The Annals of Probability
2004, Vol. 32, No. 1A, 78–99

LOCAL EXTINCTION VERSUS LOCAL EXPONENTIAL GROWTH FOR SPATIAL BRANCHING PROCESSES

BY JÁNOS ENGLÄNDER AND ANDREAS E. KYPRIANOU

University of California, Santa Barbara and Utrecht University

Let X be either the branching diffusion corresponding to the operator $Lu + \beta(u^2 - u)$ on $D \subseteq \mathbb{R}^d$ [where $\beta(x) \geq 0$ and $\beta \not\equiv 0$ is bounded from above] or the superprocess corresponding to the operator $Lu + \beta u - \alpha u^2$ on $D \subseteq \mathbb{R}^d$ (with $\alpha > 0$ and β is bounded from above but no restriction on its sign). Let λ_c denote the generalized principal eigenvalue for the operator $L + \beta$ on D. We prove the following dichotomy: either $\lambda_c \leq 0$ and X exhibits local extinction or $\lambda_c > 0$ and there is exponential growth of mass on compacts of D with rate λ_c. For superdiffusions, this completes the local extinction criterion obtained by Pinsky [*Ann. Probab.* **24** (1996) 237–267] and a recent result on the local growth of mass under a spectral assumption given by Engländer and Turaev [*Ann. Probab.* **30** (2002) 683–722]. The proofs in the above two papers are based on PDE techniques, however the proofs we offer are probabilistically conceptual. For the most part they are based on "spine" decompositions or "immortal particle representations" along with martingale convergence and the law of large numbers. Further they are generic in the sense that they work for both types of processes.

1. Introduction.

1.1. *Model.* Let $D \subseteq \mathbb{R}^d$ be a domain and denote $C^{i,\eta}(D)$ the space of i times, $i = 1, 2$, continuously differentiable functions with all their ith order derivatives belonging to $C^\eta(D)$. (Here $C^\eta(D)$ denotes the usual Hölder space with some $\eta \in (0, 1]$.) Consider $Y = \{Y(t) : t \geq 0\}$, the diffusion process with probabilities $\{\mathbb{P}_x, x \in D\}$ corresponding to the operator

$$L = \tfrac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \qquad \text{on } \mathbb{R}^d, \tag{1.1}$$

where the coefficients $a_{i,j}$ and b_i belong to $C^{1,\eta}$, $i, j = 1, \ldots, d$, for some η in (0, 1], and the symmetric matrix $a = \{a_{i,j}\}$ is positive definite for all $x \in D$. We do not assume that Y is conservative, that is, Y may get killed at the Euclidean boundary of D or run out to infinity in finite time. Furthermore let $0 \leq \beta \in C^\eta(D)$ be bounded from above on D and $\beta \not\equiv 0$. The (binary) $(L, \beta; D)$-*branching diffusion* is the Markov process with motion component Y and with spatially dependent branching rate β, replacing particles by precisely two offspring when

Received September 2001; revised November 2002.
AMS 2000 subject classification. 60J80.
Key words and phrases. Spatial branching processes, branching diffusion, superdiffusion, local extinction, spine decomposition, immortal particle decomposition, generalized principal eigenvalue.

branching. At each time $t > 0$, the process consists of a point process X_t defined on Borel sets of D.

Another closely related process is the $(L, \beta, \alpha; D)$-*superprocess*. Here L and D are as before and $\alpha, \beta \in C^\eta(D)$ with $\alpha > 0$ and β bounded above. This finite measure-valued process arises as a high density limit of certain, appropriately rescaled spatial branching processes (with random offspring numbers though). See Dawson (1993) or Etheridge (2000) for superprocesses in general and Engländer and Pinsky (1999) for this particular setting. The spatial dependence of β allows local (sub)criticality ($\beta \le 0$) in certain regions and local supercriticality ($\beta > 0$) in others; α is related to the variance of the offspring distribution.

In the sequel, and unless otherwise stated, X will denote both the $(L, \beta; D)$-branching diffusion and the $(L, \beta, \alpha; D)$-superprocess with probabilities and expectations $P_\mu,\ E_\mu$. Here the starting measure μ is always a finite measure with support compactly embedded in D if X is a superprocess and given by a finite collection of (not necessarily distinct) points in D if X is a branching process.

1.2. *Motivation.* This paper concerns the evolution of mass of X on domains B compactly embedded in D (written $B \subset\subset D$). Let us begin with some definitions and notation.

DEFINITION 1. Fix a finite μ with $\operatorname{supp}\mu \subset\subset D$.

(i) We say that X under P_μ *exhibits local extinction* if for every Borel set $B \subset\subset D$, there exists a random time τ_B such that

$$P_\mu(\tau_B < \infty) = 1 \qquad \text{and} \qquad P_\mu\big(X_t(B) = 0 \text{ for all } t \ge \tau_B\big) = 1.$$

(ii) We say that X under P_μ *exhibits weak local extinction* if for every Borel set $B \subset\subset D$, $P_\mu(\lim_{t\uparrow\infty} X_t(B) = 0) = 1$.

(iii) If there is no weak local extinction, we shall say that X *exhibits recharge*.

For the $(L, \beta; D)$-branching diffusion, note that local extinction and weak local extinction coincide.

For the $(L, \beta, \alpha; D)$-superprocess, local extinction has been studied by Pinsky (1996). [Note that in Pinsky (1996) and Engländer and Pinsky (1999) the terminology is slightly different: it is said that the *support* of the superprocess exhibits local extinction.] To explain his result, let

$$\lambda_c = \lambda_c(L + \beta, D) := \inf\{\lambda \in \mathbb{R} : \exists\, u > 0 \text{ satisfying } (L + \beta - \lambda)u = 0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L + \beta$ on D [the boundedness of β ensures that $\lambda_c < \infty$; see Section 4.4 in Pinsky (1995)]. Note that in fact for every $\lambda \ge \lambda_c$ there exists a $u > 0$ such that $(L + \beta - \lambda)u = 0$. Pinsky proved that the process exhibits local extinction if and only if $\lambda_c \le 0$. Note in particular that local extinction does not depend on the coefficient α. His proof uses quite a bit of PDE machinery. To prove local extinction for $\lambda_c \leq 0$ turned out to be the harder part. The proof that there is no local extinction when $\lambda_c > 0$ is based on the proof of the existence of a nonzero stationary solution for the equation $Lu + \beta u - \alpha u^2 = 0$ on a large smooth subdomain $B \subset\subset D$ with Dirichlet boundary condition. The domain must be so large that $\lambda = \lambda_c(L + \beta, B)$, the principal Dirichlet eigenvalue on B, is positive; this is possible because $\lambda_c > 0$. Using this, one can show that even the $(L, \beta, \alpha; B)$-superprocess survives and, with positive probability, its total mass (i.e., the total mass of B) does not tend to 0 thus implying recharge. The question about the behavior of the mass for *small* balls however is left open. On the other hand, it shows that *local extinction is in fact equivalent to weak local extinction for superprocesses* too.

REMARK 2. It turns out that using the technology developed in Engländer and Pinsky (1999), one can give an alternative proof for the part that there is no local extinction for superprocesses when $\lambda_c > 0$, which we sketch here for completeness. Let $\phi \in C^{2,\eta}(B)$ be the eigenfunction corresponding to λ which is zero on ∂B (see Section 2.1). Then a (nonlinear) h-transform (with $h = \phi$) given by $\widehat{X}^{\phi} := \phi \widehat{X}$ takes the $(L, \beta, \alpha; B)$-superprocess $\widehat{X}$ into the $(L_0^{\phi}, \lambda, \alpha\phi; B)$-superprocess (with a changed starting measure) where $L_0^{\phi} := L + a\frac{\nabla\phi}{\phi} \cdot \nabla$. It can be shown that L_0^{ϕ} corresponds to a conservative (positive recurrent) diffusion on B (see again Section 2.1). Using this and the boundedness of $\alpha\phi$, it is easy to conclude that $\widehat{X}^{\phi}$ survives with positive probability [see Theorem 3.1 and Proposition 3.1 in Engländer and Pinsky (1999) and the parabolic comparison principle given by Proposition 7.2 there]. Hence the same holds for $\widehat{X}$ too.

With some extra work one can show that *small* balls too are charged for arbitrarily large times with positive probability as follows. First, use a comparison argument and replace $\alpha\phi$ by its supremum [again, using PDE representations found in Engländer and Pinsky (1999) and the elliptic comparison stated in Proposition 7.1 there, it can be shown that the new process has a smaller or equal probability of ever charging a given compactly embedded domain]. Then use the result that for recurrent motion and constant supercritical branching, the process conditioned on survival charges all nonempty balls for arbitrarily large times almost surely [see Definition 1.4, Proposition 3.1 and Theorem 4.5(a) of Engländer and Pinsky (1999) for further elaboration].

Note however that these arguments only work for superprocesses as they are based on the analytical tools of Engländer and Pinsky (1999), and that they do not give any information about the growth of mass on (small) compacts. Also, they do not rule out weak local extinction.

In this paper we have three main goals.

- To prove that the same condition on λ_c holds for the $(L, \beta; D)$-branching diffusion with regard to local extinction versus recharge.

- To give a proof which is *generic and conceptual*. That is, it works for both classes of process and also provides an intuitive explanation of the result.
- To give new results on the growth rates of mass on small balls when $\lambda_c > 0$.

As far as the first goal is concerned, a natural idea is trying to use some kind of connection between the two types of processes in order to recycle the result for superprocesses. In particular, it may first seem to be easy to argue by "Poissonization" in order to exploit Pinsky's result. That is, to use the well-known fact that for fixed time, the distribution of a branching diffusion started from a Poisson number of particles at x is the same as that of a Poisson point process whose intensity is given by the superdiffusion. A second thought however shows that knowing the distributions for fixed times is not sufficient for investigating the large time behavior of the process. (Note that we do not know anything *quantitative* about those distributions for the superprocess, so for example it is not clear how to use Borel–Cantelli along a sequence of times.)

The other possibility is to express local extinction using PDE conditions and to try to compare those conditions for the two processes. It is indeed possible to follow this track by using certain stochastic representation theorems proven by Evans and O'Connell (1994) and Iscoe (1988). The proof is not too long but quite technical (using PDE tools) and does not give any insight into the origin of the criterion on λ_c. We will provide this proof in the appendix for completeness.

As far as the second goal is concerned, we will present a proof which uses a "*spine-decomposition*" for X. There are several similar spine (sometimes called backbone, immortal particle or immortal backbone) decomposition results in the literature which we shall discuss later. For our purposes we will need to add to this collection of decompositions with the proof of yet another theorem of that type. The novelty of this approach will be that it provides us with the following *intuitive picture*: for every nonempty open domain $B \subset\subset D$ with a smooth boundary there exists a change of probability such that under the new probability there is a particle (the spine or immortal particle) whose trajectory is that of an ergodic diffusion [different from $(Y, \mathbb{P}.)$] confined to B almost surely and along which copies of the original process under $\mathbb{P}.$ immigrate at a certain rate. Then *the sign of λ_c determines whether or not the appearance of the spine is a null-event under the original probability* for large enough B's. If it is not a null-event (in the case $\lambda_c > 0$), recharge follows. To explain this, note that B can be chosen arbitrarily large, and that the spine visits every region of B for arbitrarily large times. Therefore *any* given nonempty ball is charged for arbitrarily large times with positive probability.

Regarding superprocesses, we will only prove weak local extinction for $\lambda_c \leq 0$.

As far as the third goal is concerned, we get new results on the *local growth rate* for the case when $\lambda_c > 0$ (for both processes). Namely, we will show that the local growth rate is λ_c.

For superprocesses, as far as the $\lambda_c > 0$ regime is concerned, *local exponential growth of mass in law* has been established in the recent paper Engländer and

Turaev (2002); the rate is shown to be λ_c. Note, however, that in that paper only a particular class of superdiffusions is considered satisfying a certain specific spectral theoretical assumption.

Finally we mention that it is also possible to find older but weaker results concerning the specified three goals for a special class of Markov branching diffusions in Ogura (1983).

1.3. *Results.* In the sequel we will use the notation $\langle f, \mu \rangle := \int_D f(x)\mu(dx)$. Our main theorem is as follows.

THEOREM 3 (Weak local extinction vs. local exponential growth). *Let* $0 \neq \mu$ *be a finite measure with* $\operatorname{supp} \mu \subset\subset D$.

(i) *Under* P_μ *the process* X *exhibits weak local extinction if and only if there exists a function* $h > 0$ *satisfying* $(L + \beta)h = 0$ *on* D, *that is, if and only if* $\lambda_c \leq 0$. *In particular, the weak local extinction property does not depend on the starting measure* μ.

(ii) *When* $\lambda_c > 0$, *for any* $\lambda < \lambda_c$ *and any open* $\varnothing \neq B \subset\subset D$,

$$P_\mu\Big(\limsup_{t\uparrow\infty} e^{-\lambda t} X_t(B) = \infty\Big) > 0$$

and

$$P_\mu\Big(\limsup_{t\uparrow\infty} e^{-\lambda_c t} X_t(B) < \infty\Big) = 1.$$

REMARK 4 (Total mass). In Theorem 3 we are concerned with the *local* behavior of the population size. When considering the *total mass* process $\|X_t\| := X_t(D)$, the growth rate may actually exceed λ_c. Indeed, take for example the $(L, \beta; D)$-branching diffusion with a conservative diffusion corresponding to L on D and with $\lambda_0 := \lambda_c(L, D) < 0$, and let $\beta > 0$ be constant. Then $\lambda_c(L + \beta, D) = \beta + \lambda_0 < \beta$, but—since the branching rate is spatially constant and since there is no "loss of mass" by conservativeness—a classical theorem on Yule's processes tells us that $e^{-\beta t}\|X_t\|$ tends to a nontrivial random variable as $t \to \infty$, that is, that the growth rate of the total mass is $\beta > \lambda_c$.

1.4. *Outline.* The rest of this article is organized as follows. Section 2 concerns certain spine or immortal particle decomposition theorems which are needed for our probabilistic proofs, while Section 3 presents the proofs themselves. The results then are illustrated with examples in Section 4. Finally, the Appendix provides the promised alternative proofs for part of the results along the lines of Iscoe (1988) and Pinsky (1996).

2. Martingales, spines and immortal particles.

2.1. *A decomposition result.* We begin this section by recalling some facts about changes of measure for both diffusions and Poisson processes and then combine them to provide a change of measure for X. As before, B will always denote a nonempty open set compactly embedded in D with a smooth boundary.

Girsanov change of measure. Let $\lambda = \lambda_c(L + \beta, B)$. It is known [see, e.g., Pinsky (1995), Theorem 3.5.5] that there exists a $\phi \in C^{2,\eta}(B)$ such that

$$(L + \beta - \lambda)\phi = 0 \text{ in } B \qquad \text{with } \phi = 0 \text{ on } \partial B.$$

Let $\tau^B = \inf\{t \geq 0 : Y_t \notin B\}$ and assume that the diffusion $(Y, \mathbb{P}_x)$ is adapted to some filtration $\{\mathcal{G}_t : t \geq 0\}$. Then under the change of measure

$$\left.\frac{d\mathbb{P}_x^\phi}{d\mathbb{P}_x}\right|_{\mathcal{G}_t} = \frac{\phi(Y_{t\wedge\tau^B})}{\phi(x)} \exp\left\{-\int_0^{t\wedge\tau^B} \lambda - \beta(Y_s)\, ds\right\}$$

the process $(Y, \mathbb{P}_x^\phi)$ corresponds to the h-transformed ($h = \phi$) generator $(L + \beta - \lambda)^\phi = L + a\phi^{-1}\nabla\phi \cdot \nabla$.

For further reference we point out that the process $(Y, \mathbb{P}_x^\phi)$ is *ergodic* on B (i.e., it is positive recurrent). This follows from the following three facts [see Section 4.9 in Pinsky (1995)]. A diffusion is positive recurrent if and only if it corresponds to a so-called "product L^1-critical operator"; this latter property of operators is invariant under h-transforms and finally, the operator $L + \beta - \lambda$ on B possesses this property.

Change of measure for Poisson point processes. Suppose now that given a nonnegative bounded continuous function $g(t)$, $t \geq 0$, the Poisson process $(n, \mathbb{L}^g)$ where $n = \{\{\sigma_i : i = 1, \ldots, n_t\} : t \geq 0\}$ has instantaneous rate $g(t)$. Further, assume that n is adapted to $\{\mathcal{G}_t : t \geq 0\}$. Then under the change of measure

$$\left.\frac{d\mathbb{L}^{2g}}{d\mathbb{L}^g}\right|_{\mathcal{G}_t} = 2^{n_t} \exp\left\{-\int_0^t g(s)\, ds\right\}$$

the process $(n, \mathbb{L}^{2g})$ is also a Poisson process with rate $2g(t)$ [cf. Chapter 3, Jacod and Shiryayev (1987)].

Change of measure for spatial branching processes. Suppose for simplicity that μ is finite and $\operatorname{supp}\mu \subset B$. Let $\mathcal{F}_t$ denote the natural filtration of X up to time $t \geq 0$ and let $X^{t,B}$ denote the exit measure from $B \times [0, t)$—note that the exit measure is defined for both types of processes [see Dynkin (2001) or Dynkin (1993) for detailed discussion on exit measures].

Recall that ϕ solves $(L+\beta-\lambda)\phi=0$ on B with Dirichlet boundary condition. Define for each fixed $t\geq 0$, $\phi^t:\overline{B}\times[0,t]\to[0,\infty)$ such that $\phi^t(\cdot,u)=\phi(\cdot)$ for each $u\in[0,t]$. (Here $\overline{B}$ denotes the closure of B.) We claim that

$$\left\{M_t^\phi := e^{-\lambda t}\frac{\langle\phi^t, X^{t,B}\rangle}{\langle\phi,\mu\rangle} : t\geq 0\right\}$$

is a (mean one) P_μ-martingale. Here the inner product notation is extended for $X^{t,B}$ in the obvious way. To see this note that on account of the Markov property of exit measures [cf. Dynkin (1993), Theorem I.1.3, page 1195] and the Dirichlet boundary condition for ϕ,

$$\begin{aligned}E_\mu(M_{t+s}^\phi|\mathcal{F}_t) &= E_\mu\left(e^{-\lambda(t+s)}\frac{\langle\phi^{t+s}, X^{t+s,B}\rangle}{\langle\phi,\mu\rangle}\Big|\mathcal{F}_t\right)\\ &= e^{-\lambda t}E_{X_t}\left(e^{-\lambda s}\frac{\langle\phi^s, X^{s,B}\rangle}{\langle\phi,\mu\rangle}\right)\end{aligned}$$

for $s,t\geq 0$. We have to show that the right-hand side equals M_t^ϕ a.s. On account of the branching property it is enough to show that $E_{\delta_x}(e^{-\lambda t}\langle\phi^t, X^{t,B}\rangle)=\phi(x)$ for all $t\geq 0$ and $x\in B$. To show this latter property, note that from the backward log-Laplace equation for exit measures given in Dynkin [(1993), Theorem II.3.1] it is standard to derive that $v(x,0):=E_{\delta_x}(e^{-\lambda t}\langle\phi^t, X^{t,B}\rangle)$ where v solves $-\dot{v}=(L+\beta-\lambda)v$ on $B\times(0,t)$ with $v=\phi^t$ on $\partial^{t,B}$ and $\partial^{t,B}$ is the union of $\partial B\times(0,t)$ and $B\times\{t\}$. For branching particle processes the boundary condition follows from the regularity properties of the underlying diffusion; for superprocesses the proof goes along the lines of the proof of formula II (3.5) in Dynkin [(1993), page 1226]. Therefore, by parabolic uniqueness, we obtain that $v(\cdot,0)=\phi(\cdot)$, $t\geq 0$.

THEOREM 5. *Suppose that μ is a finite measure with* $\operatorname{supp}\mu\subset B$. *For branching particle process we can thus take $\mu=\sum_i\delta_{x_i}$ where $\{x_i\}$ is a finite set of* (*not necessarily distinct*) *points in B. Define $\widetilde{P}_\mu$ by the martingale change of measure*

$$\frac{d\widetilde{P}_\mu}{dP_\mu}\bigg|_{\mathcal{F}_t} = M_t^\phi = e^{-\lambda t}\frac{\langle\phi^t, X^{t,B}\rangle}{\langle\phi,\mu\rangle}.$$

Define

$$\mathbb{P}_{\phi\mu}^\phi = \int_B \mu(dx)\frac{\phi(x)}{\langle\phi,\mu\rangle}\mathbb{P}_x^\phi,$$

that is, *we randomize the starting point of* $(Y,\mathbb{P}_\cdot^\phi)$ *according to the probability distribution $\phi\mu/\langle\phi,\mu\rangle$. Note in particular that when $\mu=\delta_x$, $\mathbb{P}_{\phi\mu}^\phi=\mathbb{P}_x^\phi$.*

(i) *Suppose that* $g \in C_{\mathrm{b}}^{+}(D)$ *and* u_g *is the minimal nonnegative solution either to* $\dot{u} = Lu + \beta u - \alpha u^2$ *on* $D \times (0, \infty)$ (*for the superprocess*) *or to* $\dot{u} = Lu + \beta u^2 - \beta u$ *on* $D \times (0, \infty)$ (*for the branching process*) *with* $\lim_{t \downarrow 0} u(\cdot, t) = g(\cdot)$, *respectively*, $\lim_{t \downarrow 0} u(\cdot, t) = e^{-g(\cdot)}$. *Then*

$$\widetilde{E}_\mu(e^{-\langle g, X_t \rangle}) = \begin{cases} \mathbb{E}^{\phi}_{\phi\mu}\left(\exp\left\{-\int_0^t ds\, 2\alpha(Y_s) u_g(Y_s, t-s)\right\}\right) E_\mu(e^{-\langle g, X_t \rangle}) \\ \sum_i \dfrac{\phi(x_i)}{\langle \phi, \mu \rangle} \left\{\mathbb{E}^{\phi}_{x_i} \mathbb{L}^{2\beta(Y)}\left(e^{-g(Y_t)} \prod_{k=1}^{n_t} u_g(Y_{\sigma_k}, t - \sigma_k)\right) \prod_{j \neq i} u_g(x_j, t)\right\} \end{cases} \tag{2.1}$$

where the first equation applies for the case when X *is a superprocess and the second for the case when* X *is a branching process.*

(ii) *With the same order of cases, we have*

$$\langle \phi, \mu \rangle \widetilde{E}_\mu(M_t^\phi) = \begin{cases} \langle \phi, \mu \rangle + \mathbb{E}^{\phi}_{\phi\mu}\left(\int_0^t ds\, e^{-\lambda s} 2\alpha\phi(Y_s)\right) \\ \sum_i \dfrac{\phi(x_i)}{\langle \phi, \mu \rangle}\left\{\mathbb{E}^{\phi}_{x_i}\mathbb{L}^{2\beta(Y)}\left(e^{-\lambda t}\phi(Y_t) + \sum_{k=1}^{n_t} e^{-\lambda\sigma_k}\phi(\sigma_k)\right) + \sum_{j \neq i}\phi(x_j)\right\}. \end{cases}$$

2.2. *Discussion.* For superprocesses, the decompositions suggests that $(X, \widetilde{P}_\mu)$ is equal in law to the sum of two independent processes. The first is a copy of (X, P_μ) and the second is a process of immigration; at every time $t > 0$, a copy of the $(L, \beta, \alpha; D)$-superprocess is initiated at Y_t, where $Y = \{Y_t : t \geq 0\}$ is a copy of $(Y, \mathbb{P}^{\phi}_{\phi\mu})$ and further, the "rate" of immigration is $2\alpha(Y_t)$.

For branching processes the decompositions suggest that $(X, \widetilde{P}_\mu)$ has the same law as a process constructed in the following way. From the configuration $\mu = \sum_i \delta_{x_i}$ pick a point $x' \in \{x_i\}$ with probability $\phi(x')/\langle \phi, \mu \rangle$. From the remaining points, independent $(L, \beta; D)$-branching processes are initiated. From the chosen point, a $(Y, \mathbb{P}^{\phi}_{x'})$-diffusion is initiated along which $(L, \beta; D)$-branching processes immigrate at space–time points $\{(Y_{\sigma_i}, \sigma_i) : i \geq 1\}$ where $n = \{\{\sigma_i : i = 1, \ldots, n_t\} : t \geq 0\}$ is a Poisson process with law $\mathbb{L}^{2\beta(Y)}$.

Both decompositions relate to a spectrum of similar results that exist in the literature for both superprocesses and branching processes. For two different classes of critical superprocesses, Evans (1993) and Etheridge and Williams (2000) consider a change of measure based on the martingale associated with the total mass process $\|X\|$. They find a decomposition consisting of a copy of the original process together with an independent immigration process along the path of a Doob h-transformed diffusion. One sees [cf. Roelly and Rouault (1989)] that their change of measure is equivalent to conditioning the process on survival. This decomposition is known as the *Evans immortal particle picture*. For supercritical superprocesses, Evans and O'Connell (1994) and Engländer and Pinsky (1999) have demonstrated a decomposition in which a given superprocess is equal in law to a process consisting of two independent components: the first being a copy of the process conditioned on extinction, the second is an immigration process, where immigration is initiated along the trajectory of a "backbone" branching Markov diffusion that starts with a random set of points.

In the branching particle process literature, conditioning the process on survival, its equivalence with the martingale change of measure associated with total mass and a representation using a single randomly chosen genealogical line of descent with size biasing of the offspring distribution along it is the result of Lyons Pemantle and Peres (1995) for Galton–Watson processes. Changes of measure using martingales of an innerproduct form for supercritical spatial and typed branching processes have also been considered by Athreya (2000), Lyons (1997) and Biggins and Kyprianou (2001), for example. These authors found that the change of measure again induced a randomly chosen genealogical line of decent, known as a *spine*, along which the spatial reproductive distribution is size biased. The size biased behavior along the spine is analogous to Evans' immortal particle picture.

Some other decompositions are as follows. Geiger and Kersting (1998) produce a "spine-like" decomposition for finite Galton–Watson trees (different from those of the previous paragraph) by conditioning on the height of the tree being at generation n. By taking limits as n tends to infinity they produce a Galton–Watson process conditioned on survival. Overbeck (1993) considers changing measure of critical super-Brownian motion using martingales constructed from innerproducts of the process with space–time harmonic functions; again, a decomposition appears in which the immortal particle has an h-transformed Brownian motion. Salisbury and Verzani (1999) show for critical super-Brownian motion that a backbone decomposition appears when conditioning the process to hit n specified points when exiting a bounded smooth domain. See also the citations contained in all the aforementioned publications for yet more examples.

Essentially these decompositions can be thought of as types of Doob h-transforms for branching processes induced by changes of measures using martingales. These martingales are built from positive eigenfunctions (elliptic solutions) of linear (semilinear) operators associated with the branching process. The fundamental concepts for such decompositions can be routed back to ideas found in the paper Kallenberg (1977).

Theorem 5 offers decompositions with the new feature that the immortal particle or spine is represented by a diffusion conditioned to stay in the compactly embedded domain B.

PROOF OF THEOREM 5. We will only prove part (i), the proof of part (ii) follows by similar reasoning.

Superprocesses. In this proof we will work with backward solutions of the log-Laplace equation for convenience. Thus we will work with the function $u_g^t := u_g(\cdot, t-\cdot)$, where t is fixed, instead of u_g. Recall that $\partial^{t,B}$ is the union of $\partial B \times (0,t)$ and $B \times \{t\}$ and recall that for $t > 0$, $X^{t,B}$ is a measure with support on $\partial^{t,B}$ (here $\mathrm{supp}\mu \subset B$ is in force). For any nonnegative bounded continuous function $f : \partial^{t,B} \to \mathbb{R}$ we have $E_\mu(e^{-\langle f, X^{t,B}\rangle}) = e^{-\langle v_f(\cdot,0),\mu\rangle}$ where v_f solves $-\dot{v} = Lv + \beta v - \alpha v^2$ in $B \times (0,t)$ with $v = f$ on $\partial^{t,B}$ [see again Dynkin (1993), Theorem II.3.1]. Starting with the left-hand side of (2.1) and applying the Markov property of exit measures [Dynkin (1993), Theorem I.1.3], one has

$$\begin{aligned}\widetilde{E}_\mu(e^{-\langle g, X_t\rangle}) &= E_\mu\biggl(\frac{\langle \phi^t, X^{t,B}\rangle}{\langle \phi, \mu\rangle} e^{-\lambda t - \langle g, X_t\rangle}\biggr)\\ &= \frac{1}{\langle \phi, \mu\rangle} E_\mu\bigl(\langle \phi^t, X^{t,B}\rangle e^{-\lambda t - \langle u_g^t, X^{t,B}\rangle}\bigr).\end{aligned} \tag{2.2}$$

It now follows that

$$\widetilde{E}_\mu(e^{-\langle g, X_t\rangle}) = -\frac{1}{\langle \phi, \mu\rangle}\frac{\partial}{\partial\theta} E_\mu\bigl(e^{-\lambda t - \langle u_g^t + \theta\phi^t, X^{t,B}\rangle}\bigr)\Big|_{\theta=0}.$$

(The differentiation and the expectation has been interchanged using dominated convergence.) Note however that $E_\mu(e^{-\langle u_g^t + \theta\phi^t, X^{t,B}\rangle}) = e^{-\langle v^\theta, X^{0,B}\rangle}$ where v^θ is the unique solution to the system, $-\dot{v}^\theta = Lv^\theta + \beta v^\theta - \alpha(v^\theta)^2$ in $B \times (0,t)$ with $v^\theta = u_g^t + \theta\phi^t$ on $\partial^{t,B}$ (here we suppressed the dependence of v^θ on t) and $X^{0,B} = \mu \otimes \delta_0$. Using $v^0 = u_g^t$ and hence that $\langle u_g^t, X^{0,B}\rangle = \langle u_g(\cdot,t), \mu\rangle$, we obtain using again the log-Laplace equation, that

$$\widetilde{E}_\mu(e^{-\langle g, X_t\rangle}) = E_\mu(e^{-\langle g, X_t\rangle})\frac{1}{\langle \phi, \mu\rangle} e^{-\lambda t}\frac{\partial}{\partial\theta}\langle v^\theta, X^{0,B}\rangle\Big|_{\theta=0}, \tag{2.3}$$

thus leaving us the job of proving that $\langle \phi, \mu\rangle^{-1} e^{-\lambda t}\partial\langle v^\theta, X^{0,B}\rangle/\partial\theta|_{\theta=0}$ is equal to the first factor on the right-hand side of (2.1). A simple calculation, using again the fact that $v^0 = u_g^t$, shows that $\eta := \partial v^\theta/\partial\theta|_{\theta=0}$ is the unique solution to the system $-\dot{\eta} = L\eta + \beta\eta - 2\alpha u_g^t \eta$ in $B \times (0,t)$ and $\eta = \phi^t$ on $\partial^{t,B}$. Note that the boundary condition (i.e., the smoothness of η up to the boundary $\partial^{t,B}$) follows by comparing the right-hand sides of (2.2) and (2.3). Taking the aforementioned first factor on the right-hand side of (2.1), use the Girsanov density for $\mathbb{P}_x^{\phi}$ to write it as

$$\int \frac{1}{\langle \phi, \mu \rangle} \mathbb{E}_x^{\phi}\biggl(\exp\biggl\{-\int_0^t ds\, 2\alpha(Y_s) u_g^t(Y_s, s)\biggr\}\biggr)\phi(x)\mu(dx)$$

$$(2.4) \qquad = e^{-\lambda t} \int \frac{1}{\langle \phi, \mu \rangle}\biggl(\phi(Y_{t\wedge\tau^B}) \times \mathbb{E}_x \exp\biggl\{-\int_0^{t\wedge\tau^B} ds\, [2\alpha(Y_s) u_g^t(Y_s, s) - \beta(Y_s)]\biggr\}\biggr)\mu(dx).$$

Note that to bring the exponential factor outside the integral on the right-hand side we have used the fact that $\phi(Y_{\tau^B}) = 0$. Consider now the right-hand side of (2.4) and observe that the expectation is precisely the "Feynman–Kac type" probabilistic representation for $\eta(x, 0)$. Therefore the right-hand side of (2.4) equals $e^{-\lambda t}\langle \eta, X^{0,B}\rangle / \langle \phi, \mu \rangle$, as required.

Branching processes. We shall begin by considering the case $\mu = \delta_x$ where $x \in B$. Write $\xi\,(x, t)$ for the left-hand side of (2.1) and note by conditioning on the first time of fission we have after a routine application of the Markov property

$$\begin{aligned}
\xi(x,t) &= \mathbb{E}_x \mathbb{L}^{\beta(Y)}\biggl(\mathbf{1}_{(t\wedge\tau^B < \sigma_1)} \frac{\phi(Y_{t\wedge\tau^B})}{\phi\,(x)} e^{-\lambda t - g(Y_t)} \\
&\qquad + \mathbf{1}_{(t\wedge\tau^B \geq \sigma_1)} \frac{\phi(Y_{\sigma_1})}{\phi(x)} e^{-\lambda\sigma_1} \xi\bigl(Y_{\sigma_1}, t-\sigma_1\bigr) u_g\bigl(Y_{\sigma_1}, t-\sigma_1\bigr)\biggr) \\
(2.5) \qquad &= \mathbb{E}_x\biggl(\exp\biggl\{-\int_0^{t\wedge\tau^B} \beta\,(Y_s)\, ds\biggr\} \frac{\phi(Y_{t\wedge\tau^B})}{\phi\,(x)} e^{-\lambda t - g(Y_t)} \\
&\qquad + \int_0^{t\wedge\tau^B} \beta\,(Y_s) \exp\biggl\{-\int_0^s \beta\,(Y_u)\, du\biggr\} \frac{\phi\,(Y_s)}{\phi(x)} \\
&\qquad\qquad \times e^{-\lambda s} \xi(Y_s, t-s) u_g(Y_s, t-s)\, ds\biggr).
\end{aligned}$$

Now let $\rho\,(x, t)$ be the right-hand side of (2.1). Condition on σ_1 to produce

$$\rho\,(x,t) = \mathbb{E}_x^{\phi} \mathbb{L}^{2\beta(Y)}\Bigl(\mathbf{1}_{(\sigma_1 > t)} e^{-g(Y_t)} + \mathbf{1}_{(\sigma_1 \leq t)} u_g\,\bigl(Y_{\sigma_1}, t-\sigma_1\bigr)\,\rho\,\bigl(Y_{\sigma_1}, t-\sigma_1\bigr)\Bigr).$$

By changing measure back to $\mathbb{P}_x$ in the last expression, one produces a second solution to the functional equation (2.5). Let $\Pi := |\xi - \rho|$. Our goal is to show that $\Pi \equiv 0$. Suppose to the contrary that $\Pi(x_0, t_0) > 0$ for some $x_0 \in B$ and $t_0 > 0$. On account of continuity on $[0, t] \times \overline{B}$, there exist $\varepsilon, a > 0$ such that $f(t) := \Pi(x_0, t) \geq a$ on $I := [t_0, t_0 + \varepsilon]$. Then, by the boundedness of β, ϕ, u_g and Π, $f(t) \leq C \int_0^t f(s)\, ds$ on I, where C is an appropriate positive constant.

Gronwall's inequality now implies that $f \equiv 0$; contradiction. Therefore $\Pi \equiv 0$ must hold.

To complete the proof for the case that $\mu = \sum_i \delta_{x_i}$, begin by noting that under P_μ we can write $X^{t,B} = \sum_i X_i^{t,B}$ where each $X_i^{t,B}$ is independent of the others and has the same distribution as $X^{t,B}$ under $P_{\delta_{x_i}}$. We now have

$$\begin{aligned}\widetilde{E}_\mu(e^{-\langle g, X_t\rangle}) &= \sum_i E_{\delta_{x_i}}\left(e^{-\lambda t}\frac{\langle \phi^t, X^{t,B}\rangle}{\langle \phi, \mu\rangle}e^{-\langle g, X^{t,B}\rangle}\right)\prod_{j\neq i} E_{\delta_{x_j}}(e^{-\langle g, X^{t,B}\rangle})\\ &= \sum_i \frac{\phi(x_i)}{\langle \phi, \mu\rangle}\widetilde{E}_{\delta_{x_i}}(e^{-\langle g, X^{t,B}\rangle})\prod_{j\neq i} u_g(x_j, t).\end{aligned}$$

By using the earlier established expression for the case $\mu = \delta_x$ in the right-hand side above we are done. $\square$

2.3. *Mean convergence for the martingale.* Before finishing this section, we make one immediate application of the previous theorem to the martingale limit M_∞^ϕ.

LEMMA 6. *Suppose that μ is finite*, $\operatorname{supp}\mu \subset B$ *and $\lambda = \lambda_c(L + \beta, B) > 0$. Then M_t^ϕ converges to its almost sure limit M_∞^ϕ in $L^1(P_\mu)$, and furthermore $\widetilde{P}_\mu \ll P_\mu$.*

PROOF. Since $(M_t^\phi)^{-1}$ is a positive $\widetilde{P}_\mu$-martingale, it has a $\widetilde{P}_\mu$-a.s. limit and hence

$$\exists \lim_{t\uparrow\infty} M_t^\phi \in (0, \infty], \qquad \widetilde{P}_\mu\text{-a.s.}$$

We now show the finiteness of the limit. Using $\lambda > 0$, Theorem 5(ii), Fatou's lemma, the fact that ϕ, β and α are all bounded on B, and finally, that Y under $\mathbb{P}_\cdot^\phi$ ergodizes on B, one obtains that

$$\widetilde{E}_\mu\left(\lim_{t\uparrow\infty} M_t^\phi\right) \le \lim_{t\uparrow\infty} \widetilde{E}_\mu(M_t^\phi) < \infty.$$

[The existence of the limit on the right-hand side follows by the fact that interchanging the integrals and the expectations on the right-hand side of the displayed formula in Theorem 5(ii) yields convergent integrals.] Consequently, $\lim_{t\uparrow\infty} M_t^\phi < \infty$, $\widetilde{P}_\mu$-a.s. Then, by a fundamental measure theoretic result [cf. Durrett (1996), page 242] M_t^ϕ converges in $L^1(P_\mu)$ and also the (equivalent) relation $\widetilde{P}_\mu \ll P_\mu$ holds. This completes the proof. $\square$

3. Proof of Theorem 3.

3.1. *A preparatory proposition.*

PROPOSITION 7. *For any nonempty open set $B \subset D$ and finite μ,*

$$P_\mu\Big(\limsup_{t\uparrow\infty} X_t(B) \in \{0,\infty\}\Big) = 1.$$

PROOF.

Branching processes. It suffices to prove the proposition for the case $\mu = \delta_x$. Let Ω_0 denote the event that $\limsup_{t\uparrow\infty} X_t(B) > 0$. Define a sequence of finite stopping times as follows. On Ω_0, let $\tau_0 = \inf\{t > 0 : X_t(\overline{B}) \geq 1\}$ and $\tau_{n+1} = \inf\{t > \tau_n + 1 : X_t(\overline{B}) \geq 1\}$; for Ω_0^c extend the sequence in an arbitrary way so that the τ_n's are finite stopping times. Fix $K > 0$, let $A_n := \Omega_0 \cap \{X_{\tau_n+1}(B) \geq K\}$ and let $\Omega_1 := \{\omega : \omega \in A_n \text{ infinitely often}\}$. Using elementary properties of the diffusion Y along with the fact that β is assumed to be bounded away from zero in some region, it is straightforward to prove that $\varepsilon(K,B) := \inf_{x\in\overline{B}} P_{\delta_x}(X_1(B) \geq K) > 0$. Thus, by the strong Markov property,

$$\sum_{n=1}^{n} P_{\delta_x}(A_n | X_{\tau_n}, \ldots, X_{\tau_1}) = \infty, \qquad P_{\delta_x}\text{-a.s. on } \Omega_0.$$

It follows then by the extended Borel–Cantelli lemma [see Corollary 5.29 in Breiman (1992)], that P_{δ_x}-almost every $\omega \in \Omega_0$ belongs to Ω_1. Therefore $\limsup_{t\uparrow\infty} X_t(B) \geq K$, P_{δ_x}-a.s. on Ω_0, and since K can be arbitrarily large, the result follows.

Superprocesses. Let $0 < \varepsilon$ and let $\mathcal{M}_\varepsilon$ denote the following set of measures: $\mu \in \mathcal{M}_\varepsilon \Leftrightarrow$ and $\varepsilon < \mu(B)$. The proof essentially requires us to show that for all $K > 0$,

$$\inf_{\mu\in\mathcal{M}_\varepsilon} P_\mu\big(X_1(B) \geq K\big) > 0. \tag{3.1}$$

For once we are in possession of (3.1), a very similar argument to the one given for the branching process yields the statement of the proposition.

We continue then with the proof of (3.1). We first note that

$$P_{\delta_x}\big(X_1(B) \geq K\big) > 0, \qquad x \in D. \tag{3.2}$$

This follows from the fact that $X_1(B)$ is a (nonnegative) nondegenerate infinitely divisible random variable and consequently [see, e.g., Chapter 2 in Sato (1999)] its distribution has unbounded support on $\mathbb{R}_+$. (Infinite divisibility is of course a consequence of the branching property.)

Now let g be a nonnegative bounded continuous function on $\mathbb{R}_+$ such that

$$\mathbf{1}_{\{x>K\}} \leq g(x) \leq \mathbf{1}_{\{x>k\}}$$

with some $0 < k < K$. Using (3.2) and the choice of g, we see that

$$E_{\delta_x} g(X_1(B)) > 0, \qquad x \in D.$$

Note that

$$x \mapsto E_{\delta_x} g(X_1(B)) > 0, \qquad x \in D \text{ is continuous,} \tag{3.3}$$

and therefore has a positive infimum on compacts. The property in (3.3) follows from the continuity of the map $x \mapsto P_{\delta_x}(X_1(B) \in \cdot)$ with respect to the weak topology of probability distributions. (This latter fact is equivalent to the continuity of Laplace functionals in x and is thus a consequence of the log-Laplace equation.) Obviously,

$$P_\mu\big(X_1(B) \geq k\big) \geq E_\mu g(X_1(B)), \qquad \mu \in \mathcal{M}_\varepsilon. \tag{3.4}$$

Using (3.3), it follows that the infimum (over $\mathcal{M}_\varepsilon$) of the right-hand side in (3.4) is positive and thus so is the infimum of the left-hand side. We have just obtained (3.1) (with K replaced by k). $\square$

PROOF OF THEOREM 3(i). Assume that $\lambda_c \leq 0$; then there exists an $h > 0$ solving $(L + \beta)h = 0$. We claim that $\{\langle h, X_t\rangle : t \geq 0\}$ is a positive P_μ-supermartingale for all $\operatorname{supp}\mu \subset\subset D$. To see this note again that from a standard application of the Markov property and the branching property, it is enough to show that $E_{\delta_x}\langle h, X_t\rangle \leq h(x)$ for $t \geq 0$ and $x \in D$. We exemplify the case of superprocesses. Recall the discussion before Theorem 5 regarding the identity $E_{\delta_x}\langle \phi^t, X^{t,B}\rangle = \phi(x)$ for x in the nonempty open set B compactly embedded in D. For each fixed t we can define similarly to ϕ^t the function $h^t : D \times [0, t] \to (0, \infty)$ such that $h^t(\cdot, u) = h(\cdot)$ for each $u \in [0, t]$. Let $\{B_n, n \geq 1\}$ be an increasing sequence of open domains with smooth boundaries compactly embedded in D such that $D = \bigcup_{n\geq 1} B_n$. Similarly to the aforementioned identity involving ϕ we can deduce that for sufficiently large n so that $x \in B_n$, $E_{\delta_x}\langle h^t, X^{t,B_n}\rangle = h(x)$. (Here we use the obvious notation that X^{t,B_n} is the exit measure from $B_n \times [0, t)$.) Note that since the underlying motion is not necessarily conservative, we only know that

$$E_{\delta_x}\langle h, X_t\rangle \leq E_{\delta_x} \lim_{n\uparrow\infty} \big\langle h^t, X^{t,B_n}\big\rangle.$$

Fatou's lemma yields now the desired inequality $E_{\delta_x}\langle h, X_t\rangle \leq h(x)$.

Now, in the possession of this supermartingale it follows for Borel $B \subset\subset D$ that

$$\limsup_{t\uparrow\infty} X_t(B) \leq C \limsup_{t\uparrow\infty} \langle h, X_t\rangle < \infty, \tag{3.5}$$

P_μ-almost surely where C is a constant. When B is open, by Proposition 7 it follows that $\lim_{t\uparrow\infty} X_t(B) = 0$, P_μ-a.s. Since every compactly embedded Borel can be fattened up to an open $B \subset\subset D$, weak local extinction follows by comparison.

Assume now that $\lambda_c > 0$. Since it is assumed $\operatorname{supp}\mu \subset\subset D$, we can choose a large enough B for which $\operatorname{supp}\mu \subset B$ and $\lambda = \lambda_c(L+\beta, B) > 0$. Choose $0 \neq g \in C_c^+$ so that $g \le \mathbf{1}_B$; obviously, it suffices to prove that $P_\mu(\limsup_{t\uparrow\infty}\langle g, X_t\rangle > 0) > 0$. Let M_t^ϕ and $\widetilde{P}_\mu$ be as in Theorem 5. By Lemma 6, it is enough to show that $\widetilde{P}_\mu(\limsup_{t\uparrow\infty}\langle g, X_t\rangle > 0) > 0$, or equivalently, that $\widetilde{E}_\mu(e^{-\limsup_{t\uparrow\infty}\langle g, X_t\rangle}) < 1$. Let $\varepsilon > 0$. First use Fatou's lemma and then Theorem 5 (for the two classes of processes) along with the bound $u_g \le 1$ (for the case of branching processes) to obtain the estimate

$$\begin{aligned} &\widetilde{E}_\mu\Bigl(\exp\Bigl\{-\limsup_{t\uparrow\infty}\langle g, X_t\rangle\Bigr\}\Bigr) \\ &\quad\le \liminf_{t\uparrow\infty}\widetilde{E}_\mu(e^{-\langle g, X_t\rangle}) \\ &\quad\le \liminf_{t\uparrow\infty}\begin{cases}\mathbb{E}^\phi_{\phi\mu}\Bigl(\exp\Bigl\{-\displaystyle\int_{(t-\varepsilon)\vee 0}^t ds\, 2\alpha(Y_s)u_g(Y_s, t-s)\Bigr\}\Bigr) \\ \mathbb{E}^\phi_{\phi\mu}(e^{-g(Y_t)}),\end{cases} \end{aligned} \tag{3.6}$$

where the first case on the right-hand side is for superprocesses and the second is for branching processes. Before proceeding further, we need the fact that $u_g > 0$ on $D \times (0,\infty)$ (strict positivity) for the case of superprocesses. To see why this is true, suppose that $u_g(x,t) = 0$ for some $x \in D$, $t > 0$. This would imply by the log-Laplace equation that $\langle g, X_t\rangle = 0$ P_{δ_x}-almost surely. On the other hand, it is standard to derive that $v(x,t) := E_{\delta_x}\langle g, X_t\rangle = (T_t(g))(x)$, where T_t denotes the linear semigroup corresponding to the operator $L+\beta$ on D. Since $v > 0$ on $(0,\infty)\times D$, we get a contradiction. (The strict positivity of v follows from the strict positivity of the motion kernel corresponding to $\mathbb{P}$.)

Turning back to the estimate (3.6), the ergodicity of $(Y, \mathbb{P}^\phi_{\phi\mu})$ and the just mentioned strict positivity of u_g (for the case of superprocesses) imply that the right-hand side of (3.6) is less than one, which completes the proof. $\square$

Note the *intuition* behind the last part of the proof: the spine or immortal particle visits every part of B for arbitrarily large times because it is an ergodic diffusion; this forces the branching process itself to do the same.

PROOF OF THEOREM 3(ii). We may assume without loss of generality that $\lambda \in (0, \lambda_c)$. By standard theory, there exists a large enough $B^* \subset\subset D$ with a smooth boundary so that

$$\lambda^* := \lambda_c(L+\beta, B^*) \in (\lambda, \lambda_c].$$

Further, we can also choose B^* big enough so that $\operatorname{supp}\mu \subset\subset B^*$.

We first claim that if $\Omega_0 := \{\lim_{t\uparrow\infty} e^{-\lambda t} X_t(B^*) = \infty\}$, then $P_\mu(\Omega_0) > 0$. Indeed, if $\widehat{X}$ is defined as the $(L, \beta; B^*)$-branching process or the $(L, \beta, \alpha; B^*)$-superprocess, respectively, then

$$
\begin{aligned}
P_\mu(\Omega_0) &\geq P_\mu\Big(\liminf_{t\uparrow\infty} e^{-\lambda^* t} X_t(B^*) > 0\Big) \\
&\geq P_\mu\Big(\liminf_{t\uparrow\infty} e^{-\lambda^* t} \|\widehat{X}_t\| > 0\Big) \qquad (3.7)\\
&\geq P_\mu\Big(\lim_{t\uparrow\infty} e^{-\lambda^* t} \langle \phi^*, \widehat{X}_t\rangle > 0\Big),
\end{aligned}
$$

where $(L + \beta - \lambda^*)\phi^* = 0$ in B^* and $\phi^* = 0$ on ∂B^*. Note that it is implicit in the definition of $\widehat{X}$ that particles are killed on the boundary ∂B^*. Since $\lambda^* > 0$, Lemma 6 implies that the last term in (3.7) is positive.

Now let B be *any* open set with $\varnothing \neq B \subset\subset D$. From this point we consider two cases separately. Let X be the branching diffusion first. Let $p := \inf_{x\in B^*} p(1, x, B) > 0$, where $\{p(t, \cdot, dy) : t > 0\}$ is the transition measure for $(Y, \mathbb{P})$. Let $0 < q < p$ and $A_n := \{X_{n+1}(B) \geq q X_n(B^*)\}$. It follows from the law of large numbers and the Markov property that on Ω_0, $\lim_{n\uparrow\infty} P(A_n | X_n, \ldots, X_1) = 1$. Using the extended Borel–Cantelli lemma just like in the proof of Proposition 7, it follows that $\limsup_{t\uparrow\infty} e^{-\lambda t} X_t(B) = \infty$, P_{δ_x}-a.s. on Ω_0.

If X is the superprocess, the proof goes through with minor modifications as follows. Use again the branching property and split the mass in B^* into unit masses (with some possible leftover). Then, to imitate the previous proof, one only needs to know that for some $\varepsilon > 0$,

$$\inf_{\mu:\,\mathrm{supp}\mu\subset B^*,\,\|\mu\|\geq 1} P_\mu\big(X_1(B) > \varepsilon\big) > 0. \qquad (3.8)$$

Indeed, replace $\mathbf{1}_B$ by a nonnegative smooth continuous function g, such that $g \leq \mathbf{1}_B$. Recall the log-Laplace equation: $E_\mu(\exp\{-\langle g, X_t\rangle\}) = \exp\{-\langle u_g, \mu\rangle\}$ where u_g is the minimal nonnegative solution to $v_t = Lv + \beta v - \alpha v^2$ in D with $v(x, 0) = g(x)$. Note that $0 < \inf_{B^*} u_g(\cdot, 1) =: 2\varepsilon$. Thus (3.8) follows from the Markov inequality: for supp $\mu \subset B^*$, $\|\mu\| \geq 1$,

$$P_\mu(\langle g, X_t\rangle \leq \varepsilon) = P_\mu(\exp\{-\langle g, X_t\rangle\} \geq e^{-\varepsilon}) \leq \exp\{-\langle u_g, \mu\rangle\} e^{\varepsilon} \leq e^{-\varepsilon}.$$

This completes the proof of the first statement.

For the remaining statement, recall from the definition of λ_c in the Introduction that there exists an $h > 0$ such that $(L + \beta - \lambda_c)h = 0$ on D. Using this, it is easy to show [just like in the beginning of the proof of Theorem 3(i)] that $\exp(-\lambda_c t)\langle h, X_t\rangle$ is a supermartingale converging almost surely. Since $X_t(B)$ is bounded above by a constant times $\langle h, X_t\rangle$, a.s. finiteness of the lim sup follows. $\square$

4. Examples. In this section we will present four examples for branching particle diffusions which will illustrate the general results of this paper.

Let X denote the branching diffusion on D. We note the following connection with elliptic PDEs. Fix a nonempty open $B \subset\subset D$ and define $\rho(x) = P_{\delta_x}(\lim_{t\uparrow\infty} X_t(B) = 0)$. (Here we mean the equality in the *extended* sense: the left-hand side may be zero, in which case $\rho \equiv 0$.) That $\rho < 1$ follows from Theorem 3(i). The Markov and branching properties imply that

$$\rho(x) = E_{\delta_x} P_{X_t}\Big(\lim_{t\uparrow\infty} X_t(B) = 0\Big) = E_{\delta_x}\big(e^{\langle \log\rho, X_t\rangle}\big).$$

On the other hand, the function $u(x,t)$, defined to be equal to the right-hand side above, is known to solve the parabolic equation $\dot{u} = Lu + \beta(u^2 - u)$ on D [cf. Dynkin (1993), Theorem II.3.1]. It follows that ρ is a solution to the elliptic equation $Lu + \beta(u^2 - u)$ on D. Using an argument similar to the proof of Proposition 7 it is straightforward to show that one gets an almost surely equivalent event when one replaces B by any other $B^* \subset\subset D$ in the expression $\{\lim_{t\uparrow\infty} X_t(B) = 0\}$. It follows that ρ is independent of the choice of B in its definition.

After this short preparation we now present the examples.

4.1. *Branching Brownian motion* (*with drift*). Let $\varepsilon \in \mathbb{R}$, let $L = \frac{1}{2}\Delta + \varepsilon d/dx$ on $\mathbb{R}$ and let β be a positive constant. Then, for a small enough $|\varepsilon|$, the reproduction "wins" against the motion (which is transient for $\varepsilon \neq 0$), where $|\varepsilon|$ being small is captured by the condition $\lambda_c > 0$ of Theorem 3.

Indeed, a standard computation shows that $\lambda_c = \beta - (1/2)\varepsilon^2$, which is positive if and only if $|\varepsilon| < \sqrt{2\beta}$. According to the discussion at the beginning of this section, $\rho \in [0,1]$ satisfies $L\rho + \beta(\rho^2 - \rho) = 0$. Kolmogorov et al. (1937) proved that there are no nontrivial solutions bounded in [0, 1] to this, the traveling wave K–P–P equation, for $|\varepsilon| < \sqrt{2\beta}$ and otherwise there is a unique nontrivial solution valued in [0, 1]. We see that the probability that balls with positive radius become empty is either zero or one (i.e., $\rho \equiv 0$ or $\rho \equiv 1$) according to whether $\lambda_c > 0$ or $\lambda_c \le 0$, respectively.

4.2. *Transient L and compactly supported β.* Let L correspond to a transient diffusion on $D \subseteq \mathbb{R}^d$ and let β be a smooth nonnegative compactly supported function. Since the generalized principal eigenvalue coincides with the classical principal eigenvalue for smooth bounded domains, it follows that for any nonempty ball $B \subset\subset D$ one can pick such a β with $\overline{B} = \mathrm{supp}(\beta)$ and so that $L + \beta$ is supercritical on D, that is $\lambda_c > 0$ (all one has to do is to ensure that the infimum of β on a somewhat smaller ball $B' \subset\subset B$ is larger then the absolute value of the principal eigenvalue on B'; then, a fortiori, $L + \beta$ is supercritical on D as well). On the other hand, by the transience assumption, it is clear that the initial L-particle wanders out to infinity (or gets killed at the Euclidean boundary) with positive probability without ever visiting B (and thus without ever branching), when starting from a point in $D \setminus B$. This now shows that *there exists a nontrivial travelling wave solution* to $Lu + \beta(u^2 - u) = 0$ for such an L and β, namely $u = \rho$ where ρ is as in the beginning of this section. To the best of our knowledge, this is a new result concerning generalized K–P–P travelling wave equations.

4.3. *Branching Ornstein–Uhlenbeck process and generalization.* Let $L = \frac{1}{2}\Delta - kx \cdot \nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then L corresponds to the d-dimensional Ornstein–Uhlenbeck process with drift parameter k. Note that it is a (positive) recurrent process. Furthermore let β be a positive constant. Consider now the $(L, \beta; \mathbb{R}^d)$-branching diffusion X. We call X a *branching Ornstein–Uhlenbeck process*. By recurrence it follows that L is a critical operator, and thus $\lambda_c = \lambda_c(L, \mathbb{R}^d) = 0$. Consequently, $\lambda_c(L + \beta, \mathbb{R}^d) = \beta$. Obviously (by comparison with a single L-particle), the process does not exhibit local extinction. By Theorem 3(ii), X exhibits local exponential growth with rate β. In fact, as Theorem 4.6.3(i) in Pinsky (1996) shows, $\lambda_c(L + \beta, D) > 0$, whenever L corresponds to a recurrent diffusion on D and the branching rate $\beta \geq 0$ is not identically zero. Therefore, X *exhibits local exponential growth for any recurrent motion and any not identically zero branching rate.*

4.4. *Branching outward Ornstein–Uhlenbeck process.* Let $L = \frac{1}{2}\Delta + kx \cdot \nabla$ on $\mathbb{R}^d$, $d \geq 1$, where $k > 0$. Then L corresponds to the d-dimensional "outward" Ornstein–Uhlenbeck process with drift parameter k. This process is transient. Furthermore let β be a positive constant, and consider the $(L, \beta; \mathbb{R}^d)$-branching diffusion X. Following Example 2 in Pinsky (1996), we have that $\lambda_c(L + \beta, \mathbb{R}^d) = \beta - kd$. From Theorem 3(i) we conclude that if $\beta > kd$ then X exhibits local exponential growth (with rate $\beta - kd$). However if $\beta \leq kd$ then X exhibits local extinction.

It is easy to see that $h(x) = \exp\{-k|x|^2\}$ satisfies $(L + \beta - \lambda_c)h = 0$, and that making an h-transform with this h, $L + \beta - \lambda_c$ transforms into

$$(L + \beta - \lambda)^h = \tfrac{1}{2}\Delta - kx \cdot \nabla. \tag{4.1}$$

Now the operator in (4.1) corresponds to an (inward) Ornstein–Uhlenbeck process which is (positive) recurrent.

Using the associated inner-product martingale (which can again be shown to be a martingale exploiting backward parabolic equation together with boundedness) we can follow the arguments of the proof of Theorem 5 to produce (under a changed probability measure) a spine with a doubled rate of reproduction. This spine is precisely the Ornstein–Uhlenbeck process corresponding to the operator (4.1). In order to transfer statements of local survival back to the process under the original measure, we would need mean convergence of the inner-product martingale, or equivalently, the condition $\beta - kd > 0$.

APPENDIX

Local extinction criterion: analytical arguments. In this section we present an *analytical* proof of the local extinction criterion for the $(L,\beta;D)$-branching diffusion. As far as the proof of the condition for local extinction is concerned, we will show how to derive this from Pinsky's result discussed in the Introduction of this paper [Theorem 6 and Remark 1 in Pinsky (1996)] using a comparison argument between branching diffusions and superdiffusions. Our proof of the condition for local nonextinction will be essentially the same as his proof for superdiffusions.

Regarding the comparison mentioned above, it is likely that the deeper reason for it is hidden in the Evans and O'Connell (1994) "immigration picture" (see the comments after Theorem 5). For the rigorous proof we will utilize a result on the "weighted occupation time" for branching particle systems obtained by Evans and O'Connell (1994) (also used for proving the immigration picture in the same paper). In this section Z will denote the $(L,\beta;D)$-branching diffusion.

PROOF OF THE CRITERION ON LOCAL EXTINCTION. (i) Assume that $\lambda_c \le 0$. Let $(x,s)\mapsto \psi(s,x)$ be jointly measurable in (x,s) and let $\psi(s)=\psi(s,\cdot)$ be nonnegative and bounded for each $s\ge 0$. By Evans and O'Connell [(1994), Theorem 2.2], $E_x[\exp(-\int_0^t\langle\psi(s),Z_s\rangle\,ds)]=u(t,x)$, where u is the so-called *mild solution* of the evolution equation

$$\begin{aligned}&\dot u(s)=Lu(s)-\beta u(s)+\beta u^2(s)-\psi(t-s)u(s),\qquad 0<s\le t,\\&\lim_{s\downarrow 0}u(s)=1.\end{aligned}\tag{A.1}$$

[Here we used the notation $u(s)=u(s,\cdot)$ and $\dot u$ denotes the time-derivative of u.] Pick a $\psi\in C_c^+(D)$ satisfying $\psi(x)>0$ for $x\in B$ and $\psi(x)=0$ for $x\in D\setminus B$. Let $u=u_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned}&\dot u(s)=Lu(s)-\beta u(s)+\beta u^2(s)-\theta\psi\mathbf{1}_{[t,\infty)}(T-s)u(s),\qquad 0<s\le T,\\&\lim_{s\downarrow 0}u(s)=1.\end{aligned}\tag{A.2}$$

For the rest of the proof of part (i), let the starting point $x\in D$ be fixed. Using the argument given in Iscoe [(1988), page 207], we have that Z exhibits local extinction if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty}u_{t,\theta}^{(T)}(T,x)=1.\tag{A.3}$$

Let X be the (L,β,β,D)-superdiffusion and let $U=U_{t,\theta}^{(T)}$ be the mild solution of the evolution equation

$$\begin{aligned}&\dot U(s)=LU(s)+\beta U(s)-\beta U^2(s)+\theta\psi\mathbf{1}_{[t,\infty)}(T-s),\qquad 0<s\le T,\\&\lim_{s\downarrow 0}U(s)=0.\end{aligned}\tag{A.4}$$

Again, the argument given in Iscoe [(1988), page 207] shows that X exhibits local extinction if and only if

$$\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} U^{(T)}_{t,\theta}(T,x)=0. \tag{A.5}$$

In light of Theorem 6 and Remark 1 of Pinsky (1996), (A.5) follows from $\lambda_c\le 0$. We now show that (A.5) implies (A.3), which will complete the proof of this part. Making the substitution $v:=1-u$, we have that v is the mild solution of the evolution equation

$$\begin{aligned} &\dot v(s)=Lv(s)+\beta v(s)-\beta v^2(s)+\theta\psi\mathbf{1}_{[t,\infty)}(T-s)\big(1-v(s)\big),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0<s\le T,\\ &\lim_{s\downarrow 0} v(s)=0. \end{aligned}\tag{A.6}$$

By Iscoe [(1988), page 204], U and v (with t,θ fixed) have the following probabilistic representations:

$$\begin{aligned} U(T,x)&=-\log E_x\exp\bigg(-\int_0^T ds\,\langle\theta\psi\mathbf{1}_{[t,\infty)}(s),X_s\rangle\bigg),\\ v(T,x)&=-\log E_x\exp\bigg(-\int_0^T ds\,\langle\theta\psi\mathbf{1}_{[t,\infty)}(s)\big(1-v(T-s)\big),X_s\rangle\bigg). \end{aligned}\tag{A.7}$$

From (A.7) it is clear that $v\le U$. Hence $\lim_{t\uparrow\infty}\lim_{\theta\uparrow\infty}\lim_{T\uparrow\infty} v^{(T)}_{t,\theta}(T,x)=0$.

(ii) Assume now that $\lambda_c>0$. The proof of this part is almost the same as the proof of the analogous statement for superdiffusions in Pinsky [(1996), pages 262 and 263]. In that proof it is shown that the assumption $\lambda_c>0$ guarantees the existence of a (large) subdomain $D_0\subset\subset D$, and a function $v\ge 0$ defined on D_0 which is not identically zero and which satisfies

$$\begin{aligned} &Lv+\beta v-\beta v^2=0 \qquad \text{in } D_0,\\ &\lim_{x\to\partial D_0} v(x)=0,\\ &v>0 \qquad \text{in } D_0. \end{aligned}\tag{A.8}$$

[The proof of the existence of such a v relies on finding so-called lower and upper solutions for (A.8). The assumption $\lambda_c>0$ enters the stage when a positive lower solution is constructed.] Since $f\equiv 1$ also solves $Lf+\beta f-\beta f^2=0$ in D_0, the elliptic maximum principle [see Pinsky (1996), Proposition 3, and Engländer and Pinsky (1999), Proposition 7.1] implies that $v\le 1$. Let $w:=1-v$. Then $w\ge 0$ and furthermore w satisfies

$$\begin{aligned} &Lw-\beta w+\beta w^2=0 \qquad \text{in } D_0,\\ &\lim_{x\to\partial D_0} w(x)=1,\\ &w<1 \qquad \text{in } D_0. \end{aligned}\tag{A.9}$$

Let $\widehat{P}$ denote the probability for the branching diffusion $\widehat{Z}$ obtained from Z by killing the particles upon exiting ∂D_0. Obviously $\widehat{P}_x(\widehat{Z}$ survives$) \le P_x(Z(t, D_0) > 0$ for arbitrary large t's$)$, and thus, it is enough to show that

$$0 < \widehat{P}_x(\widehat{Z} \text{ survives}). \tag{A.10}$$

We now need the fact that $w > 0$ on D_0. This follows from the equation

$$\big(L - \beta(1 - w)\big)w = 0 \qquad \text{in } D_0$$

and the strong maximum principle [Theorem 3.2.6 in Pinsky (1995)] applied to the linear operator $L - \beta(1 - w)$. [Indeed, w is a nonnegative harmonic function for the operator, and thus, by the strong maximum principle it must be either everywhere zero (i.e., $v \equiv 1$) or everywhere positive; however the first case is ruled out by the second equation of (A.8).]

Since w is a positive solution to the elliptic equation and is one at the boundary, it is standard to prove that $\exp\{-\langle \log w, \widehat{Z}\rangle\}$ is a martingale and in particular,

$$\widehat{E}_x(\exp\{-\langle \log w, \widehat{Z}\rangle\}) = w(x), \qquad t \ge 0. \tag{A.11}$$

Suppose that (A.10) is not true. Then the left-hand side of (A.11) converges to 1 as $t \uparrow \infty$. On the other hand, the right-hand side of (A.11) is independent of t and is smaller than 1, which is a contradiction. Consequently, (A.10) is true. □

Acknowledgments. We have greatly benefited from Ross Pinsky's comments concerning the proof of Theorem 3(ii). We are also grateful to an anonymous referee whose remarks and observations led to considerable improvements in this paper. A substantial part of this collaborative work has been done during the first author's stay at EURANDOM, Eindhoven. The first author is grateful to EURANDOM for its hospitality.

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DEPARTMENT OF STATISTICS
AND APPLIED PROBABILITY
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106-3110
USA
E-MAIL: englander@pstat.ucsb.edu

DEPARTMENT OF MATHEMATICS
UTRECHT UNIVERSITY
BUDAPESTLAAN 6
NL-3584 CD UTRECHT
THE NETHERLANDS
E-MAIL: kyprianou@math.uu.nl