tree
connected
no cycles

forests
(not nec. conn.)

\[ G = (V, E) \]
spanning subgraph of \( G \)
is \[ G_i = (V_i, E_i) \subseteq G \]
\( V_i = V \)

\[ \Omega = \{ 0, 1 \} \]

\[ \in \{ \omega \in L \} \rightarrow \text{spanning subgraph of } G \]
Def: Uniform Spanning Tree

\[ \text{UST}[w] = \begin{cases} \frac{1}{N(G)} & \text{w sp. tree} \\ 0 & \text{otherwise} \end{cases} \]

Motivation from stat. phys.

Random cluster measures:

\[ 0 < p < 1, \quad q > 0 \]

\[ M_{pq}(w) = \frac{1}{\mathbb{Z}_{pq}} p^{n(w)} (1-p)^{K(w) - n(w)} q \]

\[ n(w) = \# \text{ of edges in } w \]

\[ K(w) = \# \text{ of components of } w \]

Ex: As \( p \to 0 \), \( q \to 0 \)

\[ M_{pq}(w) \to \text{UST}[w] \]
Some questions:

Put \( T = \{ e \in E : \omega_e = 1 \} \)

1) \( UST[e \in T] = ? \)
   \( \leadsto \) random walk
   \( \leadsto \) electric networks

2) \( K \subseteq E, K = \{ e_1, \ldots, e_k \} \)
   \( UST[KCT] = ? \)
   \( \leadsto \) \( \det(Q(e_i, e_j))_{i,j=1}^k \)
   \( \leadsto \) determinantal process

3) Infinite graphs. \( G = (\mathbb{Z}^d, E) \)
   Can one make sense of uniform sp. tree on \( G \)?
$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots \subset \mathbb{Z}$

finite $UST_n$

$\lim_{n \to \infty} \bigcup_{n \to \infty} V_n = \mathbb{Z}$

Does $UST_n[e^{e^{e^T}}] \to$ some limit?

Weak convergence of $UST_n$ to a measure on $SL = \{0, 1\}$

[Pemantle] $d \leq 4 \Rightarrow$ limiting $T$ is conn.

$d > 4 \Rightarrow$ it contains $\omega$ many trees.

$\mathbb{Z}^d$ group

$\{ \pm e_i, i = 1, \ldots, d \}$

Cayley graph of $\mathbb{Z}^d$

$\Rightarrow$ amenability (a property of groups)
4) How to simulate UST?

→ two efficient alg's to generate samples based on s.r.w. on G

\[
P[X_{n+1} = v | X_n = u] = \begin{cases} \frac{1}{\text{deg}(v)} & \text{if } \text{deg}(u) > 0 \\ 0 & \text{otherwise} \end{cases}
\]

\text{deg}(u) = \# of neighbours of } u.

4) Geometry of T

\[\pi(x,y) = \text{unique path from } x \text{ to } y\]

\[\Pi(x,y) \text{ has the same distr. as a Loop-erased Random Walk (LERW)}\]
6) Scaling limits

\[ G = D\eta^{2} \]

Does LERW converge to a random curve?

- \( d \geq 5 \) \[ \text{[Lawler]} \] LERW
- \( d = 2 \) LERW \( \rightarrow \) Brownian motion
- \( d = 3 \) LERW \( \rightarrow \) SLE_\frac{3}{2}
1) Aldous–Broder alg.

Discovered independently by Aldous and Broder.

\[ G = (V, E) \] (finite conn. gr)

\[ (X_n)_{n \geq 0} \] (simple random walk on G).

Start from any vertex \( V \)

Every time a new vertex is visited, mark the edge that was used.
\[ T_v = \inf \{ n \geq 0 : X_n = v \} \]
\[ C = \max_v T_v \quad (\text{cover time}) \]
\[ C < \infty \text{ w.p. 1} \]
\[ T = \left\{ \left( X_{T_{w-1}}, X_{T_w} \right) : w \neq X_0 \right\} \]

**Theorem** \( T \) is uniformly distributed over all spanning trees of \( G \).

**Markov chain-Tree Theorem**

\( (X_n)_{n \geq 0} \) any Markov chain on finite state space \( V \)

(irred.) \( G = (V, E) \) directed graph

\[ E = \left\{ [v,w] : p_{vw} > 0 \right\} \]

(\( p_{xy} \)) transition matrix of \( X \).
To a spanning tree \((t,v)\) we associate the weight

\[
q(t,v) = \prod_{[v,u] \in E(t,v)} P_{uw}
\]
Let \[ p(v) = \sum_{(t,v)} q(t,v) \]

**Theorem:** \( p(v) \) is proportional to the stationary distribution.

**Proof:** We build a Markov chain on rooted trees.

\((t',v')\) \hspace{1cm} \((t,v)\)

Add \([v'v]\)

Remove outgoing edge from \(v\) (say \([v,w]\))

\((t,v)\) and \(w\) uniquely determine \((t',v')\) one-to-one neighbours \(w\) of \(v\) \(\leftrightarrow\) "precursor trees" \((t',v')\)
Possible:

Let \((Y_n)_{n \geq 0}\) be the Markov chain

\[
\mathbb{P} \left[ Y_{n+1} = (t, v) \mid Y_n = (t', u) \right] = \mathbb{P} \left[ X_{n+1} = u \mid X_n = v \right]
\]

Claim: Statistical distribution of \((Y_n)\) is prop. to \(q(t, u)\)

\[
q(t, u) = \sum_{w} P_{uw} q(t', u')
\]

\[
= \sum_{(t', u')} P_{uv} q(t', u')
\]
\[ p(u) = \sum_{(t, v)} q(t, v) \]

= const. fraction of time
\[ Y_n = \text{some} (t, v) \]

= const. fraction of time
\[ X_n = u \]

The backwards tree

\[ \ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots \]
stationary tree chain

For \( w \neq X_n \)
\[ L^n_w = \text{time of last visit to } w \]

before time \( n \)

= \( \sup \{ k < n : X_k = w \} \)

\[ Y_n = \left\{ \left[ X_{L^n_w}, X_{L^n_w+1} \right] ; w \neq X_n \right\} \]
Verifying the algorithm

\( G = (V, E) \)

Simple r.w. is reversible:

\[ \Pi(u) \Pr_u = \Pi(v) \Pr_v \]

\[ \Pi(u) = \deg(u) \cdot \text{const.} \]

\[ P \left[ X_0 = x_0, \ldots, X_k = x_k \right] \]

\[ = P \left[ X_0 = x_k, \ldots, X_k = x_0 \right] \]

\[ = P \left[ X_{-k} = x_k, \ldots, X_0 = x_0 \right] \]

\[ P \left[ X_1 = x_1, \ldots, X_k = x_k \mid X_0 = x_0 \right] \]

\[ = P \left[ X_{-1} = x_{-1}, \ldots, X_{-k} = x_{-k} \mid X_0 = x_0 \right] \]
\[ q(t, v) = \prod_{w \neq v} \frac{1}{\deg(w)} = \text{const.} \deg(v) \]

backwards tree constr. from 
\[ \ldots, x_2, x_1, v = (t, v) \]
\[ \iff \text{forward tree constr. from } \]
\[ v_1, x_1, x_2, \ldots = (t, v) \]

\[ P \left[ T = (t, v) \mid X_0 = v \right] \]
\[ = P \left[ Y_0 = (t, v) \mid X_0 = v \right] \]
\[ = \frac{\text{const. } q(t, v)}{\text{const. } \deg(v)} = \text{const.} \]

Forgetting the root we get a uniformly distrib. tree.

Running time: \( EC = O(h^3) \)
\[ |V| = h. \]
\[ |EC| \leq 2|E|(n-1) \]
Wilson's algorithm

\[ P = [u_0, \ldots, u_n] \]

\[ \text{LE}(P) = [y_0, \ldots, y_2] \] where

\[ y_0 = u_0 \]
\[ \alpha = \max \{ k : u_k = y_0 \} \]

\[ y_1 = u_\lceil \alpha + 1 \rceil \]
\[ \beta = \max \{ k : u_k = y_1 \} \]

\[ y_2 = u_\lceil \beta + 1 \rceil \]

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**Algorithm**

Pick revV (root)

Pick any \( x_1 \in V \)

Run LERW from \( x_1 \) for \( r \)

Pick any \( x_2 \in V \)

Run LERW until it hits path \( y_1 \) \( \rightarrow \) call its loop-erasure \( y_2 \).