

EXIT PROBLEMS FOR SPECTRALLY NEGATIVE LÉVY PROCESSES AND APPLICATIONS TO (CANADIZED) RUSSIAN OPTIONS

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We consider spectrally negative Lévy process and determine the joint Laplace transform of the exit time and exit position from an interval containing the origin of the process reflected in its supremum. In the literature of fluid models, this stopping time can be identified as *the time to buffer-overflow*. The Laplace transform is determined in terms of the scale functions that appear in the two sided exit problem of the given Lévy process. The obtained results together with existing results on two sided exit problems are applied to solving optimal stopping problems associated with the pricing of Russian options and their Canadized versions.

1. Introduction In this paper we consider the class of spectrally negative Lévy processes. These are real valued random processes with stationary independent increments which have no positive jumps. Amongst others Emery [12], Suprun [28], Bingham [5] and Bertoin [4] have all considered fluctuation theory for this class of processes. Such processes are often considered in the context of the theories of dams, queues, insurance risk and continuous branching processes; see for example [7, 5, 6, 22]. Following the exposition on two sided exit problems in Bertoin [4] we study first exit from an interval containing the origin for spectrally negative Lévy processes reflected in their supremum (equivalently spectrally positive Lévy processes reflected in their infimum). In particular we derive the joint Laplace transform of the time to first exit and the overshoot. The aforementioned stopping time can be identified in the literature of fluid models as *the time to buffer overflow* (see for example [1, 14]). Together with existing results on exit problems we apply our results to certain optimal stopping problems that are now classically associated with mathematical finance.

In sections 2 and 3 we introduce notation and discuss and develop existing results concerning exit problems of spectrally negative Lévy processes. In section 4 an expression is derived for the joint Laplace transform of the exit time and exit position of the reflected process from an interval containing the origin. This Laplace transform can be written in terms of scale functions that already appear in the solution to the two sided exit problem. In Section 5 we outline an optimal stopping problem which is associated with the pricing of Russian options. Section 6 is devoted to solving this optimal stopping problem in terms of scale functions that appear in the afore mentioned exit problems. In Section 7 we consider a modification of the optimal stopping problem known as Canadization (corresponding to the case that the expiry date of the option contract is randomized with an independent

AMS 1991 subject classifications. Primary 60J99; secondary 60G40, 91B70.

Key words and phrases. Reflected Lévy processes, Exit problems, scale functions, American option, Russian option, Canadized option, optimal stopping.

Short title: **Exit problems and Russian options.**

exponential distribution) and show that an explicit solution is also available in terms of scale functions. Finally we conclude the paper with some examples of the optimal stopping problems under consideration.

2. Spectrally negative Lévy processes Let $X = \{X_t, t \geq 0\}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. Restricting ourselves to spectrally negative Lévy processes, the process X may be represented as

$$X_t = \mu t + \sigma W_t + J_t^{(-)}, \quad (1)$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion and $J^{(-)} = \{J_t^{(-)}, t \geq 0\}$ is a non-Gaussian spectrally negative Lévy process. Both processes are independent. We exclude the case that X has monotone paths.

The jumps of $J^{(-)}$ are all nonpositive and hence the moment generating function $\mathbb{E}[e^{\theta X_t}]$ exists for all $\theta \geq 0$. A standard property of Lévy processes, following from the independence and stationarity of their increments, is that, when the moment generating function of the process at time t exists, it satisfies

$$\mathbb{E}[e^{\theta X_t}] = e^{t \psi(\theta)} \quad (2)$$

for some function $\psi(\theta)$, the cumulant, which is well defined at least on the non-negative complex half plane and will be referred to as the Lévy exponent of X . It can be checked that this function is strictly convex and tends to infinity as θ tends to infinity, see Bertoin [2, p. 188].

We restrict ourselves to the Lévy processes which have unbounded variation or have bounded variation and a Lévy measure which is absolutely continuous with respect to the Lebesgue measure

$$\Lambda(dx) \ll dx. \quad (\text{AC})$$

We conclude this section by introducing for any Lévy process having $X_0 = 0$ the family of martingales

$$\exp(cX_t - \psi(c)t),$$

defined for any c for which $\psi(c) = \log \mathbb{E}[\exp cX_1]$ is finite, and further the corresponding family of measures $\{\mathbb{P}^c\}$ with Radon-Nikodym derivatives:

$$\left. \frac{d\mathbb{P}^c}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp(cX_t - \psi(c)t). \quad (3)$$

For all such c (including $c = 0$) the measure \mathbb{P}_x^c will denote the translation of \mathbb{P}^c under which $X_0 = x$.

REMARK 1. *Under the measure \mathbb{P}^c the characteristics of the process X have changed. How they have changed can be found out by looking at the cumulant of X*

under \mathbb{P}^c :

$$\begin{aligned}
 \psi_c(\theta) &:= \log(\mathbb{E}^c[\exp(\theta X_1)]) \\
 &= \log(\mathbb{E}[\exp((\theta + c)X_1 - \psi(c))]) \\
 (4) \qquad &= \psi(\theta + c) - \psi(c), \quad \theta \geq \min\{-c, 0\}.
 \end{aligned}$$

3. Two sided exit problems for Lévy processes

3.1. *Scale functions* Bertoin [4] studies two sided exit problems of spectrally negative Lévy processes in terms of a class of functions known as q -scale functions. Here we give a slightly modified definition of these objects (Definition 2).

DEFINITION 1. *Let $q \geq 0$ and then define $\Phi_c(q)$ as the largest root of $\psi_c(\theta) = q$.*

DEFINITION 2. *For $q \geq 0$, the q -scale function $W^{(q)} : (-\infty, \infty) \rightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q)$$

and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_c^{(q)}(x)$ to mean the q -scale function as defined above for (X, \mathbb{P}^c) .

It is known that the q -scale function is increasing on $(0, \infty)$. Furthermore, if X has unbounded variation or if X has bounded variation and satisfies (AC), the restricted function $W_v^{(q)}|_{(0, \infty)}$ is continuously differentiable. See Lambert [16] and Bertoin [4]. For every $x \geq 0$, we can extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane by the identity

$$W_v^{(q)}(x) = \sum_{k \geq 0} q^k W_v^{*(k+1)}(x) \tag{5}$$

where W_v^{*k} denotes the k -th convolution power of $W_v = W_v^{(0)}$. The convergence of this series is plain from the inequality

$$W_v^{*k+1}(x) \leq x^k W_v(x)^{k+1}/k!, \quad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of W_v .

REMARK 2. *For each $q > 0$, a spectrally negative Lévy process X has an absolutely continuous potential measure $U^q(dx) = \int_0^\infty e^{-qt} \mathbb{P}(X_t \in dx) dt$. Its density, say u^q , is related to the q -scale function $W^{(q)}$. Indeed, in Bingham [5] it is shown that there exists a version of the potential density u^q such that*

$$W^{(q)}(x) = u^q(-x) + u^q(x) = u^q(-x) + \Phi'(q) \exp(-\Phi(q)x)$$

where $\Phi(q) = \Phi_0(q)$.

REMARK 3. *By Corollary VII.1.5 in Bertoin [2] $\lim_{x \downarrow 0} W_v(x) = 0$ if and only if X has unbounded variation. By the expansion (5) it also follows that, under the same condition, $\lim_{x \downarrow 0} W_v^{(q)}(x) = 0$.*

REMARK 4. *We have the following relationship between scale functions*

$$W^{(u)}(x) = e^{vx} W_v^{(u-\psi(v))}(x)$$

for v such that $\psi(v) < \infty$ and $u \geq \psi(v)$. To see this, simply take Laplace transforms of both sides. By analytical extension, we see that the identity remains valid for all $u \in \mathbb{C}$.

Equally important as far as the following discussion is concerned is the function $Z^{(q)}$ which is defined as follows.

DEFINITION 3. *For $q \geq 0$ we define $Z^{(q)} : \mathbb{R} \rightarrow [1, \infty)$ by*

$$Z^{(q)}(x) = 1 + q \int_{-\infty}^x W^{(q)}(z) dz. \quad (6)$$

Keeping with our earlier convention, we shall use $Z_c^{(q)}(x)$ in the obvious way. Just like $W^{(q)}$, the function $Z^{(q)}$ may be characterised by its Laplace transform and continuity on $(0, \infty)$. Indeed, we can check that

$$\int_0^\infty e^{-\theta x} Z^{(q)}(x) dx = \psi(\theta) / (\theta(\psi(\theta) - q)), \quad \theta > \Phi(q).$$

Note that when $q > 0$ this function inherits some properties from $W^{(q)}(x)$. Specifically it is strictly increasing, is equal to the constant 1 for $x \leq 0$ and $Z^{(q)}|_{(0, \infty)} \in C^2(0, \infty)$. When $q = 0$ then $Z^{(0)}(x) = Z(x) = 1$. Also, by working with the analytic extension of $q \mapsto W_v^{(q)}(x)$ we can define $q \mapsto Z_v^{(q)}(x)$ for all $q \in \mathbb{C}$.

We state the following result for the limit of $Z^{(q)}(x)/W^{(q)}(x)$ as x tends to infinity. For the formulation of this result and in the sequel, we shall understand $0/\Phi(0)$ to mean $\lim_{\theta \downarrow 0} \theta/\Phi(\theta)$.

LEMMA 1. *For $q \geq 0$, $\lim_{x \rightarrow \infty} Z^{(q)}(x)/W^{(q)}(x) = q/\Phi(q)$.*

PROOF. First suppose $q > 0$. The fact that $\theta \mapsto \psi(\theta)$ is increasing for $\theta \geq \Phi(0)$ in conjunction with equation (4) implies that $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) > 0$. Recalling that $1/\psi_{\Phi(q)}$ is the Laplace transform of $W_{\Phi(q)}$, we now deduce from a Tauberian theorem (e.g. [2, p. 10]) that

$$0 < W_{\Phi(q)}(\infty) := \lim_{x \rightarrow \infty} W_{\Phi(q)}(x) = 1/\psi'_{\Phi(q)}(0) < \infty. \quad (7)$$

Recall from Remark 4 that $W^{(q)}(x)$ is equal to $\exp(\Phi(q)x)$ times $W_{\Phi(q)}(x)$. By partial integration, we then find for $x > 0$

$$Z^{(q)}(x) = 1 + q(W^{(q)}(x) - W^{(q)}(0^+))/\Phi(q) - q \int_0^x e^{\Phi(q)y} W'_{\Phi(q)}(y) dy / \Phi(q),$$

where $W^{(q)}(0^+) := \lim_{x \downarrow 0} W^{(q)}(x)$. Then equation (7) in conjunction with dominated convergence implies that the integral $\int_0^x e^{\Phi(q)(y-x)} W'_{\Phi(q)}(y) dy / W_{\Phi(q)}(x)$ converges to zero as x tends to ∞ ; hence $Z^{(q)}(x) / W^{(q)}(x)$ converges to $q / \Phi(q)$.

Consider now the case $q = 0$. We know from e.g. Bertoin [2] that $\Phi(0) > 0$ if and only if X drifts to $-\infty$. Recalling that by Remark 4 $W(x) = \exp(\Phi(0)x) W_{\Phi(0)}(x)$, we see that, if X drifts to $-\infty$, the limit $\lim_{x \rightarrow \infty} W(x)^{-1} = 0$. If X does not drift to $-\infty$, we find by the same Tauberian theorem mentioned in the previous paragraph that $W(x)^{-1} \sim x \psi(x^{-1})$ as $x \rightarrow \infty$. We finish the proof by noting that $\psi'(0^+) = 1/\Phi'(0^+)$, since $\psi(\Phi(q)) = q$. \square

3.2. *Exit from a finite interval* The following Proposition gives a complete account of the two sided exit problem for the class of spectrally negative Lévy processes we are interested in. Before stating the result, we first introduce the following passage times.

DEFINITION 4. We denote the passage times above and below k for X by

$$T_k^- = \inf\{t > 0 : X_t \leq k\} \text{ and } T_k^+ = \inf\{t > 0 : X_t \geq k\}. \quad (8)$$

PROPOSITION 1. Let $q \geq 0$. The Laplace transform of the two-sided exit time $T_a^- \wedge T_b^+$ on the part of the probability space where X , starting in $x \in (a, b)$, exits the interval (a, b) above and below are respectively given by

$$\mathbb{E}_x \left[e^{-qT_b^+} I_{(T_b^+ < T_a^-)} \right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}; \quad (9)$$

$$\mathbb{E}_x \left[e^{-qT_a^-} I_{(T_b^+ > T_a^-)} \right] = Z^{(q)}(x-a) - W^{(q)}(x-a) \frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}. \quad (10)$$

PROOF. This result can be extracted directly out of existing literature. See for example Bertoin [2, Thm. VII.8] for a proof of (9). Combining this with Bertoin [4, Cor. 1], we find equation (10). Note, in Bertoin [4] there is a small typographic mistake so that in equation (10) the function $\int_0^{x-a} W^{(q)}(y) dy$ is used instead of $Z^{(q)}(x-a) - 1$. \square

REMARK 5. The strong Markov property, in conjunction with equation (9), is enough to prove that

$$e^{-q(T_b^+ \wedge T_a^- \wedge t)} W^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) \quad (11)$$

is a martingale. To see this let $\tau = T_b^+ \wedge T_a^-$ and note that $W^{(q)}(X_\tau - a)/W^{(q)}(b - a)$ is another way of writing the indicator of $\{T_b^+ < T_a^-\}$. Thus, by (9)

$$\begin{aligned} & W^{(q)}(x - a) \\ &= \mathbb{E}_x \left[e^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x \left[e^{-q\tau} W^{(q)}(X_\tau - a) \mid \mathcal{F}_t \right] \right] \\ &= \mathbb{E}_x \left[I_{(t \leq \tau)} e^{-qt} \mathbb{E}_{X_t} \left[e^{-q\tau} W^{(q)}(X_\tau - a) \right] + I_{(t > \tau)} e^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &= \mathbb{E}_x \left[I_{(t \leq \tau)} e^{-qt} W^{(q)}(X_t - a) + I_{(t > \tau)} e^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &= \mathbb{E}_x \left[e^{-q(\tau \wedge t)} W^{(q)}(X_{\tau \wedge t} - a) \right]. \end{aligned}$$

Now that this constant expectation has been established, the martingale property follows by a similar manipulation of the expression (11). Similarly, this technique can also be employed to prove that

$$e^{-q(T_b^+ \wedge T_a^- \wedge t)} \left(Z^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) - \frac{Z^{(q)}(b - a)}{W^{(q)}(b - a)} W^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) \right)$$

and hence (by linearity) $e^{-q(T_b^+ \wedge T_a^- \wedge t)} Z^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a)$ is a martingale.

4. Exit problems for reflected Lévy processes Denote by $\overline{X} = \{\overline{X}_t, t \geq 0\}$, with

$$\overline{X}_t = \max \left\{ s, \sup_{0 \leq u \leq t} X_u \right\},$$

the non-decreasing process representing the current maximum of X given that, at time zero, the maximum from some arbitrary prior point of reference in time is s . Further, let us alter slightly our notation so that now $\mathbb{P}_{s,x}$ refers to the Lévy process X which at time zero is given to have a current maximum s and position x . The notation $\mathbb{P}_{s,x}^c$ is also used in the obvious way. Further in the sequel, we shall frequently exchange between $\mathbb{P}_{s,x}^c$, $\mathbb{P}_{(s-x),0}^c$ and $\mathbb{P}_{-(s-x)}^c$ as appropriate.

We can address similar questions to those of the previous section of the process $Y = \overline{X} - X$. In this case, problems of two sided exit from a finite interval $[a, b] \subset (0, \infty)$ for the process Y are the same as for the process X . In this section we study one sided exit problems centred around the stopping time

$$\tau_k := \inf\{t \geq 0 : Y_t \notin [0, k]\}$$

defined for $k > 0$.

THEOREM 1. For $u \geq 0$ and v such that $\psi(v) < \infty$, the joint Laplace transform of τ_k and Y_{τ_k} is given by

$$\mathbb{E}_{s,x} [e^{-u\tau_k - vY_{\tau_k}}] = e^{-vz} \left(Z_v^{(p)}(k - z) - W_v^{(p)}(k - z) \frac{pW_v^{(p)}(k) + vZ_v^{(p)}(k)}{W_v^{(p)'}(k) + vW_v^{(p)}(k)} \right), \quad (12)$$

where $z = s - x \geq 0$ and $p = u - \psi(v)$.

PROOF. Suppose first that u, v are such that $u \geq \psi(v) \vee 0$ and let $z = s - x$. Denote by $\tau_{\{0\}}$ the first time that Y hits zero. An application of the strong Markov property of Y at $\tau_{\{0\}}$ yields that $\mathbb{E}_{s,x}[e^{-u\tau_k - vY_{\tau_k}}]$ is equal to

$$\mathbb{E}_{s,x} \left[e^{-u\tau_k - vY_{\tau_k}} I_{(\tau_k < \tau_{\{0\}})} \right] + C \mathbb{E}_{s,x} \left[e^{-u\tau_{\{0\}}} I_{(\tau_k > \tau_{\{0\}})} \right], \quad (13)$$

where $C = \mathbb{E}_{s,s}[e^{-u\tau_k - vY_{\tau_k}}] = \mathbb{E}_{0,0}[e^{-u\tau_k - vY_{\tau_k}}]$. Since

$$\{Y_t, t \leq \tau_{\{0\}}, \mathbb{P}_{s,x}\} \stackrel{d}{=} \{-X_t, t \leq T_0^+, \mathbb{P}_{-z}\} \quad (14)$$

and $\exp(vX_{T_{-k}^- \wedge T_0^+} - \psi(v)(T_{-k}^- \wedge T_0^+) + vz)$ is an equivalent change of measure under \mathbb{P}_{-z} (since $T_{-k}^- \wedge T_0^+$ is almost surely finite), we can rewrite the first expectation on the right hand side of (13) as $\exp(-vz)$ times

$$\mathbb{E}_{-z}^v \left[e^{(\psi(v)-u)T_{-k}^-} I_{(T_{-k}^- < T_0^+)} \right] = Z_v^{(p)}(k-z) - W_v^{(p)}(k-z) \frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \quad (15)$$

from Proposition 1. By (14), Remark 4 and again Proposition 1 we find for the second expectation on the right-hand side of (13)

$$\mathbb{E}_{-z} \left[e^{-uT_0^+} I_{(T_{-k}^- > T_0^+)} \right] = \frac{W^{(u)}(k-z)}{W^{(u)}(k)} = e^{-vz} \frac{W_v^{(p)}(k-z)}{W_v^{(p)}(k)}. \quad (16)$$

We compute C by excursion theory. To be more precise, we are going to make use of the compensation formula of excursion theory. For this we shall use standard notation (see Bertoin [2, Ch. 4]). Specifically, we denote by \mathcal{E} the set of excursions away from zero of finite length

$$\mathcal{E} = \{\epsilon \in D : \exists \zeta = \zeta(\epsilon) > 0 \text{ such that } \epsilon(\zeta) = 0 \text{ and } \epsilon(x) > 0 \text{ for } 0 < x < \zeta\},$$

where $D = D([0, \infty))$ denotes the space of all càdlàg functions on $[0, \infty)$. Analogously, $\mathcal{E}^{(\infty)}$ denotes the set of excursions ϵ away from zero with infinite length $\zeta = \infty$. We are interested in the excursion process $e = \{e_t, t \geq 0\}$ of Y , which takes values in the space of excursions $\mathcal{E} \cup \mathcal{E}^{(\infty)}$ and is given by

$$e_t = \{Y_s, L^{-1}(t^-) \leq s < L^{-1}(t)\} \quad \text{if } L^{-1}(t^-) < L^{-1}(t),$$

where L^{-1} is the right inverse of a local time L of Y at 0. We take the running supremum of X to be this local time (c.f. Bertoin [2, Ch. VII]). The space \mathcal{E} is endowed with the Itô-excursion measure n . A famous theorem of Itô now states that, if Y is recurrent, $\{e_t, t \geq 0\}$ is a Poisson point process taking values in \mathcal{E} with characteristic measure n ; if Y is transient, $\{e_t, t < L(\infty)\}$ is a Poisson point process stopped at the first point in $\mathcal{E}^{(\infty)}$. This stopped Poisson point process has the same characteristic measure n and is independent of $L(\infty)$, an exponentially distributed random variable with parameter $\Phi(0)$. For an excursion $\epsilon \in \mathcal{E}$ with lifetime $\zeta = \zeta(\epsilon)$, we denote by $\bar{\tau}$ the supremum of ϵ , that is, $\bar{\tau} = \sup_{s \leq \zeta} \epsilon(s)$. The point process of maximum heights $h = \{h_t : t \leq L(\infty)\}$ of excursions appearing in the process e is a

Poisson point process (respectively stopped Poisson point process) if Y is recurrent (respectively transient).

Following the line of reasoning in Bertoin [2] concerning Proposition 1 we can also deduce the characteristic measure of the process h . Suppose first Y is recurrent. The event that X starting in 0 exits the interval $(-x, y)$ at y is equal to the event $A = \{h_t \leq t + x \forall t \leq x + y\}$. Hence from Proposition 1 we find by differentiation that

$$W(x)/W(x+y) = \exp\left(-\int_0^y n(\bar{\tau} \geq t+x)dt\right) \Rightarrow n(\bar{\tau} \geq k) = W'(k)/W(k).$$

If Y is transient, we replace the event A by $A' = \{h_t \leq t + x \forall t \leq x + y, x + y < L(\infty)\}$. Denoting by $n^{(\infty)}$ the characteristic function of e on $\mathcal{E} \cup \mathcal{E}^{(\infty)}$, we find that $n^{(\infty)}(\bar{\tau} \geq k) = \Phi(0) + n(\bar{\tau} \geq k)$. Since $\Phi(0) > 0$ precisely if Y is transient and the stopped Poisson point process has the same characteristic measure n , we see that above display remains valid if we replace everywhere n by $n^{(\infty)}$, irrespective of whether Y is transient or not. Hence in the sequel, we shall also write n for $n^{(\infty)}$ to lighten the notation.

Now let $\rho_k = \inf\{t \geq 0 : \epsilon(t) \geq k\}$ and denote by ϵ_g the excursion starting at real time g , that is, $\epsilon_g = \{Y_{g+t}, 0 \leq t < \zeta(\epsilon_g)\}$. The promised calculation involving the compensation formula is as follows.

$$\begin{aligned} \mathbb{E}(e^{-u\tau_k - vY_{\tau_k}}) &= \mathbb{E}\left(\sum_g \left\{e^{-ug} I_{(\sup_{h < g} \bar{\tau}_h < k)}\right\} \left\{I_{(\bar{\tau}_g \geq k)} e^{-u(\tau_k - g) - vY_{\tau_k}}\right\}\right) \\ &= \mathbb{E}\left(\int_0^\infty e^{-us} I_{(\sup_{h < s} \bar{\tau}_h < k)} L(ds)\right) \int_{\mathcal{E}} I_{(\bar{\tau} \geq k)} e^{-u\rho_k - v\epsilon(\rho_k)} n(d\epsilon) \\ &= \int_0^\infty \mathbb{E}\left(e^{-uL_t^{-1}} I_{(\sup_{l < L_t^{-1}} \bar{\tau}_l < k, t < L(\infty))}\right) dt \\ &\quad \times \int_{\mathcal{E}} e^{-u\rho_k - v\epsilon(\rho_k)} n(d\epsilon | \bar{\tau} \geq k) n(\bar{\tau} \geq k). \end{aligned}$$

The suprema and the sum are taken over left starting points g of excursions. The desired expectation is now identified as the product of the two items in the last equality, say I_1 and I_2 which can now be evaluated separately. For the first, note that L_t^{-1} is a stopping time and hence an argument involving a change of measure yields

$$\begin{aligned} I_1 &= \int_0^\infty \mathbb{E}\left(e^{-uL_t^{-1} + \Phi(u)t} I_{(\sup_{l < L_t^{-1}} \bar{\tau}_l < k, t < L(\infty))}\right) e^{-\Phi(u)t} dt \\ &= \int_0^\infty \mathbb{P}^{\Phi(u)}\left(\sup_{l < L_t^{-1}} \bar{\tau}_l < k, t < L(\infty)\right) e^{-\Phi(u)t} dt. \end{aligned}$$

Since $\psi'_{\Phi(u)}(0) = \psi(\Phi(u)) > 0$, the process X drifts to infinity under $\mathbb{P}^{\Phi(u)}$. Thus, under $\mathbb{P}^{\Phi(u)}$ the reflected process Y is recurrent and $L(\infty) = \infty$. Thus, the probability in the previous integral is the chance that, in the Poisson point process of

excursions (indexed by local time), the first excursion of height greater or equal to k occurs after time s . The intensity of the Poisson process (again indexed by local time) counting the number of excursions with height not smaller than x associated with measure $\mathbb{P}^{\Phi(u)}$ is $W'_{\Phi(u)}(x)/W_{\Phi(u)}(x)$. We deduce that

$$\mathbb{P}^{\Phi(u)} \left(\sup_{t < L_t^{-1}} \bar{e}_t < k \right) = \exp \left\{ -t \frac{W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)} \right\},$$

so that

$$\begin{aligned} I_1 &= \int_0^\infty \exp \left\{ -t \frac{\Phi(u)W_{\Phi(u)}(k) + W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)} \right\} dt \\ &= \frac{W_{\Phi(u)}(k)}{\Phi(u)W_{\Phi(u)}(k) + W'_{\Phi(u)}(k)} = \frac{W^{(u)}(k)}{W^{(u)'}(k)}, \end{aligned}$$

where the final identity follows from Remark 4. Note that $I_1 > 0$, since $W^{(u)}$ is an increasing non-negative on $(0, \infty)$. Now turning to I_2 , we begin by noting from before that $n(\bar{\tau} \geq k) = W'(k)/W(k)$. Our aim is now to prove that

$$\int_{\mathcal{E}} e^{-u\rho_k - v\epsilon(\rho_k)} n(d\epsilon | \bar{\tau} \geq k) = \frac{Z_v^{(p)}(k) W_v^{(p)'}(k) / W_v^{(p)}(k) - pW_v^{(p)}(k)}{W'(k) / W(k)} \quad (17)$$

and hence that

$$I_2 = Z_v^{(p)}(k) W_v^{(p)'}(k) / W_v^{(p)}(k) - pW_v^{(p)}(k). \quad (18)$$

We start with setting the function f on $(0, \infty)$ equal to

$$f(z) := \frac{Z_v^{(p)}(k-z) - W_v^{(p)}(k-z) Z_v^{(p)}(k) / W_v^{(p)}(k)}{1 - W(k-z) / W(k)} \quad \text{for } z > 0 \quad (19)$$

and $f(0) := \lim_{z \downarrow 0} f(z)$. By de l'Hôpital's rule, we find that

$$f(0) = \frac{Z_v^{(p)}(k) W_v^{(p)'}(k) / W_v^{(p)}(k) - pW_v^{(p)}(k)}{W'(k) / W(k)}. \quad (20)$$

To prove (17), we will show that, with $\rho_\theta = \inf \{t \geq 0 : \epsilon(t) \geq \theta\}$,

$$M_\theta = e^{-u\rho_\theta - v\epsilon(\rho_\theta)} f(\epsilon(\rho_\theta)) \quad \theta \in (0, k]$$

is a martingale under the measure $n(\cdot | \bar{\tau} \geq k)$ with respect to the filtration $\{\mathcal{G}_\theta : \theta \in (0, k]\}$, where $\mathcal{G}_\theta = \sigma(\epsilon(t) : t \leq \rho_\theta)$. Note the exclusion of 0 in the martingale parameter sequence is deliberate.

Let $\eta(\cdot) = n(\cdot | \bar{\tau} \geq k)$. To show that the sequence $\{M_\theta : \theta \in (0, k]\}$ is a martingale consider first that

$$\eta(M_k | \mathcal{G}_\theta) = \frac{n(e^{-u\rho_k - v\epsilon(\rho_k)} \mathbf{1}_{(\rho_k < \infty)} | \mathcal{G}_\theta)}{n(\rho_k < \infty | \mathcal{G}_\theta)}.$$

Using the strong Markov property for excursions, we have that given \mathcal{G}_θ the law of the continuing excursion is that of $-X$ killed on entering $(-\infty, 0)$ with entrance law being that of $\epsilon(\rho_\theta)$. Thus, we find that

$$\begin{aligned}
& n \left(e^{-u\rho_k - v\epsilon(\rho_k)} \mathbf{1}_{(\rho_k < \infty)} \middle| \mathcal{G}_\theta \right) \\
&= e^{-u\rho_\theta} \mathbb{E}_{-\epsilon(\rho_\theta)} \left(e^{-uT_{-k}^- + vX_{T_{-k}^-}} \mathbf{1}_{(T_{-k}^- < \infty)} \mathbf{1}_{(T_{-k}^- < T_0^+)} \right) \\
&= e^{-u\rho_\theta} \mathbb{E}_{-\epsilon(\rho_\theta)}^v \left(e^{-pT_{-k}^-} \mathbf{1}_{(T_{-k}^- < T_0^+)} \right) e^{-v\epsilon(\rho_\theta)} \\
(21) \quad &= e^{-u\rho_\theta - v\epsilon(\rho_\theta)} \left(Z_v^{(p)}(k - \epsilon(\rho_\theta)) - W_v^{(p)}(k - \epsilon(\rho_\theta)) \frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \right)
\end{aligned}$$

and choosing $u = v = 0$ in the above calculation

$$n(\rho_k < \infty | \mathcal{G}_\theta) = 1 - W(k - \epsilon(\rho_\theta)) / W(k).$$

The martingale status of $\{M_\theta : \theta \in (0, k]\}$ is proved. By this martingale property

$$\int_{\mathcal{E}} e^{-u\rho_k - v\epsilon(\rho_k)} n(d\epsilon | \bar{\epsilon} \geq k) = n(M_\theta | \bar{\epsilon} \geq k), \quad \text{for all } \theta \in (0, k].$$

If X has unbounded variation, almost all excursion ϵ leave continuously from zero and by right-continuity of the paths $\epsilon(\rho_\theta) \rightarrow \epsilon(\rho_0) = 0$ $n(\cdot | \bar{\epsilon} \geq k)$ -almost surely as θ tends to zero. Noting that the function f defined in (19) and (20) is continuous and bounded (since it takes the value 1 for $z \geq k$), we find by bounded convergence that

$$n(M_k | \bar{\epsilon} \geq k) = \lim_{\theta \downarrow 0} n(M_\theta | \bar{\epsilon} \geq k) = n(M_0 | \bar{\epsilon} \geq k).$$

Putting the pieces together from I_1 and I_2 and noting Remark 4 implies

$$W^{(u)}(k) / W^{(u)'}(k) = W_v^{(p)}(k) / (W_v^{(p)'}(k) + vW_v^{(p)}(k)),$$

we find

$$C = -W_v^{(p)}(k) \frac{pW_v^{(p)}(k) + vZ_v^{(p)}(k)}{W_v^{(p)'}(k) + vW_v^{(p)}(k)} + Z_v^{(p)}(k) \quad (22)$$

and by substitution of (15), (16) and (22) in (13) a weaker version (in view of the restrictions on u and v) of the theorem is proved for X having unbounded variation.

Suppose now we are still under the regime that $u \geq \psi(v) \vee 0$ and that X has bounded variation. Note that one may now deduce that $e^{vx} W_v^{(p)}(x)$ and $e^{vx} Z_v^{(p)}(x)$ are positive eigenfunctions of the infinitesimal generator of X restricted to domains of the form $(0, a)$ for any $a > 0$. To see this apply the change of variable formula (e.g. [23, Thm. II.31]) to the martingales mentioned in Remark 5. Next, use these facts when applying the change of variable formula again to the process

$$e^{-u(t \wedge \tau_k) - vY_{t \wedge \tau_k}} \left(Z_v^{(p)}(k - Y_{t \wedge \tau_k}) - \frac{vZ_v^{(q)}(k) + pW_v^{(p)}(k)}{W_v^{(p)'}(k) + vW_v^{(p)}(k)} W_v^{(p)}(k - Y_{t \wedge \tau_k}) \right), \quad (23)$$

$t \geq 0$, to deduce that it is a martingale. The expectation of the terminal value of this martingale must be equal to its initial value. This is the statement of the theorem.

The result is now established for X having both bounded and unbounded variation and u, v such that $u \geq \psi(v) \vee 0$. Now fix any v such that $0 < \psi(v) < \infty$ and note that the right-hand side of (12) can be extended (as a function of u) to a neighbourhood of the strictly positive reals. The left-hand side of (12) is also analytic in u on the same domain. Indeed this follows by virtue of the fact that it is finite. This is clear when $v \geq 0$. To see finiteness when $v < 0$ note that for any $M > k$ the left hand side can also be bounded above by

$$\begin{aligned} & e^{-v(k+M)} + \mathbb{E}_{s,x} \left[\sum_{t \geq 0} e^{-ut - vY_t} I_{(\inf_{s < t} Y_s < k, \Delta Y_t > k + M - Y_{t-})} \right] \\ & \leq e^{-v(k+M)} + e^{-vk} \mathbb{E}_{s,x} \left[\sum_{t \geq 0} e^{-ut - v\Delta Y_t} I_{(\Delta Y_t > M)} \right] \\ & = e^{-v(k+M)} + e^{-vk} \int_0^\infty dt e^{-ut} \int_{-\infty}^{-M} e^{vy} \Lambda(dy) \end{aligned}$$

where Λ is the Lévy measure of X . Since it was assumed that $\psi(v) < \infty$ it follows that the second integral on the right hand side above is finite. It now follows by the Identity Theorem that (12) holds for $u > 0$ and v such that $\psi(v) < \infty$. To get the result with $u = 0$, take limits on both sides of (12) using Monotone Convergence for the left-hand side. \square

REMARK 6. *When X has unbounded variation, one cannot use the method in the proof used for the case of bounded variation on account of the fact that the function $W_v^{(p)}$ is not necessarily smooth enough to use in conjunction with Itô's formula. Having proved Theorem 1 however, following the comments in Remark 5 it is not difficult to show that for all u, v as in Theorem 1, (23) is again martingale, where, as before, $p = u - \psi(v)$.*

When X has bounded variation, the method of proof used for the case of unbounded variation is valid up to establishing the identity (18) for I_2 . The method can be pushed through in a similar way to the case of unbounded variation but, as we shall now explain, the given technique in the proof is considerably quicker.

For the case of bounded variation it is known that (e.g. [24] and more recently [29, 20]) an excursion ϵ starts with a jump almost surely and $n(\epsilon(\rho_0) \in dx) = \mathbf{d}^{-1}\Lambda(-dx)$ where Λ and \mathbf{d} are the Lévy measure and drift of X respectively. The law of an excursion ϵ under n is then that of $-X$ killed upon entering the negative half-line with entrance law $n(\epsilon(\rho_0) \in dx)$. Then by the computation in (21),

$$I_2 = \int_{-\infty}^0 e^{vx} \left(Z_v^{(p)}(k+x) - W_v^{(p)}(k+x) \frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \right) \mathbf{d}^{-1}\Lambda(dx). \quad (24)$$

By showing that the right hand side of (24) and the right hand side of (20) are continuous in k and their Laplace transform with respect to k coincide, one checks

that these expressions are equal. But this boils down to the fact that $e^{vx}W_v^p(x)$ and $e^{vx}Z_v^p(x)$ are positive eigenfunctions of the infinitesimal generator of X on finite open intervals, which lead to the quicker martingale proof that was presented.

5. Russian options Consider a financial market consisting of a riskless bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 \exp(rt) \quad B_0 > 0, \quad r \geq 0, \quad t \geq 0. \quad (25)$$

The price of the risky asset is modeled as the exponential spectrally negative Lévy process

$$S_t = S_0 \exp(X_t), \quad S_0 > 0, t \geq 0. \quad (26)$$

If $X_t = \mu t + \sigma W_t$ where, as before, $W = \{W_t, t \geq 0\}$ is a standard Brownian motion, we get the standard Black-Scholes model for the price of the asset. Extensive empirical research has shown that this (Gaussian) model is not capable of capturing certain features (such as skewness, asymmetry and heavy tails) which are commonly encountered in financial data, for example returns of stocks. To accommodate for these problems, an idea, going back to Merton [17], is to replace the Brownian motion as model for the log-price by a general Lévy process X . In this paper, we will restrict ourselves to the model where X is given by the spectrally negative Lévy process given in (1). This restriction is mainly motivated by analytical tractability and the availability of many results (such as those given in the previous sections) which exploit the fact that X is spectrally negative. It is worth mentioning however that in a recent study, Carr and Wu [9] have offered empirical evidence (based on a study of implied volatility) to support the case of a model in which the risky asset is driven by a spectrally negative Lévy process. Specifically, a spectrally negative stable process of index $\alpha \in (1, 2)$. See the examples in the final section for further discussion involving this class of Lévy process.

The model (25) – (26) for our market is free of arbitrage since there exists an equivalent martingale measure, that is, there exists a measure (equivalent to the implicit measure of the risky asset) under which the process $\{S_t/B_t : t \geq 0\}$ is a martingale. We can choose this measure so that X remains a spectrally negative Lévy process under this measure. If $\sigma > 0$ and $J^{(-)} \neq 0$ or $\sigma = 0$ and $J^{(-)}$ has more than one jump-size the model is incomplete and has infinitely many equivalent martingale measures. Which one to choose for pricing, is an important issue in which we do not indulge in in this article. We refer the interested reader to the paper of Chan [10] and references therein. We thus assume that some martingale measure has been chosen and let \mathbb{P} take the role of this measure. Note that this necessarily implies that $\psi(1) = r$.

Russian options were originally introduced by Shepp and Shiryaev [25, 26] within the context of the Black-Scholes market (the case that the underlying Lévy process is a Brownian motion with drift). In this paper we shall consider perpetual Russian options under the given model of spectrally negative Lévy processes. This option gives the holder the right to exercise at any almost surely finite \mathbf{F} -stopping time τ

yielding payouts

$$e^{-\alpha\tau} \max \left\{ M_0, \sup_{0 \leq u \leq \tau} S_u \right\}, \quad M_0 \geq S_0, \alpha > 0.$$

The constant M_0 can be viewed as representing the “starting” maximum of the stock price (say, the maximum over some previous period $(-t_0, 0]$). The discount factor α is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite (cf. [25, 26]).

Standard theory of pricing American-type options in the original Black-Scholes market directs one to solving optimal stopping problems. For the Russian option, the analogy in this context involves evaluating

$$V_r(M_0, S_0) := B_0 \sup_{\tau} \mathbb{E}_{\text{log } S_0} \left[B_{\tau}^{-1} \cdot e^{-\alpha\tau} \max \left\{ M_0, \sup_{0 \leq u \leq \tau} S_u \right\} \right], \quad (27)$$

where the supremum is taken over all almost surely finite respectively all \mathbf{F} -stopping times. That is, to find a stopping time which optimizes the expected discounted claim under the chosen risk neutral measure. We refer to the optimal stopping problems (27) as the *Russian optimal stopping problem*. In Section 6 we will solve (27) by combining well known optimal stopping theory with the results on exit problems from Section 4. The real object of interest is of course the finite time version with the extra constraint $\tau \leq T$, where T is a given expiration time (this is closely related to the lookback option). Note however that Carr [8] has shown that a close relative of the perpetual version lies at the basis of a very efficient approximation for the finite time expiration option, justifying therefore the interest in perpetuums. We shall address this matter in more detail in Section 7.

6. The Russian optimal stopping problem When dealing with Russian options, our method leans on the experience of Shepp and Shiryaev [25, 26], Duffie and Harrison [11], Graversen and Peškir [13] and Kyprianou and Pistorius [15]; all of which deal with the perpetual Russian option within the standard Black-Scholes market. The first thing to note is that the optimal stopping problem (27), depending on the two-dimensional Markov process (X, \bar{X}) , can be reduced to an optimal stopping problem depending only on the one-dimensional Markov process $Y = \bar{X} - X$, the reflection of X at its supremum. Indeed, by Shepp and Shiryaev’s technique of performing a change of measure using the \mathbb{P}_x -martingale $\exp\{-rt - x\}S_t$, we get for all \mathbb{P}_x -a.s. finite \mathbf{F} -stopping times τ

$$\begin{aligned} & B_0 \mathbb{E}_x \left[B_{\tau}^{-1} \cdot e^{-\alpha\tau} \max \left\{ M_0, \sup_{0 \leq u \leq \tau} S_u \right\} \right] \\ &= S_0 \mathbb{E}_x \left[\frac{B_0 S_{\tau}}{B_{\tau} S_0} \times e^{-\alpha\tau} \max \left\{ \frac{M_0}{S_{\tau}}, \sup_{0 \leq u \leq \tau} \frac{S_u}{S_{\tau}} \right\} \right] \\ &= e^x \mathbb{E}_x^1 \left[e^{-\alpha\tau + \max\{\bar{X}_{\tau}, s\} - X_{\tau}} \right] \\ &= e^x \mathbb{E}_{x-s}^1 \left[e^{-\alpha\tau + Y_{\tau}} \right], \end{aligned}$$

where $x = \log S_0$ and $s = \log M_0$. Note that under \mathbb{P}_{x-s}^1 the process Y starts in $Y_0 = s - x$. In this way we are lead to the problem of finding a function w^R and an almost surely finite stopping time τ^* such that

$$w^R(z) = \sup_{\tau} \mathbb{E}_{-z}^1 [e^{-\alpha\tau + Y_{\tau}}] = \mathbb{E}_{-z}^1 [e^{-\alpha\tau^* + Y_{\tau^*}}]. \quad (28)$$

The value function $V_r(M_0, S_0)$ of the optimal stopping problem (27) is related to w^R by $V_r(M_0, S_0) = S_0 \times w^R(\log(M_0/S_0))$.

In view of the fact that the payoff in the modified optimal stopping problem (28) is now Markovian, well known theory of optimal stopping suggests that we should now expect the optimal stopping time to be an upcrossing time of the reflected process Y at a certain constant (positive) level k . Appealing to standard techniques using martingale optimality and exploiting the fluctuation theory discussed in the previous sections we are able to prove that this is indeed the case.

Our study of exit problems for the reflected Lévy processes in Section 4 yields an expression for the value of stopping at τ_k .

COROLLARY 1. *Suppose X is as in Theorem 1 with $\psi(1) = r$. Then, for $k \geq 0, z \geq 0, \alpha > 0$ and $q = \alpha + r$*

$$\mathbb{E}_{-z}^1 (e^{-\alpha\tau_k + Y_{\tau_k}}) = e^z \left(Z^{(q)}(k - z) + \frac{Z^{(q)}(k) - qW^{(q)}(k)}{W^{(q)'}(k) - W^{(q)}(k)} W^{(q)}(k - z) \right). \quad (29)$$

PROOF. Noting that $\psi_1(-1) = \psi(0) - \psi(1) = -r$, we may apply Theorem 1 with $u = \alpha, v = -1$ and \mathbb{P} replaced by \mathbb{P}^1 . The proof is finished once we note that $p = \alpha + r = q$ and

$$e^x e^x (W_1)_{-1}^{(\alpha+r)} \equiv e^x W_1^{(\alpha)} \equiv W^{(q)}$$

each time by Remark 4. □

To complete the solution of the optimal stopping problem (28), we need to find the optimal level $k = \kappa^*$. It turns out that the optimal level is given by

$$\kappa^* = \inf \{x : Z^{(q)}(x) \leq qW^{(q)}(x)\}. \quad (30)$$

Write w_{κ} for the function of z on $[0, \infty)$ given in (29). Since under (AC) $W^{(q)}$ and $Z^{(q)}$ are differentiable on $(0, \infty)$, the function w_{κ} is so on $\mathbb{R} \setminus \{k\}$. In the case of *bounded* variation and $W^{(q)}(0^+) < q^{-1}$, $W^{(q)}(0^+) \in (0, q^{-1})$ and one notes that the level κ^* can be achieved by a principle of *continuous fit*:

$$\lim_{z \uparrow \kappa^*} w_{\kappa^*}(z) = \lim_{z \downarrow \kappa^*} w_{\kappa^*}(z).$$

This principle was discovered by Peškir and Shiryaev [21] in their study of a sequential testing problem for Poisson processes. If X has bounded variation and $W^{(q)}(0^+) > q^{-1}$, we see that $\kappa^* = 0$ and it is optimal to stop immediately.

On the other hand, if X has *unbounded* variation, $W^{(q)}(0^+) = 0$ and we find that κ^* can be recovered by a principle of *smooth fit*:

$$\lim_{z \uparrow \kappa^*} \frac{1}{z - \kappa^*} (w_{\kappa^*}(z) - w_{\kappa^*}(\kappa^*)) = \lim_{z \downarrow \kappa^*} \frac{1}{z - \kappa^*} (w_{\kappa^*}(z) - w_{\kappa^*}(\kappa^*))$$

For the aforementioned sequential testing problem involving a Wiener process, the principle of smooth fit was first discovered in 1955 by Mikhalevich which later appeared in the publication [18] (see also Chapter 4 of Shiryaev [27]). We see that by this choice of κ^* the function w_{κ^*} is of class C^2 on $\mathbb{R} \setminus \{\kappa^*\}$ and differentiable and continuous in κ^* respectively according to whether X has unbounded or bounded variation. The next theorem summarises the solution of the optimal stopping problem (28).

THEOREM 2. *Define $u : [0, \infty) \rightarrow [0, \infty)$ by $u(z) = e^z Z^{(q)}(\kappa^* - z)$ with κ^* given in (30). Then the solution to (28) is given by $w^R = u$ where $\tau^* = \tau_{\kappa^*}$ is the optimal stopping time.*

Before we start the proof we collect some useful facts:

LEMMA 2. *Define the function $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = Z^{(q)}(x) - qW^{(q)}(x)$ and let κ^* be as in Theorem 2. Then the following two assertions hold true:*

(i) *For $q > r$, f decreases monotonically to $-\infty$.*

(ii) *If $W^{(q)}(0^+) \geq q^{-1}$, $\kappa^* = 0$; otherwise $\kappa^* > 0$ is the unique root of $f(x) = 0$.*

PROOF. (i) By Remark 4, the function f has derivative in $x > 0$

$$f'(x) = qW^{(q)}(x) - qW^{(q)'}(x) = qe^{\Phi(q)x} \left((1 - \Phi(q))W_{\Phi(q)}(x) - W'_{\Phi(q)}(x) \right).$$

For $x > 0$ and $q \geq r$, this derivative is seen to be negative, since $W_{\Phi(q)}$ is positive and increasing on $(0, \infty)$ and $\Phi(q) \geq \Phi(r) = 1$ for $q \geq r$. By Lemma 1, $f(x)/(qW^{(q)}(x))$ tends to $\Phi(q)^{-1} - 1$ as $x \rightarrow \infty$. By Remark 4, $W^{(q)}(x) = \exp(\Phi(q)x)W_{\Phi(q)}(x)$ tends to infinity and the statement follows.

(ii) If $W^{(q)}(0^+) \geq q^{-1}$, (i) implies that $\kappa^* = 0$, whereas if $W^{(q)}(0^+) < q^{-1}$, we have existence and uniqueness of a positive root of $Z^{(q)}(x) = qW^{(q)}(x)$. \square

PROOF OF THEOREM 2. Suppose now first $W^{(q)}(0^+) = 0$ (that is, X has unbounded variation). From the properties of $Z^{(q)}$, we see that u lives in $C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{\kappa^*\})$. Hence Itô's lemma implies that $\exp\{-\alpha t\}u(\bar{X}_t - X_t)$ can be written as the sum of stochastic and Stieltjes integrals. The non-martingale component of these integrals can be expressed as $\exp\{-\alpha t\}$ times

$$(\hat{\Gamma}_1 - \alpha)u(\bar{X}_t - X_t)dt + u'(\bar{X}_t - X_t)d\bar{X}_t = (\hat{\Gamma}_1 - \alpha)u(Y_t)dt, \quad (31)$$

where $\hat{\Gamma}_1$ is the infinitesimal generator corresponding to the process $-X$ under \mathbb{P}^1 and the equality follows from the fact that the process \bar{X}_t only increments when $Y_t = 0$ (since Y reaches zero always by creeping in the absence of positive jumps of X) and $u'(0) = 0$. From Remark 6 we know that $\exp\{-\alpha(t \wedge \tau_{\kappa^*})\}u(Y_{t \wedge \tau_{\kappa^*}})$ is a martingale, which implies that on $\{t \leq \tau_{\kappa^*}\}$, and hence on $\{X_t < \kappa^*\}$,

$$(\hat{\Gamma}_1 - \alpha)u(z) = 0 \text{ for } z \in [0, \kappa^*).$$

Now recall that under the measure $\mathbb{P}_{s,x}^1$ the process $\exp\{-X_t + rt\}$ is a martingale. By a similar reasoning to the above, we can deduce that $(\widehat{\Gamma}_1 + r)(\exp\{z\}) = 0$. Specifically, this implies for $z > \kappa^*$ that

$$(\widehat{\Gamma}_1 - \alpha)u(z) = \left(\widehat{\Gamma}_1 + r - (r + \alpha)\right)(\exp\{z\}) \leq 0.$$

By the expression (31) for the non-martingale part of $d(\exp\{-\alpha t\}u(Y_t))$, we deduce that

$$\mathbb{E}_{s,x}^1 \left[e^{-\alpha t + Y_t} Z^{(q)}(\kappa^* - Y_t) \right] \leq e^{(s-x)} Z^{(q)}(\kappa^* - s + x).$$

A argument similar to the one presented in Remark 5, now shows that $\exp\{-\alpha t\}u(Y_t)$ is a $\mathbb{P}_{s,x}^1$ -supermartingale. Doob's optional stopping theorem for supermartingales together with the fact that $\exp\{z\} \leq u(z)$ implies that for all almost surely finite stopping times τ ,

$$\mathbb{E}_{s,x}^1 [e^{-\alpha\tau + Y_\tau}] \leq \mathbb{E}_{s,x}^1 [e^{-\alpha\tau} u(Y_\tau)] \leq u(s - x).$$

Since the inequalities above can be made equalities by choosing $\tau = \tau_{\kappa^*}$, the proof is complete for the case of unbounded variation.

If $W^{(q)}(0^+) \in (0, q^{-1})$ (X has bounded variation) we see that u lives in $C^0(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{\kappa^*\})$. Itô's lemma for this case is nothing more than the change of variable formula for Stieltjes integrals (cf. Protter [23]) and the rest of the proof follows exactly the same line of reasoning as above.

Finally the case $W^{(q)}(0^+) \geq q^{-1}$ (again X has bounded variation). Recall from Lemma 2 that, if $W^{(q)}(0^+) \geq q^{-1}$, then for $x > 0$

$$Z^{(q)}(x) - qW^{(q)}(x) < 0 \quad \text{and} \quad W^{(q)}(x) - W^{(q)'}(x) < 0.$$

Hence, recalling that $Z^{(q)}(x) = 1$ for $x \leq 0$, we see from Corollary 1 that for any $k \leq 0$

$$\begin{aligned} \mathbb{E}_{s,x}^1 (e^{-\alpha\tau_k + Y_{\tau_k}}) &= \mathbb{E}_{s,x}^1 (e^{-\alpha\tau_k + Y_{\tau_k}} Z(k - Y_{\tau_k})) \\ &\leq e^{(s-x)} Z^{(q)}(k - s + x). \end{aligned}$$

As before we conclude that, for any $k \geq 0$, $\{e^{-\alpha(\tau_k \wedge t)} u(Y_{\tau_k \wedge t})\}_{t \geq 0}$ is a supermartingale and hence using similar reasoning to the previous case it is still the case that $\{e^{-\alpha t} u(Y_t)\}_{t \geq 0}$ is a supermartingale. It follows that for all almost surely finite \mathbf{F} -stopping times τ ,

$$\mathbb{E}_{s,x}^1 (e^{-\alpha\tau + Y_\tau}) \leq \mathbb{E}_{s,x}^1 (e^{-\alpha\tau + Y_\tau} Z(k - Y_\tau)) \leq e^{s-x} Z^{(q)}(k - s + x).$$

Taking $k = 0$, we see that for any a.s. finite stopping time τ ,

$$\mathbb{E}_{s,x}^1 (e^{-\alpha\tau + Y_\tau}) \leq e^{s-x}$$

with equality for $\tau = 0$, which completes the proof. \square

REMARK 7. *Given that the optimal stopping time in (28) is of the form τ_k , here is another way of finding the optimal level κ^* if $W^{(q)}$ is twice differentiable. Let*

$\eta(q)$ be an independent exponential random variable. Since $\bar{X}_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ which is larger than 1 for $q > r$,

$$\mathbb{E}[e^{-q\tau_k + \bar{X}_{\tau_k}}] = \mathbb{E}[e^{\bar{X}_{\tau_k}} \mathbf{1}_{\tau_k < \eta(q)}] \leq \mathbb{E}[e^{\bar{X}_{\eta(q)}}] < \infty,$$

where $q > r$. Thus, there exists a finite κ^* such that for all $z \geq 0$ the right-hand side of (29) has its maximum at κ^* . By elementary optimization using the assumed differentiability combined with Lemma 2, one then deduces that κ^* is given by (30).

7. Canadized Russian options Suppose now we consider a claim structure in which the holder again receives a payout like that of the Russian option. However, we also impose the restriction that the holder must claim before some time $\eta(\lambda)$, where $\eta(\lambda)$ is an \mathbf{F} -independent exponential random variable with parameter λ . If the holder has not exercised by time $\eta(\lambda)$, then he/she is forced a rebate equal to the claim evaluated at time $\eta(\lambda)$. This is what is known in the literature as Canadization (c.f. Carr [8]).

We are thus interested in a solution to the optimal stopping problem

$$w^{CR}(z) = \sup \mathbb{E}_{-z}^1 \left[e^{-\alpha(\tau \wedge \eta(\lambda)) + Y_{(\tau \wedge \eta(\lambda))}} \right], \quad (32)$$

where the supremum is taken over almost surely finite stopping times τ . Using the fact that $\eta(\lambda)$ is independent of the Lévy process, we can rewrite this problem in the following form,

$$w^{CR}(z) = \sup_{\tau} \mathbb{E}_{-z}^1 \left[e^{-(\alpha+\lambda)\tau + Y_{\tau}} + \lambda \int_0^{\tau} e^{-(\alpha+\lambda)t + Y_t} dt \right].$$

Given the calculations in Kyprianou and Pistorius [15], one should again expect to see that the optimal stopping time is of the form τ_k for some $k > 0$.

From now we write $p = \alpha + \lambda + r$.

LEMMA 3. For each $k > 0$,

$$\begin{aligned} \mathbb{E}_{-z}^1 \left[e^{-\alpha(\tau_k \wedge \eta(\lambda)) + Y_{(\tau_k \wedge \eta(\lambda))}} \right] &= \left(\frac{p-\lambda}{p} \right) e^z Z^{(p)}(k-z) + \frac{\lambda}{p} e^z \\ &+ e^z \frac{(p-\lambda)(Z^{(p)}(k) - pW^{(p)}(k)) + \lambda}{p(W^{(p)'}(k) - W^{(p)}(k))} W^{(p)}(k-z). \end{aligned} \quad (33)$$

PROOF. Consider the Itô Lemma applied to the process $\exp\{-(\alpha+\lambda)t + Y_t\}$ on the event $\{t \leq \tau_k\}$. Denote by Γ_1 the infinitesimal generator of X under \mathbb{P}^1 . Standard calculations making use of the fact that $(\Gamma_1 + r)(\exp\{-x\}) = 0$ yield

$$\begin{aligned} d \left(e^{-(\alpha+\lambda)t + Y_t} \right) &= -(\alpha+\lambda) e^{-(\alpha+\lambda)t + Y_t} dt - r e^{-(\alpha+\lambda)t + Y_t} dt \\ &+ e^{-(\alpha+\lambda)t + Y_t} d\bar{X}_t + dM_t, \end{aligned}$$

where dM_t is a martingale term. Taking expectations of the stochastic integral given by the above equalities we have

$$(34) \quad p\mathbb{E}_{s,x}^1 \left[\int_0^{\tau_k} e^{-(\alpha+\lambda)t+Y_t} dt \right] = e^{(s-x)} - \mathbb{E}_{s,x}^1 \left[e^{-(\alpha+\lambda)\tau_k+Y_{\tau_k}} \right] \\ + \mathbb{E}_{s,x}^1 \left[\int_0^{\tau_k} e^{-(\alpha+\lambda)t+Y_t} d\bar{X}_t \right].$$

The last term in the previous expression can be dealt with by taking account of the fact that $\bar{X} = L$, the local time at the supremum of the process X . Recall that $\tau_{\{0\}}$ is the first time that Y reaches 0 and note that $d\bar{X}_t = 0$ on the set where $\{\tau_k < \tau_{\{0\}}\}$. Letting $A \in \mathcal{F}_t$ be the set

$$A = \left\{ \sup_{0 \leq u \leq L_t^{-1}} Y_u < k, t < L(\infty) \right\},$$

we have by the strong Markov property of (X, \bar{X}) and Propostion 1

$$\mathbb{E}_{s,x}^1 \left[\int_0^{\tau_k} e^{-(\alpha+\lambda)t+Y_t} d\bar{X}_t I_{(\tau_k > \tau_{\{0\}})} \right] \\ = \mathbb{E}_{-(s-x)}^1 \left[e^{-(\alpha+\lambda)\tau_{\{0\}}} I_{(\tau_k > \tau_{\{0\}})} \right] \mathbb{E}^1 \left[\int_0^\infty I_{(t < \tau_k)} e^{-(\alpha+\lambda)t+Y_t} dL_t \right] \\ = \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)} \mathbb{E}^1 \left[\int_0^\infty I_A e^{-(\alpha+\lambda)L_t^{-1}} dt \right] \\ = \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)} \int_0^\infty e^{-\Phi_1(\alpha+\lambda)t} \mathbb{P}^{1+\Phi_1(\alpha+\lambda)}(A) dt, \quad (35)$$

where in the last equality we have applied a change of measue with respect to \mathbb{P}^1 using the exponential density $\exp\{\Phi_1(\alpha+\lambda)X_t - (\alpha+\lambda)t\}$.

We can apply now techniques from excursion theory, similarly as in the proof of Theorem 1. The number of heights of the excursions of Y away from zero that exceed height k forms a Poisson process with intensity given by $W_{1+\Phi_1(\alpha+\lambda)}'(k)/W_{1+\Phi_1(\alpha+\lambda)}(k)$. The probability in the last line of (35) can now be re-written as

$$\mathbb{P}^{1+\Phi_1(\alpha+\lambda)} \left(\sup_{0 \leq u \leq L_t^{-1}} Y_u < k, t < L(\infty) \right) = \exp \left\{ -t \frac{W_{1+\Phi_1(\alpha+\lambda)}'(k)}{W_{1+\Phi_1(\alpha+\lambda)}(k)} \right\}.$$

Completing the integral in (35) much in the same way the integral I_1 was computed in Theorem 1 we end up with

$$\mathbb{E}_{s,x}^1 \left[\int_0^{\tau_k} e^{-(\alpha+\lambda)t+Y_t} d\bar{X}_t \right] = e^{(s-x)} \frac{W^{(p)}(k-s+x)}{W^{(p)'}(k) - W^{(p)}(k)}.$$

Substituting this term back in (34) and combining with Corollary 1, we end up with the expression stated. \square

Using continuous and smooth fit suggests that at the level

$$\kappa_* = \inf \{ x \geq 0 : Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/(p-\lambda) \}$$

it is optimal to exercise the Canadized Russian. Next result shows this is indeed the case:

THEOREM 3. *Define $h : [0, \infty) \rightarrow [0, \infty)$ by*

$$h(z) = (p - \lambda)e^z Z^{(p)}(\kappa_* - z)/p + \lambda e^z/p.$$

Then the solution to the optimal stopping problem (32) is $w^{CR} = h$ where $\tau^ = \tau_{\kappa_*}$ is the optimal stopping time.*

The proof of the theorem uses the following observation:

LEMMA 4. *Let h and κ_* be as in Theorem 3. If $W^{(p)}(0^+) \geq p^{-1}$ then $\kappa_* = 0$. If $W^{(p)}(0^+) < p^{-1}$, $\kappa_* > 0$ is the unique root of $Z^{(p)}(x) - pW^{(p)}(x) = -\lambda/p$ and for $t \geq 0$*

$$e^{-(\alpha+\lambda)(\tau_{\kappa_*} \wedge t)} h(Y_{\tau_{\kappa_*} \wedge t}) + \lambda \int_0^{\tau_{\kappa_*} \wedge t} e^{-(\alpha+\lambda)s+Y_s} ds$$

is a $\mathbb{P}_{s,x}^1$ -martingale.

PROOF. The statements involving κ_* follow from Lemma 2. Note that $h(s - x) = \exp\{s - x\}$ when $s - x \geq \kappa_*$. Let for $t \geq 0$

$$U_t = e^{-(\alpha+\lambda)t} h(Y_t) + \lambda \int_0^t e^{-(\alpha+\lambda)s+Y_s} ds.$$

It is a matter of checking that the special choice of κ_* together with Lemma 3 imply that $h(s - x) = \mathbb{E}_{s,x}^1[U_{\tau_{\kappa_*}}]$ for all $s - x \geq 0$.

Starting from this fact and making use of the strong Markov property, we can prove that $h(s - x)$ is equal to $\mathbb{E}_{s,x}^1[U_{\tau_{\kappa_*} \wedge t}]$, in the same vein as Remark 5. The martingale property of $U_{t \wedge \tau_{\kappa_*}}$ will follow in a fashion similar to the proof of this fact. \square

PROOF OF THEOREM 3. First suppose $W^{(p)}(0^+) = 0$. We know that $U_{\tau_{\kappa_*} \wedge t}$ is a $\mathbb{P}_{s,x}^1$ -martingale from the previous lemma. As earlier seen, $Z^{(p)}$ is twice differentiable everywhere with continuous derivatives except in κ_* where it is just continuously differentiable. The Itô formula applied to $U_{\tau_{\kappa_*} \wedge t}$ implies now that necessarily on $\{t \leq \tau_{\kappa_*}\}$, and hence on $\{Y_t < \kappa_*\}$,

$$\left(\widehat{\Gamma}_1 - (\alpha + \lambda)\right) h(Y_t) dt + \lambda e^{-(\alpha+\lambda)t+Y_t} dt + h'(Y_{t-}) d\overline{X}_t = 0$$

$\mathbb{P}_{s,x}^1$ -almost surely, where as before $\widehat{\Gamma}_1$ denotes the infinitesimal generator of $-X$. It can be easily checked that $h'(0) = 0$ by simple differentiation and use of the definition of κ_* . Since, as before, \overline{X}_t only increments when $\overline{X}_{t-} = X_{t-}$ (and this when the process creeps) it follows that the integral with respect to $d\overline{X}_t$ above is zero.

Recall that $(\widehat{\Gamma}_1 + r)(\exp\{y\}) = 0$. Since in the regime $z \geq \kappa_*$ $h(z)$ is equal to $\exp\{z\}$, we have on $\{Y_t \geq \kappa_*\}$

$$\left(\widehat{\Gamma}_1 - (\alpha + \lambda)\right) h(Y_{t-})dt + \lambda e^{-(\alpha+\lambda)t+Y_t}dt = e^{Y_t}(\lambda e^{-(\alpha+\lambda)t} - p)dt,$$

which is non-positive. From these inequalities we now have, as before, that $\mathbb{E}_{s,x}^1(U_t) \leq h(s-x)$ for all $t \geq 0$ and $s-x \geq 0$. Computations along the lines in the previous Lemma show that this is sufficient to conclude that U_t is a $\mathbb{P}_{s,x}^1$ -supermartingale.

We finish the proof of optimal stopping as in the previous optimal stopping problem. Note that

$$h(z) = e^z + (p-\lambda)e^z \int_0^{\kappa_*-z} W^{(p)}(y) dy \geq e^z.$$

By the supermartingale property and Doob's optional stopping theorem, for all almost surely finite stopping times τ , it follows that

$$\mathbb{E}_{s,x}^1 \left[e^{-(\alpha+\lambda)\tau+Y_\tau} + \lambda \int_0^\tau e^{-(\alpha+\lambda)t+Y_t} dt \right] \leq \mathbb{E}_{s,x}^1(U_\tau) \leq h(s-x).$$

Since we can make these inequalities equalities by choosing $\tau = \tau_{\kappa_*}$, we are done.

If $W^{(p)}(0^+) \in (0, (p-\lambda)^{-1})$, the use of the Change of Variable Formula is justified by the same arguments as used in the proof of Theorem 2. The proof then goes the same as above.

Finally, if $W^{(p)}(0^+) \geq (p-\lambda)^{-1}$, we see from Lemma 4 that $Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/p$ for all x positive and the proof runs analogously as the one of Theorem 2. Indeed, one should find for all $k \geq 0$ and almost surely finite \mathbf{F} -stopping times τ ,

$$\mathbb{E}_{s,x}^1 \left[e^{-\alpha(\tau \wedge \eta(\lambda)) + Y_{(\tau \wedge \eta(\lambda))}} \right] \leq \frac{p-\lambda}{p} e^{s-x} Z^{(p)}(k-s+x) + \frac{\lambda}{p} e^{s-x}.$$

Taking $k=0$ in the previous display, we conclude that for all almost surely finite stopping times τ

$$\mathbb{E}_{s,x}^1 \left[e^{-\alpha(\tau \wedge \eta(\lambda)) + Y_{(\tau \wedge \eta(\lambda))}} \right] \leq e^{s-x}$$

with equality for $\tau=0$. □

8. Examples In this section we provide some explicit examples of the foregoing theory. We concentrate on new expressions that can be now be produced for Canadized Russian options. Needless to say, one can also recover existing expressions in the literature (cf. [26, 19]) via similar calculations.

8.1. Exponential Brownian motion In the case of the classical Black-Scholes geometric Brownian motion model the functions $W^{(q)}$ and $Z^{(q)}$ are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2 \epsilon} e^{\gamma x} \sinh(\epsilon x), \quad Z^{(q)}(x) = e^{\gamma x} \cosh(\epsilon x) - \frac{\gamma}{\epsilon} e^{\gamma x} \sinh(\epsilon x)$$

on $x \geq 0$ where $\epsilon = \epsilon(q) = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}}$ and $\gamma = \frac{1}{2} - \frac{r}{\sigma^2}$. Note $\gamma \pm \epsilon$ are the roots of $\frac{\sigma^2}{2}\theta^2 + (r - \frac{\sigma^2}{2})\theta - q = 0$.

Let κ_* be the unique positive root of

$$(\epsilon - \gamma + 1)(\epsilon + \gamma)e^{-\epsilon x} - (\epsilon + \gamma - 1)(\epsilon - \gamma)e^{\epsilon x} - 2q^{-1}\epsilon\lambda e^{-\gamma x} = 0,$$

where $\epsilon = \epsilon(p)$ for $p = r + \alpha + \lambda$. Then we find the value function to be given by

$$w^{CR}(s, x) = e^s \left[\frac{q}{q + \lambda} \left(\frac{\epsilon + \gamma}{2\epsilon} \left(\frac{e^{s-x}}{e^{\kappa_*}} \right)^{\epsilon - \gamma} + \frac{\epsilon - \gamma}{2\epsilon} \left(\frac{e^{s-x}}{e^{\kappa_*}} \right)^{-\epsilon - \gamma} \right) + \frac{\lambda}{q + \lambda} \right]$$

for $s - x \in [0, \kappa_*)$ and e^s otherwise.

8.2. Jump-diffusion with hyperexponential jumps Let $X = \{X_t, t \geq 0\}$ be a jump-diffusion given by

$$X_t = (a - \sigma^2/2)t + \sigma W_t - \sum_{i=1}^{N_t} Y_i,$$

where $\sigma > 0$, N is a Poisson process and $\{Y_i\}$ is a sequence of i.i.d. random variables with hyper-exponential distribution

$$F(y) = 1 - \sum_{i=1}^n A_i e^{-\alpha_i y}, \quad y \geq 0,$$

where $A_i > 0$; $\sum_i A_i = 1$; and $0 < \alpha_1 < \dots < \alpha_n$. The processes W, N, Y are independent. We claim that for $x \geq 0$ the function $Z^{(q)}$ of X is given by

$$Z^{(q)}(x) = \sum_{i=0}^{n+1} D_i(q) e^{\theta_i x}$$

where $\theta_i = \theta_i(q)$ are the roots of $\psi(\theta) = q$, where $\theta_{n+1} > 0$ and the rest of the roots are negative, and where

$$D_i(q) = \prod_{k=1}^n (\theta_i(q)/\alpha_k + 1) \bigg/ \prod_{k=0, k \neq i}^{n+1} (\theta_i(q)/\theta_k(q) - 1).$$

Indeed, recall that $\psi(\lambda)/\lambda(\psi(\lambda) - q)$ is the Laplace transform of $Z^{(q)}$ and note that

$$\begin{aligned} D_i(q) &= \frac{1}{\theta_i(q)} \frac{\prod_{k=0}^{n+1} (-\theta_k(q))}{\prod_{k=0, k \neq i}^{n+1} (\theta_k(q) - \theta_i(q))} \frac{\prod_{k=1}^n (\theta_i(q) + \alpha_k)}{\prod_{k=1}^n \alpha_k} \\ &= \frac{q}{\theta_i(q)} \frac{\prod_{k=1}^n (\theta_i(q) + \alpha_k)}{\prod_{k=0, k \neq i}^{n+1} (\theta_k(q) - \theta_i)} = \frac{\psi(\theta_i(q))}{\theta_i(q)} \frac{1}{\psi'(\theta_i(q))} \end{aligned}$$

are the coefficients in the partial fraction expansion of $\psi(\lambda)/(\lambda(\psi(\lambda) - q))$. Hence we find for the value function of the Canadized Russian option

$$w^{CR}(s, x) = e^x \begin{cases} \frac{\alpha + \lambda}{\alpha + \lambda + r} \sum_{i=0}^{n+1} D_i(\alpha + \lambda) \left(\frac{e^{s-x}}{e^{\kappa_*}} \right)^{1 - \theta_i(\alpha + \lambda)} + \frac{r}{\alpha + \lambda + r} e^{s-x} & s - x \in [0, \kappa_*), \\ e^{s-x} & s - x > \kappa_*; \end{cases}$$

where κ_* is the root of

$$\frac{\alpha + \lambda}{\alpha + \lambda + r} \sum_{i=0}^{n+1} (\theta_i(\alpha + \lambda) - 1) D_i e^{k\theta_i(\alpha + \lambda)} + \frac{r}{\alpha + \lambda + r} e^k = -\frac{\lambda}{\alpha + r}.$$

8.3. *Stable jumps* We model X as

$$X_t = \sigma Z_t,$$

where Z is a standard stable process of index $\gamma \in (1, 2]$. Its cumulant is given by $\psi(\theta) = (\sigma\theta)^\gamma$. Note the martingale restriction amounts to $1 = \sigma^\gamma$. By inverting the Laplace transform $(\psi(\theta) - q)^{-1}$, Bertoin [3] found that the q -scale function is given by

$$W^{(q)}(x) = \gamma \frac{x^{\gamma-1}}{\sigma^\gamma} E'_\gamma \left(q \frac{x^\gamma}{\sigma^\gamma} \right), \quad x > 0$$

and hence $Z^{(q)}(x) = E_\gamma(q(x/\sigma)^\gamma)$ for $x > 0$, where E_γ is the Mittag-Leffler function of index γ

$$E_\gamma(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1 + \gamma n)}, \quad y \in \mathbb{R}.$$

From Theorem 2, 3 we can find closed formulas for the (Canadized) Russian option.

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