Discrete-Time Low-Gain Control of Linear Systems with Input/Output Nonlinearities *

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Abstract: Discrete-time low-gain control strategies are presented for tracking of constant reference signals for finite-dimensional, discrete-time, power-stable, single-input, single-output, linear systems subject to a globally Lipschitz, non-decreasing input nonlinearity and a locally Lipschitz, non-decreasing, affinely sector-bounded output nonlinearity (the conditions on the output nonlinearities may be relaxed if the input nonlinearity is bounded). Both non-adaptive and adaptive gain sequences are considered. In particular, it is shown that applying error feedback using a discrete-time 'integral' controller ensures asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive gain sequence is ultimately sufficiently small and (c) the reference value is feasible in a very natural sense. The classes of input and output nonlinearities under consideration contain standard nonlinearities important in control engineering such as saturation and deadzone. The discrete-time results are applied in the development of sampled-data low-gain control strategies for finite-dimensional, continuous-time, exponentially stable, linear systems with input and output nonlinearities.

Keywords: Adaptive control; discrete-time systems; input/output nonlinearities; integral control; sampled-data systems; robust control; tracking.

AMS subject classifications: 93B52, 93C10, 93C55, 93C57, 93D09, 93D10, 93D21.

1 Introduction

The present paper continues a sequence [5, 6, 7, 8] of recent investigations pertaining to the problem of tracking constant reference signals, by low-gain integral control, for linear (uncertain) systems subject to input and/or output nonlinearities. These investigations extend the well-known principle that closing the loop around an exponentially stable, linear, finite-dimensional, continuous-time, single-input, single-output plant Σ_c, with transfer function G_c, compensated by an integral controller (see Fig. 1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that the modulus $|k|$ of the integrator gain k is sufficiently small and $kG_c(0) > 0$ (see [4, 13, 16]). Therefore, if a plant is exponentially stable and if the sign of $G_c(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design approach ('tuning regulator theory'[4]) has been successfully applied in process control, see, for example, [3] and [12]. Furthermore, the problem of tuning the integrator gain adaptively has been addressed in a number

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of papers for finite-dimensional [2, 5, 14, 15] and infinite-dimensional systems [6, 7, 11], with input nonlinearities considered in [6, 7, 14] and both input and output nonlinearities treated in [5].

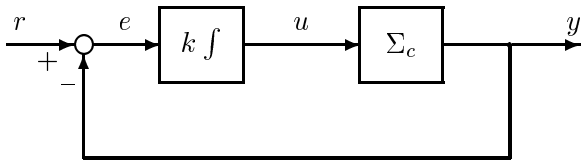

Fig. 1: Low-gain control system

The purpose of this paper is twofold:
(i) to provide the discrete-time counterparts of the results in [5];
(ii) to apply the discrete-time theory in the derivation of sampled-data control strategies for continuous-time systems.

Whilst the structure of the discrete-time analysis parallels that of the continuous-time analysis in [5], there are several points where these analyses differ in an essential manner. The discrete-time low-gain results should, nonetheless, be seen primarily as the main tool in the derivation of sampled-data low-gain integral control strategies for a class of continuous-time nonlinear systems which is the second and main aim of the paper.

With reference to (i) we show that the principle alluded to in the opening paragraph remains true if the plant to be controlled is a discrete-time, power-stable, single-input, single-output, linear system with transfer function G satisfying $G(1) > 0$ and with nonlinearities, φ and ψ, in the input and output channel which belong to a certain class. Figure 2 depicts the control structure schematically, wherein DI denotes the discrete-time 'integral' controller

$$u_{n+1} = u_n + k_n e_n\,, \qquad u_0 \in \mathbb{R}. \tag{1}$$

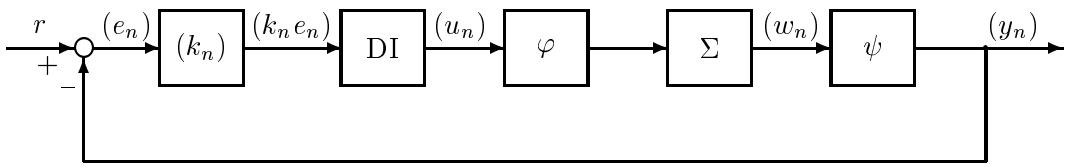

Fig. 2: Discrete-time low-gain control with input and output nonlinearities

Precisely, we prove that, if the constant reference signal r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all non-decreasing, globally Lipschitz input nonlinearities φ and all non-decreasing, locally Lipschitz and affinely sector-bounded output nonlinearities ψ (the sector-bound assumption on ψ can be removed if φ is bounded) the following holds: for all positive and bounded gain sequences (k_n) (thus in particular for positive constant gains), the output y_n of the closed-loop system converges to r as $n \to \infty$ provided that $\limsup_{n\to\infty} k_n < k^*$ and (k_n) is not of class l^1 (under some additional assumptions on the nonlinearities and the gain sequence (k_n), results concerning the rate of convergence are derived).

Moreover, we show that the following simple adaptation law (a generalization of that introduced in [7])

$$k_n = 1/l_n, \qquad l_{n+1} = l_n + \chi(|r - y_n|), \quad l_0 > 0\,,$$

with $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ suitably chosen, produces a gain sequence (k_n) such that the output y_n of the closed-loop system converges to r as $n \to \infty$.

The number k^* is closely related to the supremum κ^* of the set of all numbers $\kappa > 0$ such that the function

$$z \mapsto 1 + \kappa \frac{G(z)}{z-1}$$

is positive-real: for example, if both φ and ψ are globally Lipschitz with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then $k^* = \kappa^*/(\lambda_1\lambda_2)$.

In the sampled-data context (ii), our aim is to establish the efficacy of the control structure in Fig. 3, wherein S and H denote standard sampling and hold operations, respectively, and LG represents the (discrete-time) low-gain 'integral' controller given by (1), where the gain sequence $(k_n) \subset \mathbb{R}_+$ is either prescribed or updated adaptively and the nonlinearities in the input and output channel, φ and ψ, have the same properties as above.

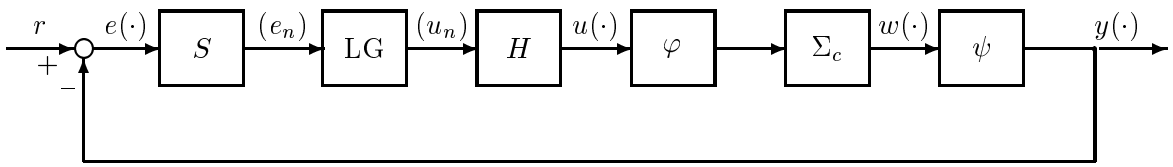

Fig. 3: Sampled-data low-gain control with input and output nonlinearities

We show that tracking of feasible reference values r is achieved by (1) for continuous-time, exponentially stable, single-input, single-output, linear systems Σ_c with transfer function G_c satisfying $G_c(0) > 0$ provided that the positive gain sequence (k_n) is chosen according to the strategies derived in the purely discrete-time setting. The performance of the sampled-data controller is illustrated by means of an example.

2 The class $\mathcal{N}$ of input/output nonlinearities

For $\lambda > 0$, we introduce the following sets of monotone, non-decreasing nonlinearities:

$$\mathcal{M} := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ locally Lipschitz and non-decreasing}\}\,,$$

$$\mathcal{M}(\lambda) := \{f \in \mathcal{M} \mid 0 \le (f(\xi) - f(0))\xi \le \lambda\xi^2 \quad \forall \xi \in \mathbb{R}\}\,,$$

$$\mathcal{M}_L(\lambda) := \{f \in \mathcal{M} \mid f \text{ is globally Lipschitz with Lipschitz constant } \lambda\}\,.$$

Clearly, $\mathcal{M}_L(\lambda) \subset \mathcal{M}(\lambda) \subset \mathcal{M}$.

Remarks 2.1
(i) Let $f \in \mathcal{M}(\lambda)$. Then $f(\xi) = f(0) + \gamma(\xi)\xi$ for all ξ, where

$$\gamma : \xi \mapsto \begin{cases} (f(\xi) - f(0))/\xi \ , & \xi \neq 0 \\ \lambda \qquad\quad , & \xi = 0 \end{cases}$$

and $\gamma(\xi) \in [0, \lambda]$ for all $\xi \in \mathbb{R}$.

(ii) If $f \in \mathcal{M}(\lambda)$, then, for each $\nu \in \mathbb{R}$, there exists $\tilde{\lambda} \geq 0$ such that the function $\xi \mapsto f(\xi + \nu) - f(\nu)$ is of class $\mathcal{M}(\tilde{\lambda})$.

(iii) If $f \in \mathcal{M}_L(\lambda)$, then, for each $\nu \in \mathbb{R}$, $\xi \mapsto f(\xi + \nu) - f(\nu)$ is also of class $\mathcal{M}_L(\lambda)$.

For locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$, define

$$f^- : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \liminf_{\substack{\theta \to \xi \\ \varepsilon \downarrow 0}} \frac{f(\theta) - f(\theta - \varepsilon)}{\varepsilon} .$$

Note that $-(f^-)(\xi)$ is the Clarke generalized derivative of f at ξ in direction -1 (see [1]). If f is C^1 with derivative f', then $f^- \equiv f'$. Let $f \in \mathcal{M}$: a point $\xi \in \mathbb{R}$ is said to be a *critical point* (and $f(\xi)$ is said to be a *critical value*) of f if $f^-(\xi) = 0$.[†] We denote, by $\mathcal{C}(f)$, the set of critical values of f.

The following two lemmas will be used later. The first result appeared originally in [7], while a proof of the second lemma may be found in [5].

Lemma 2.2 *Let $f \in \mathcal{M}_L(\lambda)$ for some $\lambda > 0$. Let $(u_n) \subset \mathbb{R}$ and define $(\delta_n) \subset [0, \lambda]$ by*

$$\delta_n := \begin{cases} (f(u_{n+1}) - f(u_n))/(u_{n+1} - u_n) \ , & u_{n+1} \neq u_n \\ \lambda \qquad\qquad\qquad\quad , & u_{n+1} = u_n \, . \end{cases}$$

If (u_n) is convergent and its limit is not a critical point of f, then there exist $\mu > 0$ and $n_0 \in \mathbb{N}_0$ such that

$$\delta_n \geq \mu \quad \forall n \geq n_0 \, .$$

Lemma 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Then*

$$f^-(\xi) \leq \liminf_{h \to 0} \frac{f(\xi + h) - f(\xi)}{h} \qquad \forall \xi \in \mathbb{R}.$$

Finally, we make precise the class $\mathcal{N}$ (originally introduced in [5]) of input/output nonlinearities considered in this paper: a pair (φ, ψ) is in $\mathcal{N}$ if $\varphi \in \mathcal{M}_L(\lambda_1)$ for some $\lambda_1 > 0$, $\psi \in \mathcal{M}$ and at least one of the following holds:

$$\text{(i) } \varphi \text{ is bounded, or (ii) } \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0 \, .$$

Equivalently,

$$\mathcal{N} := \{(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M} \mid \lambda_1 > 0, \ \varphi \text{ unbounded } \Rightarrow \ \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\} \, .$$

[†] If f is merely locally Lipschitz, but not in $\mathcal{M}$, then it would be natural to deem $\xi \in \mathbb{R}$ a critical point of f if 0 belongs to the (Clarke [1]) generalized gradient $\partial f(\xi)$ of f at ξ. We remark that, in the case of $f \in \mathcal{M}$, $0 \leq \min\{\partial f(\xi)\} = f^-(\xi)$. Therefore, for functions $f \in \mathcal{M}$, the latter concept of a critical point coincides with that given above.

3 Discrete-time low-gain control

In this section we address the low-gain tracking problem in discrete time. The results may be seen as the discrete-time analogues of those in [5].

3.1 The class of discrete-time systems and the control objective

We consider a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u_n \in \mathbb{R}$), single-output ($y_n \in \mathbb{R}$), discrete-time (time domain $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), real linear systems $\Sigma = (A, B, C, D)$ having a nonlinearity in the input and output channel:

$$x_{n+1} = Ax_n + B\varphi(u_n)\,, \quad x_0 \in \mathbb{R}^N\,, \tag{2a}$$

$$w_n = Cx_n + D\varphi(u_n)\,, \tag{2b}$$

$$y_n = \psi(w_n) = \psi(Cx_n + D\varphi(u_n))\,. \tag{2c}$$

In (2), A is assumed to be power-stable, *i.e.*, each eigenvalue of A has modulus strictly less than one, and (φ, ψ) is assumed to belong to $\mathcal{N}$. Furthermore, the transfer function G of the associated linear system, given by

$$G(z) = C(zI - A)^{-1}B + D\,,$$

is assumed to satisfy $G(1) > 0$. The underlying class of linear discrete-time systems $\Sigma = (A, B, C, D)$ is denoted by

$$\mathcal{S} := \{\Sigma = (A, B, C, D) \mid A \text{ power-stable, } G(1) = C(I - A)^{-1}B + D > 0\}.$$

Given $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $(\varphi, \psi) \in \mathcal{N}$, the control objective is to track constant reference values $r \in \mathbb{R}$, *i.e.*, to determine, by feedback, an input sequence $(u_n) \subset \mathbb{R}$ such that, for given $r \in \mathbb{R}$, the output y_n of (2) has the property $y_n \to r$ as $n \to \infty$. Clearly, if this objective is achievable, then r is necessarily in the closure of $\operatorname{im}\psi$. We will impose a stronger condition, namely,

$$\Psi^r \cap \overline{\Phi} \neq \emptyset\,, \quad \text{where } \Psi^r := \{v \in \mathbb{R} \mid \psi(G(1)v) = r\}, \quad \Phi := \operatorname{im}\varphi, \quad \overline{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \overline{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. It will be shown later (see Subsection 3.4) that $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\varphi(u_n)$, then $r \in \mathcal{R}$.

To achieve the objective of tracking feasible reference values $r \in \mathcal{R}$, we shall investigate discrete-time 'integral' control action of the form

$$u_{n+1} = u_n + k_n(r - \psi(Cx_n + D\varphi(u_n))) = u_n + k_n(r - y_n)\,, \tag{3}$$

where $(k_n) \subset \mathbb{R}$ is a sequence of gain parameters (possibly constant) which is either prescribed or determined adaptively.

3.2 Prescribed gain

Henceforth, we assume that the gain sequence (k_n) satisfies

$$(k_n) \in \mathcal{G} := \{g \mid g : \mathbb{N}_0 \to (0,\infty) \quad \text{bounded}\}\,.$$

An application of the control law (3) leads to the following system of nonlinear difference equations

$$x_{n+1} = Ax_n + B\varphi(u_n)\,, \quad x_0 \in \mathbb{R}^N\,, \tag{4a}$$
$$u_{n+1} = u_n + k_n(r - \psi(Cx_n + D\varphi(u_n)))\,, \quad u_0 \in \mathbb{R}\,. \tag{4b}$$

If G is the transfer function of a system $\Sigma = (A,B,C,D) \in \mathcal{S}$, then it is readily shown that

$$1 + \kappa \operatorname{Re} \frac{G(z)}{z-1} \geq 0 \qquad \forall z \in \mathbb{C} \quad \text{with } |z| > 1 \tag{5}$$

for all sufficiently small $\kappa > 0$, see [10, Theorem 2.5]. Define

$$\kappa^* := \sup\{\kappa > 0 \,|\, (5) \text{ holds}\}\,. \tag{6}$$

The following lemma will play a crucial rôle.

Lemma 3.1 *Assume that $\Sigma = (A,B,C,D) \in \mathcal{S}$ and let $\Delta > 1/\kappa^*$. Then there exists $P \in \mathbb{R}^{N\times N}$ such that $P = P^T \geq 0$ and*

$$\begin{bmatrix} A^TPA - P & A^TP(I-A)^{-1}B + C^T \\ ((I-A)^{-1}B)^TPA + C & ((I-A)^{-1}B)^TP(I-A)^{-1}B - 2\Delta + G(1) \end{bmatrix} < 0\,.$$

Proof. Define

$$H(z) := -C(zI-A)^{-1}(I-A)^{-1}B - G(1)/2 = \begin{cases} \dfrac{G(z)-G(1)}{z-1} - G(1)/2\,, & z \neq 1 \\ G'(1) - G(1)/2\,, & z = 1\,. \end{cases}$$

Let $\tilde{\Delta} \in (1/\kappa^*, \Delta)$. Clearly, H is holomorphic in $\{z \in \mathbb{C} \mid |z| > \alpha\}$ for some $\alpha \in (0,1)$ and the positive-real condition (5) holds for $\kappa = 1/\tilde{\Delta}$. Notice that $\operatorname{Re}(1-e^{i\theta})^{-1} = 1/2,\ \theta \in (0,2\pi)$. Therefore, $\tilde{\Delta} + \operatorname{Re} H(e^{i\theta}) \geq 0$ for all $\theta \in [0,2\pi)$ and so

$$\Delta + \operatorname{Re} H(e^{i\theta}) > 0 \qquad \forall \theta \in [0,2\pi)\,.$$

Writing

$$M := \begin{bmatrix} 0 & C^T \\ C & -2\Delta + G(1) \end{bmatrix}\,,$$

we may conclude that, for all $\theta \in [0,2\pi)$,

$$\begin{bmatrix} (e^{i\theta}I - A)^{-1}(I-A)^{-1}B \\ 1 \end{bmatrix}^* M \begin{bmatrix} (e^{i\theta}I - A)^{-1}(I-A)^{-1}B \\ 1 \end{bmatrix} = -2(\operatorname{Re} H(e^{i\theta}) + \Delta) < 0\,,$$

and the assertion of the lemma immediately follows from a variant of the Kalman-Yakubovich-Popov Lemma as given by Rantzer ([17, Theorem 2]). □

Before presenting the main results, we describe a convenient family of projection operators. Specifically, with each $p \in [0,\infty]$, we associate an operator $\Pi_p : \mathcal{M} \to \mathcal{M}$, with the property $\Pi_p \circ \Pi_p = \Pi_p$ (whence the terminology projection operator), defined as follows:

$$\text{if} \quad p < \infty, \ \text{then} \quad \Pi_p f : \xi \mapsto \begin{cases} f(-p) \ , & \xi < -p \\ f(\xi) \ , & |\xi| \le p \\ f(p) \ , & \xi > p \end{cases} ; \qquad \text{if} \quad p = \infty, \ \text{then} \quad \Pi_p f = \Pi_\infty f := f \, .$$

Furthermore, we denote the l^∞-gain of $\Sigma = (A,B,C,D) \in \mathcal{S}$ by Γ_Σ, and so

$$0 < G(1) \le \Gamma_\Sigma = \sum_{j=0}^{\infty} |CA^jB| + |D| < \infty \, . \tag{7}$$

The next theorem forms the core of the paper.

Theorem 3.2 *Let $\Sigma = (A,B,C,D) \in \mathcal{S}$, $(\varphi,\psi) \in \mathcal{N}$, $r \in \mathcal{R}$ and $(k_n) \in \mathcal{G}$. Define*

$$p^* := \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \in (0,\infty] \, .$$

For $(x_0,u_0) \in \mathbb{R}^N \times \mathbb{R}$, let $n \mapsto (x_n,u_n)$ be the solution of the initial-value problem (4).

A. *Assume that $K_n := \sum_{j=0}^n k_j \to \infty$ as $n \to \infty$ and that $\lim_{n\to\infty} \varphi(u_n)$ exists and is finite. Then*

   (a1) $\lim_{n\to\infty} \varphi(u_n) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

   (a2) $\lim_{n\to\infty} x_n = (I-A)^{-1}B\varphi^r$,

   (a3) $\lim_{n\to\infty} y_n = r$, *where* $y_n = \psi(Cx_n + D\varphi(u_n))$,

   (a4) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{n\to\infty} \operatorname{dist}(u_n, \varphi^{-1}(\varphi^r)) = 0$,

   (a5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* (u_n) *is bounded.*

B. *There exists $k^* > 0$, independent of (x_0,u_0) and (k_n), such that the following hold.*

   (b1) *If* $\limsup_{n\to\infty} k_n < k^*$, *then* $\lim_{n\to\infty} \varphi(u_n)$ *exists and is finite.*

   (b2) *If* $\limsup_{n\to\infty} k_n < k^*$ *and* $K_n := \sum_{j=0}^n k_j \to \infty$ *as* $n \to \infty$, *then statements* (a1) *to* (a5) *hold. If, in addition,* (k_n) *is ultimately non-increasing and the nonlinearities* φ *and* ψ *are such that* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ *and* $r \notin \mathcal{C}(\psi)$, *then the convergence in* (a1) *to* (a3) *is of order* ρ^{-K_n} *for some* $\rho > 1$.

C. *Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$, then* (b1) *and* (b2) *are valid with $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by* (6).

Remarks 3.3
(i) The essence of part A of Theorem 3.2 is an assertion, via (a1) to (a3), that the tracking objective is achieved by the control law (3), provided that $(\varphi(u_n))$ converges and that the gain

sequence $(k_n) \in \mathcal{G}$ is *not* of class l^1; moreover, under additional hypotheses on the nonlinearities, (a4) and (a5) capture aspects of the qualitative behaviour of the sequence (u_n) (for example, (a4) implies that (u_n) converges if the set $\varphi^{-1}(\varphi^r)$ is a singleton, which, in turn, is true if φ^r is not a critical value of φ). Part A of Theorem 3.2 is of importance in the analysis of the adaptive control proposed in Subsection 3.3 below.

(ii) Part B of Theorem 3.2 contains the main tracking result in the non-adaptive situation. It asserts the existence of a positive constant $k^* > 0$ (independent of $(k_n) \in \mathcal{G}$ and the initial data (x_0, u_0)) such that, if $(k_n) \in \mathcal{G}$ is chosen to be ultimately strictly bounded above by k^* and, in addition, (k_n) is *not* of class l^1, then statements (a1) to (a5) hold, in particular, tracking of arbitrary feasible reference values is achieved; furthermore, if (k_n) is ultimately non-increasing, if the sets $\Psi^r \cap \overline{\Phi}$ and $\Psi^r \cap \Phi$ coincide, if Ψ^r contains no critical values of φ and if r is not a critical value of ψ, then (b2) provides an estimate of the rate of convergence in (a1) to (a3). An immediate consequence of part B of Theorem 3.2 is the following: if $(k_n) \in \mathcal{G}$ is chosen such that, as $n \to \infty$, k_n tends to zero *sufficiently slowly* (in the sense that $(k_n) \notin l^1(\mathbb{R}_+)$), then the tracking objective is achieved.

(iii) Under the assumption that $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta, \lambda_2 > 0$ (which, since $(\varphi, \psi) \in \mathcal{N}$, is equivalent to requiring that $\psi \in \mathcal{M}_L(\lambda_2)$ for some $\lambda_2 > 0$ if φ is unbounded), part C of Theorem 3.2 provides a formula for k^*, namely $k^* = \kappa^*/(\lambda_1\lambda_2)$. If $p^* < \infty$ (or, equivalently, if φ is bounded) and if ψ is continuously differentiable in $(p^*-\varepsilon, p^*+\varepsilon)$ and in $(-p^*-\varepsilon, -p^*+\varepsilon)$ for some $\varepsilon > 0$, then part C of Theorem 3.2 remains true with $\delta = 0$. Furthermore, under the hypotheses of part C of Theorem 3.2, we may infer that the tracking objective is achievable by *constant* gain $k \in (0, \kappa^*/(\lambda_1\lambda_2))$ and, moreover, under additional conditions on the nonlinearities, the convergence is of exponential order.

Proof of Theorem 3.2. A. By hypothesis, there exists $\varphi^r \in \mathbb{R}$ such that $\lim_{n\to\infty} \varphi(u_n) = \varphi^r$ which, together with the power-stability of A, implies $\lim_{n\to\infty} w_n = G(1)\varphi^r$. Evidently, $\varphi^r \in \overline{\Phi}$ and so, to establish (a1), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that

$$\theta := \frac{r - \psi(G(1)\varphi^r)}{2} \neq 0\,.$$

Using continuity of ψ, we obtain for sufficiently large $n_0 \in \mathbb{N}_0$

$$|y_n - \psi(G(1)\varphi^r)| \leq |\theta| \qquad \forall n \geq n_0\,.$$

As a consequence, and noticing that $u_{n+1} - u_n = k_n(r - y_n) = k_n(2\theta - y_n + \psi(G(1)\varphi^r))$, we have

$$-|\theta|k_n \leq u_{n+1} - u_n - 2\theta k_n \leq |\theta|k_n \qquad \forall n \geq n_0\,.$$

Since $\theta \neq 0$, either $\theta > 0$ or $\theta < 0$. Assume $\theta > 0$. Then

$$u_{n+1} - u_n \geq \theta k_n \qquad \forall n \geq n_0\,,$$

which, on summation, yields

$$u_{n+1+n_0} - u_{n_0} \geq \theta(K_{n+n_0} - K_{n_0-1}) \qquad \forall n \in \mathbb{N}_0\,.$$

Since $K_n \to \infty$ as $n \to \infty$, we conclude that $u_n \to \infty$ as $n \to \infty$, hence

$$\sup \overline{\Phi} = \lim_{n\to\infty} \varphi(u_n) = \varphi^r\,. \tag{8}$$

Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. By definition of Ψ^r this means in particular that $\psi(G(1)\varphi^*) = r$. Since $\theta > 0$, the monotonicity of ψ implies $\varphi^* > \varphi^r$ and, consequently, by (8), $\sup \overline{\Phi} < \varphi^* \in \overline{\Phi}$, a contradiction. A similar argument shows that the assumption $\theta < 0$ also leads to a contradiction. Therefore, we may conclude $\varphi^r \in \Psi^r \cap \overline{\Phi}$ which is statement (a1). Statement (a2) follows from (a1) and the power-stability of A. Statement (a3) is a consequence of (a1), (a2) and continuity of ψ. Next, we establish statement (a4). Assume $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ which, together with (a1), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\operatorname{dist}(u_n, \varphi^{-1}(\varphi^r)) \not\to 0$ as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence (u_{n_k}) of (u_n) such that

$$\operatorname{dist}(u_{n_k}, \varphi^{-1}(\varphi^r)) \geq \varepsilon \quad \forall k\,. \tag{9}$$

If the sequence (u_{n_k}) is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (9). Therefore, we may assume that (u_{n_k}) is unbounded. Extracting a subsequence if necessary, we may then assume that either $u_{n_k} \to \infty$ or $u_{n_k} \to -\infty$ as $k \to \infty$: if the former holds, then $u_{n_k} > \xi^*$ for all k sufficiently large; if the latter holds, then $u_{n_k} < \xi^*$ for all k sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u_{n_k}) = \varphi(\xi^*) = \varphi^r$ for all k sufficiently large. Clearly, this contradicts (9) and so statement (a4) must hold.

Now, assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$ and, for contradiction, suppose that (u_n) is unbounded. Then there exists a sequence $(n_k) \subset \mathbb{N}$ with $n_k \to \infty$ and $|u_{n_k}| \to \infty$ as $k \to \infty$. By monotonicity of φ and (a1), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \operatorname{int}(\Phi) \subset \operatorname{int}(\Phi)$. Therefore, (u_n) is bounded. This completes the proof of part A.

B. Let $\varepsilon > 0$. Then power-stability of A implies that

$$|w_n| = |Cx_n + D\varphi(u_n)| \leq |CA^n x_0| + \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \leq p^* + \varepsilon =: q \in (0, \infty]$$

for all n sufficiently large. Hence, there exists $n_0 \in \mathbb{N}_0$ such that

$$\psi(w_n) = \Pi_q \psi(w_n) \qquad \forall n \geq n_0\,. \tag{10}$$

Define $\hat{\Psi}^r := \{\xi \in \mathbb{R} \,|\, \Pi_q \psi(G(1)\xi) = r\}$. We claim that

$$\hat{\Psi}^r \cap \overline{\Phi} = \Psi^r \cap \overline{\Phi}\,. \tag{11}$$

To see this, note that, using (7), we have $G(1)|\xi| \leq p^*$ for all $\xi \in \overline{\Phi}$: therefore, $\Pi_q \psi(G(1)\xi) = \psi(G(1)\xi)$ for all $\xi \in \overline{\Phi}$ and so $\Psi^r \cap \overline{\Phi} = \hat{\Psi}^r \cap \overline{\Phi}$. This establishes (11).

Since $r \in \mathcal{R}$, we may conclude from (11) that $\hat{\Psi}^r \neq \emptyset$. Let $\varphi^* \in \hat{\Psi}^r$; since $\Pi_q \psi \in \mathcal{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_q \psi(\xi + G(1)\varphi^*) - r \tag{12}$$

is in $\mathcal{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ (cf. Remark 2.1(ii)). Therefore, by Remark 2.1(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi \qquad \forall \xi \in \mathbb{R}, \tag{13}$$

where

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi & , \quad \xi \neq 0 \\ \tilde{\lambda}_2 & , \quad \xi = 0\,. \end{cases} \tag{14}$$

By Remark 2.1(i), $0 \le \gamma(\xi) \le \tilde{\lambda}_2$ for all $\xi \in \mathbb{R}$. Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\lambda := \lambda_1 \tilde{\lambda}_2 \,, \qquad k^* := \kappa^*/\lambda \,,$$

where κ^* is given by (6).

Let $(k_n) \in \mathcal{G}$ be such that $\limsup_{n\to\infty} k_n < k^*$. Choose $k_* > 0$ such that

$$\limsup_{n\to\infty} k_n < k_* < k^* \,. \tag{15}$$

By Lemma 3.1 there exists a $P \in \mathbb{R}^{N\times N}$, with $P = P^T \ge 0$, such that

$$\Lambda := \begin{bmatrix} A^T P A - P & A^T P (I-A)^{-1} B + C^T \\ ((I-A)^{-1}B)^T P A + C & ((I-A)^{-1}B)^T P (I-A)^{-1} B - 2/(k_*\lambda) + G(1) \end{bmatrix} < 0 \,.$$

Introduce new variables

$$z_n := x_n - (I-A)^{-1} B \varphi(u_n) \,, \quad v_n := \varphi(u_n) - \varphi^* \qquad \forall n \ge 0 \,,$$

and notice that, in terms of the new variables, w_n is given by

$$w_n = C x_n + D\varphi(u_n) = C z_n + G(1)(v_n + \varphi^*) = \tilde{w}_n + G(1)\varphi^* \,, \quad \text{where } \tilde{w}_n := C z_n + G(1) v_n \,.$$

Define the sequence $(\delta_n) \subset [0, \lambda_1]$ (as in Lemma 2.2) by

$$\delta_n := \begin{cases} (\varphi(u_{n+1}) - \varphi(u_n))/(u_{n+1} - u_n) & , \quad u_{n+1} \ne u_n \\ \lambda_1 & , \quad u_{n+1} = u_n \,. \end{cases}$$

Using (10) and (12), an easy calculation yields

$$\begin{aligned} z_{n+1} &= A z_n + \delta_n k_n (I-A)^{-1} B \tilde{\psi}(\tilde{w}_n) \qquad \forall n \ge n_0 \,, \\ v_{n+1} &= v_n - \delta_n k_n \tilde{\psi}(\tilde{w}_n) \qquad \forall n \ge n_0 \,. \end{aligned}$$

Defining

$$\zeta_n := k_n \eta_n \tilde{w}_n \quad \text{ with } \quad \eta_n := \delta_n \gamma(\tilde{w}_n) \tag{16}$$

and invoking (13), we obtain

$$z_{n+1} = A z_n + (I-A)^{-1} B \zeta_n \qquad \forall n \ge n_0 \,, \tag{17a}$$

$$v_{n+1} = v_n - \zeta_n \qquad \forall n \ge n_0 \,. \tag{17b}$$

Note that

$$0 \le \eta_n \le \lambda_1 \tilde{\lambda}_2 = \lambda \qquad \forall n \ge 0 \,. \tag{18}$$

We shall investigate asymptotic properties of (z_n, v_n) using a Lyapunov approach. Define the sequence (V_n) by

$$V_n = \langle z_n, P z_n \rangle + G(1) v_n^2 \,.$$

Then, using (17) and (16)

$$\begin{aligned} V_{n+1} - V_n &= \langle z_n, (A^T P A - P) z_n \rangle + 2((I-A)^{-1}B)^T P A z_n \zeta_n + \\ &\qquad [((I-A)^{-1}B)^T P (I-A)^{-1} B + G(1)] \zeta_n^2 - 2G(1) v_n \zeta_n \\ &= \left\langle [z_n^T, \zeta_n]^T, \Lambda [z_n^T, \zeta_n]^T \right\rangle + (2/(k_* \lambda)) \zeta_n^2 - 2[C z_n + G(1) v_n] \zeta_n \\ &\le -\alpha \left[\|z_n\|^2 + \zeta_n^2 \right] - 2 k_n \eta_n \left[1 - \frac{k_n \eta_n}{k_* \lambda} \right] \tilde{w}_n^2 \quad \forall n \ge n_0 \,, \end{aligned}$$

where $\alpha = 1/\|\Lambda^{-1}\|$. Invoking (15) and (18), we see that there exists $n_1 \ge n_0$ such that

$$\sup_{n \ge n_1} k_n \eta_n < k_* \lambda \,.$$

Consequently, there exists $\beta > 0$ such that

$$V_{n+1} - V_n \le -\alpha \|z_n\|^2 - \alpha \zeta_n^2 - \beta k_n \eta_n \tilde{w}_n^2 \qquad \forall n \ge n_1 . \tag{19}$$

It follows that $(z_n) \in l^2(\mathbb{R}^N)$ and $(\zeta_n) \in l^2(\mathbb{R})$. Since $k_n \eta_n v_n = -(G(1))^{-1}[k_n \eta_n C z_n - \zeta_n]$, we may conclude that $(k_n \eta_n v_n) \in l^2(\mathbb{R})$ and thus

$$(k_n \eta_n v_n C z_n) \in l^1(\mathbb{R}) \,. \tag{20}$$

Moreover, using (19) and the definition of $\tilde{w}_n$, we obtain

$$V_{n+1} - V_n \le -2\beta G(1) k_n \eta_n v_n C z_n - \beta G^2(1) k_n \eta_n v_n^2 \qquad \forall n \ge n_1 \,,$$

and so we have

$$0 \le V_{m+1+n_1} \le V_{n_1} - 2\beta G(1) \sum_{n=n_1}^{m+n_1} k_n \eta_n v_n C z_n - \beta G^2(1) \sum_{n=n_1}^{m+n_1} k_n \eta_n v_n^2 \quad \forall m \in \mathbb{N}_0 \,,$$

which, together with (20), implies that

$$(k_n \eta_n v_n^2) \in l^1(\mathbb{R}) \,. \tag{21}$$

From (20) and (21), we may infer that $v_n \zeta_n = k_n \eta_n v_n [C z_n + G(1) v_n] \in l^1(\mathbb{R})$. By (17b), summation yields

$$\lim_{m \to \infty} v_{m+1+n_1}^2 = v_{n_1}^2 + \lim_{m \to \infty} \sum_{n=n_1}^{m+n_1} \zeta_n^2 - 2 \lim_{m \to \infty} \sum_{n=n_1}^{m+n_1} v_n \zeta_n = c \tag{22}$$

for some constant $c \in [0, \infty)$. If $c = 0$, then $\lim_{n \to \infty} v_n = 0$ and so $\lim_{n \to \infty} \varphi(u_n) = \varphi^*$. Now consider the remaining case $c > 0$. Since $(\zeta_n) \in l^2(\mathbb{R})$, we may conclude from (17b) that

$$\lim_{n \to \infty} |v_{n+1} - v_n| = \lim_{n \to \infty} |\zeta_n| = 0 \,. \tag{23}$$

Since $c \neq 0$, (22) and (23) imply that v_n is of constant sign for all n sufficiently large. Thus $\lim_{n \to \infty} v_n$ exists and takes one of the values $\pm\sqrt{c}$. This establishes statement (b1).

To prove (b2), assume that $\limsup_{n \to \infty} k_n < k^*$ and $K_n \to \infty$ as $n \to \infty$. Then it immediately follows from A and (b1) that (a1) to (a5) hold; in particular, there exists $\varphi^r \in \Psi^r \cap \overline{\Phi}$ such that $\lim_{n \to \infty} \varphi(u_n) = \varphi^r$.

Now assume that (k_n) is ultimately non-increasing, $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$. To show that the convergence is of order ρ^{-K_n} for some $\rho > 1$, notice that, by hypothesis, $\varphi^r \in \Phi$ and $\varphi^r \notin \mathcal{C}(\varphi)$. Thus, by monotonicity of φ, the preimage $\varphi^{-1}(\varphi^r)$ is a singleton $\{u^r\}$. Moreover, by (a4), $u_n \to u^r$ as $n \to \infty$. By Lemma 2.2, there exist $\mu > 0$ and $n_2 \geq n_1$ such that

$$0 < \mu \leq \delta_n \qquad \forall n \geq n_2 \,. \tag{24}$$

In the argument leading to (b1), replace φ^* by φ^r. Since by assumption $r \notin \mathcal{C}(\psi)$, we have that

$$\tilde{\psi}^-(0) = (\Pi_q \psi)^-(G(1)\varphi^r) = \psi^-(G(1)\varphi^r) > 0 \,.$$

We claim that there exist $n_3 \geq n_2$ and $0 < \tilde{\mu} < \lambda$ such that $\tilde{\mu} \leq \eta_n := \delta_n \gamma(\tilde{w}_n)$ for all $n \geq n_3$. Seeking a contradiction, suppose the claim is not true. Then, by (24), there is a sequence (n_k) with $n_k \to \infty$ as $k \to \infty$ such that

$$0 \leq \gamma(\tilde{w}_{n_k}) \leq 1/k \,.$$

By (a1) and (a2) we know that $\tilde{w}_{n_k} \to 0$ for $k \to \infty$. Now, using (14) and Lemma 2.3,

$$0 = \lim_{k\to\infty} \gamma(\tilde{w}_{n_k}) \geq \liminf_{h\to 0} \frac{\tilde{\psi}(h)}{h} \geq \tilde{\psi}^-(0) > 0 \,,$$

a contradiction. Thus, there exist $\tilde{\mu} \in (0, \lambda)$ and $n_3 \geq n_2$ such that $\tilde{\mu} \leq \eta_n$ for all $n \geq n_3$.

Let $\alpha = 1/\|\Lambda^{-1}\|$ and β be such that (19) holds. Fix $\nu > 0$ and $\rho > 1$ such that

$$0 < \beta\tilde{\mu}k^*\|C\|^2\nu \leq \frac{\alpha}{4} \tag{25}$$

and

$$0 < 2\ln\rho \leq \min\left\{\frac{\alpha}{4\|P\|k^*}, \frac{\nu}{1+\nu}G(1)\beta\tilde{\mu}\right\} . \tag{26}$$

Introduce weighted variables by defining, for all $n \in \mathbb{N}_0$,

$$z_n^\rho := \rho^{K_n}(x_n - (I-A)^{-1}B\varphi(u_n)) = \rho^{K_n} z_n \,, \quad v_n^\rho := \rho^{K_n}(\varphi(u_n) - \varphi^r) = \rho^{K_n} v_n \,,$$
$$\tilde{w}_n^\rho := \rho^{K_n}\tilde{w}_n \,, \quad \zeta_n^\rho := \rho^{K_n}\zeta_n = k_n\eta_n\tilde{w}_n^\rho \,.$$

We use the non-negative sequence (W_n) given by

$$W_n = \langle z_n^\rho, Pz_n^\rho\rangle + G(1)(v_n^\rho)^2$$

to investigate asymptotic properties of (z_n^ρ, v_n^ρ). Since (k_n) is ultimately non-increasing and $\limsup_{n\to\infty} k_n < k^*$, there exists $n_4 \geq n_3$ such that

$$(1 - 1/\rho^{2k_{n+1}}) \leq (1 - 1/\rho^{2k_n}) \leq 2k_n\ln\rho \leq 2k^*\ln\rho \qquad \forall n \geq n_4 \,. \tag{27}$$

Invoking inequalities (19),(26),(27) and recalling that $\tilde{\mu} \leq \eta_n$, we conclude that for all $n \geq n_4$

$$\begin{aligned}
W_{n+1} - W_n &= \langle z_{n+1}^\rho, Pz_{n+1}^\rho\rangle + G(1)(v_{n+1}^\rho)^2 - \langle z_n^\rho, Pz_n^\rho\rangle - G(1)(v_n^\rho)^2 \\
&= \rho^{2k_{n+1}}\Big[\langle z_{n+1}^\rho/\rho^{k_{n+1}}, Pz_{n+1}^\rho/\rho^{k_{n+1}}\rangle + G(1)(v_{n+1}^\rho/\rho^{k_{n+1}})^2 - \langle z_n^\rho, Pz_n^\rho\rangle - G(1)(v_n^\rho)^2 \\
&\qquad + (1 - 1/\rho^{2k_{n+1}})\left(\langle z_n^\rho, Pz_n^\rho\rangle + G(1)(v_n^\rho)^2\right)\Big] \\
&\leq \rho^{2k_{n+1}}\left[\rho^{2K_n}(V_{n+1} - V_n) + 2k_n\ln\rho\,\left(\langle z_n^\rho, Pz_n^\rho\rangle + G(1)(v_n^\rho)^2\right)\right] \\
&\leq -\rho^{2k_{n+1}}\left[\alpha\|z_n^\rho\|^2 + \alpha(\zeta_n^\rho)^2 + \beta\tilde{\mu}k_n(\tilde{w}_n^\rho)^2 - 2k^*\ln\rho\,\langle z_n^\rho, Pz_n^\rho\rangle - 2k_n\ln\rho\,G(1)(v_n^\rho)^2\right] \\
&\leq -\rho^{2k_{n+1}}\left[(3/4)\alpha\|z_n^\rho\|^2 + \beta\tilde{\mu}k_n(\tilde{w}_n^\rho)^2 - 2k_n\ln\rho\,G(1)(v_n^\rho)^2\right] .
\end{aligned} \tag{28}$$

Observing that

$$\begin{aligned}(\tilde{w}_n^\rho)^2 &= (Cz_n^\rho + G(1)v_n^\rho)^2 \\ &= -\nu(Cz_n^\rho)^2 + \nu(1+\nu)^{-1}G^2(1)(v_n^\rho)^2 + (1+\nu)\left(Cz_n^\rho + (1+\nu)^{-1}G(1)v_n^\rho\right)^2 \\ &\geq -\nu\|C\|^2\|z_n^\rho\|^2 + \nu(1+\nu)^{-1}G^2(1)(v_n^\rho)^2\,,\end{aligned}$$

it now follows from (25), (26) and (28) that for all $n \geq n_4$

$$\begin{aligned}W_{n+1} - W_n \leq -\rho^{2k_{n+1}}\big[&(3\alpha/4)\|z_n^\rho\|^2 - \beta\tilde{\mu}k_n\nu\|C\|^2\|z_n^\rho\|^2 \\ &+\beta\tilde{\mu}k_n\nu(1+\nu)^{-1}G^2(1)(v_n^\rho)^2 - 2k_n\ln\rho\, G(1)(v_n^\rho)^2\big] \\ \leq -\frac{\alpha}{2}&\|z_n^\rho\|^2\,.\end{aligned}$$

This implies $\sum_{n=n_4}^{\infty}\|z_n^\rho\|^2 \leq (2/\alpha)W_{n_4}$ and hence boundedness of (z_n^ρ). Since, moreover, $(v_n^\rho)^2 \leq W_{n_4}/G(1)$ for all $n \geq n_4$, we also have boundedness of (v_n^ρ) and so, convergence of order ρ^{-K_n} in (a1) and (a2) follows. Next we show convergence of order ρ^{-K_n} in (a3). The hypothesis $r \notin \mathcal{C}(\psi)$ implies that $\psi^{-1}(r)$ is a singleton: $\psi^{-1}(r) = \{w^r\}$, where $w^r = G(1)\varphi^r$ and $w_n \to w^r$ as $n \to \infty$. Convergence of order ρ^{-K_n} in (a1) and (a2) implies the existence of $L > 0$ such that

$$|w^r - w_n| \leq L\rho^{-K_n} \quad \forall n \in \mathbb{N}_0\,.$$

Let λ be a Lipschitz constant for ψ restricted to the compact set $\{w_n \,|\, n \in \mathbb{N}_0\} \cup \{w^r\}$. Then

$$|r - \psi(w_n)| \leq \lambda|w^r - w_n| \leq \lambda L\rho^{-K_n} \quad \forall n \in \mathbb{N}_0\,,$$

implying convergence of order ρ^{-K_n} in (a3).

C. Let $\lambda_1 > 0$ be a Lipschitz constant for φ. By hypothesis, $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$. By Remark 2.1(iii) it follows that, for any fixed $r \in \mathcal{R}$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_{(p^*+\delta)}\psi(\xi + G(1)\varphi^r) - r$$

is also in $\mathcal{M}_L(\lambda_2)$. The argument used in the proof of part B above applies *mutatis mutandis* (specifically, on replacing ε by δ, φ^* by φ^r and $\tilde{\lambda}_2$ by λ_2) to conclude that (b1) and (b2) hold with $k^* = \kappa^*/(\lambda_1\lambda_2)$. □

3.3 Adaptive gain

Whilst Theorem 3.2B (see also Remark 3.3(ii)) identifies conditions under which the tracking objective is achieved through the use of a prescribed gain sequence, the resulting control strategy is somewhat unsatisfactory insofar as the gain sequence is selected *a priori:* no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain sequence. Let $\mathcal{L}$ denote the class of locally Lipschitz functions mapping from $\mathbb{R}_+$ into $\mathbb{R}_+$ with value zero only at zero and which satisfy a particular growth condition near zero, specifically:

$$\mathcal{L} := \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ locally Lipschitz},\ f^{-1}(0) = \{0\},\ \liminf_{\xi \downarrow 0}\xi^{-1}f(\xi) > 0\}.$$

Let $\chi \in \mathcal{L}$ and let the gain sequence (k_n) be generated by the following adaptation law:

$$k_n = \frac{1}{l_n}, \quad l_{n+1} = l_n + \chi(|r - y_n|), \quad l_0 > 0\,. \tag{29}$$

This leads to the feedback system

$$x_{n+1} = Ax_n + B\varphi(u_n)\,, \quad x_0 \in \mathbb{R}^N\,, \tag{30a}$$
$$u_{n+1} = u_n + l_n^{-1}(r - \psi(Cx_n + D\varphi(u_n)))\,, \quad u_0 \in \mathbb{R}, \tag{30b}$$
$$l_{n+1} = l_n + \chi(|r - \psi(Cx_n + D\varphi(u_n))|)\,, \quad l_0 \in (0,\infty)\,. \tag{30c}$$

We now arrive at the main adaptive tracking result.

Corollary 3.4 *Let $\Sigma = (A,B,C,D) \in \mathcal{S}$, $(\varphi,\psi) \in \mathcal{N}$, $\chi \in \mathcal{L}$ and $r \in \mathcal{R}$. Assume furthermore that, if φ is unbounded, then there exists $\delta > 0$ such that $\chi(\xi) \geq \delta\xi$ for all $\xi \in \mathbb{R}_+$.*
For each $(x_0,u_0,l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0,\infty)$, the solution $n \mapsto (x_n,u_n,l_n)$ of the initial-value problem (30) is such that statements (a1) to (a5) of Theorem 3.2 hold. Moreover, if $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$, then the non-increasing gain sequence $(k_n) = (l_n^{-1})$ converges to a positive value.

Proof. Let $(x_0,u_0,l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0,\infty)$. Since (l_n) is non-decreasing, either $l_n \to \infty$ as $n \to \infty$ (*Case* 1), or $l_n \to \hat{l} \in (0,\infty)$ as $n \to \infty$ (*Case* 2). We consider these two cases separately.

Case 1: In this case, $k_n \downarrow 0$ as $n \to \infty$ and by Theorem 3.2(b1), $(\varphi(u_n))$ and consequently (x_n) converge as $n \to \infty$, and so, in particular, are bounded sequences. Therefore, there exists $\alpha > 0$ such that

$$k_n = \frac{1}{l_n} \geq \frac{1}{l_0 + \alpha n} \qquad \forall n \in \mathbb{N}_0\,, \tag{31}$$

and so $K_n \to \infty$ as $n \to \infty$. Therefore, by Theorem 3.2(b2), statements (a1) to (a5) of Theorem 3.2 hold.

Case 2: In this case, $k_n \to \hat{k} := 1/\hat{l} > 0$ as $n \to \infty$ and so $K_n \to \infty$ as $n \to \infty$. In order to conclude that statements (a1) to (a5) of Theorem 3.2 hold, it suffices to show (by part A of Theorem 3.2) that (u_n) converges to a finite limit as $n \to \infty$: in view of (30b), we will establish the latter convergence by showing that $(e_n) := (r - y_n)$ is of class $l^1(\mathbb{R})$. Note first that, using (30c) and boundedness of (l_n), we may conclude that

$$\sum_{n=0}^{\infty} \chi(|e_n|) < \infty\,. \tag{32}$$

First assume that φ is unbounded. Then, by hypothesis, $\chi(|e_n|) \geq \delta|e_n|$ for all $n \in \mathbb{N}_0$ which, together with (32), implies that $(e_n) \in l^1(\mathbb{R})$. Next, assume φ is bounded. Then, by the power-stability of A, it follows that (e_n) is bounded. By (32) we have $\chi(|e_n|) \to 0$ as $n \to \infty$ which, recalling that $\chi^{-1}(0) = \{0\}$ and invoking boundedness of (e_n), implies that $|e_n| \to 0$ as $n \to \infty$. Since $\liminf_{\xi \downarrow 0} \xi^{-1}\chi(\xi) > 0$, we may infer the existence of $n_0 \in \mathbb{N}_0$ and $\delta > 0$ such that $\chi(|e_n|) \geq \delta|e_n|$ for all $n \geq n_0$ which, together with (32), implies that $(e_n) \in l^1(\mathbb{R})$. Therefore, (u_n) converges to a finite limit as $n \to \infty$. Hence, the hypotheses of part A of Theorem 3.2 are satisfied and so statements (a1) to (a5) of Theorem 3.2 hold.

Finally, assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$. We will show that the monotone gain sequence (k_n) converges to a positive value. Seeking a contradiction, suppose

that the monotone sequence (l_n) is unbounded (equivalently, $k_n \downarrow 0$ as $n \to \infty$). Then the hypothesis of Theorem 3.2(b1) is satisfied: therefore, $(\varphi(u_n))$ and (x_n) are bounded and so (31) holds for some $\alpha > 0$. Therefore, $K_n \to \infty$ as $n \to \infty$ and so, by Theorem 3.2(b2), $e_n \to 0$ as $n \to \infty$, and the convergence is of order ρ^{-K_n} for some $\rho > 1$. This, together with the fact that χ is locally Lipschitz with $\chi(0) = 0$, implies the existence of a constant $L > 0$ such that

$$\chi(|e_n|) \le L\rho^{-K_n} \qquad \forall n \in \mathbb{N}_0 \,. \tag{33}$$

Choose $\beta \ge \alpha$ such that $(\ln \rho)/\beta < 1$. By (31), $k_n = 1/l_n \ge (l_0 + \beta n)^{-1}$ for all $n \in \mathbb{N}_0$. Therefore,

$$K_n \ge \sum_{j=0}^{n-1} \frac{1}{l_0 + \beta j} \ge \sum_{j=0}^{n-1} \int_j^{j+1} \frac{1}{l_0 + \beta t} dt = \int_0^n \frac{1}{l_0 + \beta t} dt = \ln\left[((l_0 + \beta n)/l_0)^{1/\beta}\right] . \tag{34}$$

By (33),

$$l_{n+1} \le l_n + L\rho^{-K_n} \qquad \forall n \in \mathbb{N}_0 \,,$$

and so, using (34)

$$l_n \le l_0 + L \sum_{j=0}^{n-1} \exp\left(\ln\left[((l_0 + \beta j)/l_0)^{-\eta}\right]\right) = l_0 + L_1 \sum_{j=0}^{n-1} (l_0 + \beta j)^{-\eta} \quad \forall n \in \mathbb{N},$$

where $\eta := (\ln \rho)/\beta \in (0, 1)$ and $L_1 := L l_0^{\eta}$. Therefore,

$$l_n \le l_0 + L_1 \left[l_0^{-\eta} + \sum_{j=0}^{n-1} \int_{j-1}^{j} (l_0 + \beta t)^{-\eta} dt \right] = l_0 + L_1 \left[l_0^{-\eta} + \int_0^{n-1} (l_0 + \beta t)^{-\eta} dt \right] \quad \forall n \in \mathbb{N},$$

and we see that, for some suitable constant $L_2 > 0$,

$$l_n \le L_2 (l_0 + \beta(n-1))^{1-\eta} \le L_2 (l_0 + \beta n)^{1-\eta} \quad \forall n \in \mathbb{N}.$$

From this we may conclude that

$$-K_n = -L_2^{-1} \sum_{j=0}^{n} (l_0 + \beta j)^{\eta - 1} \le -L_2^{-1} \sum_{j=0}^{n-1} (l_0 + \beta j)^{\eta - 1} \le -L_2^{-1} \sum_{j=0}^{n-1} \int_j^{j+1} (l_0 + \beta t)^{\eta - 1} dt \,,$$

and thus

$$-K_n \le -L_2^{-1} \int_0^n (l_0 + \beta t)^{\eta - 1} = -a(l_0 + \beta n)^{\eta} + b \,,$$

with $a := 1/(L_2 \beta \eta) > 0$ and $b := a l_0^{\eta} > 0$. Therefore,

$$\rho^{-K_n} \le \rho^b \exp(-a(\ln \rho)((l_0 + \beta n)^{\eta}) = \rho^b \exp(-\gamma (l_0 + \beta n)^{\eta}) \,, \tag{35}$$

where $\gamma := a \ln \rho > 0$. Using a change of variables it is readily verified that the non-increasing positive-valued function f defined by

$$f : t \mapsto \rho^b \exp(-\gamma (l_0 + \beta t)^{\eta}) \quad \forall t \ge 0 \,,$$

is in $L^1(\mathbb{R}_+, \mathbb{R})$. By the integral test for the convergence of an infinite series, it follows that the sequence $(f(n))$ is summable. By (35) we may conclude that the sequence (ρ^{-K_n}) is summable. Combining this with (33) shows that (l_n) is bounded, contradicting the supposition of unboundedness of (l_n). □

3.4 Necessity of the feasibility condition

In Theorem 3.2 and Corollary 3.4 we imposed the condition $r \in \mathcal{R}$. In the next proposition we show that this condition is close to being necessary insofar as, if tracking of $r \in \mathbb{R}$ is achievable whilst maintaining boundedness of $(\varphi(u_n))$, then $r \in \mathcal{R}$.

Proposition 3.5 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and let ψ be continuous and monotone. Let $(u_n) \subset \mathbb{R}$ be such that $\varphi(u_n)$ is bounded. For $x_0 \in \mathbb{R}^N$, let $n \mapsto x_n$ denote the bounded solution of the initial-value problem* (2a)*. If*

$$\lim_{n\to\infty} [\psi(Cx_n + D\varphi(u_n))] = r\,, \tag{36}$$

then $r \in \Psi^r \cap \overline{\Phi}$, where $\Psi^r := \{v \in \mathbb{R} \mid \psi(G(1)v) = r\}$ and $\overline{\Phi} := \mathrm{clos}(\mathrm{im}\,\varphi)$.

Proof. The sequence (w_n), where $w_n = Cx_n + D\varphi(u_n)$, is bounded and so has non-empty and compact (but not necessarily connected) ω-limit set Ω. From the definition of ω-limit sets, continuity of ψ and (36), it follows that

$$\psi(\omega) = r \qquad \forall \omega \in \Omega\,. \tag{37}$$

Denote by $\mathrm{co}\,(\Omega)$ the convex hull of Ω. We claim that $\mathrm{co}\,(\Omega) \cap G(1)\overline{\Phi} \neq \emptyset$. Seeking a contradiction, suppose $\mathrm{co}\,(\Omega) \cap G(1)\overline{\Phi} = \emptyset$. Then $\mathrm{co}\,(\Omega)$ and $G(1)\overline{\Phi}$ are disjoint closed intervals and so there exist $\varepsilon > 0$ and $\beta = \pm 1$ such that

$$\inf\{\beta\omega \mid \omega \in \mathrm{co}\,(\Omega)\} - \sup\{\beta G(1)v \mid v \in \overline{\Phi}\} = 2\varepsilon\,.$$

Moreover, since w_n approaches its compact ω-limit set Ω (and thus $\mathrm{co}\,(\Omega)$) as $n \to \infty$, there exists $n_0 \in \mathbb{N}_0$ such that

$$\inf \mathrm{co}\,(\Omega) - \varepsilon < w_n < \sup \mathrm{co}\,(\Omega) + \varepsilon \quad \forall n \geq n_0\,,$$

and so $\beta w_n > \inf\{\beta\omega \mid \omega \in \mathrm{co}\,(\Omega)\} - \varepsilon$ for all $n \geq n_0$. Hence, we obtain

$$\beta C(A - I)^{-1}(x_{n+1} - x_n) = \beta(w_n - G(1)\varphi(u_n)) \geq \varepsilon \qquad \forall n \geq n_0\,,$$

and thus

$$\beta C(A - I)^{-1}(x_{n+1+n_0} - x_{n_0}) = \beta C(A - I)^{-1} \sum_{j=n_0}^{n+n_0} (x_{j+1} - x_j) \geq n\varepsilon \quad \forall n \in \mathbb{N}_0\,,$$

which contradicts the boundedness of (x_n). Therefore, $\mathrm{co}\,(\Omega) \cap G(1)\overline{\Phi} \neq \emptyset$.
Although the set $\mathrm{co}\,(\Omega) \cap G(1)\overline{\Phi}$ cannot be empty, $\Omega \cap G(1)\overline{\Phi}$ may (*Case* 1) or may not (*Case* 2) be empty. We consider these cases separately.

Case 1: If $\Omega \cap G(1)\overline{\Phi} = \emptyset$, then, since $\mathrm{co}\,(\Omega) \cap G(1)\overline{\Phi} \neq \emptyset$, we infer the existence of $\omega_1, \omega_2 \in \Omega$ such that

$$\omega_1 < \inf\{G(1)\overline{\Phi}\} \leq \sup\{G(1)\overline{\Phi}\} < \omega_2\,. \tag{38}$$

Equation (37) together with (38) and monotonicity of ψ gives $\overline{\Phi} \subset \Psi^r$ and so $\Psi^r \cap \overline{\Phi} \neq \emptyset$.

Case 2: If $\Omega \cap G(1)\overline{\Phi} \neq \emptyset$, then there exist $\omega \in \Omega$ and $v \in \overline{\Phi}$ such that $\omega = G(1)v$. Therefore, by (37),

$$\psi(G(1)v) = \psi(\omega) = r$$

and so $v \in \Psi^r \cap \overline{\Phi}$ or, equivalently, $\Psi^r \cap \overline{\Phi} \neq \emptyset$. ∎

4 Sampled-data low-gain control of continuous-time systems

In this section we apply the results of Section 3 to solve the continuous-time low-gain tracking problem, by sampled-data 'integral' control, for the class of systems introduced below.

4.1 The class of continuous-time systems and the tracking objective

Tracking results will be derived for a class of finite-dimensional (state space $\mathbb{R}^N$) single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$), continuous-time (time domain $\mathbb{R}_+ := [0, \infty)$), real linear systems $\Sigma_c = (A_c, B_c, C_c, D_c)$ having a nonlinearity in the input and output channel:

$$\dot{x} = A_c x + B_c \varphi(u)\,, \quad x_0 := x(0) \in \mathbb{R}^N\,, \tag{39a}$$

$$y = \psi(C_c x + D_c \varphi(u))\,. \tag{39b}$$

In (39), it is assumed that A_c is Hurwitz, *i.e.*, each eigenvalue of A_c has negative real part and $(\varphi, \psi) \in \mathcal{N}$. Furthermore, the transfer function G_c, given by

$$G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c\,,$$

is assumed to satisfy $G_c(0) > 0$. The underlying class of real continuous-time linear systems $\Sigma_c = (A_c, B_c, C_c, D_c)$ is denoted

$$\mathcal{S}_c := \{\Sigma_c = (A_c, B_c, C_c, D_c) \mid A_c \text{ Hurwitz},\ G_c(0) = D_c - C_c A_c^{-1} B_c > 0\}.$$

Given $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{S}_c$ and $(\varphi, \psi) \in \mathcal{N}$, the objective is to determine a sampling period $\tau > 0$ and, by sampled-data control, a sequence $(u_n) \subset \mathbb{R}$ such that, for given $r \in \mathbb{R}$, the output y of (39), resulting from the input u given by

$$u(t) = u_n \qquad \text{for } t \in [n\tau, (n+1)\tau)\,,\ n \in \mathbb{N}_0\,, \tag{40}$$

has the property $y(t) \to r$ as $t \to \infty$. As in Section 3, it will not be possible to track arbitrary $r \in \mathbb{R}$. Instead, we require $r \in \mathcal{R}_c$, where

$$\mathcal{R}_c := \{r \in \mathbb{R} \mid \Psi_c^r \cap \overline{\Phi} \neq \emptyset\}$$

and

$$\Psi_c^r := \{v \in \mathbb{R} \mid \psi(G_c(0)v) = r\}, \quad \Phi := \operatorname{im} \varphi, \quad \overline{\Phi} := \operatorname{clos}(\Phi)\,.$$

Again, it may be shown that $r \in \mathcal{R}_c$ is close to being necessary for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\varphi(u)$, then $r \in \mathcal{R}_c$ (see [5]).

Given a sequence $(u_n) \subset \mathbb{R}$, we define a continuous-time signal $u(\cdot)$ by the standard hold operation (40), where $\tau > 0$ denotes the sampling period. If this signal is applied to the continuous-time system (39), then, defining

$$x_n := x(n\tau) \quad \text{and} \quad y_n := y(n\tau) \qquad \forall n \in \mathbb{N}_0\,, \tag{41}$$

it follows that

$$x_{n+1} = Ax_n + B\varphi(u_n)\,, \quad x_0 := x(0) \in \mathbb{R}^N\,, \tag{42a}$$

$$y_n = \psi(Cx_n + D\varphi(u_n))\,, \tag{42b}$$

where

$$A = e^{A_c\tau},\ B = (e^{A_c\tau} - I)A_c^{-1}B_c,\ C = C_c \quad \text{and} \quad D = D_c. \tag{43}$$

Since A_c is Hurwitz, the eigenvalues of A have, for arbitrary $\tau > 0$, modulus strictly less than one, *i.e.*, the matrix A is power-stable. Furthermore, the transfer function G of $\Sigma = (A, B, C, D)$ satisfies

$$G(1) = C(I - A)^{-1}B + D = C_c(I - e^{A_c\tau})^{-1}(e^{A_c\tau} - I)A_c^{-1}B_c + D_c = G_c(0)\,. \tag{44}$$

It follows from (44) that

$$\Psi^r = \Psi_c^r \quad \text{and} \quad \mathcal{R} = \mathcal{R}_c\,,$$

where $\Psi^r = \{v \in \mathbb{R} \,|\, \psi(G(1)v) = r\}$ and $\mathcal{R}$ denotes the set of feasible reference values for the discretization (42) of (39).

4.2 Prescribed gain

We first treat the case of a prescribed gain sequence $(k_n) \in \mathcal{G}$. For this purpose, let $\tau > 0$ be arbitrary and consider the following sampled-data low-gain controller for (39)

$$u(t) = u_n\,, \qquad \text{for } t \in [n\tau, (n+1)\tau)\,,\ n \in \mathbb{N}_0\,, \tag{45a}$$

$$y_n = y(n\tau)\,, \qquad n \in \mathbb{N}_0\,, \tag{45b}$$

$$u_{n+1} = u_n + k_n(r - y_n)\,, \qquad u_0 \in \mathbb{R}. \tag{45c}$$

Theorem 4.1 *Let $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{S}_c$, $(\varphi, \psi) \in \mathcal{N}$, $\tau > 0$ and $r \in \mathcal{R}_c$. Define*

$$p^* := \Gamma_\Sigma \sup_{\xi \in \Phi} |\xi| \in (0, \infty]\,,$$

where $\Sigma = (A, B, C, D)$ denotes the discretization (43) of $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{S}_c$ and Γ_Σ is given by (7).

A. *There exists $k^* > 0$, independent of (x_0, u_0) and $(k_n) \in \mathcal{G}$, such that for all $(k_n) \in \mathcal{G}$ with the properties $\limsup_{n\to\infty} k_n < k^*$ and $K_n := \sum_{j=0}^{n} k_j \to \infty$ as $n \to \infty$, and for every $(x_0, u_0) \in \mathbb{R}^N \times \mathbb{R}$ the following hold for the solution $(x, u) : \mathbb{R}_+ \to \mathbb{R}^N \times \mathbb{R}$ of the closed loop system given by (39) and (45):*

   (i) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi_c^r \cap \overline{\Phi}$,

   (ii) $\lim_{t\to\infty} x(t) = -A_c^{-1}B_c\varphi^r$,

   (iii) $\lim_{t\to\infty} y(t) = r$, *where* $y(t) = \psi(C_c x(t) + D_c\varphi(u(t)))$,

   (iv) *if* $\Psi_c^r \cap \overline{\Phi} = \Psi_c^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

   (v) *if* $\Psi_c^r \cap \overline{\Phi} = \Psi_c^r \cap \operatorname{int}(\Phi)$, *then $u(\cdot)$ is bounded,*

   (vi) *if (k_n) is ultimately non-increasing, $\Psi_c^r \cap \overline{\Phi} = \Psi_c^r \cap \Phi$, $\Psi_c^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$, then the convergence in* (i) *to* (iii) *is of order $\exp(-\alpha K(t))$ for some $\alpha > 0$, where $K : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $K(t) = K_n$ for $t \in [n\tau, (n+1)\tau)$.*

B. *Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta > 0$ and $\lambda_2 > 0$, then statement* A *holds with $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by (6).*

Remarks 4.2
(i) An immediate consequence of Theorem 4.1 is the following: if $(k_n) \in \mathcal{G}$ is chosen such that, as $n \to \infty$, k_n tends to zero sufficiently slowly in the sense that $(k_n) \notin l^1(\mathbb{R}_+)$, then the tracking objective is achieved.

(ii) Let $\lambda_1 > 0$ be a Lipschitz constant for φ. If $\Pi_{(p^*+\delta)}\psi \in \mathcal{M}_L(\lambda_2)$ for some $\delta > 0$ and some $\lambda_2 > 0$, we may infer that the tracking objective is achievable by a *constant* gain sequence (k_n) with $k_n = k \in (0, \kappa^*/(\lambda_1\lambda_2))$ for all $n \in \mathbb{N}_0$ and, moreover, if the extra assumptions of statement (vi) are satisfied, the convergence is of exponential order.

Proof of Theorem 4.1. A. It follows from the previous subsection that x_n, y_n (given by (41)) and u_n (generated by (45c)) satisfy (42) with (A, B, C, D) given by (43). Moreover, A is power-stable and $G(1) = G_c(0) > 0$. Hence $\Sigma = (A, B, C, D) \in \mathcal{S}$. Since by assumption $K_n := \sum_{j=0}^{n} k_j \to \infty$ as $n \to \infty$, part B of Theorem 3.2 furnishes the existence of a number $k^* > 0$ such that statements (a1) to (a5) of Theorem 3.2 hold for $\Sigma = (A, B, C, D)$, whenever $\limsup_{n\to\infty} k_n < k^*$. In particular,

$$\lim_{n\to\infty} \varphi(u_n) = \varphi^r \in \Psi^r \cap \overline{\Phi} = \Psi_c^r \cap \overline{\Phi}\,.$$

Since the continuous-time signal u is given by (45a), assertions (i), (iv) and (v) follow. Assertion (ii) is a consequence of (i) and the Hurwitz property of A_c. Finally, assertion (iii) follows from (i), (ii) and continuity of ψ.

To show (vi), note that by Theorem 3.2(b2), there exist $M > 0$ and $\rho > 1$ such that

$$|\varphi(u_n) - \varphi^r| \le M\rho^{-K_n} \qquad \text{and} \qquad \|x_n - x^r\| \le M\rho^{-K_n}$$

for all $n \in \mathbb{N}_0$, where $x^r = -A_c^{-1}B_c\varphi^r$. For all $t \in [n\tau, (n+1)\tau)$, $\varphi(u(t)) = \varphi(u_n)$ and so $|\varphi(u(t)) - \varphi^r| \le M\rho^{-K_n} = M\exp(-\alpha K_n)$, where $\alpha = \ln\rho > 0$. Therefore, the convergence in (i) is of order $\exp(-\alpha K(t))$. Define $\zeta(t) := x(t) - x^r$. Then $\zeta(\cdot)$ satisfies

$$\dot{\zeta}(t) = A_c\zeta(t) + B_c(\varphi(u(t)) - \varphi^r)$$

and thus

$$\zeta(t) = e^{A_c(t-n\tau)}\zeta(n\tau) + \int_{n\tau}^{t} e^{A_c(t-s)}B_c(\varphi(u_n) - \varphi^r)ds\,, \qquad t \in [n\tau, (n+1)\tau)\,.$$

We may now conclude that

$$\|\zeta(t)\| \le \max_{0\le s\le\tau} \|e^{A_c s}\| M e^{-\alpha K_n} + \tau \max_{0\le s\le\tau} \|e^{A_c s}\|\|B_c\| M e^{-\alpha K_n}\,, \qquad t \in [n\tau, (n+1)\tau)$$

which proves that convergence in (ii) is of order $\exp(-\alpha K(t))$. The claim that convergence in (iii) is also of order $\exp(-\alpha K(t))$ follows by an argument analogous to that used in the proof of the latter part of (b2) in Theorem 3.2.

B. This follows from part C of Theorem 3.2. □

4.3 Adaptive gain

In this subsection, our goal is to establish the efficacy of the adaptation law (29) to generate an appropriate gain sequence in the sampled-data setting. Hence, consider the following adaptive

sampled-data low-gain controller for (39)

$$u(t) = u_n\,, \qquad \text{for } t \in [n\tau, (n+1)\tau)\,,\ n \in \mathbb{N}_0\,, \tag{46a}$$
$$y_n = y(n\tau)\,, \qquad n \in \mathbb{N}_0\,, \tag{46b}$$
$$u_{n+1} = u_n + l_n^{-1}(r - y_n)\,, \qquad u_0 \in \mathbb{R}, \tag{46c}$$
$$l_{n+1} = l_n + \chi(|r - y_n|)\,, \qquad l_0 \in (0, \infty)\,. \tag{46d}$$

Theorem 4.3 *Let $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{S}_c$, $(\varphi, \psi) \in \mathcal{N}$, $\tau > 0$, $\chi \in \mathcal{L}$ and $r \in \mathcal{R}_c$. Assume furthermore that, if φ is unbounded, then there exists $\delta > 0$ such that $\chi(\xi) \geq \delta\xi$ for all $\xi \in \mathbb{R}_+$. For all $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the solution $(x(\cdot), u(\cdot), (l_n))$ of the closed-loop system given by* (39) *and* (46) *is such that statements* (i) *to* (v) *of Theorem* 4.1 *hold. If moreover $\Psi_c^r \cap \overline{\Phi} = \Psi_c^r \cap \Phi$, $\Psi_c^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$, the monotone gain sequence $(k_n) = (l_n^{-1})$ converges to a positive value.*

Proof. First recall that the discretization $\Sigma = (A, B, C, D)$ of $\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{S}_c$ is of class $\mathcal{S}$. Let $(x_0, u_0, l_0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$. It follows from Subsection 4.1 that x_n (given by (41)) and (u_n, l_n) (given by (46c),(46d)) satisfy (30). By Corollary 3.4, statements (a1) to (a5) of Theorem 3.2 hold for (x_n, u_n) and $y_n = \psi(Cx_n + D\varphi(u_n))$. From this it follows as in the proof of Theorem 4.1 that statements (i) to (v) of Theorem 4.1 hold. If, moreover, $\Psi_c^r \cap \overline{\Phi} = \Psi_c^r \cap \Phi$, $\Psi_c^r \cap \mathcal{C}(\varphi) = \emptyset$ and $r \notin \mathcal{C}(\psi)$, then, again as an immediate consequence of Corollary 3.4, we conclude that the gain sequence $(k_n) = (l_n^{-1})$ converges to a positive limit. ∎

4.4 Example

Consider the second-order system

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -ax_2 - bx_1 + \varphi(u)\,,$$
$$w = x_1\,, \quad y = \psi(w)\,,$$

where $a, b > 0$, the output nonlinearity $\psi \in \mathcal{M}$ is the cubic $\psi : w \mapsto w^3$, and the input nonlinearity $\varphi \in \mathcal{M}_L(1)$ is of saturation type, defined as follows

$$u \mapsto \varphi(u) := \begin{cases} -1\ , & u < -1 \\ u\ , & u \in [-1, 1] \\ 1\ , & u > 1\,, \end{cases}$$

and so $\overline{\Phi} = \Phi = [-1, 1]$. The transfer function G_c of the associated linear system Σ_c is given by

$$G_c(s) = \frac{1}{s^2 + as + b}\,, \qquad \text{with } G_c(0) = 1/b > 0\,.$$

Since $\Psi_c^r = \{v \in \mathbb{R} \mid \psi(G_c(0)v) = r\} = \{br^{1/3}\}$, we have $\Psi_c^r \cap \overline{\Phi} \neq \emptyset$ if and only if $-b^{-3} \leq r \leq b^{-3}$. Thus, the set $\mathcal{R}_c$ of feasible reference values is given by $\mathcal{R}_c = [-b^{-3}, b^{-3}]$.

Let $\chi \in \mathcal{L}$. By Theorem 4.3, it follows that the adaptive sampled-data controller (46) achieves the tracking objective for each feasible reference value $r \in \mathcal{R}_c = [-b^{-3}, b^{-3}]$. Moreover, if r is both non-zero (so that $r \notin \mathcal{C}(\psi) = \{0\}$) and in the interior of $\mathcal{R}_c$ (so that $\Psi_c^r \cap \mathcal{C}(\varphi) = \emptyset$), then the

adapting gain sequence (k_n) converges to a positive value. For purposes of illustration, let $a = 2$, $b = 1$ (in which case $\mathcal{R}_c = [-1, 1]$), $\tau = 1$ and let $\chi = \varphi|_{\mathbb{R}_+}$ (the restriction of the saturation function to $\mathbb{R}_+$). For initial data $(x_1(0), x_2(0), u_0, l_0) = (0, 0, 0, 1)$ and the feasible reference value $r = 0.75$, Fig. 4 (generated using SIMULINK Simulation Software under MATLAB) depicts the system performance under adaptive sampled-data control. The convergence of the gain to a positive limiting value is evident.

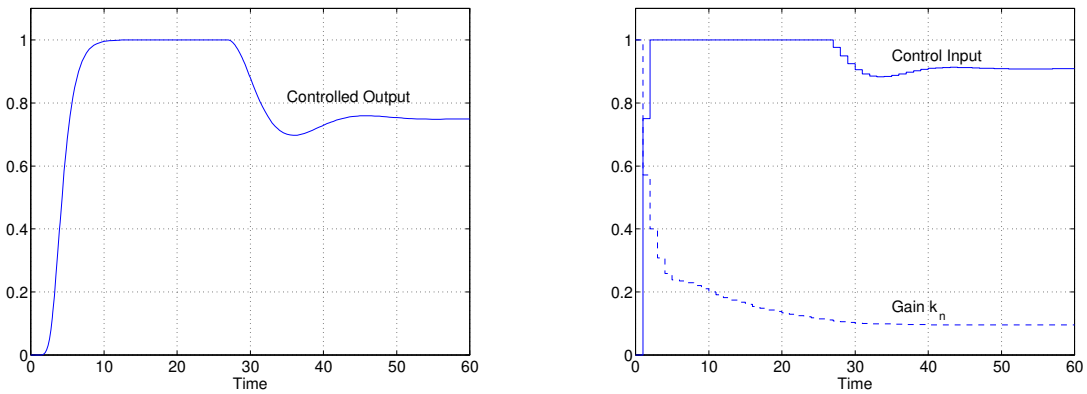

Fig. 4: Performance under adaptive sampled-data control

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