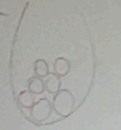


Polya urns (and smoothing equations)



$0 \rightarrow 0000$
 $0 \rightarrow 0000$
 $R = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

$U(n) = \begin{pmatrix} U_1(n) \\ U_2(n) \end{pmatrix} \leftarrow n \text{ total balls}$

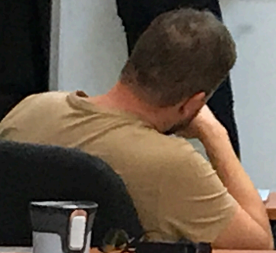
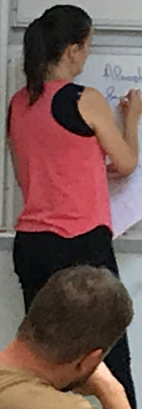
two pieces
 $U(0)$ initial composition
 R transition matrix

$U(n) \stackrel{?}{=} \text{what } n \text{ is large?}$

(the) The original Polya urn (Polya's Urn) (B&S)
 like 3 colors & take $U(0) = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$

$R = \text{diag} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_k \end{pmatrix}$

Almost surely converge



Polige ones (and smoking equations)

Q: Data



$$\begin{aligned} 0 &\rightarrow 0000 \\ 0 &\rightarrow 0000 \\ R &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

$$U(n) = \begin{pmatrix} U(n) \\ U(n) \end{pmatrix} \leftarrow \text{a Markov process}$$

two years

$U(0)$ initial condition
 R transition matrix

$$U(n) \stackrel{?}{=} \text{what } n \text{ is large?}$$

Q: The original Polige one (Polige's Eigenvalue 13.6's)
Take 4 states & take $U(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$R = \text{SIB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If Almost surely converge
& in $U(p, \infty)$

$$\frac{U(n)}{n} \rightarrow V \sim \text{Dirichlet} \left(\frac{1}{2}, \dots, \frac{1}{2} \right)$$

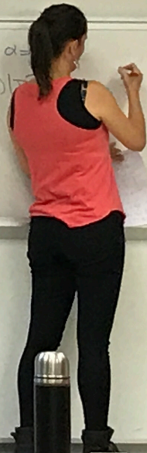
The Dirichlet int. is $\text{let } \Sigma = \{(x_1, \dots, x_n) \in [0, 1]^n \mid \sum x_i = 1\}$
The density of V in Σ is

$$\frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \prod_{i=1}^n x_i^{\alpha_i - 1} \mathbb{1}_{\Sigma}(x_1, \dots, x_n)$$

Ex: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^d$ if Dirichlet $(\alpha_1, \dots, \alpha_n)$

$$\mathbb{E}[V] = \mathbb{E}[(V_1, \dots, V_n)] = \left(\frac{\alpha_1}{\alpha_1 + \dots + \alpha_n}, \dots, \frac{\alpha_n}{\alpha_1 + \dots + \alpha_n} \right) \text{ where } \alpha = \sum \alpha_i$$

Prop. Let $\alpha =$
 $\mathbb{E}[U(n)] =$



$\frac{U(n)}{n} \rightarrow V \sim \text{Dirichlet}(\frac{\alpha_1}{\alpha_1 + \dots + \alpha_k})$
 The Dirichlet dist. for $\Sigma = \{(x_1, \dots, x_k) \in [0,1]^k \mid \sum x_i = 1\}$
 Re. conv. of $V = \Sigma$

$$\frac{P(x \rightarrow y)}{P(x) - P(y)} = \prod_{i=1}^k x_i^{\alpha_i - 1} \frac{\alpha \Sigma(x_1, \dots, x_k)}{\uparrow \text{Laplace } \Sigma}$$

Exercise: $\forall p = (p_1, \dots, p_k) \in \mathcal{N}^k$ if Dirichlet $(\alpha_1, \dots, \alpha_k)$

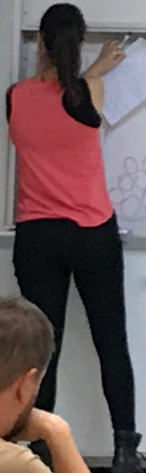
$$E[X^p] = E[X^p - \alpha_1^p] = \frac{P(p)}{(p_1 + p_1)!} \frac{P(\alpha_1 + p_1)}{P(\alpha_1)}$$
 where $\alpha = \sum_{i=1}^k \alpha_i$

Prod. Let $\alpha = \sum_{i=1}^k \alpha_i$

$$E\left[\prod_{i=1}^k X_i^{\alpha_i}\right] = U(n) + \sum_{i=1}^k \frac{U(n)}{\alpha_i} \frac{\alpha_i}{\alpha}$$

$$= U(n) + \frac{\alpha}{\alpha^2} \sum_{i=1}^k \alpha_i U(n)$$

$$= \frac{\alpha U(n) + \alpha U(n)}{\alpha^2}$$



(and modeling equations)
 $0 \rightarrow 0000$
 $0 \rightarrow 0000$
 $R = \begin{pmatrix} 3 & 1 \\ & 2 & 2 \end{pmatrix}$
 $(R = \begin{pmatrix} 3 \\ 2 \end{pmatrix})$

$U(n) = \begin{pmatrix} U(n) \\ U(n) \end{pmatrix} \leftarrow \text{a Markov process}$
 two piece
 $U(n)$ initial condition
 R replacement matrix

$U(n) \approx$ when $n \rightarrow \infty$?
 The original Polya urn (Polya's Urn) with replacement matrix R 's.
 Rate of colors & take $U(n) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$
 $R = \text{IID} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$

Almost surely when $n \rightarrow \infty$
 $\mathbb{E} = \mathbb{E}^*$ (p.s.)
 $\frac{U_n}{n} \rightarrow V \sim \text{Dirichlet}(\frac{\alpha_1}{n}, \dots, \frac{\alpha_k}{n})$

The Dirichlet dist? Let $\Sigma = \{(x_1, \dots, x_k) \in [0,1]^k \mid \sum x_i = 1\}$
 the supp of $V = \Sigma$

$$\frac{f(x_1, \dots, x_k)}{f(x_1, \dots, x_k)} \prod_{i=1}^k x_i^{\alpha_i - 1} \cdot d\Sigma(x_1, \dots, x_k)$$

↑
 (keep on Σ)

Exercise: $\forall p = (p_1, \dots, p_k) \in \mathbb{N}^k$ if Dirichlet dist $(\alpha_1, \dots, \alpha_k)$

$$\mathbb{E}[x_i^p] = \mathbb{E}[x_i^{p_i} \prod_{j \neq i} x_j^{p_j}] = \frac{f(x_1, \dots, x_k)}{f(0, \dots, 0)} \prod_{j \neq i} \frac{f(x_j, \dots, x_k)}{f(x_j, \dots, x_k)}$$

where $\alpha_i = \sum_{j=1}^k \alpha_j$
 $(1 - \sum_{j \neq i} \alpha_j)$

Proof: Let $\alpha = \sum_{i=1}^k \alpha_i$

$$\mathbb{E}[U(n) | \mathcal{F}_n] = U(n) + \sum_{i=1}^k \frac{U(n)}{n + \alpha_i} \mathbb{E}^* \frac{1}{\alpha_i}$$

$$\frac{(n + \sum \alpha_i) U(n)}{\alpha_i n} = U(n) + \sum_{i=1}^k \frac{U(n)}{n + \alpha_i} \frac{\alpha_i}{\alpha_i}$$

$$= \frac{\alpha_i n + \alpha_i U(n)}{\alpha_i n}$$

$\frac{U(n)}{\alpha_i n} \xrightarrow{\text{a.s.}}$ is a multiple thing also in $[0,1]^k$
 $\xrightarrow{\text{a.s.}}$ V + more

25) When R is invertible

Ex: R is inv if $\forall i, j \exists a$ st $R_{ij}^2 > 0$
 $\text{det} R = \det R \neq 0$ $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
 $a \neq 0$ $a \neq 0$ $bc \neq 0$ $a \neq 0$ $a \neq 0$ $a \neq 0$
 Assume that the inv "element" is $a \neq 0$ $a \neq 0$ $a \neq 0$ $a \neq 0$ $a \neq 0$ $a \neq 0$

$\Rightarrow S$ is the signed square of R
 $\Rightarrow a$ is the word on $CA = ad - S$

eigenvalues $\frac{\alpha_1 + \alpha_2}{2} \pm \frac{\alpha_1 - \alpha_2}{2}$

det inv $U(x,y) = \frac{xy}{2} \quad U(x,y) = \frac{bx-cy}{2}$

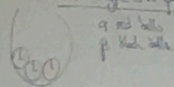
$\frac{U(x,y)}{n} \rightarrow \frac{1}{2}$ "SUS"

R 2) R is not a det
 3) \dagger does not depend on $U(0)$

Proof: embedding in \mathbb{C}^n true

② Embedding in \mathbb{C} line

① Multiple type Grobman



a real particle is equipped with an $Sp(1)$ clock
 a clock a clock says, the ball goes to the clock, writes into
 → (a) real balls if it was red
 and blue balls
 → a red ball & (b) blue balls if it was black

$U(1)$ is the circle of the unit circle

Exerc. if τ is the line of \mathbb{R}^2 with τ
 then $(U(1))_{\tau,0} \cong (U(1))_{\tau,1}$

② Properties of τ

Def. $N(\tau) = \#$ balls in \mathbb{R}^2 on τ of track.

Lemma $e^i N(t) \xrightarrow{a} \mathbb{R}^2$

Def. (a) $(e^i N(t))_{t=0}$ is a multiple (a) $\rightarrow \mathbb{R}$

Exerc. any being parallel \exists a mapping φ

$P(N(t) \geq x) = P(\tau \leq t)$

① When R is invertible

Def R is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\det R > 0$

② R is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\det R < 0$

Assume that the unit ball is $a+b=c+d=5$

$\Rightarrow S$ is the diagonal eigenvalue of R
 Def. n is the second one. $c \leq a-d-S$

eigenvalues

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

dual basis

$u(x,y) = \frac{ax+by}{c}$ $v(x,y) = \frac{bx-cy}{c}$

Th (Hodge & Kohn) A is other $n \rightarrow 0$

$\frac{U(n)}{n} \rightarrow \mathbb{R}^n$

Rt

- ① \mathbb{R}^2 has a det
- ② \neq det not depend on $U(\cdot)$

Def. embedding in \mathbb{C}^n line

② Embedding in \mathbb{C} line

① Multi-type G.W. process

of red balls
p blue balls



each particle is equipped with an Exp(1) clock
+ when a clock rings, the ball goes to the clock, splits into
→ (a) red balls if it was red and blue balls if it was blue
→ c red balls & (d) blue balls if it was blue

$U^{\sigma}(t)$ is the colors of the urn at time t

Exerc. if τ_n is the time of the n-th split, then $(U^{\sigma}(t))_{t \geq 0} \stackrel{d}{=} (U^{\sigma}(\tau_n))_{n \geq 0}$

② Properties of τ_n

def $N(t) = \#$ balls in the urn at time t

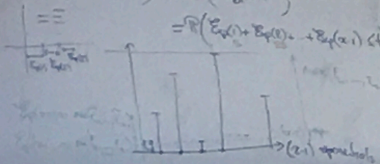
Lemma $e^{\alpha N(t)} \xrightarrow{as} \text{Gamma}(\frac{c+d}{\alpha}, \frac{c+d}{\alpha})$

def: (a) $(e^{\alpha N(t)})_{t \geq 0}$ is a martingale (exer.)
 $\xrightarrow{as} \mathbb{Z}$

Exerc. any bounded process X is a martingale w.r.t. \mathbb{F}

$P(N(t) \geq x) = P(\tau_x \leq t)$

$= P(\text{Exp}(1) + \text{Exp}(1) + \dots + \text{Exp}(1) \leq t)$



$P(\max(\tau_1, \dots, \tau_n) \leq t)$

$= P(\text{Exp}(1) \leq t)^n$

If $N(0) = 2 \rightsquigarrow \tau \sim \text{Exp}(1/2)$

If $N(0) = \alpha\beta \rightsquigarrow \tau \sim \text{Gamma}(\alpha, \beta)$

generators

$\begin{matrix} \sigma_1, \sigma_2 \\ \downarrow \\ \sigma \\ \downarrow \\ 1 \end{matrix}$

dual basis

$U(\sigma) = \frac{\sigma_1}{\sigma} \quad U(\sigma_2) = \frac{\sigma_2}{\sigma}$

Th. (Allouche & Kreweras) \exists a choice $n \rightarrow \infty$

$\frac{U(n)}{1} \rightarrow \sigma_1$

Rt ① E_c but not det

② it does not depend on $U(0)$

Prop. embedding in \mathbb{C}^n line

