(Very rough)[1] notes on C-M-J processes

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C-M-J processes are short for Crump-Mode-Jagers processes and are also referred to as general, or age-dependent branching processes (particularly by Jagers himself). The main reference for background reading and this is the classic book of Jagers (1975).

1 Informal definition of a C-M-J process

- A C-M-J process is an elaboration on the classical GW process in which birth times and death times are introduced for individuals.
- I will use the usual Ulam-Harris labelling notation for GW trees. Namely the initial ancestor is labelled $\emptyset$. An individual $u = (u_1, ..., u_n)$ is understood to be the u_n-th descendend of .... of u_1-th descendent of $\emptyset$. Such an individual has the property that $|u| = n$ meaning that it belongs to the n-th generation. We say that $u < v$ if individual v is a descendent of u and one may concatenate labels so that if $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_m)$ then $uv = (u_1, ..., u_n, v_1, ..., v_m)$ so that the individual uv may be identified as individual v in the tree rooted at u.
- Definition of a C-M-J process. Each individual u in the GW is labelled with the pair (λ_u, ξ_u) which is an iid copy of the generic pair (λ, ξ) such that
  - λ is a random variable whose distribution is concentrated on $(0, \infty)$,
  - ξ is a point process on $(0, \infty)$ (specifically $\xi(\{0\}) = 0$) such that $\xi(0, \infty)$ is equal in distribution card$\{u : |u| = 1\}$such that $\xi_u(0, \infty) =$ card$\{uv : |v| = 1\}$. Further we shall assume that $\mathbb{E}\xi(0, \infty) < \infty$ which implies that $\inf\{t > 0 : \xi(t, \infty) = 0\} < \infty$ almost surely. Henceforth we shall denote by μ the intenstity of ξ so that $\mathbb{E}(\xi(A)) =: \mu(A)$ for Borel A.
- If the points of ξ are $0 \leq \sigma_1 \leq ... \leq \sigma_{\xi(0,\infty)} < \infty$ then we may think of them as the birth times of the offspring of a typical individual relative to her own birth time. (Note that multiple births at the same time are allowed). With a slight abuse of notation, we shall henceforth refer to the birth time of individual u as σ_u. Using obvious (and well used) notation we can say for example that $\sigma_{uv} = \sigma_u + \sigma_v^u$ where σ_v^u is the birth time of uv relative to the tree rooted at u.

[1]But less rough than the first version.[2]

[2]And less rough than the second version

Figure 1: Part of a sample path of a C-M-J process.

2 Counting with characteristics

What is a natural way to monitor the evolution of the C-M-J process. There are a number of obvious candidates. One may simply look at the numbers in the n generation (ie the embedded GW process), but then this would make the introduction of birth and death times redundant. One may look at the population alive at time t, which can be written as

$$z_t = \sum_{\sigma_u \le t} \mathbf{1}_{\{\sigma_u \le t < \sigma_u + \lambda_u\}}.$$

Another possibility is the population alive at time t but not older than $a > 0$. This is written

$$z_t^a = \sum_{\sigma_u \le t} \mathbf{1}_{\{\sigma_u \le t < (\sigma_u + \lambda_u) \wedge (\sigma_u + a)\}}.$$

Jagers' work on C-M-J processes advocated a general point of view, namely 'counting with characteristics' as we shall now explain. In addition to the marks (λ_u, ξ_u) to each u in the Galton-Watson tree, Jagers proposed a third mark $\phi_u : \mathbb{R} \to [0, \infty)$ such that the triple $(\lambda_u, \xi_u, \phi_u)$ are again iid across individuals u. Moreover, $\phi_u(t) = 0$ for $t \le 0$ and (typically) one may think of ϕ_u is a measurable function of (λ_u, ξ_u). We shall always denote by ϕ the typical representative of the ϕ_u-s. In that case we may talk about the ϕ-counted process or the process counted with characteristic ϕ as the process

$$z_t^\phi := \sum_u \phi_u(t - \sigma_u).$$

Note that it would do no harm to sum instead over $\{u : \sigma_u \le t\}$ on account of the fact that $\phi(t) = 0$ for all $t \le 0$. Here are some examples of characterisitcs.

1. $\phi(t) = \mathbf{1}_{\{0\le t<\lambda\}}$ in which case $z_t^\phi = z_t$, the total population alive at time t.
2. $\phi(t) = \mathbf{1}_{\{0\le t<\lambda\wedge a\}}$ in which case $z_t^\phi = z_t^a$, the total population alive at time t which is no older than a.
3. $\phi(t) = \mathbf{1}_{\{t\ge 0\}}$ in which case z_t^ϕ = total progeny up to time t.
4. $\phi(t) = (t \wedge \lambda)\mathbf{1}_{\{t\ge 0\}}$ in which case $z_t^\phi = \int_0^t z_s ds$.
5. $\phi(t) = \xi(t,\infty)\mathbf{1}_{\{t\ge 0\}}$ in which case

   $$z_t^\phi = \sum_{\sigma_u\le t} \xi_u(t-\sigma_u,\infty) = \text{card}\{\text{those born after } t \text{ whose parents were born before } t\}.$$

   We call the the process of the collected individuals in the curly brackets above the *coming generation* and henceforth we shall denote it by $\mathcal{C}(t)$.
6. A more elaborate version of the last example is to take $\phi(t) = \xi(t, t + a)\mathbf{1}_{\{t\ge 0\}}$ in which cae one can check that z_t^ϕ counts those in the coming generation who are born before $t + a$. We denote this set by $\mathcal{C}(t, a)$.
7. A yet more elaborate version of the last example is to choose a constant $\alpha > 0$ and define

   $$\phi(t) = \mathbf{1}_{\{t\ge 0\}} e^{\alpha t} \int_{(t,t+a)} e^{-\alpha u}\xi(du) = \mathbf{1}_{\{t\ge 0\}} e^{\alpha t} \sum_{i=1}^{\xi(0,\infty)} e^{-\alpha\sigma_i}\mathbf{1}_{\{\sigma_i\in(t,t+a)\}}.$$

   In this case we have

   $$\begin{aligned} z_t^\phi &= \sum_{\sigma_u\le t} e^{\alpha(t-\sigma_u)} \sum_{ui:|i|=1} e^{-\alpha\sigma_i^u}\mathbf{1}_{\{\sigma_i^u\in(t-\sigma_u,t-\sigma_u+a)\}} \\ &= e^{\alpha t} \sum_{u\in\mathcal{C}(t,a)} e^{-\alpha\sigma_u}. \end{aligned}$$

3 Malthusian growth

In this section we shall address the point alluded to earlier that all processes counted within an appropriate class of characteristics grow at the same rate. We need an assumption first which remains active throughout the remainder of this text.

Assumption 1 There exists an $\alpha > 0$ such that

$$\mathbb{E}\int_{(0,\infty)} e^{-\alpha u}\xi(du) = \int_{(0,\infty)} e^{-\alpha u}\mu(du) = 1.$$

Note that this assumption implies supercriticality (ie $\mathbb{E}(\xi(0,\infty)) > 1$). The parameter α is called the Malthusian parameter (as it will characterise the rate of growth) and note also for future reference that

$$\mu_\alpha(du) := e^{-\alpha u}\mu(du)$$

is a probability measure concentrated on $(0,\infty)$.

In order to reach some classical results on Malthusian growth of C-M-J processes counted with characteristics, we shall temporarily recall some classical renewal theory. (Note this is also covered in Jagers' book but another good reference is the book of Feller (1972)).

3.1 Renewal equations

Suppose that F is a probability measure concentrated on $(0,\infty)$ and $f:[0,\infty)\to[0,\infty)$ is a given measurable function. We are interested in positive solutions to the *renewal equation.*

$$x(t) = f(t) + \int_{(0,t)} x(t-u)F(du) = f(t) + x * F(t),\ t \geq 0$$

For convenience we shall assume that F does not have a lattice support. Classical theory tells us how to solve the renewal equation uniquely in terms of the so-called renewal measure

$$U(dy) = \sum_{n=0}^{\infty} F^{*n}(dy)$$

for $y \geq 0$, where $F^{*n}(dy)$ is the n-fold convolution[2] of F and we undertand in particular $F^{*0}(dy) = \delta_0(dy)$.

Theorem 2 *If f is uniformly bounded then the renewal equation has a unique soluion in the class of positive functions which are bounded on bounded intervals and the latter solution is given by*

$$x(t) = f * U(t) = \int_{[0,t)} f(t-y)U(dy).$$

We do not prove this result here, but naively, the following string of manipulations make it easy enough to see why $f * U$ is a solution to the renewal equation;

$$\begin{aligned} f * U(t) &= f * \sum_{n=0}^{\infty} F^{*n}(t) \\ &= f * \delta_0(t) + f * \sum_{n=1}^{\infty} F^{*n}(t) \\ &= f(t) + (f * U) * F(t) \end{aligned}$$

[2]For example $F^{*n}(0,x) = \int_{(0,x)} F(x-y)F^{*(n-1)}(dy)$.

The classical Renewal theorem tells us for bounded intervals A

$$\lim_{x\uparrow\infty} U(A+x) = \frac{|A|}{\int_{(0,\infty)} uF(du)}$$

where $|A|$ is the Lebesgue measure of A and we interpret ther right hand side to be zero if F has infinite mean. A variant of the following theorem will be important in understanding Malthusian growth of C-M-J processes. It is sometimes called the Key Renewal Theorem although Feller advocates that it should be called the Alternative Renewal Theorem. It says that since the solution to the renewal equation, $x(t)$, is a convolution of f and U and since the latter behaves, up to a constant, asymptotically like Lebesgue measure, then the asymptotic behaviour of $x(t)$ should behave like an integral of f with respect to Lebesgue measure normalised by the mean of F.

Theorem 3 *Suppose that* $\int_{(0,\infty)} uF(du) < \infty$ *and*

$$\sum_{k=0}^{\infty} \sup_{0\leq t<1} |f(k+t)| < \infty \tag{1}$$

then

$$\lim_{t\uparrow\infty} x(t) = \frac{\int_0^\infty f(u)du}{\int_{(0,\infty)} uF(du)}$$

Note that this theorem is stated in a slighly different way to its usual formulation. The condition (1) implies that f is *directly Riemann integrable* and the latter condition is what is usually stated in place of (1). The reason for the above formulation is that it allows one to state more convenient conditions in later results.

3.2 Malthusian growth

Returning to the C-M-J process, we can now make the connection between renewal theory and growth rates of the process counted with appropriate characteristics.

First note that for a given characteristic ϕ,

$$e^{-\alpha t} z_t^{\phi} = e^{-\alpha t}\phi_{\emptyset}(t) + \sum_{|u|=1} \mathbf{1}_{\{\sigma_u \leq t\}} e^{-\alpha(t-\sigma_u)} z^{\phi}_{t-\sigma_u}(u) e^{-\alpha\sigma_u}$$

where $z^{\phi}_{\cdot}(u)$ is the copy of $z^{\phi}_{\cdot}$ relative to the tree rooted at u. Defining $f(t) = e^{-\alpha t}\mathbb{E}\phi(t)$ and $x^{\phi}(t) = e^{-\alpha t}\mathbb{E}(z^{\phi}_t)$ and assuming that the latter two expectations are finite for all $t \geq 0$, we have

$$\begin{aligned}
x^\phi(t) &= f(t) + \mathbb{E}\sum_{|u|=1} e^{-\alpha\sigma_u}\mathbf{1}_{\{\sigma_u \le t\}}\mathbb{E}\left(z^\phi_{t-\sigma_u}(u)e^{-\alpha(t-\sigma_u)}|\xi_\emptyset\right) \\
&= f(t) + \mathbb{E}\sum_{|u|=1} e^{-\alpha\sigma_u}\mathbf{1}_{\{\sigma_u \le t\}}x^\phi(t-\sigma_u) \\
&= f(t) + \mathbb{E}\int_{(0,t)} e^{-\alpha u}x^\phi(t-u)\xi(du) \\
&= f(t) + \int_{(0,t)} e^{-\alpha u}x^\phi(t-u)\mu(du) \\
&= f(t) + \int_{(0,t)} x^\phi(t-u)\mu_\alpha(du).
\end{aligned}$$

With some additional computations (omitted), it is now clear that one may apply Theorem 3 to obtain the following intuitively appealing result.

Theorem 4 *Suppose that $\mathbb{E}\phi(t)$ is Lebesgue almost everywhere continuous,*

$$\int_{(0,\infty)} u\mu_\alpha(du) < \infty \tag{2}$$

and

$$\sum_{k=0}^{\infty} \sup_{k\le t<k+1} e^{-\alpha t}\mathbb{E}\phi(t) < \infty, \tag{3}$$

then

$$\lim_{t\uparrow\infty} x^\phi(t) = \lim_{t\uparrow\infty}\mathbb{E}[e^{-\alpha t}z^\phi_t] = x^\phi_\infty := \frac{\int_0^\infty e^{-\alpha u}\mathbb{E}\phi(u)du}{\int_{(0,\infty)} u\mu_\alpha(du)}.$$

4 A martingale

There is a special case of the last renewal analysis for which things are much nicer. Suppose that we take

$$\phi(t) = \mathbf{1}_{\{t\ge 0\}}e^{\alpha t}\int_{(t,\infty)} e^{-\alpha u}\xi(du)$$

such that (2) is satisfied in which case the first condition of the last theorem and (3) are also satisfied. (Note that ultimately these are conditions on the underlying point process ξ). Then the renewal equation for $x^\phi(t)$ becomes

$$x^\phi(t) = \mu_\alpha(t,\infty) + \int_{(0,t)} x^\phi(t-u)\mu_\alpha(du)$$

in which case it is clear that the unique solution (bounded on bounded intervals) is given by $x^\phi(t) = 1$ for all $t \geq 0$. In particular this means that for all $t \geq 0$ we have

$$\mathbb{E}\sum_{u\in\mathcal{C}(t)} e^{-\alpha\sigma_u} = 1.$$

This leads us to the following result.

Lemma 5 *Suppose that ϕ is given as above such that (2) holds. Then*

$$Y_t := e^{-\alpha t} z_t^\phi = \sum_{u \in \mathcal{C}(t)} e^{-\alpha \sigma_u}$$

is a martingale.

Proof. This is more of a sketch proof. Define

$$\mathcal{F}_{\mathcal{C}(t)} = \sigma(\xi_u, \lambda_u : u < v \text{ for some } v \in \mathcal{C}(s) \text{ with } s \leq t).$$

This is the filtration with respect to which we have our martingale. Next note that we have the non-intersecting, exhaustive partition of $\mathcal{C}(t+s)$

$$\mathcal{C}(t+s) = \{\mathcal{C}(t+s) \cap \mathcal{C}(t)\} \cup \{\mathcal{C}(t+s) \backslash C(t)\}.$$

This allows us to write (using obvious notation)

$$\begin{aligned}
\mathbb{E}\left[\sum_{u \in \mathcal{C}(t+s)} e^{-\alpha \sigma_u} | \mathcal{F}_{\mathcal{C}(t)}\right] &= \sum_{u \in \mathcal{C}(t+s) \cap \mathcal{C}(t)} e^{-\alpha \sigma_u} \\
&\quad + \mathbb{E}\left[\sum_{u \in \mathcal{C}(t)} \sum_{v \in \mathcal{C}^u(t+s-\sigma_u)} e^{-\alpha \sigma_{uv}} | \mathcal{F}_{\mathcal{C}(t)}\right] \\
&= \sum_{u \in \mathcal{C}(t+s) \cap \mathcal{C}(t)} e^{-\alpha \sigma_u} \\
&\quad + \sum_{u \in \mathcal{C}(t)} e^{-\alpha \sigma_u} \mathbb{E}\left[\sum_{v \in \mathcal{C}^u(t+s-\sigma_u)} e^{-\alpha \sigma_v^u} | \mathcal{F}_{\mathcal{C}(t)}\right] \\
&= \sum_{u \in \mathcal{C}(t+s) \cap \mathcal{C}(t)} e^{-\alpha \sigma_u} + \sum_{u \in \mathcal{C}(t)} e^{-\alpha \sigma_u} x^\phi(t+s-\sigma_u) \\
&= \sum_{u \in \mathcal{C}(t)} e^{-\alpha \sigma_u}
\end{aligned}$$

thus completing the (sketch) proof. ■

Since Y_t is a positive martingale, it converges almost surely. In fact, subject to very familiar conditions, namely that $\mathbb{E}(X \log X) < \infty$ where $X = \int_{(0,\infty)} e^{-\alpha u} \xi(du)$, it is known that this martinagle converges in $L^1(\mathbb{P})$. Naturally a spine proof is easily put together to demonstrate this fact, see Olofsson (1998)[3] for example.

However, the convergence of $e^{-\alpha t} z_t^\phi$ for the particular choice of ϕ discussed here, and the convergence of $\mathbb{E}(e^{-\alpha t} z_t^\phi)$ for a general class of ϕ in Theorem 4,

[3]A student of P. Jagers.

begs the question as to whether it is possible that, at least for a similar class of characteristics, one may deduce that $e^{-\alpha t} z_t^{\phi}$ converges almost surely in general. Said, another way, under what conditions is it possible to deduce a strong law of large numbers in the sense

$$\lim_{t\uparrow\infty} \frac{z_t^{\phi_1}}{z_t^{\phi_2}} = c$$

almost surely where ϕ_1 and ϕ_2 are two different characteristics and $c = c(\phi_1, \phi_2)$ is a constant? An answer to this question is provided by the classical and magnificent result established by Nerman (1981)[4]. (Note that there is no need for an $X \log X$ condition!)

Theorem 6 *Suppose that ϕ_1 and ϕ_2 are characteristics which are cadlag in their argument. Suppose further that there exists a $\beta < \alpha$ such that*

$$\mathbb{E} \int_{(0,\infty)} e^{-\beta u} \xi(du) < \infty \text{ and } \mathbb{E} \sup_{t \geq 0} e^{-\beta t} \phi_i(t) < \infty \text{ for } i = 1, 2.$$

Then almost surely on the survival set of the embedded GW process,

$$\lim_{t\uparrow\infty} \frac{z_t^{\phi_1}}{z_t^{\phi_2}} = \frac{\int_0^\infty e^{-\alpha t} \mathbb{E}\phi_1(t)dt}{\int_0^\infty e^{-\alpha t} \mathbb{E}\phi_2(t)dt}.$$

In particular, note that under mild assumptions this implies that for a large class of characteristics ϕ,

$$\lim_{t\uparrow\infty} e^{-\alpha t} z_t^{\phi} = x_\infty^{\phi} Y_\infty,$$

almost surely where Y_∞ is the martingale limit of Y_t.

References

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[4] A student of P. Jagers.