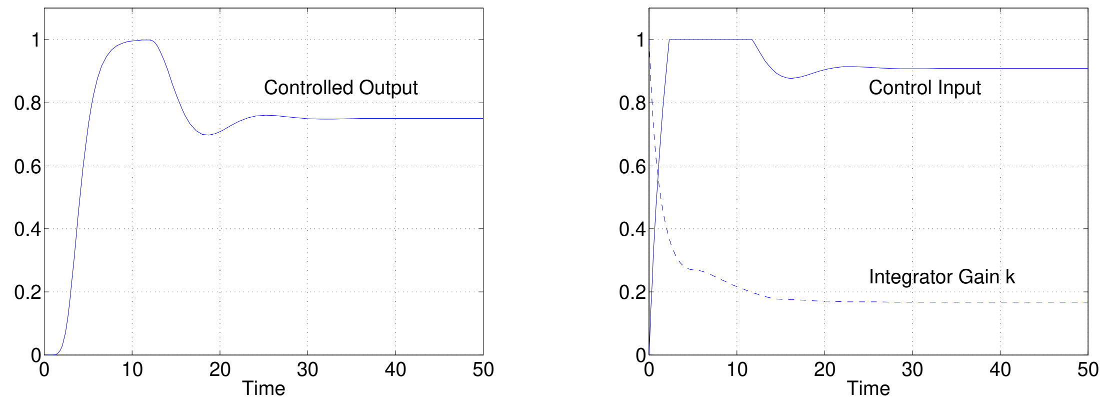

Fig. 4: Performance under adaptive control

Appendix

Proof of Lemma 2.3. It is clear that $f^\diamond \in L^\infty_{loc}(\mathbb{R},\mathbb{R})$ if f is locally Lipschitz, and that $f^\diamond \in L^\infty(\mathbb{R},\mathbb{R})$ if f is globally Lipschitz. Moreover, as the limsup of a sequence of Borel functions, $f^\diamond$ is a Borel function. Let g be absolutely continuous. Setting $q = f \circ g$, it follows from the Lipschitz continuity of f and the absolute continuity of g that q is absolutely continuous. If $t \in [0,\infty)$ is such that g is differentiable at t, then we have

$$q(t+h) - q(t) = f(g(t) + h\dot{g}(t)) - f(g(t)) + f(g(t+h)) - f(g(t) + h\dot{g}(t)) \,. \tag{36}$$

Moreover, by Lipschitz continuity of f, there exists a constant $L > 0$ such that, for all sufficiently small $|h|$,

$$\left| \frac{1}{h} \left[f(g(t+h)) - f(g(t) + h\dot{g}(t)) \right] \right| \le L \left| \frac{1}{h} \left[g(t+h) - g(t) \right] - \dot{g}(t) \right| \,. \tag{37}$$

Let $\mathcal{D} \subset \mathbb{R}$ be the set of all points t such that both g and q are differentiable at t. Then $\mathcal{D}$ is of full measure, and combining (36) and (37) yields

$$\lim_{h\to 0} \frac{1}{h}[q(t+h) - q(t)] = \lim_{h\to 0} \frac{1}{h}[f(g(t) + h\dot{g}(t)) - f(g(t))] \qquad \forall t \in \mathcal{D} \,.$$

Therefore, for every $t \in \mathcal{D}$,

$$\dot{q}(t) = 0 \,, \qquad \text{if } \dot{g}(t) = 0 \,, \tag{38a}$$

$$\dot{q}(t) = \lim_{h\to 0} \frac{f(g(t) + h\dot{g}(t)) - f(g(t))}{h\dot{g}(t)} \, \dot{g}(t) = f'(g(t))\dot{g}(t) \,, \qquad \text{if } \dot{g}(t) \neq 0 \,. \tag{38b}$$

In particular, if $t \in \mathcal{D}_0 := \{t \in \mathcal{D} \,|\, \dot{g}(t) \neq 0\}$, then f is differentiable at $g(t)$. For $t \in \mathcal{D}_0$ we have, of course, $f^\diamond(g(t)) = f'(g(t))$ and thus it follows from (38) that

$$\dot{q}(t) = f^\diamond(g(t))\dot{g}(t) \qquad \text{for a.a. } t \in [0,\infty) \,.$$

$\square$

Proof of Lemma 2.4. For all $\xi \in \mathbb{R}$ we have

$$\begin{aligned} f^-(\xi) &= -\limsup_{\substack{\theta\to\xi \\ h\downarrow 0}} \frac{f(\theta - h) - f(\theta)}{h} = \liminf_{\substack{\theta\to\xi \\ h\downarrow 0}} \frac{f(\theta - h) - f(\theta)}{-h} = \liminf_{\substack{\theta\to\xi \\ h\uparrow 0}} \frac{f(\theta + h) - f(\theta)}{h} \\ &\le \liminf_{h\uparrow 0} \frac{f(\xi + h) - f(\xi)}{h} \,, \end{aligned} \tag{39}$$

and

$$\begin{aligned} f^-(\xi) &= -\limsup_{\substack{\theta\to\xi \\ h\downarrow 0}} \frac{f(\theta - h) - f(\theta)}{h} = -\limsup_{\substack{\zeta\to\xi \\ h\downarrow 0}} \frac{f(\zeta) - f(\zeta + h)}{h} = \liminf_{\substack{\zeta\to\xi \\ h\downarrow 0}} \frac{f(\zeta + h) - f(\zeta)}{h} \\ &\le \liminf_{h\downarrow 0} \frac{f(\xi + h) - f(\xi)}{h} \,. \end{aligned} \tag{40}$$

It follows from (39) and (40) that

$$f^-(\xi) \le \liminf_{h\to 0} \frac{f(\xi + h) - f(\xi)}{h} \qquad \forall \xi \in \mathbb{R} \,,$$

which is the first inequality in (5). The second inequality in (5) is an immediate consequence of the definition of $f^\diamond$.

If $f \in \mathcal{M}_L(\lambda)$ for some $\lambda \geq 0$, then (6) follows from (5), the monotonicity of f and the definition of $f^\diamond$. $\square$

References

[1] I. Barbalăt. Systèmes d'équations différentielles d'oscillations non linéaires, *Revue de Mathématique Pures et Appliquées*, Bucharest, **IV** (1959), 267-270.

[2] F.H. Clarke. *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[3] P.A. Cook. Controllers with universal tracking properties, *Proc. of Int. IMA Conf. on Control: Modelling, Computation, Information*, Manchester, 1992.

[4] G.W.M. Coppus, S.L. Sha and R.K. Wood. Robust multivariable control of a binary distillation column, *IEE Proceedings, Pt. D* **130** (1983), 201–208.

[5] E.J. Davison. Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem, *IEEE Trans. Auto. Control* **21** (1976), 35–47.

[6] H.K. Khalil. *Nonlinear Systems*, 2nd ed., Prentice Hall, New Jersey, 1996.

[7] H. Logemann and E.P. Ryan. Time-varying and adaptive integral control of infinite-dimensional regular systems with input nonlinearities, Mathematics Preprint 98/12, University of Bath, 1998[†], (to appear in *SIAM J. Control & Optim.*).

[8] H. Logemann and E.P. Ryan. Time-Varying and Adaptive Discrete-Time Low-Gain Control of Infinite-Dimensional Linear Systems with Input Nonlinearities, Mathematics Preprints 98/20, University of Bath, 1998[†], (submitted for publication).

[9] H. Logemann, E.P. Ryan and S.Townley. Integral control of infinite-dimensional linear systems subject to input saturation, *SIAM J. Control & Optim.* **36** (1998), 1940–1961.

[10] H. Logemann, E.P. Ryan and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain, *IEEE Trans. Auto. Control* **44** (1999), 1315–1319.

[11] H. Logemann and S. Townley. Low-gain control of uncertain regular linear systems, *SIAM J. Control & Opt.*, **35** (1997), 78–116.

[12] J. Lunze. Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln* **30** (1987), 2-6.

[13] J. Lunze. *Robust Multivariable Feedback Control*, Prentice Hall, London, 1988.

[14] D.E. Miller and E.J. Davison. An adaptive tracking problem with a control input constraint, *Automatica* **29** (1993), 877–887.

[15] D.E. Miller and E.J. Davison. The self-tuning robust servomechanism problem, *IEEE Trans. Auto. Control* **34** (1989), 511–523.

[16] M. Morari. Robust stability of systems with integral control, *IEEE Trans. Auto. Control* **30** (1985), 574–577.

[17] A. Rantzer. On the Kalman-Yakubovich-Popov lemma, *Systems & Control Letters* **28** (1996), 7–10.

[†] Preprints in this series are available by anonymous ftp from `ftp.maths.bath.ac.uk` in the directory `/pub/preprints`.