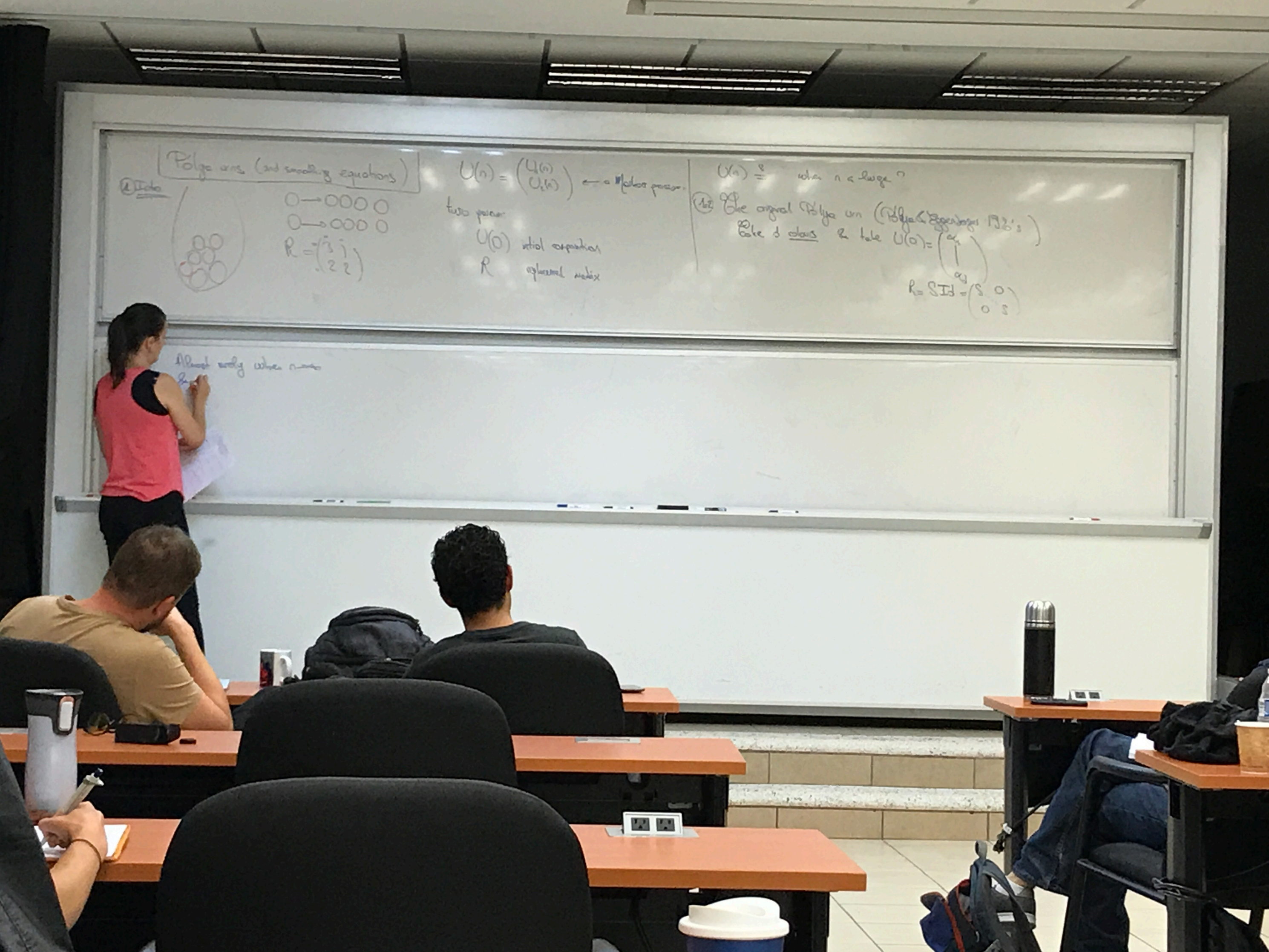
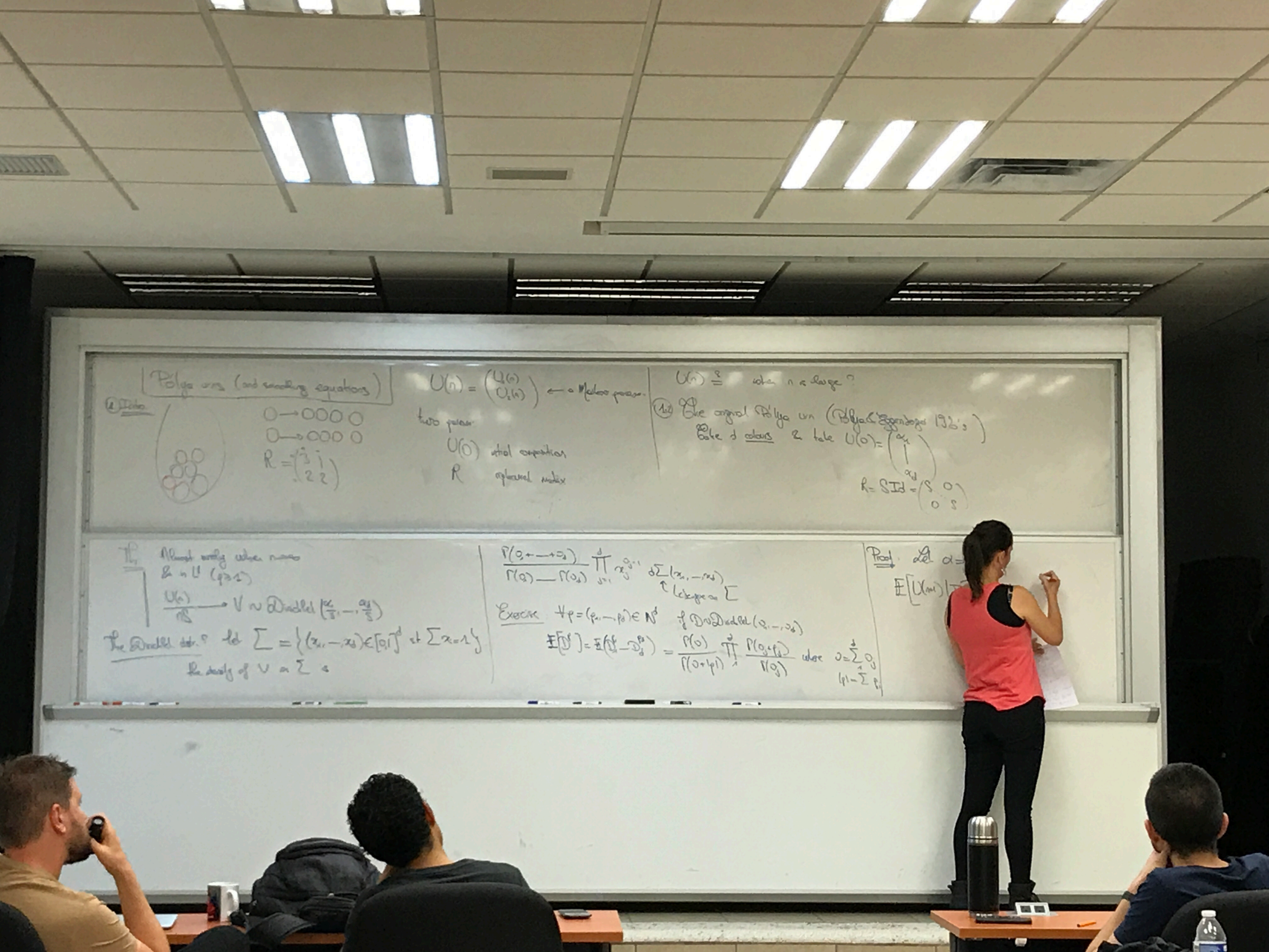
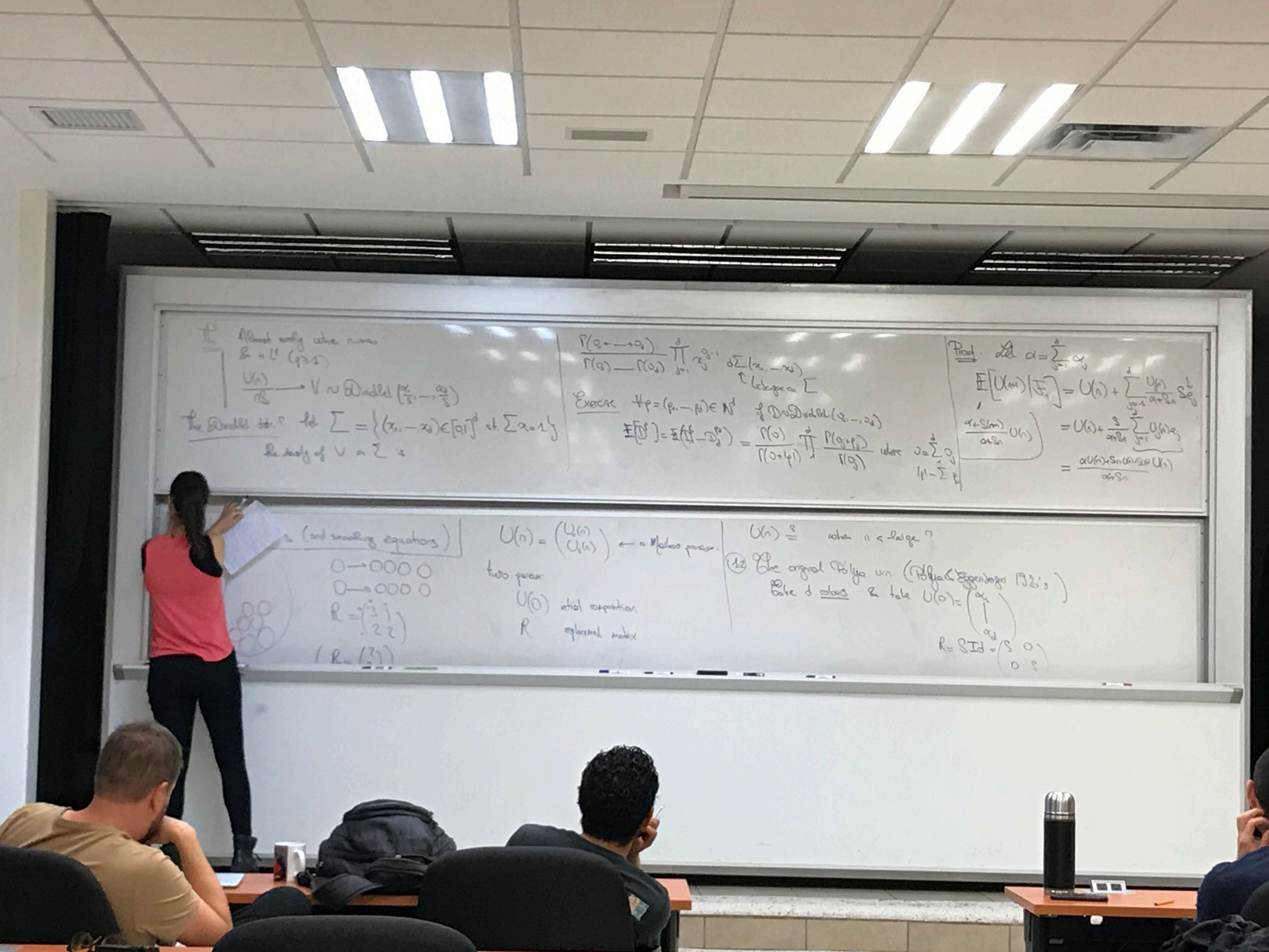
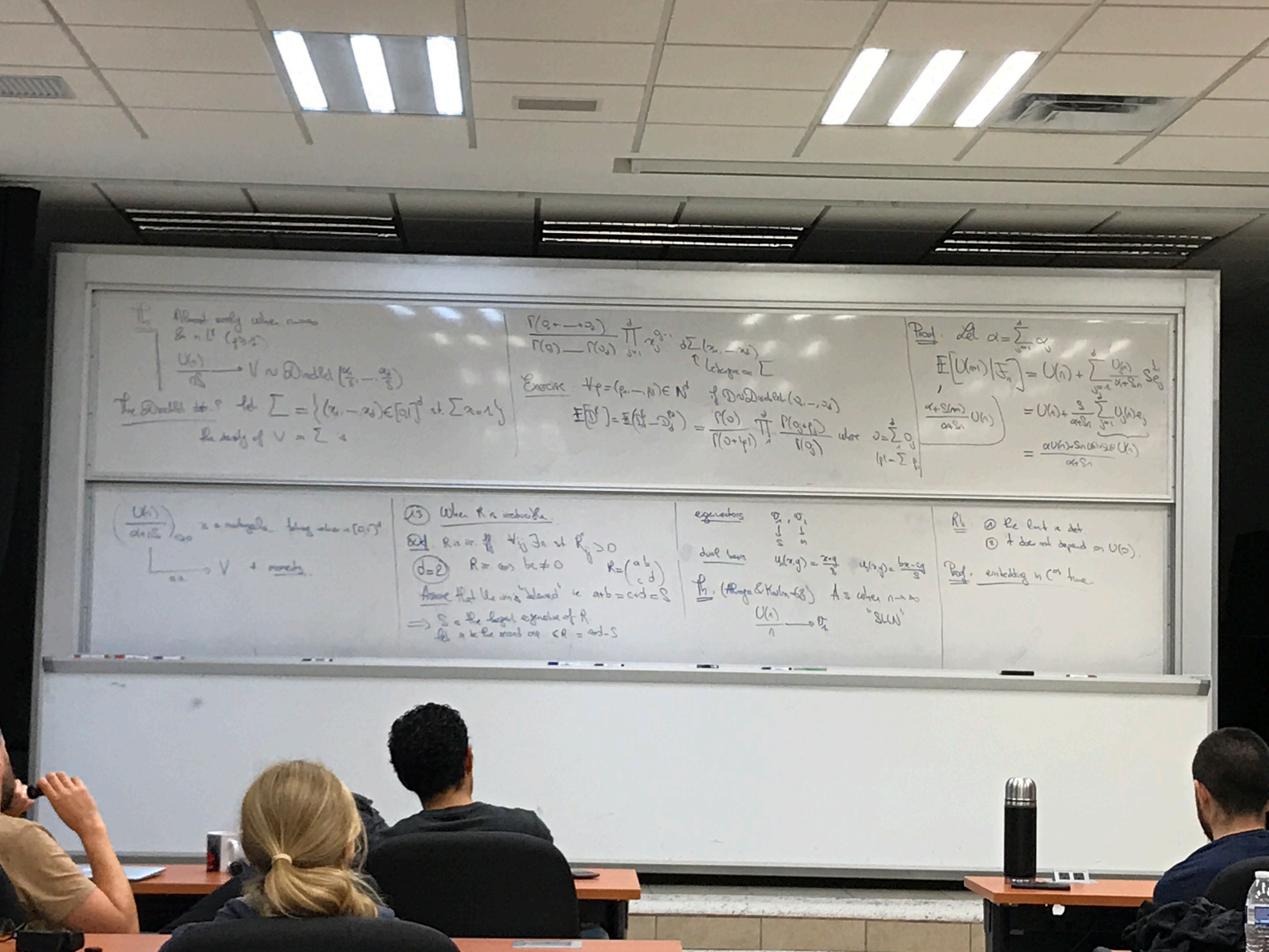

Pólya urns (and smoothing equations)
1 Intro
two param
U(0) initial composition
R replacement matrix
1.2 The original Pólya urn
R = S Id

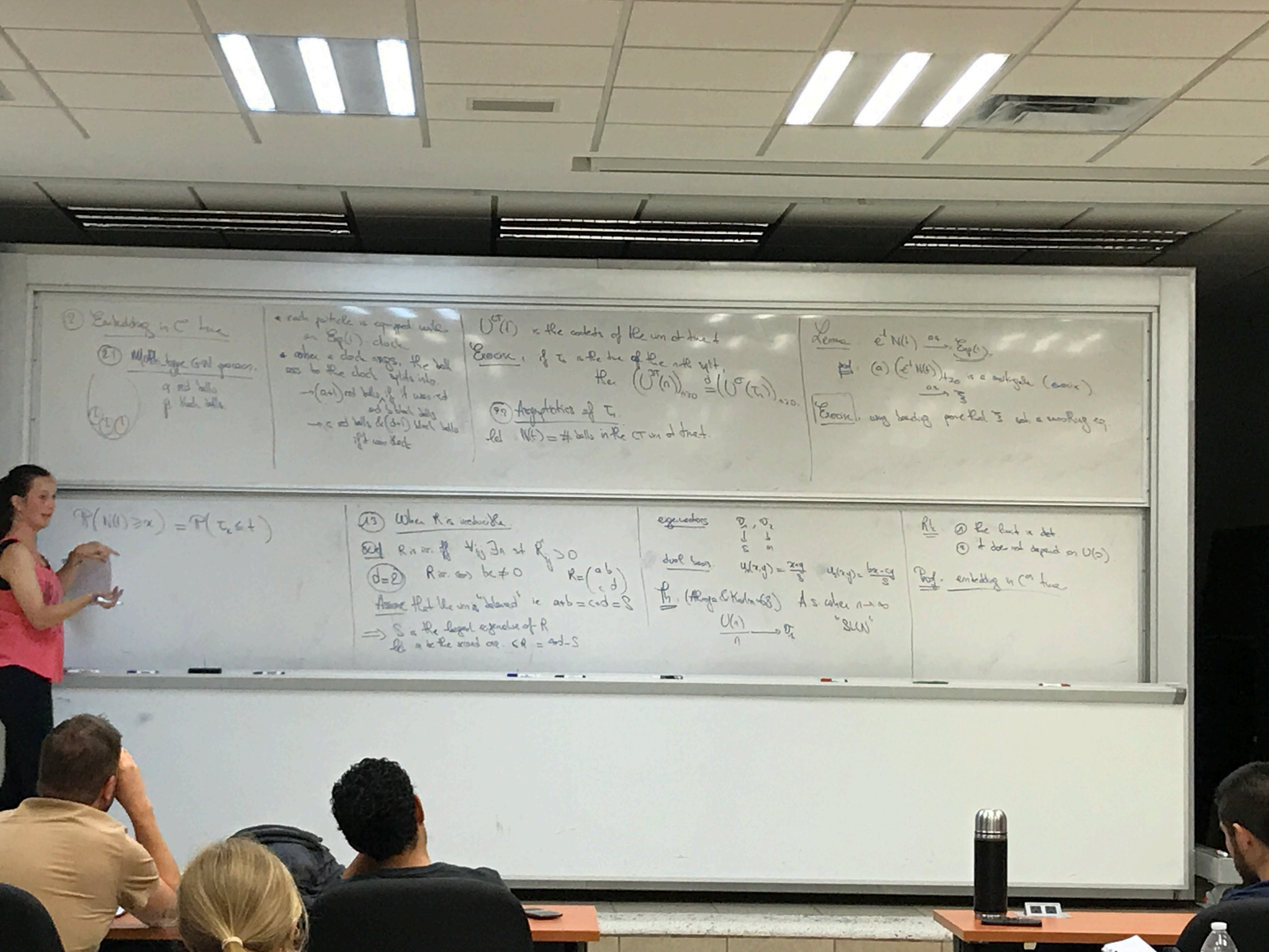


Pólya urns (and stochastic equations)
U(n) = (U1(n), U2(n)) ← a Markov process
two param:
U(0) initial composition
R replacement matrix
R = (3 1; 2 2)
U(n) ≈ ? when n is large?
(1.2) The original Pólya urn (Pólya & Eggenberger 1920's)
Take d colours & take U(0) = (α1, …, αd)
R = S Id = (S 0; 0 S)
Thm
Almost surely when n → ∞ & in L^p (p ≥ 1)
U(n)/nS → V ~ Dirichlet(α1/S, …, αd/S)
The Dirichlet distr.: let Σ = {(x1, …, xd) ∈ [0,1]^d s.t. Σ xi = 1}
the density of V on Σ is
Γ(α1 + … + αd) / (Γ(α1) … Γ(αd)) ∏_{j=1}^{d} xj^{αj − 1} dΣ(x1, …, xd)
↑ Lebesgue on Σ
Exercise ∀ p = (p1, …, pd) ∈ N^d if D ~ Dirichlet(α1, …, αd)
E[D^p] = E(D1^{p1} … Dd^{pd}) = Γ(α)/Γ(α+|p|) ∏_{1}^{d} Γ(αj+pj)/Γ(αj) where α = Σ_{1}^{d} αj, |p| = Σ pj
Proof. Let α =


Almost surely when n → ∞
& in L^p (p ≥ 1)
U(n)/nS → V ~ Dirichlet(α1/S, ..., αd/S)
The Dirichlet distrib.n let Σ = {(x1, ..., xd) ∈ [0,1]^d st Σ x = 1}
the density of V on Σ is
Lebesgue on Σ
Exercise ∀ p = (p1, ..., pd) ∈ N^d if D ~ Dirichlet(a1, ..., ad)
Proof: Let α = Σ αj
(and smoothing equations)
U(n) = (U1(n), U2(n)) ← a Markov process.
two param:
U(0) initial composition
R replacement matrix
when n is large?
1.2 The original Pólya urn (Pólya & Eggenberger 1920's)
Take d colours & take U(0) =
R = S Id

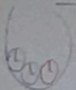


The Dirichlet dist.
Lebesgue on Σ
Exercise
Proof. Let
when Ω_1 is irreducible
eigenvectors
dual basis
Thm (Athreya & Karlin 68)
"SLLN"
Rk
Proof. embedding in C^n time


Embedding in CT time
Multi-type GW process
a red balls
β black balls
each particle is equipped with an Exp(1) clock
when a clock rings, the ball attached to the clock splits into
→ (a+1) red balls & b black balls if it was red
→ c red balls & (d+1) black balls if it was black
Exercise
Asymptotics of τn
let N(t) = # balls in the CT urn at time t
Lemma
Exp(1)
(a) is a martingale
Exercise
P(N(t) ≥ x) = P(τx ≤ t)
When R is irreducible
Def R is irr. iff ∀ i,j ∃ n st R^n_ij > 0
d=2 R irr. ⟺ bc ≠ 0
R = (a b / c d)
Assume that the urn is "balanced" ie a+b = c+d = S
⟹ S is the largest eigenvalue of R
eigenvectors
dual basis
Th. (Athreya & Karlin 68) A.s. when n→∞
"SLLN"
Rk
1 the limit is det.
2 it does not depend on U(0)
Proof. embedding in CT time

② Embedding in C^t time

②.1 Multi-type GW process.



α red balls
β black balls

- each particle is equipped with an $Exp(1)$-clock
- when a clock rings, the ball associated to the clock splits into
  - $\to (a+1)$ red balls & b black balls if it was red
  - $\to c$ red balls & $(d+1)$ black balls if it was black

$U^{CT}(t)$ is the contents of the urn at time t

Exercise. If τ_n is the time of the n-th split, then $\left(U^{DT}(n)\right)_{n\geq 0} \overset{d}{=} \left(U^{CT}(\tau_n)\right)_{n\geq 0}$.

②.2 Asymptotics of τ_n

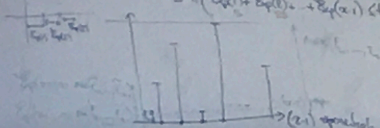
Let $N(t) = \#$ balls in the CT urn at time t.

Lemma. $e^{-St} N(t) \xrightarrow{a.s.} \text{Gamma}\left(\frac{\alpha+\beta}{S}\right)$

Proof. (a) $\left(e^{-St} N(t)\right)_{t\geq 0}$ is a martingale (exercise) $\xrightarrow{a.s.} \xi$

Exercise. Using branching property that ξ satisfies a recursive eq.

$$\mathbb{P}(N(t) \geq x) = \mathbb{P}(\tau_x \leq t)$$
$$= \mathbb{P}\left(Exp(1) + Exp(2) + \dots + Exp(x-1) \leq t\right)$$
$$= \mathbb{P}\left(\max(\xi_1, \dots, \xi_{x-1}) \leq t\right)$$
$$= \mathbb{P}\left(Exp(1) \leq t\right)^{x-1}$$



If $N(0) = S \leadsto \xi \sim Exp(1/S)$

If $N(0) = \alpha+\beta \leadsto \xi \sim \text{Gamma}\left(\frac{\alpha+\beta}{S}\right)$

eigenvectors v_1, v_2
$\downarrow \quad \downarrow$
$S \quad m$

dual basis $u_1(x,y) = \frac{x+y}{S}$ $\quad u_2(x,y) = \frac{bx-cy}{S}$

Th. (Athreya & Karlin 68) A.s. when $n \to \infty$

$$\frac{U(n)}{n} \longrightarrow v_1 \qquad \text{"SLLN"}$$

Rk. ① The limit is det.
② It does not depend on $U(0)$

Proof. embedding in C^{ts} time