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The Gapeev–Kühn stochastic game driven by a spectrally positive Lévy process

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Abstract

In Gapeev and Kühn (2005) [8], the Dynkin game corresponding to perpetual convertible bonds was considered, when driven by a Brownian motion and a compound Poisson process with exponential jumps. We consider the same stochastic game but driven by a spectrally positive Lévy process. We establish a complete solution to the game indicating four principle parameter regimes as well as characterizing the occurrence of continuous and smooth fit. In Gapeev and Kühn (2005) [8], the method of proof was mainly based on solving a free boundary value problem. In this paper, we instead use fluctuation theory and an auxiliary optimal stopping problem to find a solution to the game.

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1. Introduction

Let $X = (X_t, t \geq 0)$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see, for instance, Definition 1.3.38 in [4]). For $x \in \mathbb{R}$ denote by \mathbb{P}_x the law of X when it starts at x , and write simply $\mathbb{P}_0 = \mathbb{P}$. Accordingly, we shall write \mathbb{E}_x and \mathbb{E} for the associated expectation

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operators. In this paper we shall assume throughout that X is *spectrally positive* meaning here that it has no negative jumps and that it is not a subordinator. It is well known that the latter allows us to talk about the Laplace exponent $\psi(\theta) : [0, \infty) \rightarrow \mathbb{R}$, i.e.

$$\mathbb{E}\left[e^{-\theta X_t}\right] =: e^{t\psi(\theta)}, \quad t, \theta \geq 0,$$

and the Laplace exponent is given by the Lévy–Khintchine formula

$$\psi(\theta) = \mu\theta + \frac{b^2}{2}\theta^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x \mathbb{1}_{\{x < 1\}}) \Pi(dx), \tag{1.1}$$

where $\mu \in \mathbb{R}$, $b^2 \geq 0$ and Π is a measure on $(0, \infty)$ called the Lévy measure of X and satisfies

$$\int_{(0,\infty)} (1 \wedge x^2) \Pi(dx) < \infty.$$

The reader is referred to Bertoin [3] and Sato [14] for a complete introduction to the theory of Lévy processes.

Denote by $\mathcal{T}_{0,\infty}$ the family of all $[0, \infty]$ -valued stopping times with respect to \mathbb{F} . We are interested in establishing a solution to a special class of stochastic games which are driven by spectrally positive Lévy processes. Specifically, for $\alpha > 0$ and $\beta \geq 0$ and $q, K > 0$, let

$$L_t := e^{-qt+X_t} + \int_0^t e^{-qs} (\alpha + \beta e^{X_s}) ds,$$

and

$$U_t := e^{-qt} (e^{X_t} \vee K) + \int_0^t e^{-qs} (\alpha + \beta e^{X_s}) ds.$$

We are interested in the Dynkin game consisting of two players and expected payoff given by

$$M_x(\tau, \sigma) := \mathbb{E}_x \left[L_\tau \mathbb{1}_{\{\tau < \sigma\}} + U_\sigma \mathbb{1}_{\{\sigma \leq \tau\}} \right], \tag{1.2}$$

for $x \geq 0$. See [6] for the original source of this type of problem. The inf-player’s objective is to choose some $\sigma \in \mathcal{T}_{0,\infty}$ which minimizes (1.2), whereas the sup-player chooses some $\tau \in \mathcal{T}_{0,\infty}$ which maximizes this quantity. We are principally interested in showing the existence of a stochastic saddle point. That is, we want to find τ^* and σ^* such that

$$M_x(\tau, \sigma^*) \leq M_x(\tau^*, \sigma^*) \leq M_x(\tau^*, \sigma) \quad \text{for all } \tau, \sigma \in \mathcal{T}_{0,\infty}. \tag{1.3}$$

Note that (1.3) implies for each x ,

$$V(x) := \sup_\tau \inf_\sigma M_x(\tau, \sigma) = \inf_\sigma \sup_\tau M_x(\tau, \sigma), \tag{1.4}$$

the value of the game. Recent work on stochastic games refers to the stochastic saddle point as the Nash equilibrium and the value of the game the Stackelberg equilibrium; cf. [7,11]. Note that for $x \geq \log K$

$$M_x(\tau, 0) = e^x = M_x(0, 0) = M_x(0, \sigma)$$

for any τ, σ , i.e. $\tau^* = 0$ and $\sigma^* = 0$ form a stochastic saddle point whenever $x \geq \log K$ with $V(x) = e^x$. In what follows, we assume

$$(A) : \psi(-1) < q.$$

In that case, the Laplace exponent ψ is well defined on $[-1, \infty)$ and moreover the Lévy–Khintchine formula can be extended to the interval $[-1, 0)$ (see for instance Lemma 26.4 in [14]). Without this condition the gain in the expectations in (1.2), that is to say U_∞ , is infinity on the event $\{\tau = \sigma = \infty\}$. Note however, the general theory of Dynkin games as described in [7,11] also allows us to consider adaptations of the above stochastic games in which we allocate a third payoff, say Z , on the event that $\{\tau = \sigma\}$ which is valued between L and U .

The problem of finding a stochastic saddle point solution to (1.4) was originally considered in the setting of compound Poisson and exponentially distributed jumps by Gapeev and Kühn [8]. In that paper the authors motivate the stochastic game from the point of view of financial modelling as follows. One assumes that a firm issues a convertible bond at time zero. At each subsequent time, the bondholder can decide whether to continue to hold the bond, thereby collecting coupons at the rate $\alpha + \beta e^{X_t}$, where we think of the process $\{e^{X_t} : t \geq 0\}$ as the value of some underlying risky asset, or to convert it into a unit of the underlying. On the other hand, at any time the issuing firm can redeem the bond at some call price K , but at the same time it has to offer the holder to convert the bond instantly. Put differently, the firm can terminate the contract by paying the amount $\max\{K, e^{X_t}\}$.

Our interest in this stochastic game however comes more from a theoretical point of view and in particular the possible effects of the more complex jump structure that can be found in the underlying Lévy process. Technically speaking, the specific jump structure in [8] allowed them to convert the problem into an integro-differential free boundary problem. In the present setting, there is limited assistance one can gain from the theory of integro-differential free boundary problems on account of the fact that they are poorly understood at the required degree of generality. We turn instead to a mixture of probabilistic reasoning using the general theory of optimal stopping problems together with potential analytic considerations for spectrally negative Lévy processes which manifest themselves in the modern theory of scale functions.

2. Main results

Below, in Theorems 1–4 we give a qualitative and quantitative exposition of the solution to (1.3). Before doing so, we need to give a brief reminder of a class of special functions which appear commonly in connection with the study of spectrally positive Lévy processes. For each $p \geq 0$ we introduce the functions $W^{(p)} : \mathbb{R} \rightarrow [0, \infty)$ which are known to satisfy for all $x, y \geq 0$,

$$\mathbb{E} \left[e^{-p\tau_{-x}^-} \mathbb{1}_{\{\tau_{-x}^- < \tau_y^+\}} \right] = \frac{W^{(p)}(y)}{W^{(p)}(x + y)}, \tag{2.5}$$

where

$$\tau_y^+ := \inf\{t > 0 : X_t > y\} \quad \text{and} \quad \tau_{-x}^- := \inf\{t > 0 : X_t < -x\}$$

(cf. Chapter 8 of Kyprianou [9]). In particular $W^{(p)}(x) = 0$ for all $x < 0$ and further, it is known that on $(0, \infty)$, $W^{(p)}$ is almost everywhere differentiable, there is right continuity at zero and

$$\int_0^\infty e^{-\beta x} W^{(p)}(x) dx = \frac{1}{\psi(\beta) - p}$$

for all $\beta > \Phi(p)$, where $\Phi(p)$ is the largest root of the equation $\psi(\theta) = p$ (of which there are at most two). For convenience, we write W instead of $W^{(0)}$.

Associated to the functions $W^{(p)}$ are the functions $Z^{(p)} : \mathbb{R} \rightarrow [1, \infty)$ defined by

$$Z^{(p)}(x) = 1 + p \int_0^x W^{(p)}(y)dy$$

for $p \geq 0$. Together, the functions $W^{(p)}$ and $Z^{(p)}$ are collectively known as scale functions and predominantly appear in almost all fluctuation identities for spectrally positive Lévy processes. For example, it is known that for all $y \in \mathbb{R}$

$$\mathbb{E} \left[e^{-p\tau_y^+} \mathbb{1}_{\{\tau_y^+ < \infty\}} \right] = Z^{(p)}(y) - \frac{p}{\Phi(p)} W^{(p)}(y). \tag{2.6}$$

We make the very mild assumption that Π has no atoms when X has paths of bounded variation. This suffices to deduce (cf. [5]) that $W^{(p)} \in C^1(0, \infty)$ and hence $Z^{(p)} \in C^2(0, \infty)$ and further, if X has a Gaussian component they both belong to $C^2(0, \infty)$. It is also known that if X has bounded variation with drift d , then $\lim_{x \downarrow 0} W^{(q)}(x) =: W^{(p)}(0+) = 1/d$ and otherwise $W^{(p)}(0+) = 0$. (Here and in what follows we take the canonical representation of a bounded variation spectrally positive Lévy processes $X_t = S_t - dt$ for $t \geq 0$ where $(S_t, t \geq 0)$ is a driftless subordinator and d is a strictly positive constant which is referred to as the drift.) Further, when X has unbounded variation,

$$W^{(p)'}(0+) = 2/b^2 \tag{2.7}$$

which is understood to be $+\infty$ when $b^2 = 0$. Consider the exponential change of measure

$$\left. \frac{d\mathbb{P}^{(\lambda)}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{-\lambda X_t - \psi(\lambda)t}, \quad \text{for } \lambda \geq -1. \tag{2.8}$$

Under $\mathbb{P}^{(\lambda)}$, the process X is still a spectrally positive Lévy process and we mark its Laplace exponent and scale functions with the subscript λ . It holds that

$$\psi_\lambda(\theta) = \psi(\lambda + \theta) - \psi(\lambda)$$

for $\theta \geq 0$ and, by taking Laplace transforms, we find

$$W_\lambda^{(p)}(x) = e^{-\lambda x} W^{(p+\psi(\lambda))}(x)$$

for $p \geq 0$. The reader is otherwise referred to Chapter VII of Bertoin [3] or Chapter 8 of Kyprianou [9] for a general overview of one-sided Lévy processes and scale functions.

It turns out that the solution to the stochastic game can fall in four different regimes, depending on the value of the discount factor q . We remind the reader of the standing assumption (A).

Theorem 1. *Suppose $q \leq \alpha/K$. Then a saddle point for the stochastic game (1.3) is given by $\sigma^* = 0, \tau^* = \tau_{\log K}^+$. In particular $V(x) = K \vee e^x$ for all x .*

Theorem 2.

(i) *As a function of q ,*

$$a^*(q) := \frac{\alpha(\Phi(q) + 1)}{\Phi(q)(q - \psi(-1) - \beta)},$$

defined for $q > \beta + \psi(-1)$, is strictly monotone decreasing with $a^*(\beta + \psi(-1)+) = \infty$ and $a^*(\infty) = 0$. Define

$$q_0 = \sup \left\{ q \in (0, \infty) : a^*(q) < K \right\}$$

(noting that $q_0 > \beta + \psi(-1)$ necessarily). It holds that $q_0 > \alpha/K$.

(ii) For all $q \in [q_0, \infty)$ a stochastic saddle point is given by the pair

$$\tau^* = \inf \left\{ t \geq 0 : X_t > \log a^*(q) \right\} \quad \text{and} \quad \sigma^* = \inf \left\{ t \geq 0 : X_t > \log K \right\}.$$

In particular,

$$V(x) = e^x + \frac{\alpha}{\Phi(q)} g(\log(a^*(q)) - x),$$

where

$$g(z) = (\Phi(q) + 1) \int_0^z e^{y-z} W^{(q)}(y) dy - \Phi(q) \int_0^z W^{(q)}(y) dy.$$

(iii) For $q \in [q_0, \infty)$, there is smooth fit at $\log a^*(q)$ if and only if X has paths of unbounded variation and otherwise there is continuous fit.

Theorem 3.

(i) Assume that $b^2 > 0$. The set

$$\left\{ q \in (0, q_0) : 0 < K \frac{b^2}{2} + \frac{\alpha}{\Phi(q)} \left(\frac{K}{a^*(q)} - 1 \right) \right\}$$

is an interval whose infimum we denote by q_1 . It holds that $q_1 > \alpha/K$.

(ii) When $b^2 > 0$ and $q \in [q_1, q_0]$, a saddle point for the stochastic game (1.3) is given by

$$\tau^* = \sigma^* = \inf \left\{ t \geq 0 : X_t > \log K \right\}.$$

In particular,

$$\begin{aligned} V(x) = e^x & \left(1 + (q - \psi(-1)) \int_0^{\log K - x} e^y W^{(q)}(y) dy \right. \\ & \left. - K \frac{q - \psi(-1)}{\Phi(q) + 1} e^{-x} W^{(q)}(\log K - x) \right) \\ & + \frac{W^{(q)}(\log K - x)}{K \Phi(q)} \int_{-\infty}^{\log K} (\alpha + \beta e^y) e^{\Phi(q)y} dy \\ & - \int_x^{\log K} (\alpha + \beta e^y) W^{(q)}(y - x) dy. \end{aligned}$$

(iii) When $b^2 > 0$ and $q \in [q_1, q_0]$ there is smooth fit at $\log K$ if and only if $q = q_0$ or $q = q_1$.

(iv) When $b^2 = 0$, then the strategies $\tau^* = \sigma^* = \inf\{t \geq 0 : X_t > \log K\}$ do not form a stochastic saddle point when $q < q_0$.

When $b^2 = 0$ the above theorem does not define a value of q_1 . For convenience of use in the next theorem we shall simply set $q_1 := q_0$ when $b^2 = 0$.

Theorem 4. Suppose that $\alpha/K < q < q_1$.

(i) *The functional equation in a*

$$\frac{q}{\Phi(q)} = \frac{1}{K} \left(\frac{\alpha}{\Phi(q)} + \frac{\beta}{\Phi(q)+1} e^a \right) - \frac{1}{\Phi(q)} \int_0^\infty \Pi(dz + \log K - a)(1 - e^{-\Phi(q)z}) + \frac{1}{\Phi(q)+1} \int_0^\infty \Pi(dz + \log K - a)e^z(1 - e^{-(\Phi(q)+1)z})$$

has a unique solution in $(-\infty, \log K)$ which we denote by $c^*(q)$.

(ii) *A stochastic saddle point is given by the pair*

$$\tau^* = \inf\{t \geq 0 : X_t > \log K\} \quad \text{and} \quad \sigma^* = \inf\{t \geq 0 : X_t > c^*(q)\}.$$

It holds that, for $x < c^*(q)$

$$\begin{aligned} V(x) = & K \left(Z^{(q)}(c^*(q) - x) - \frac{q}{\Phi(q)} W^{(q)}(c^*(q) - x) \right) \\ & + \int_{-\infty}^c (\alpha + \beta e^y) \left(e^{-\Phi(q)(c^*(q)-y)} \right. \\ & \times W^{(q)}(c^*(q) - x) - W^{(q)}(y - x) \Big) dy + e^{\Phi(q)(x-c^*(q))} \int_0^\infty e^{-\Phi(q)u} \Pi(du) \\ & \times \int_{-\infty}^0 dy \left(W^{(q)}(c^*(q) - x) - e^{-\Phi(q)y} W^{(q)}(c^*(q) - x + y) \right) \\ & \times e^{\Phi(q)(u+y)} \left(e^{c^*(q)+u+y} - K \right) \mathbb{1}_{\{u+y+c^*(q) > \log K\}} \end{aligned}$$

and for $x \geq c^*(q)$

$$V(x) = e^x \vee K.$$

(iii) *There is smooth fit at $c^*(q)$ if and only if X has paths of unbounded variation and otherwise there is continuous fit.*

The order in which we present these statements above (first $q \leq \alpha/K$, followed by $q \geq q_0$, $q \in [q_1, q_0]$ when $b^2 > 0$, and finally $q \in (\alpha/K, q_1)$) is convenient with regard to the dependency between their proofs. We also remark that careful consideration of the proof of [Theorem 2](#) reveals that we can also include the parameter choices $\alpha = \beta = 0$; albeit that the corresponding saddle point is somewhat uninteresting in that $\tau^* = 0$ and $\sigma^* = \tau_{\log K}^+$. In particular, one may think of $a^*(q) = 0$, $q_0 = \infty$ and $q_1 = 0$ in this case.

The four theorems above can also be understood intuitively in terms of the discount rate q . Because of the Markovian nature of the Dynkin game, as well as the fact that there is an infinite horizon, it makes heuristic sense that both the inf-player and sup-player will seek threshold strategies for the underlying process X which are time invariant. (This is also consistent with other optimal stopping problems of this type.) From the sup-player's point of view, the larger the value of q the less time they are prepared to wait for a large value of X (and hence L) before stopping as this severely discounts the ultimate payout they will receive. This results in a lower threshold tolerance the higher the value of q . From the inf-player's point of view, a small value of q is bad as it means that payouts to the sup-player are relatively large and hence, in that case, it makes sense to intervene and force a payout early before a large value of X becomes accessible. In particular, [Theorem 1](#) identifies α/K as a cut-off value in q below which the inf-player finds

it preferential to simply stop immediately rather than incur any risk on the part of the sup-player. **Theorem 4** indicates that if q is still small enough but bigger than the aforementioned cut-off, namely $\alpha/K < q < q_1$, then the inf-player has a lower degree of tolerance with regard to their stopping threshold than the sup-player and the saddle point is controlled entirely by the behaviour of the inf-player. In light of **Theorem 3** one may think of the constant q_1 as the level at which the threshold for the inf-player, $c^*(q)$, equals $\log K$. As the value of q rises above q_1 , **Theorem 3** also says that, in the presence of a Gaussian coefficient, as long as q does not exceed the value $q_0 > q_1$ both inf- and sup-player agree on stopping the game once the threshold $\log K$ has been exceeded. If there is no Gaussian component, this agreement occurs uniquely at q_0 which is determined by **Theorem 2**. For q larger than q_0 , the stochastic game is controlled entirely by the sup-player in the sense this player has a lower threshold tolerance on account of the heavier discounting and is prepared to stop the game at $\log a^*(q) < \log K$.

By comparison with the original paper of Gapeev and Kühn [8] the above qualitative summary is essentially the same despite the fact that there is now a much more general class of processes driving the stochastic game. This follows on account of the structure of the solution lying largely with threshold tolerances which are largely determined by the degree of discounting. A difference that the analysis here does bring out however is the extent to which smooth and continuous pasting is determined by the small-time behaviour of the underlying Lévy process.

Recall that the issue of smooth or continuous pasting is a phenomenon which occurs in most optimal stopping problems driven by Markov processes with jumps. Consistently with the general theory given in [7,11], the value of the Dynkin game in this particular setting is upper bounded by the function $e^x \vee K$ and lower bounded by e^x . According to the statements of **Theorems 1, 4, 3** and **2**, when the inf-player has a lower threshold tolerance than the sup-player, the value function ‘pastes’ on to the curve $e^x \vee K$ at the inf-player’s stopping threshold. That is to say, the value function is equal to the function $e^x \vee K$ on the upper half-line that describes the inf-player’s stopping region and otherwise it is strictly bounded from above and below by $e^x \vee K$ and e^x respectively. Similarly, when the sup-player has a lower threshold tolerance than the inf-player, the value function pastes on to the function e^x at the sup-player’s stopping threshold. For many optimal stopping problems which are driven by diffusions it has been observed that a principle of smooth pasting holds. In other words, the first derivative of the value function is continuous at the point of pasting. Recent work on optimal stopping problems driven by Markov processes with jumps suggests that the phenomenon of smooth pasting is not always observed. When smooth pasting does not occur it is seen that there is just continuous pasting. See the discussion in [1] for a recent overview of this phenomenon as well as a heuristic of why it occurs.

Consistently with the ‘rule of thumb’ conjectured in [1], it is the regularity of the boundary of the stopping region for its interior which determines whether smooth pasting or just continuous pasting occurs. Since all stopping regions correspond to an upper half-line and X is spatially homogeneous, this boils down to the regularity of 0 for $(0, \infty)$. Since X is a spectrally positive Lévy process, the latter regularity happens to coincide with the situation that X has unbounded variation. There is one exception to this rule however, the case that both the inf- and sup-player agree on stopping at the same threshold level $\log K$ and a Gaussian component is present. Here the interior of the stopping region is regular for its boundary point, however no smooth pasting occurs when $q \in (q_1, q_0)$ due to the kink in the pay-off function there.

The remainder of this paper is dedicated to proving **Theorems 1–4** and is structured as follows. In the next section we state a lemma which will be repeatedly used to implement proofs on the basis of ‘guess and verify’ such as is common with solving optimal stopping problems. Thereafter we prove the four main theorems above in the order that they are stated.

3. Guess and verify

Following classical ideas in optimal stopping, we verify that a candidate solution solves the stochastic game (1.3) by checking certain associated bounds and martingales properties. Specifically, we will use the following verification lemma which is of a similar form in statement and proof to Lemma 5 in [2]. For the sake of brevity we omit its proof.

Lemma 1 (Verification Lemma). Fix $x \in \mathbb{R}$. Suppose that $\tau^* \in \mathcal{T}_{0,\infty}$ and $\sigma^* \in \mathcal{T}_{0,\infty}$ are candidate optimal strategies for the stochastic game (1.4) such that

$$\sup_{\sigma \in \mathcal{T}_{0,\infty}} e^{-q\sigma + X_\sigma} \mathbb{1}_{\{\sigma \leq \tau^*\}} \tag{3.9}$$

has finite mean under \mathbb{P}_x . Let

$$V^*(x) = \mathbb{E}_x \left(L_{\tau^*} \mathbb{1}_{\{\tau^* < \sigma^*\}} + U_{\sigma^*} \mathbb{1}_{\{\sigma^* \leq \tau^*\}} \right).$$

Then (τ^*, σ^*) is a stochastic saddle point of (1.3) with value V^* if

- (i) $V^*(x) \geq e^x$,
- (ii) $V^*(x) \leq e^x \vee K$,
- (iii) $V^*(X_{\tau^*}) = e^{X_{\tau^*}}$ almost surely on $\{\tau^* < \infty\}$,
- (iv) $V^*(X_{\sigma^*}) = e^{X_{\sigma^*}} \vee K$ almost surely on $\{\sigma^* < \infty\}$,
- (v) the process

$$\left(e^{-q(t \wedge \tau^*)} V^*(X_{t \wedge \tau^*}) + \int_0^{t \wedge \tau^*} e^{-qs} (\alpha + \beta e^{X_s}) ds, t \geq 0 \right)$$

is a right continuous submartingale and

- (vi) the process

$$\left(e^{-q(t \wedge \sigma^*)} V^*(X_{t \wedge \sigma^*}) + \int_0^{t \wedge \sigma^*} e^{-qs} (\alpha + \beta e^{X_s}) ds, t \geq 0 \right)$$

is a right continuous supermartingale.

4. Proof of Theorem 1

Suppose $q \leq \alpha/K$. We claim that the process $(Z_t, t \geq 0)$ defined by

$$Z_t = \left(e^{-q(t \wedge \tau_{\log K}^+)} (K \vee e^{X_{t \wedge \tau_{\log K}^+}}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds, t \geq 0 \right)$$

is a submartingale. Indeed, when $x < \log K$, we have on $\{t < \tau_{\log K}^+\}$

$$dZ_t = \left[(\alpha - qK) e^{-qt} + \beta e^{-qt + X_t} \right] dt$$

and

$$Z_{\tau_{\log K}^+} - Z_{\tau_{\log K}^+ -} = e^{-q\tau_{\log K}^+} \left(e^{X_{\tau_{\log K}^+}} - K \right),$$

showing that, as $\beta \geq 0$, Z is an adapted, strictly increasing process, i.e. a submartingale.

We may now invoke the Verification Lemma, since the other properties are automatically satisfied by taking $\sigma^* = 0$. Note in particular that the condition (3.9) is automatically satisfied since

$$\sup_{\sigma \in \mathcal{T}_{0,\infty}} e^{-q\sigma + X_\sigma} \mathbb{1}_{\{\sigma \leq \tau_{\log K}^+\}} \leq e^{-q\tau_{\log K}^+ + X_{\tau_{\log K}^+}} \mathbb{1}_{\{\tau_{\log K}^+ < \infty\}} + K$$

and

$$\mathbb{E}_x \left(e^{-q\tau_{\log K}^+ + X_{\tau_{\log K}^+}} \mathbb{1}_{\{\tau_{\log K}^+ < \infty\}} \right) = e^x \mathbb{E}_x^{(-1)} \left(e^{-(q-\psi(-1))\tau_{\log K}^+} \right) \leq e^x.$$

5. Proof of Theorem 2

The basis of the proof of Theorem 2 is the assumption that the optimal strategies take the form $\sigma^* = \inf\{t > 0 : X_t > \log K\}$ and $\tau^* = \inf\{t > 0 : X_t > y^*\}$ for some optimally chosen y^* . Establishing the value function, V , would boil down to computing H_{y^*} where for any $-\infty < y \leq \log K$, $H_y(x) := \mathbb{E}_x(L_{\tau_y^+})$, that is to say,

$$H_y(x) := \mathbb{E}_x \left(e^{-q\tau_y^+ + X_{\tau_y^+}} \right) + \int_0^\infty \mathbb{E}_x \left(e^{-qs} (\alpha + \beta e^{X_s}) \mathbb{1}_{\{s \leq \tau_y^+\}} \right) ds. \tag{5.10}$$

We thus proceed by evaluating the above expression in terms of scale functions, then we choose the value of y^* by blindly applying the principle of smooth and continuous fit respectively to the cases that X has paths of unbounded and bounded variation and finally we verify that the established strategy is indeed optimal with the help of the Verification Lemma.

With the help of the exponential change of measure, (2.8) and (2.6), the first term of the right-hand side of the above expression for H_y satisfies

$$\begin{aligned} \mathbb{E}_x \left(e^{-q\tau_y^+ + X_{\tau_y^+}} \right) &= e^x \mathbb{E} \left(e^{-q\tau_{y-x}^+ + X_{\tau_{y-x}^+}} \right) \\ &= e^x \mathbb{E}^{(-1)} \left(e^{-(q-\psi(-1))\tau_{y-x}^+} \right), \\ &= e^x \left(Z_{-1}^{(q-\psi(-1))}(y-x) - \frac{q-\psi(-1)}{\Phi_{-1}(q-\psi(-1))} e^{y-x} W^{(q)}(y-x) \right) \\ &= e^x \left(1 + (q-\psi(-1)) \int_0^{y-x} e^y W^{(q)}(z) dz \right. \\ &\quad \left. - \frac{q-\psi(-1)}{\Phi_{-1}(q-\psi(-1))} e^{y-x} W^{(q)}(y-x) \right), \end{aligned}$$

where Φ_{-1} is the right inverse of ψ_{-1} .

On the other hand from Theorem 8.7 in [9], the second term of the right-hand side of (5.10) satisfies

$$\begin{aligned} \int_0^\infty \mathbb{E}_x \left(e^{-qs} (\alpha + \beta e^{X_s}) \mathbb{1}_{\{s \leq \tau_y^+\}} \right) ds &= \int_0^\infty \widehat{\mathbb{E}} \left(e^{-qs} (\alpha + \beta e^{x-X_s}) \mathbb{1}_{\{s \leq \tau_{x-y}^-\}} \right) ds \\ &= \int_0^\infty (\alpha + \beta e^{y-z}) \left(e^{-\Phi(q)z} W^{(q)}(y-x) - W^{(q)}(y-x-z) \right) dz \\ &= \int_{-\infty}^y (\alpha + \beta e^z) \left(e^{-\Phi(q)(y-z)} W^{(q)}(y-x) - W^{(q)}(z-x) \right) dz, \end{aligned}$$

where $\widehat{\mathbb{P}}$ denotes the law of the dual process $\widehat{X} = -X$. Finally noting that $\Phi_{-1}(q - \psi(-1)) = \Phi(q) + 1$, we get

$$H_y(x) = e^x \left(1 + (q - \psi(-1)) \int_0^{y-x} e^z W^{(q)}(z) dz - \frac{q - \psi(-1)}{\Phi(q) + 1} e^{y-x} W^{(q)}(y-x) \right) + \int_{-\infty}^y (\alpha + \beta e^z) e^{-\Phi(q)(y-z)} W^{(q)}(y-x) dz - \int_x^y (\alpha + \beta e^z) W^{(q)}(z-x) dz.$$

We also see in particular, making use of the fact that $W^{(q)}(0-) = 0$ and $Z^{(q)}(0) = 1$, that

$$H_y(x) = e^x$$

for all $x > y$.

Having expressed H_y in terms of scale functions, we now turn our attention to making the choice of y^* using the principle of smooth and continuous fit in line with the ‘rule of thumb’ conjectured in [1].

Bounded variation and continuous fit: In this case it is known that $W^{(q)}(0+) = 1/d$ where $d > 0$ is the drift term of the process X . It follows that

$$\begin{aligned} H_{\log a}(\log a-) &= a \left(1 - \frac{q - \psi(-1)}{\Phi(q) + 1} W^{(q)}(0+) \right) \\ &\quad + W^{(q)}(0+) a^{-\Phi(q)} \int_{-\infty}^{\log a} (\alpha + \beta e^y) e^{\Phi(q)y} dy \\ &= a + a W^{(q)}(0+) \left(\frac{\alpha}{a \Phi(q)} + \frac{\beta}{\Phi(q) + 1} - \frac{q - \psi(-1)}{\Phi(q) + 1} \right). \end{aligned} \tag{5.11}$$

In order to avoid a discontinuity at a we choose it equal to the value a^* which satisfies

$$\frac{q - \psi(-1)}{\Phi(q) + 1} = \frac{\alpha}{a \Phi(q)} + \frac{\beta}{\Phi(q) + 1}.$$

Note that this is equivalent to requiring that

$$a^* = \frac{\alpha(\Phi(q) + 1)}{\Phi(q)(q - \psi(-1) - \beta)} \tag{5.12}$$

provided $q > \psi(-1) + \beta$. In order to see the values of q for which $a^* \leq K$ we also need to check how the function $a^* = a^*(q)$ varies with q . To this end, note that, for $q > \psi(-1) + \beta$,

$$\frac{d}{dq} a^*(q) = \frac{-(q - \psi(-1) - \beta)\alpha \Phi'(q) - \alpha(\Phi(q) + 1)\Phi(q)}{(\Phi(q)(q - \psi(-1) - \beta))^2} < 0,$$

hence $a^*(\cdot)$ is strictly decreasing. Note also that

$$\lim_{q \downarrow \beta + \psi(-1)} a^*(q) = \infty \quad \text{and} \quad \lim_{q \rightarrow \infty} a^*(q) = 0,$$

which implies the existence of a unique $q_0 > \beta + \psi(-1)$ such that $a^*(q_0) = K$. Note that it also turns out that $q_0 > \alpha/K$ on account of the fact that for $q \leq \alpha/K$

$$a^*(q) \geq K \frac{q}{\Phi(q)} \frac{\Phi(q) + 1}{q - \psi(-1) - \beta} \geq K \frac{q}{\Phi(q)} \frac{\Phi(q) + 1}{q - \psi(-1)} = K \mathbb{E}(e^{\bar{X}_{e_q}}) > K$$

where $\bar{X}_t = \sup_{s \leq t} X_s$ and e_q is an exponential random variable with parameter q which is independent of X . It is important to note that in the equality we have appealed to the well-known identity for one of the Wiener–Hopf factors of X (cf. Chapter 8 of [9]).

Unbounded variation and smooth fit: In this case it is known that $W^{(q)}(0+) = 0$ and hence in the above analysis one sees that $H_{\log a}(\log a-) = a = H_{\log a}(\log a+)$. In that case, the principle of smooth fit can be implemented and we insist on there being no discontinuity in $H'_{\log a}$ at $\log a$. We have

$$\begin{aligned} H'_{\log a}(x) &= e^x Z_{-1}^{(q-\psi(-1))}(\log a - x) - e^x (q - \psi(-1)) W_{-1}^{(q-\psi(-1))}(\log a - x) \\ &\quad + a \frac{q - \psi(-1)}{\Phi(q) + 1} W^{(q)'}(\log a - x) - W^{(q)'}(\log a - x) a^{-\Phi(q)} \\ &\quad \times \int_{-\infty}^{\log a} (\alpha + \beta e^y) e^{\Phi(q)y} dy \\ &\quad + (\alpha + \beta e^x) W^{(q)}(0+). \end{aligned} \tag{5.13}$$

Recall that $W^{(q)}(0+) = 0$ and that $W^{(q)'}(0+) = 2/b^2$ which should be interpreted as $+\infty$ in the case that the Gaussian coefficient $b^2 = 0$. We find

$$\begin{aligned} H'_{\log a}(\log a-) &= a + \left[a \frac{q - \psi(-1)}{\Phi(q) + 1} - a^{-\Phi(q)} \int_{-\infty}^{\log a} (\alpha + \beta e^y) e^{\Phi(q)y} dy \right] W^{(q)'}(0+). \end{aligned} \tag{5.14}$$

In order to obtain the smooth fit $H'_{\log a}(\log a+) = a$ we must thus have that

$$a \frac{q - \psi(-1)}{\Phi(q) + 1} = a^{-\Phi(q)} \int_{-\infty}^{\log a} (\alpha + \beta e^y) e^{\Phi(q)y} dy,$$

which, after a simple integration on the right-hand side, gives the same expression of a^* as in the bounded variation case. The same bounds on q_0 are thus still applicable in this case too.

In both cases, we obtain our candidate value function

$$\begin{aligned} H_{\log a^*}(x) &= e^x Z_{(-1)}^{(q-\psi(-1))}(\log a^* - x) - \int_x^{\log a^*} (\alpha + \beta e^y) W^{(q)}(y - x) dy \\ &= e^x \left(1 + (q - \psi(-1)) \int_0^{\log a^* - x} e^y W^{(q)}(y) dy \right) \\ &\quad - \int_0^{\log a^* - x} (\alpha + \beta e^{y+x}) W^{(q)}(y) dy. \end{aligned}$$

We now proceed to verify our candidate solution when $q \geq q_0$. That is to say, we shall verify that

$$\tau^* = \tau_{\log a^*}^+, \quad \sigma^* = \tau_{\log K}^+ \quad \text{and} \quad V^*(x) = H_{\log a^*}(x)$$

fulfil the conditions of the Verification Lemma. Note in particular that $\tau^* \leq \sigma^*$.

Submartingale and supermartingale properties: To this end, note that (5.10) together with an application of the Markov property gives us for all $t \geq 0$,

$$\begin{aligned} \Lambda_t &:= \mathbb{E}_x \left[e^{-q\tau_{\log a^*}^+} V^*(X_{\tau_{\log a^*}^+}) + \int_0^{\tau_{\log a^*}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds \middle| \mathcal{F}_t \right] \\ &= e^{-q(t \wedge \tau_{\log a^*}^+)} V^*(X_{t \wedge \tau_{\log a^*}^+}) + \int_0^{t \wedge \tau_{\log a^*}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds. \end{aligned}$$

That is to say, $\Lambda = (\Lambda_t : t \geq 0)$ is a martingale. This confirms the submartingale property (v) in the Verification Lemma.

An easy computation shows that

$$\begin{aligned} V^{*''}(x) &= e^x Z_{-1}^{(q-\psi(-1))} (\log a^* - x) - a^*(q - \psi(-1)) \\ &\quad \times \left[W^{(q)}(\log a^* - x) + W^{(q)' }(\log a^* - x) \right] \end{aligned}$$

and hence V^* belongs to $C^2(-\infty, \log a^*)$. Moreover, the latter conclusion is sufficient to show that $\Gamma V^*(x)$ is continuous on $(-\infty, \log a^*)$ where Γ is the infinitesimal generator of X , and in particular,

$$\begin{aligned} \Gamma V^*(x) &= \mu V^{*'}(x) + \frac{b^2}{2} V^{*''}(x) \\ &\quad + \int_{(0, \infty)} \left(V^*(x + y) - V^*(x) - yV^{*'}(x) \mathbb{1}_{\{y < 1\}} \right) \Pi(dy). \end{aligned}$$

(See for example the argument in Lemma 4.1 of [10]). For any $-n \leq x \leq a < \log a^*$, where $n \in \mathbb{N}$, the aforementioned facts concerning smoothness and continuity allow us to apply Itô's formula to Λ , but stopped at τ_{-n}^- , where $\tau_{-n}^- = \inf\{t > 0 : X_t < -n\}$, and deduce that

$$\Lambda_{t \wedge \tau_{-n}^-} = V^*(x) + \int_0^{t \wedge \tau_{-n}^- \wedge \tau_a^+} e^{-qs} \left[(\Gamma - q)V^*(X_s) + (\alpha + \beta e^{X_s}) \right] ds + m_t, \tag{5.15}$$

where

$$\begin{aligned} m_t &= \frac{b^2}{2} \int_0^{t \wedge \tau_{-n}^- \wedge \tau_a^+} e^{-qs} V^{*''}(X_s) dB_s + \int_0^{t \wedge \tau_{-n}^- \wedge \tau_a^+} e^{-qs} V^{*'}(X_s) dX_s^{(1)} \\ &\quad + \sum_{s \leq t \wedge \tau_{-n}^- \wedge \tau_a^+} e^{-qs} \left[V^*(X_s) - V^*(X_{s-}) - \Delta X_s V^{*'}(X_{s-}) \mathbb{1}_{\{\Delta X_s < 1\}} \right] \\ &\quad - \int_0^{t \wedge \tau_{-n}^- \wedge \tau_a^+} e^{-qs} \int_{(0, \infty)} \left(V^*(X_{s-} + y) - V^*(X_{s-}) \right. \\ &\quad \left. - yV^{*'}(X_{s-}) \mathbb{1}_{\{y < 1\}} \right) \Pi(dy) ds \end{aligned}$$

is a local martingale such that B is the Gaussian component in X and $X^{(1)}$ is the martingale consisting of compensated jumps of size strictly less than unity. In fact, thanks to the boundedness of $V^{*'}$ and ΓV^* on $[-n, a]$, the process $\{m_t : t \geq 0\}$ is a martingale. The latter, together with the fact that Λ is a martingale, implies that the drift term in (5.15) must almost surely be equal to zero. Taking expectations and writing $R^{(q)}(x, dy; a, -n)$ for the q -resolvent measure of the

process X when issued from x and killed on first entry into $(-\infty, -n) \cup (a, \infty)$ we have for all $-n \leq x \leq a < \log a^*$,

$$\int_{[-n, a]} \left[(\Gamma - q)V^*(y) + (\alpha + \beta e^y) \right] R^{(q)}(x, dy; a, -n) = 0.$$

As $R^{(q)}(x, dy; a, -n)$ is absolutely continuous with respect to Lebesgue measure with a strictly positive density in $(-\infty, 0)$ (cf. Chapter 8 of Kyprianou [9]) it follows that

$$(\Gamma - q)V^*(x) + (\alpha + \beta e^x) = 0 \tag{5.16}$$

for Lebesgue almost every $x < \log a^*$. The latter can be upgraded to every $x < \log a^*$ as the left-hand side of (5.16) is continuous. It is also trivial to check that $V^*(x) = e^x$ on $(\log a^*, \infty)$ and hence it follows from $q > \psi(-1) + \beta$ and the definition of a^* that

$$(\Gamma - q)V^*(x) + (\alpha + \beta e^x) = (\psi(-1) - q + \beta)e^x + \alpha \leq (\psi(-1) - q + \beta)a^* + \alpha \leq 0$$

on $(\log a^*, \infty)$.

Next, note that it is straightforward to see that V^* is twice continuously differentiable on $(-\infty, \log a^*) \cup (\log a^*, \infty)$ with the existence of a left and right derivative at $\log a^*$. We may thus apply the Meyer–Itô formula (cf. Theorem 70 of Protter [13]) to the process $V^*(X_{t \wedge \tau_{\log K}^+})$ and then integrate by parts to obtain, in a similar vein to (5.15), that

$$\begin{aligned} & e^{-q(t \wedge \tau_{\log K}^+)} V^*(X_{t \wedge \tau_{\log K}^+}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds \\ &= V^*(x) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} \left[(\Gamma - q)V^*(X_s) + (\alpha + \beta e^{X_s}) \right] ds \\ & \quad + \frac{1}{2} \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} \left(V^{*'}(\log a^*+) - V^{*'}(\log a^*-) \right) d\ell_s + M_t, \end{aligned}$$

where $M := (M_t : t \geq 0)$ is a local martingale and $\ell := (\ell_t : t \geq 0)$ is the semi-martingale local time of X at $\log a^*$. Note that when $b^2 = 0$, the final integral is identically zero owing to the fact that the local time process ℓ is also identically zero and otherwise, when $b^2 > 0$, the final integral is still identically zero thanks to smooth pasting. Note also that although the quantity $(\Gamma - q)V^*(x) + (\alpha + \beta e^x)$ is not defined at $x = \log a^*$, this is not a problem in the context of the above calculus as the Lebesgue measure of the time that the process X spends at $\log a^*$ is zero.

Recalling that $(\Gamma - q)V^*(x) + (\alpha + \beta e^x) \leq 0$ on $(-\infty, \log a^*) \cup (\log a^*, \infty)$, by taking expectations with the help of a suitable localizing sequence of stopping times $\{T_n : n \geq 1\}$ for M , Fatou’s lemma and monotone convergence, we obtain

$$\begin{aligned} & \mathbb{E}_x \left[e^{-q(t \wedge \tau_{\log K}^+)} V^*(X_{t \wedge \tau_{\log K}^+}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ & \leq \lim_{n \uparrow \infty} \mathbb{E}_x \left[e^{-q(t \wedge T_n \wedge \tau_{\log K}^+)} V^*(X_{t \wedge T_n \wedge \tau_{\log K}^+}) + \int_0^{t \wedge T_n \wedge \tau_{\log K}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ & \leq V^*(x) + \lim_{n \uparrow \infty} \mathbb{E}_x \left[\int_0^{t \wedge T_n \wedge \tau_{\log K}^+} e^{-qs} \left[(\Gamma - q)V^*(X_s) + (\alpha + \beta e^{X_s}) \right] ds \right] \\ & \leq V^*(x). \end{aligned}$$

The last inequality above together with the Markov property is sufficient to deduce the supermartingale property (vi) in the Verification Lemma. Note that right continuity follows immediately from the continuity of V^* and the fact that X has càdlàg paths.

Lower and upper bounds: The bounds (i) and (ii) in the Verification Lemma can be deduced directly from the expression for V^* . To this end, write

$$V^*(x) = e^x + \frac{\alpha}{\Phi(q)} g(\log a^* - x),$$

where

$$g(z) = (\Phi(q) + 1) \int_0^z e^{y-z} W^{(q)}(y) dy - \Phi(q) \int_0^z W^{(q)}(y) dy.$$

Note that $g(0) = 0$. Since $V^*(x) = e^x$ for all $x \geq \log a^*$, we have the required lower bound for V^* if we can prove that $g'(z) > 0$ for all $z > 0$. To this end we differentiate and find that

$$\begin{aligned} g'(z) &= e^{-z} \left[e^z W^{(q)}(z) - (\Phi(q) + 1) \int_0^z e^y W^{(q)}(y) dy \right] \\ &= e^{-z} \left[\mathcal{W}^{(p)}(z) - \varphi(p) \int_0^z \mathcal{W}^{(p)}(y) dy \right], \end{aligned}$$

where $p = q - \psi(-1)$, $\mathcal{W}^{(p)}(z) = e^z W^{(q)}(z) = W_{-1}^{q-\psi(-1)}(z)$ and

$$\begin{aligned} \varphi(p) &= \sup\{\theta \geq 0 : \psi_{-1}(\theta) = p\} \\ &= \sup\{\theta \geq 0 : \psi(\theta - 1) - \psi(-1) = q - \psi(-1)\} \\ &= \sup\{\theta \geq 0 : \psi(\theta - 1) = q\} \\ &= \Phi(q) + 1. \end{aligned}$$

Finally, to show that $g'(z) > 0$ we note from (8.20) of Kyprianou [9] that

$$0 < \frac{\varphi(p)}{p} \mathbb{P}^{(-1)}(-\underline{X}_{\mathbf{e}_p} \leq z) = \mathcal{W}^{(p)}(z) - \varphi(p) \int_0^z \mathcal{W}^{(p)}(y) dy,$$

where $\underline{X}_t = \inf_{s \leq t} X_s$.

For the upper bound on V^* it suffices to show in a similar vein to the lower bound that $V^{*'}(x) \geq 0$. Calculations in the spirit of the ones above show that

$$\begin{aligned} V^{*'}(x) &= \frac{e^x}{a^*} \left[a^* - \frac{\alpha}{\Phi(q)} \frac{\Phi(q) + 1}{q - \psi(-1)} \mathbb{P}^{(-1)}(-\underline{X}_{\mathbf{e}_p} \leq x - \log a^*) \right] \\ &\geq \frac{e^x}{a^*} \frac{\alpha}{\Phi(q)} \frac{\Phi(q) + 1}{q - \psi(-1)} [1 - \mathbb{P}^{(-1)}(-\underline{X}_{\mathbf{e}_p} \leq x - \log a^*)] \\ &\geq 0, \end{aligned}$$

where we have made use of (5.12).

Stopped values: Note that since $V^*(x) = e^x$ for $x \geq \log a^*$ both conditions (iii) and (iv) are automatically satisfied.

Having now checked properties (i)–(vi) of the Verification Lemma, and noting that the justification for (3.9) is the same as in the proof of Theorem 1, we may conclude that the proposed triple (τ^*, σ^*, V^*) is a stochastic saddle point. \square

6. Proof of Theorem 3

The proof of Theorem 3 relies on the following optimal stopping problem. Recall that for $q > 0$

$$U_t = e^{-qt}(e^{X_t} \vee K) + \int_0^t e^{-qs}(\alpha + \beta e^{X_s})ds.$$

Lemma 2. Let $\alpha/K < q \leq q_0$. Define the function $w(x) : \mathbb{R} \rightarrow \mathbb{R}^+$ by

$$w(x) := \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[U_{\sigma \wedge \tau_{\log K}^+} \right]. \tag{6.17}$$

Then w has the following properties,

- (i) w is non-decreasing,
- (ii) $w(x) \leq e^x \vee K$ for $x \in \mathbb{R}$,
- (iii) there exists a $c^* \leq \log K$ such that

$$w(x) = \mathbb{E}_x \left[e^{-q\tau_{c^*}^+} (e^{X_{\tau_{c^*}^+}} \vee K) + \int_0^{\tau_{c^*}^+} e^{-qs}(\alpha + \beta e^{X_s})ds \right],$$

- (iv) w is continuous in x and $w(c^*) = K$,
- (v) $w(x) \geq e^x$ for $x \in \mathbb{R}$,
- (vi) $w(X_{\tau_{\log K}^+}) = e^{X_{\tau_{\log K}^+}}$ almost surely on $\{\tau_{\log K}^+ < \infty\}$,
- (vii) $w(X_{\tau_{c^*}^+}) = e^{X_{\tau_{c^*}^+}} \vee K$ almost surely on $\{\tau_{c^*}^+ < \infty\}$,
- (viii) the process

$$\left(e^{-q(t \wedge \tau_{\log K}^+)} w(X_{t \wedge \tau_{\log K}^+}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs}(\alpha + \beta e^{X_s})ds, t \geq 0 \right)$$

is a right continuous submartingale and

- (ix) the process

$$\left(e^{-q(t \wedge \tau_{c^*}^+)} w(X_{t \wedge \tau_{c^*}^+}) + \int_0^{t \wedge \tau_{c^*}^+} e^{-qs}(\alpha + \beta e^{X_s})ds, t \geq 0 \right)$$

is a right continuous supermartingale.

Proof. (i) Denote $X_t^* = X_{t \wedge \tau_{\log K}^+}$ for all $t \geq 0$, and introduce the functional

$$A_t := \int_0^{t \wedge \tau_{\log K}^+} e^{-qs}(\alpha + \beta e^{X_s})ds, \quad \text{for all } t \geq 0.$$

Then the process $Z := (Z_t, t \geq 0)$ given by

$$Z_t := (t, A_t, X_t^*) \quad \text{for all } t \geq 0,$$

is Markovian and starts from $(0, 0, x)$ under the measure \mathbb{P}_x . Thus the optimal stopping problem (6.17) reads as follows

$$w(x) := W(0, 0, x) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[F(\sigma \wedge \tau_{\log K}^+, A_\sigma, X_\sigma^*) \right],$$

where $F(t, l, x) = e^{-qt}(e^x \vee K) + l$. Since $F : \mathbb{R}_+^2 \times \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous and X^* is quasi-left continuous we can deduce that w is upper semicontinuous. Furthermore, we have

$$\begin{aligned} \mathbb{E}_x \left[\sup_{t \geq 0} F(t \wedge \tau_{\log K}^+, A_t, X_t^*) \right] &\leq \mathbb{E}_x \left[\int_0^\infty e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ &\quad + \mathbb{E}_x \left[e^{-q\tau_{\log K}^+} e^{X_{\tau_{\log K}^+}} \right] + K \\ &\leq K + \frac{\alpha}{q} + \frac{\beta}{q - \psi(-1)} + e^x Z_{-1}^{(q-\psi(-1))} (\log K - x) \\ &< \infty, \end{aligned}$$

so we can apply a variant of Theorem 3.3 on p. 127 of Shiryaev [15] (see also Corollary 2.9 on p. 46 of Peskir and Shiryaev [12]) to conclude that

$$\tau_D = \inf\{t \geq 0 : Z_t \in D\},$$

where $D = \{(t, l, x) \in \mathbb{R}_+^2 \times \mathbb{R} : W(t, l, x) = F(t, l, x)\}$, is an optimal stopping time and

$$W(t, l, x) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[F(t + \sigma \wedge \tau_{\log K}^+, l + A_\sigma, X_\sigma^*) \right].$$

Note that for all $(t, l, x) \in \mathbb{R}_+^2 \times \mathbb{R}$, the following identity holds

$$W(t, \ell, x) = e^{-qt} W(0, 0, x) + l,$$

and thus we deduce that $D = \{x \in \mathbb{R} : w(x) = e^x \vee K\}$ and $\tau_D = \tau_{\log K}^+ \wedge \inf\{t \geq 0 : X_t \in D\}$.

In what follows, if ζ is a stopping time for X we shall write $\zeta(x)$ to show the dependence of the stopping time on the value of $X_0 = x$. Similarly, we denote

$$U_t^{(x)} = e^{-qt}(e^{X_t+x} \vee K) + \int_0^t e^{-qs} (\alpha + \beta e^{X_s+x}) ds, \quad t \geq 0.$$

For $y \geq x$, we have that $U_t^{(y)} \geq U_t^{(x)}$ for all $t \geq 0$ and thus, also appealing to the definition of w as an infimum,

$$w(x) - w(y) = \mathbb{E} \left[U_{\tau_D(x)}^{(x)} - U_{\tau_D(y)}^{(y)} \right] \leq \mathbb{E} \left[U_{\tau_D(y)}^{(x)} - U_{\tau_D(y)}^{(y)} \right] \leq 0,$$

which implies that w is non-decreasing.

(ii) This property follows directly from the definition of w as an infimum and taking for instance the stopping time $\sigma = 0$.

(iii) Recall that w is upper semicontinuous. Thus the set

$$C := \{x \in \mathbb{R} : w(x) < e^x \vee K\}$$

is open. From (ii), we deduce that $C = D^c$ and therefore D is a closed set. The fact that w is non-decreasing and that D is a closed set implies that there exists a $c^* \leq \log K$ such that $D = [c^*, \infty)$. In that case $\tau_D = \tau_{c^*}^+$.

(iv) We first note that from the definition of w as an infimum, we have

$$\mathbb{E} \left[U_{\tau_{c^*-x}^+}^{(y)} - U_{\tau_{c^*-y}^+}^{(y)} \right] \geq 0.$$

Now, for $y \geq x$, it holds that $e^x \vee K - e^y \vee K \geq e^x - e^y$ and $\tau_{c^*-x}^+ \geq \tau_{c^*-y}^+$. Therefore

$$\begin{aligned} w(x) - w(y) &= \mathbb{E} \left[U_{\tau_{c^*-x}^+}^{(x)} - U_{\tau_{c^*-x}^+}^{(y)} + U_{\tau_{c^*-x}^+}^{(y)} - U_{\tau_{c^*-y}^+}^{(y)} \right] \\ &\geq (e^x - e^y) \mathbb{E} \left[e^{-q\tau_{c^*-x}^+ + X_{\tau_{c^*-x}^+}} + \beta \int_0^\infty e^{-qs + X_s} ds \right] \\ &\quad + \mathbb{E} \left[U_{\tau_{c^*-x}^+}^{(y)} - U_{\tau_{c^*-y}^+}^{(y)} \right] \\ &\geq \mathfrak{K}(c^*, \beta)(e^x - e^y), \end{aligned}$$

for some constant $\mathfrak{K}(c^*, \beta) > 0$ which depends on c^* and β . Therefore, using part (i), we deduce that w is continuous and moreover that $w(c^*) = K$.

(v) In what follows, for $q > 0$, it is convenient to denote the function w by $w(x, q)$ and $U_t = U_t(q)$ for all $t \geq 0$. Note that for any $t \geq 0$, $U_t(q)$ is non-increasing in q . Hence,

$$w(x, q) \geq w(x, q_0) = \inf_{\sigma \in \mathcal{T}_{0, \infty}} \mathbb{E}_x \left[U_{\sigma \wedge \tau_{\log K}^+}(q_0) \right] \quad \text{for } q < q_0.$$

On the other hand, recall from [Theorem 2](#) that when $q = q_0$, a saddle point for the stochastic game (1.3) is given by $\tau^* = \sigma^* = \tau_{\log K}^+$, and in particular the value function satisfies

$$V(x, q_0) = \mathbb{E}_x \left[U_{\tau_{\log K}^+}(q_0) \right].$$

Therefore, appealing to the definition of V as an infimum and using the lower bound on the solution to [Theorem 2](#), we have

$$w(x, q) \geq w(x, q_0) = M(\tau_{c^*}^+, \tau_{\log K}^+) \geq V(x, q_0) \geq e^x.$$

(vi) and (vii) These are trivial statements.

(viii) and (ix) These are standard results from the theory of optimal stopping. See for example [Theorem 2.2](#) on p. 29 or [Theorem 2.4](#) p. 37 of [Peskir and Shiryaev \[12\]](#). \square

According to the previous lemma and the Verification Lemma, a stochastic saddle point of the Gapeev–Kühn game exists and is given by $\tau^* = \tau_{\log K}^+$ and $\sigma^* = \tau_{c^*}^+$, for a given $c^* \leq \log K$. (Note that the condition (3.9) is dealt with in the same way as before). Therefore the associated value function is given by

$$V(x) = \mathbb{E}_x \left[e^{-q\tau_{c^*}^+} \left(e^{X_{\tau_{c^*}^+}} \vee K \right) + \int_0^{\tau_{c^*}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds \right].$$

The proof of [Theorem 3](#) is thus complete as soon as we can characterize c^* as given in the statement of the theorem.

Suppose that $b^2 > 0$. Our objective is to show that $\tau^* = \sigma^* = \tau_{\log K}^+$ is the stochastic saddle point providing q is smaller than q_0 but not too small (to be made precise below). We again do this with the help of the Verification Lemma.

We show that $c^* = \log K$ if and only if $H'_{\log K}(\log K -) \geq 0$. Note that from (5.13) we find that

$$H'_{\log K}(\log K -) = K + \frac{2\alpha}{\Phi(q)b^2} \left(K e^{-a^*(q)} - 1 \right), \tag{6.18}$$

where we have used the fact that $W^{(q)}(0+) = 0$ and $W^{(q)'}(0+) = 2/b^2$ when $b^2 > 0$ (cf. Chapter 8 of Kyprianou [9]). Taking account of the monotonicity of $H_{\log K}(x, q)$ in q this implies that those $q \in (0, q_0)$ for which

$$K + \frac{2\alpha}{\Phi(q)b^2} \left(Ke^{-a^*(q)} - 1 \right) \geq 0$$

form an interval the left end point of which we shall denote by q_1 . First consider $q > q_1$. It then holds that $H'_{\log K}(\log K-) > 0$ and hence $H_{\log K}(x, q) < H_{\log K}(\log K, q) = K$ for $x \in [\log K - \varepsilon, \log K)$ for some $\varepsilon > 0$. Now any choice of $c^* < \log K$ would imply $H_{\log K}(x, q) < w(x)$ for some $x < \log K$, since $w(x) = K$ for all $x \in [c^*, \log K]$. This leads to an immediate contradiction due to the fact that $\tau_{\log K}^+$ is a feasible strategy for the optimal stopping problem (6.17). We conclude that for $q > q_1$ we have that $c^* = \log K$.

Next, we show that $c^* = \log K$ also in the case when $q = q_1$. For any $q > q_1$ it holds that $H_{\log K}(x, q) \leq K \vee e^x$ for all x and thus we find that $H_{\log K}(x, q_1) \leq K \vee e^x$ due to continuity of $H_{\log K}(x, q)$ in q .

Furthermore, note that

$$\left(e^{-q(t \wedge \tau_{\log K}^+)} w(X_{t \wedge \tau_{\log K}^+}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-qs} (\alpha + \beta e^{X_s}) ds, t \geq 0 \right) \tag{6.19}$$

is a martingale for $q \in (q_1, q_0)$, as it is both a submartingale and a supermartingale due to items (viii) and (ix) of Lemma 2. From monotone convergence it follows that (6.19) is also a martingale when $q = q_1$.

Next, we show that $q_1 > \alpha/K$. It seems unclear how to prove this inequality directly using the definition of q_1 and instead we argue by contradiction, hence suppose that $q_1 \leq \alpha/K$. Due to monotonicity in q in the definition of w and Theorem 1 it would then follow that $K \vee e^x \geq w(x, q_1) \geq V(x, \alpha/K) = K \vee e^x$ for all x . Hence in this case

$$\left(e^{-q_1(t \wedge \tau_{\log K}^+)} (K \vee e^{X_{t \wedge \tau_{\log K}^+}}) + \int_0^{t \wedge \tau_{\log K}^+} e^{-q_1s} (\alpha + \beta e^{X_s}) ds, t \geq 0 \right)$$

is a martingale. Recall from the proof of Theorem 1 however that when $x < \log K$, the process above is strictly increasing. We get a contradiction with the martingale property and thus conclude that $q_1 > \alpha/K$.

From (6.18) it is clear that smooth pasting can only occur when $H'_{\log K}(\log K-) = 0$ or K . This occurs precisely at the end points of the interval $[q_1, q_0]$.

We conclude the proof by noting that when $b^2 = 0$, by considering (5.11) and (5.13) with $a = \log K$ and recalling that $W^{(q)}(0+) > 0$ if X has bounded variation and $W^{(q)'}(0+) = +\infty$ if X has unbounded variation, the strategies $\tau^* = \sigma^* = \inf\{t \geq 0 : X_t > \log K\}$ do not constitute a stochastic saddle point when $q < q_0$ as otherwise the necessary upper bound, $K \vee e^x$ on the value function V will not be respected. \square

7. Proof of Theorem 4

The proof of Theorem 4 again relies on the optimal stopping problem introduced in the previous section. Assume that $\alpha/K < q < q_1$.

Let us first address the issue of continuous and smooth fit. We know from Lemma 2 that the value function V is always continuous and hence in particular there is always continuous fit at the

point c^* . Note that necessarily $c^* < \log K$ as otherwise $c^* = \log K$ and then from the previous theorem, $q = q_1$ which is a contradiction. As we shall see, this will be sufficient to uniquely characterize the value c^* in the case that X has paths of bounded variation. When X has paths of unbounded variation, consistently with prior experience, continuous fit is not enough and the following lemma will be needed instead.

Lemma 3. *When X has paths of unbounded variation it holds that $V'(c^* -) = V'(c^* +) = 0$.*

Proof. Thanks to monotonicity of the value function we know that $V(x) \leq V(c^*)$ for all $x \leq c^*$ and hence

$$\liminf_{x \uparrow c^*} \frac{V(c^*) - V(x)}{c^* - x} \geq 0.$$

The proof is thus complete as soon as we show that

$$\limsup_{x \uparrow c^*} \frac{V(c^*) - V(x)}{c^* - x} \leq 0. \tag{7.20}$$

To this end, let $\epsilon > 0$ and introduce $\tau_{c^*+\epsilon}^+ = \inf\{t > 0 : X_t > c^* + \epsilon\}$, $\tau_{c^*-\epsilon}^- = \inf\{t > 0 : X_t < c^* - \epsilon\}$ and $\tau = \tau_{c^*+\epsilon}^+ \wedge \tau_{c^*-\epsilon}^-$. From parts (iv) and (viii) of Lemma 2, we have

$$\begin{aligned} & \mathbb{E}_{c^*} \left[e^{-q\tau} V(X_\tau) + \int_0^\tau e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ & \geq V(c^*) \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*-\epsilon}^-} \mathbb{1}_{\{\tau_{c^*-\epsilon}^- < \tau_{c^*+\epsilon}^+\}} \right] + K \left(1 - \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*-\epsilon}^-} \mathbb{1}_{\{\tau_{c^*-\epsilon}^- < \tau_{c^*+\epsilon}^+\}} \right] \right). \end{aligned} \tag{7.21}$$

On the other hand, we have with the help of spectral positivity of X , the fact that $V(c^*) = K$ and the upper bound on V that

$$\begin{aligned} & \mathbb{E}_{c^*} \left[e^{-q\tau} V(X_\tau) + \int_0^\tau e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ & \leq V(c^* - \epsilon) \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*-\epsilon}^-} \mathbb{1}_{\{\tau_{c^*-\epsilon}^- < \tau_{c^*+\epsilon}^+\}} \right] + K \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*+\epsilon}^+} \mathbb{1}_{\{\tau_{\log K}^+ \neq \tau_{c^*+\epsilon}^+ < \tau_{c^*-\epsilon}^-\}} \right] \\ & \quad + \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*+\epsilon}^+} e^{X_{\tau_{c^*+\epsilon}^+}} \mathbb{1}_{\{\tau_{\log K}^+ = \tau_{c^*+\epsilon}^+ < \tau_{c^*-\epsilon}^-\}} \right] + \mathbb{E}_{c^*} \left[\int_0^\tau e^{-qs} (\alpha + \beta e^{X_s}) ds \right] \\ & \leq V(c^* - \epsilon) \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*-\epsilon}^-} \mathbb{1}_{\{\tau_{c^*-\epsilon}^- < \tau_{c^*+\epsilon}^+\}} \right] + K \mathbb{E}_{c^*} \left[e^{-q\tau_{c^*+\epsilon}^+} \mathbb{1}_{\{\tau_{\log K}^+ \neq \tau_{c^*+\epsilon}^+ < \tau_{c^*-\epsilon}^-\}} \right] \\ & \quad + K \mathbb{P}_{c^*}^{(-1)} \left(\tau_{\log K}^+ = \tau_{c^*+\epsilon}^+ < \tau_{c^*-\epsilon}^- \right) + \frac{(\alpha + \beta e^{c^*+\epsilon})}{q} \left(1 - \mathbb{E}_{c^*} [e^{-q\tau}] \right). \end{aligned} \tag{7.22}$$

Next, we claim that the last two terms on the right-hand side above are $o(\epsilon)$ as $\epsilon \downarrow 0$. For the first of these two terms, the claim follows by Lemma 10 of Baurdoux and Kyprianou [2]. The second of these two terms is proportional to (cf. Chapter 8 of Kyprianou [9])

$$1 - \mathbb{E}_{c^*} [e^{-q\tau}] = q \frac{W^{(q)}(\epsilon)}{W^{(q)}(2\epsilon)} \int_0^{2\epsilon} W^{(q)}(y) dy - q \int_0^\epsilon W^{(q)}(y) dy$$

which is $o(\epsilon)$ as $\epsilon \downarrow 0$ on account of the fact that $W^{(q)}$ is monotone increasing with $W^{(q)}(0+) = 0$ (the latter is due to the assumption that X has paths of unbounded variation).

Taking this into account and combining the inequalities (7.21) and (7.22) we get

$$\begin{aligned} \frac{V(c^*) - V(c^* - \epsilon)}{\epsilon} &\leq K \frac{\mathbb{E}_{c^*}[e^{-q\tau}] - 1}{\epsilon \mathbb{E}_{c^*}[e^{-q\tau_{c^*-\epsilon}^-} \mathbb{1}_{\{\tau_{c^*-\epsilon}^- < \tau_{c^*+\epsilon}^+\}]}] + \frac{o(\epsilon)}{\epsilon} \\ &= \frac{qK}{\epsilon} \left(\frac{W^{(q)}(2\epsilon)}{W^{(q)}(\epsilon)} \int_0^\epsilon W^{(q)}(y)dy - \int_0^{2\epsilon} W^{(q)}(y)dy \right) + \frac{o(\epsilon)}{\epsilon}. \end{aligned}$$

Lemma 11 in [2] states that $\limsup_{\epsilon \downarrow 0} W^{(q)}(2\epsilon)/W^{(q)}(\epsilon) \leq 2$ and hence the expression in the brackets on the right-hand side above is $o(\epsilon)$. This in turn implies (7.20) and hence the proof is complete. \square

Define, for each $c < \log K$, $G_c(x) := \mathbb{E}_x[U_{\tau_c^+}]$, that is to say

$$G_c(x) = \mathbb{E}_x[e^{-q\tau_c^+} (e^{X_{\tau_c^+}} \vee K)] + \int_0^\infty \mathbb{E}_x[e^{-qs} (\alpha + \beta e^{X_s}) \mathbb{1}_{\{s \leq \tau_c^+\}}] ds, \tag{7.23}$$

and note that for $x \geq c$, we have

$$G_c(x) = e^x \vee K.$$

We may now put the features of continuous and smooth fit to use and characterize the value of c^* . Our immediate aim is to give an explicit form of $G(x)$, for $x < c$, in terms of scale functions and the characteristics of X . We first note that the integral on the right-hand side of (7.23) has been computed before and is equal to

$$\int_{-\infty}^c (\alpha + \beta e^y) \left(e^{-\Phi(q)(c-y)} W^{(q)}(c-x) - W^{(q)}(y-x) \right) dy.$$

The first term on the right-hand side of (7.23) satisfies

$$\mathbb{E}_x[e^{-q\tau_c^+} (e^{X_{\tau_c^+}} \vee K)] = K \mathbb{E}_x[e^{-q\tau_c^+}] + \mathbb{E}_x[e^{-q\tau_c^+} (e^{X_{\tau_c^+}} - K) \mathbb{1}_{\{X_{\tau_c^+} > \log K\}}]. \tag{7.24}$$

Recall that $\widehat{\mathbb{P}}$ denotes the law of $\widehat{X} = -X$. By Theorem 8.1 in [9], we get that the first term on the right-hand side of (7.24) satisfies

$$\mathbb{E}_x[e^{-q\tau_c^+}] = \widehat{\mathbb{E}}_{c-x}[e^{-q\tau_0^-}] = Z^{(q)}(c-x) - \frac{q}{\Phi(q)} W^{(q)}(c-x).$$

Now using the exponential change of measure (2.8) with $\lambda = \Phi(q)$, we write the second term in the right-hand side of (7.24) as follows

$$\begin{aligned} \mathbb{E}_x[e^{-q\tau_c^+} (e^{X_{\tau_c^+}} - K) \mathbb{1}_{\{X_{\tau_c^+} > \log K\}}] &= \mathbb{E}_x^{\Phi(q)} \left[e^{\Phi(q)(X_{\tau_c^+} - x)} (e^{X_{\tau_c^+}} - K) \mathbb{1}_{\{X_{\tau_c^+} > \log K\}} \right] \\ &= \mathbb{E}^{\Phi(q)} \left[e^{\Phi(q)X_{\tau_{c-x}^+}} (e^{x+X_{\tau_{c-x}^+}} - K) \mathbb{1}_{\{X_{\tau_{c-x}^+} + x > \log K\}} \right]. \end{aligned}$$

Let $f(y) = e^{\Phi(q)y} (e^{x+y} - K) \mathbb{1}_{\{y+x > \log K\}}$. From Theorem 4.4 in [9] and since $x < c < \log K$, we deduce

$$\begin{aligned} \mathbb{E}^{\Phi(q)} \left[f(X_{\tau_{c-x}^+}) \right] &= \mathbb{E}^{\Phi(q)} \left[f(X_{\tau_{c-x}^+}) \mathbb{1}_{\{X_{\tau_{c-x}^+} > c-x\}} \right] \\ &= \mathbb{E}^{\Phi(q)} \left[\int_0^\infty dt f(X_t) \mathbb{1}_{\{\bar{X}_{t-} < c-x\}} \mathbb{1}_{\{X_t > c-x\}} \right] \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}_{x-c}^{\Phi(q)} \left[\int_0^\infty dt \int_0^\infty \Pi^{\Phi(q)}(du) f(u + X_{t-} + c - x) \mathbb{1}_{\{\tau_0^+ > t\}} \mathbb{1}_{\{u + X_{t-} > 0\}} \right] \\
 &= \int_0^\infty \Pi^{\Phi(q)}(du) \int_0^\infty dt \mathbb{E}_{x-c}^{\Phi(q)} \left[f(u + X_{t-} + c - x) \mathbb{1}_{\{\tau_0^+ > t\}} \mathbb{1}_{\{u + X_{t-} > 0\}} \right] \\
 &= \int_0^\infty \Pi^{\Phi(q)}(du) \int_{-\infty}^0 R_{\Phi(q)}(x - c, dy; 0) f(u + y + c - x) \mathbb{1}_{\{u+y>0\}}
 \end{aligned}$$

where $R_{\Phi(q)}(z, dy; 0)$ plays the role of $R(z, dy; 0)$ but under the measure $\mathbb{P}_z^{\Phi(q)}$. Therefore, by Corollary 8.8 in [9] we get

$$\begin{aligned}
 \mathbb{E}^{\Phi(q)} \left[f(X_{\tau_{c-x}^+}) \right] &= \int_0^\infty \Pi^{\Phi(q)}(du) \int_{-\infty}^0 dy \left(W_{\Phi(q)}(c - x) - W_{\Phi(q)}(c - x + y) \right) \\
 &\quad \times f(u + y + c - x) \mathbb{1}_{\{u+y>0\}}.
 \end{aligned}$$

Finally, putting the pieces together, using in particular that $\Pi^{\Phi(q)}(dx) = e^{-\Phi(q)x} \Pi(dx)$ and $W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x)$, we obtain the following formula for $G_c(x)$, when $x < c$,

$$\begin{aligned}
 G_c(x) &= K \left(Z^{(q)}(c - x) - \frac{q}{\Phi(q)} W^{(q)}(c - x) \right) \\
 &\quad + \int_{-\infty}^c (\alpha + \beta e^y) \left(e^{-\Phi(q)(c-y)} W^{(q)}(c - x) - W^{(q)}(y - x) \right) dy \\
 &\quad + e^{\Phi(q)(x-c)} \int_0^\infty e^{-\Phi(q)u} \Pi(du) \int_{-\infty}^0 dy \\
 &\quad \times \left(W^{(q)}(c - x) - e^{-\Phi(q)y} W^{(q)}(c - x + y) \right) \\
 &\quad \times e^{\Phi(q)(u+y)} \left(e^{c+u+y} - K \right) \mathbb{1}_{\{u+y+c > \log K\}}.
 \end{aligned}$$

Now that we have an expression for G_c we may find the one which corresponds to the optimal solution by choosing $c = c^*$ so that there is smooth or continuous fit accordingly with the path variation of X .

Bounded variation case: In this case we know that $W^{(q)}(0+) = 1/d > 0$. Hence, checking for a discontinuity at c we find that

$$\begin{aligned}
 G_c(c-) &= K \left(1 - \frac{q}{\Phi(q)} \frac{1}{d} \right) + \frac{e^{-\Phi(q)c}}{d} \left(\frac{\alpha}{\Phi(q)} e^{\Phi(q)c} + \frac{\beta}{\Phi(q) + 1} e^{(\Phi(q)+1)c} \right) \\
 &\quad + \frac{K}{d} \int_0^\infty \Pi(dz + \log K - c) f(z),
 \end{aligned} \tag{7.25}$$

where

$$f(z) = \begin{cases} \frac{1}{\Phi(q) + 1} e^z (1 - e^{-(\Phi(q)+1)z}) - \frac{1}{\Phi(q)} (1 - e^{-\Phi(q)z}) & \text{if } z \geq 0 \\ 0 & \text{if } z < 0. \end{cases} \tag{7.26}$$

It is important to note that

$$f(z) \sim z^2 \quad \text{as } z \rightarrow 0, \quad \text{and} \quad f(z) \sim \frac{1}{\Phi(q) + 1} e^z \quad \text{as } z \rightarrow \infty,$$

thus from the hypothesis (A) and the fact that Π is a Lévy measure, we have

$$\int_0^\infty \Pi(dz) f(z) < \infty.$$

So, we take

$$\begin{aligned} \frac{q}{\Phi(q)} &= \frac{1}{K} \left(\frac{\alpha}{\Phi(q)} + \frac{\beta}{\Phi(q) + 1} e^c \right) - \frac{1}{\Phi(q)} \int_0^\infty \Pi(dz + \log K - c)(1 - e^{-\Phi(q)z}) \\ &\quad + \frac{1}{\Phi(q) + 1} \int_0^\infty \Pi(dz + \log K - c)e^z(1 - e^{-(\Phi(q)+1)z}). \end{aligned} \tag{7.27}$$

In order to show that this expression has a unique solution, it is more convenient to note from (7.25) that

$$\lim_{c \downarrow -\infty} G_c(c-) = K - \frac{Kq}{\Phi(q)d} \left(1 - \frac{\alpha}{Kq} \right) < K$$

on account of the assumption that $q > \alpha/K$. Moreover, as in the case of bounded variation paths,

$$\psi(\theta) = d\theta - \int_0^\infty (1 - e^{-\theta x})\Pi(dx),$$

and $\psi(\Phi(q)) = q$, we may compute from (7.25)

$$\lim_{c \uparrow \log K} G_c(c-) = K + \frac{1}{d} \left(\frac{\alpha}{\Phi(q)} - K \frac{(q - \psi(-1) - \beta)}{(\Phi(q) + 1)} \right) > K,$$

where the strict inequality follows from the fact $q < q_1 = q_0$. Thus, we get the existence of the unique solution if we prove that $G_c(\cdot-)$ is continuous and increasing in $(-\infty, \log K]$. The continuity of $G_c(\cdot-)$ follows from (7.25) and the fact that when the measure Π has an atom at $\log K - c$, the integrand on the right-hand side of (7.25) is equal to 0 at $z = 0$.

Now, note that

$$f'(z) = \frac{1}{\Phi(q) + 1} (e^z - e^{-\Phi(q)z}) > 0 \quad \text{for all } z > 0,$$

which implies that f is positive and increasing. Then from (7.25) it is clear that $G_c(\cdot-)$ is increasing in $(-\infty, \log K]$.

Unbounded variation case: In this case $W^{(q)}(0+) = 0$ and hence in the above analysis one sees that $G_c(c-) = K = G_c(c+)$. In that case, the principle of smooth fit can be implemented and we insist on choosing c such that there is no discontinuity in $G'_c(c-)$.

Recall that $W^{(p)} \in C^1(0, \infty)$ and let $x < c$. Therefore, using a standard argument involving dominated convergence to differentiate through the integral in the last term of $G_c(x)$, we have

$$\begin{aligned} G'_c(x) &= K \left(\frac{q}{\Phi(q)} W^{(q)'}(c-x) - q W^{(q)}(c-x) \right) \\ &\quad + \int_{-\infty}^c (\alpha + \beta e^y) (W^{(q)'}(y-x) - W^{(q)'}(c-x)) \\ &\quad \times e^{-\Phi(q)(c-y)} dy + \Phi(q) e^{\Phi(q)(x-c)} \int_0^\infty e^{-\Phi(q)u} \Pi(du) \int_{-\infty}^0 dy \\ &\quad \times \left(W^{(q)}(c-x) - e^{-\Phi(q)y} W^{(q)}(c-x+y) \right) \end{aligned}$$

$$\begin{aligned} & \times e^{\Phi(q)(u+y)} (e^{c+u+y} - K) \mathbb{1}_{\{u+y+c > \log K\}} \\ & + e^{\Phi(q)(x-c)} \int_0^\infty e^{-\Phi(q)u} \Pi(du) \\ & \times \int_{-\infty}^0 dy (e^{-\Phi(q)y} W^{(q)'}(c-x+y) - W^{(q)'}(c-x)) \\ & \times e^{\Phi(q)(u+y)} (e^{c+u+y} - K) \mathbb{1}_{\{u+y+c > \log K\}}. \end{aligned}$$

Also, recall that $W^{(q)}(0+) = 0$ and that $W^{(q)'}(0+) = 2/b^2$ which should be interpreted as $+\infty$ in the case that the Gaussian coefficient $b^2 = 0$,

$$\begin{aligned} G'_c(c-) &= K \frac{q}{\Phi(q)} W^{(q)'}(0+) - W^{(q)'}(0+) e^{-\Phi(q)c} \int_{-\infty}^c (\alpha + \beta e^y) e^{\Phi(q)y} dy \\ & - W^{(q)'}(0+) \int_0^\infty e^{-\Phi(q)u} \Pi(du) \int_{-\infty}^0 dy e^{\Phi(q)(u+y)} (e^{c+u+y} - K) \mathbb{1}_{\{u+y+c > \log K\}}. \end{aligned}$$

In order to obtain the smooth fit $G'_c(c+) = 0$ we must have:

$$\begin{aligned} \frac{q}{\Phi(q)} &= \frac{e^{-\Phi(q)c}}{K} \int_{-\infty}^c (\alpha + \beta e^y) e^{\Phi(q)y} dy + \frac{1}{K} \int_0^\infty e^{-\Phi(q)u} \Pi(du) \int_{-\infty}^0 dy e^{\Phi(q)(u+y)} \\ & \times (e^{c+u+y} - K) \mathbb{1}_{\{u+y+c > \log K\}}. \end{aligned}$$

After some algebra, we get

$$\begin{aligned} \frac{q}{\Phi(q)} &= \frac{1}{K} \left(\frac{\alpha}{\Phi(q)} + \frac{\beta}{\Phi(q) + 1} e^c \right) - \frac{1}{\Phi(q)} \int_0^\infty \Pi(dz + \log K - c) (1 - e^{-\Phi(q)z}) \\ & + \frac{1}{\Phi(q) + 1} \int_0^\infty \Pi(dz + \log K - c) e^z (1 - e^{-(\Phi(q)+1)z}), \end{aligned} \tag{7.28}$$

which is the same identity as in (7.27). In order to prove the existence of a unique solution of the above identity we will follow similar arguments as those used in the bounded variation case. Let us define

$$\begin{aligned} F(c) &= K \left(1 - \frac{q}{\Phi(q)} \right) + e^{-\Phi(q)c} \left(\frac{\alpha}{\Phi(q)} e^{\Phi(q)c} + \frac{\beta}{\Phi(q) + 1} e^{(\Phi(q)+1)c} \right) \\ & + K \int_0^\infty \Pi(dz + \log K - c) f(z), \end{aligned} \tag{7.29}$$

where f is defined as in (7.26). Note that c is a solution to (7.28) if and only if c solves $F(c) = K$. Similarly to the bounded variation case, we have that

$$\int_0^\infty \Pi(dz) f(z) < \infty.$$

Now, we note from (7.29) that

$$\lim_{c \downarrow -\infty} F(c) = K - \frac{Kq}{\Phi(q)} \left(1 - \frac{\alpha}{Kq} \right) < K$$

on account of the assumption that $q > \alpha/K$. Moreover, recall that

$$\psi(\theta) = a\theta + \frac{b^2}{2}\theta^2 + \int_0^\infty (e^{-\theta x} - 1 + \theta x \mathbb{1}_{\{x < 1\}}) \Pi(dx),$$

and $\psi(\Phi(q)) = q$, then after some straightforward computations we get

$$\int_0^\infty \Pi(dz) f(z) = \frac{\psi(-1) + a - b^2/2}{\Phi(q) + 1} + \frac{q - a\Phi(q) - b^2/2\Phi^2(q)}{\Phi(q)(\Phi(q) + 1)}.$$

Hence from (7.29)

$$\lim_{c \uparrow \log K} F(c) = K + \left(\frac{\alpha}{\Phi(q)} - K \frac{(q - \psi(-1) - \beta)}{(\Phi(q) + 1)} - K \frac{b^2}{2} \right) > K,$$

where the strict inequality follows from the fact $q < q_1$ (recall that $q_1 = q_0$ when $b^2 = 0$). The existence of the unique solution now follows from the continuity and the monotonicity of F which can be proved as in the bounded variation case.

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References

- [1] L. Alili, A.E. Kyprianou, Some remarks on the first passage of Lévy processes, the American put and smooth pasting, *Ann. Appl. Probab.* 15 (2005) 2062–2080.
- [2] E.J. Baurdoux, A.E. Kyprianou, The McKean stochastic game driven by a spectrally negative Lévy process, *Electron. J. Probab.* 13 (2008) 173–197.
- [3] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, 1996.
- [4] K. Bichteler, *Stochastic Integration with Jumps*, Cambridge University Press, Cambridge, 2002.
- [5] T. Chan, A.E. Kyprianou, M. Savov, Smoothness of scale functions for spectrally negative Lévy processes, *Probab. Theory Related Fields* (2010) (in press).
- [6] E.B. Dynkin, A game-theoretic version of an optimal stopping problem, *Dokl. Akad. Nauk SSSR* 185 (1969) 16–19.
- [7] E. Ekström, G. Peskir, Optimal stopping games for Markov processes, *SIAM J. Control Optim.* 47 (2008) 684–702.
- [8] P.V. Gapeev, C. Kühn, Perpetual convertible bonds in jump–diffusion models, *Statist. Decisions* 23 (2005) 15–31.
- [9] A.E. Kyprianou, *Introductory Lectures of Fluctuations of Lévy Processes with Applications*, Springer, Berlin, 2006.
- [10] A.E. Kyprianou, V. Rivero, R. Song, Convexity and smoothness of scale functions and de Finetti’s control problem, *J. Theoret. Probab.* 23 (2010) 547–564.
- [11] G. Peskir, Optimal stopping games and Nash equilibrium, *Theory Probab. Appl.* 53 (2008) 623–638.
- [12] G. Peskir, A. Shiryaev, *Optimal Stopping and Free Boundary Value Problems*, Birkhäuser Verlag, Basel, 2006.
- [13] P.E. Protter, *Stochastic Integration and Differential Equations*, in: *Applications of Mathematics, Stochastic Modelling and Applied Probability*, Springer, 2004.
- [14] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.
- [15] A. Shiryaev, *Optimal Stopping Rules*, Springer-Verlag, Berlin, 2008.