1) Examples

\[ \mathbb{Z}^d \quad V_n = [-n,n]^d \cap \mathbb{Z}^d \quad \mathcal{L} = \{0,1\} \]

\[ G_n \quad \text{FSF} \quad \text{WSF} \quad G_n^w \]

We will see that on \( \mathbb{Z}^d \) \( \text{FSF} = \text{WSF} \)

When are they the same?

3-regular tree:

FSF concentrates on the single point \( \{0\} \)

\[ G_n^w \quad V_n = \{ v \in V : d(0,v) \leq n \} \]

\[ P_\alpha [\text{r.w. never visits } 0] \geq \alpha > 0 \]

Use Wilson rooted at \( 0 \)

\[ P_\alpha [\text{r.w. never visits } 0] \geq \alpha > 0 \]

\[ P \left[ \overline{\omega \in T_{G_n^w}} \right] \geq \alpha^2 > 0 \]
2) Wilson's method on transient graphs. [BLPS, 2001]

\((G, L)\) infinite transient network
"root at infinity"

\[ F_0 = \emptyset \quad v_1, v_2, \ldots \text{ enumeration of } V \]
\[ P_n = \text{path of network r.w. from } v_n \]
stopped if \( F_{n-1} \) hit

\[ F_n = F_{n-1} \cup \text{ULE}(P_n) \quad F = \bigcup_n F_n \]

Theorem: [BLPS, 2001]

\( F \) has the distribution of WSF

\[ \ln Z \equiv d_{25} \quad P[\text{two independent r.w.'s started at } x \neq y \text{ never intersect}] > 0 \]
Proof: If \( P = \langle x_k : k \geq 0 \rangle \) is any path that visits every vertex finitely often, then \( LE(P) \) is well-defined, and
\[
LE(\langle x_k : k \leq K \rangle) \xrightarrow{K \to \infty} LE(P)
\]
For a r.w. path
\[
LE(\langle x_k : k \leq K \rangle) \xrightarrow{K \to \infty} LE(P) \text{ - a.s.}
\]

\( G_n^W \) network r.w. \( G_n^W \) up to the time \( \tau_n \) is hit

\( \downarrow \)

network r.w. on \( G \) up to the time \( V_n^c \) is hit

\( T(n) = \text{random spanning tree on } G_n^W \)
\( G = \text{limit of } T(n) \) (WSF)

Fix \( e_1, \ldots, e_M \in E \) let \( u_1, \ldots, u_L \) s.t. all endpoints of the \( e_j \) appear.
\[ \langle X_k(u_j) \rangle \text{ r.w. started at } u_j \]

Wilson on \( G_n \) : \( T_j = \text{first time the earlier part is hit} \)

Wilson on \( G \) : \( T_j = \text{first time } F_{j-1} \) is hit

\[ P_j = \langle X_k(u_j) : k \leq T_j \rangle \quad Y_j = LE(P_j) \]

\[ Y_j = LE(P_j) \]

\[ P_j \]

\[ F_{j-1} \]

\[ P[e_1, \ldots, e_n \in T(n)] = \mathbb{P}\left[ e_i, \ldots, e_n \in \bigcup_{j=1}^{L} Y_j^h \right] \]

Claim: A.s. \( T_j^h \rightarrow T_j, Y_j^h \rightarrow Y_j \)

\( j = 1, \ldots, L \)

Induction: \( j = 1 \) \( T_j = \infty \), \( T_j^h \rightarrow \infty \)

\( Y_j^h \rightarrow Y_j \)

\( j \geq 2 \): If \( T_j = \infty \) Fix \( N \) \( G_n \)

By ind. hyp. \( Y_i^h \wedge G_N = Y_i \wedge G_N \)

for \( n \) large enough, \( i < j \)

\[ F_{j-1}^h \wedge G_N = \bigcup_{i < j} Y_i^h \wedge G_N = \bigcup_{i < j} Y_i \wedge G_N = F_{j-1} \]

\[ T_j \geq \text{ exist time from } G_N \]

\[ \rightarrow \infty \text{ as } N \rightarrow \infty. \]
If $T_i < \infty$ for large $N$

$p_j \subset G_N$

$F_j = \bigcap G_j = \bigcap G_N$

for $n$ large $\Rightarrow \bar{T}_j = T_j$, $\bar{Y}_j = Y_j$

LHS $\rightarrow \mathbb{P}[e_i, \ldots, e_n \in G_j]$

RHS $\rightarrow \mathbb{P}[e_i, \ldots, e_m \in \bigcup_{j=1}^{m} Y_j]$

3) Automorphisms

\[
\psi: V \rightarrow V \text{ bij} \\
\psi: E \rightarrow E \text{ bij}
\]

network in addition:

\[
C(\psi(e)) = C(e)
\]

\[
\nu: \mathbb{R} \rightarrow \mathbb{R} \quad e = \psi(f) \quad \Rightarrow \quad \psi^{-1}(e) \in F
\]

Proposition: RSF and WSF are invariant under any network automorphism.

Proof: $\langle G_n \rangle$ an exhaustion $\Rightarrow \langle \psi^{-1}(G_n) \rangle$ is also an exhaustion.
$$\text{WSF}(\varphi(J) \cap K = B) = \text{WSF}(J \cap \varphi^{-1}(B))$$

$$= \lim_{n} \mu_{\varphi^{-1}(G_n)}[T \cap \varphi^{-1}(K) = \varphi^{-1}(B)]$$

$$= \lim_{n} \mu_{G_n}[T \cap K = B]$$

$$= \text{WSF}(J \cap K = B)$$


Equality.

Wilson rooted at $r \in V$ makes sense.

Prop: Wilson yields a tree with distribution $FSF = WSF$.

Proof: $B$ a cylinder event.

$K_0 =$ endpoints of edges in $B$

$K = \{y_j : \exists i \geq j, y_i \in K_0\}$

$\text{int } G_n = \{x \in V_n : \exists y \in V \setminus V_n, x \sim y\}$

Run Wilson on $G_n$ rooted at $r \rightarrow T_{G_n}$

Run Wilson on $G$ rooted at $r \rightarrow T_G$

$$C_1 := \{T_G \in B\}$$

$$C_2 := \{T_{G_n} \in B\}$$
\[
\left| P(T_G \leq B) - \mu^n_B (B) \right| = \left| P(C_1) - P(C_2) \right| \leq P(C_1 \cap C_2) \\
\leq P[\text{some r.w. started in } K \text{ hits } \partial_G G] \\
\leq \sum_{v \in K} P_{v_0} \left(T_{\text{dist}(G_0)} < \tau_v \right) \xrightarrow{n \to \infty} 0
\]

by recurrence. Similarly for FSF.

4) Trees are infinite

Prop: All components are infinite WSF-a.s. and FSF-a.s. \( \Box \)
6) **Stochastic domination**

\[ \langle G_n \rangle \text{ an exhaustion of } G \]

\[ G_n \text{ is a subgraph of } G^w \]

\[ M_n^F[e \in T] \geq M_n^w[e \in T] \]

\[ \text{FSF}[e \in F] \geq \text{WSF}[e \in F] \]

**Def:** An event \( A \) is called **increasing** (upwardly closed) if

\[ F_1 \in A, F_2 \supseteq F_1 \implies F_2 \in A. \]

**Proposition:**

\[ M_n^F(A) \geq M_n^w(A) \]

for all \( A \) increasing.

(without proof)

**Example:**

\[ \left\{ \begin{array}{l}
\text{at least one of } e_1, e_2, e_3 \text{ are present} \\
\end{array} \right. \]

Hence \( \text{FSF}(A) \geq \text{WSF}(A) \) (*)

for any increasing cylinder event \( A \).
A Theorem of [Strassen, 1965] says that (\(\#\)) implies that there exists a measure \(\nu\) on \(\mathbb{R} \times \mathbb{R}\) s.t.

\[ \nu\left( (F_1, F_2) : F_1 \subset F_2 \right) = 1 \]  

(\(\#\#\)) \[ \nu(B_1 \times \mathbb{R}) = \text{WSF}(B_1) \]

\[ \nu(\mathbb{R} \times B_2) = \text{FSF}(B_2) \]

\(\nu\) is called a (monotone) coupling of \(\text{WSF}\) and \(\text{FSF}\).

It is easy to see that (\(\#\)) is also necessary for (\(\#\#\)): if \(\nu\) exists, then

\[ \text{WSF}(A) = \nu\left( (F_1, F_2) : F_1 \in A \right) \]

\[ = \nu\left( (F_1, F_2) : F_2 \in A, F_1 \subset F_2 \right) \]

\[ \leq \nu\left( (F_1, F_2) : F_2 \in A \right) \]

\[ = \text{FSF}(A) \]
**Proposition:** If $\text{WSF} (e \in E) = \text{FSF} (e \in E)$ for every $e \in E$ then $\text{WSF} = \text{FSF}$.

**Proof:** Use $V \cdot \{ e \in F_1 \} \subset \{ e \in F_2 \}$

$u(e \in F_1) = \text{WSF} (e \in F) = \text{FSF} (e \in F)$

$= u(e \in F_2) \Rightarrow \{ e \in F_1 \} = \{ e \in F_2 \}$

Hence $F_1 = F_2$ $\text{u-a.s.}$

**Proposition:** If $E[\deg_f (v)]$ is the same under $\text{WSF}$ and $\text{RSF}$, then $\text{WSF} = \text{FSF}$.

**Proof:** $\text{WSF}$

$\checkmark$

$\times$

edges in $F_1$

$\subset$

$\times$

edges in $F_2$