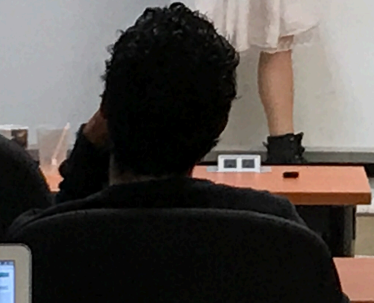
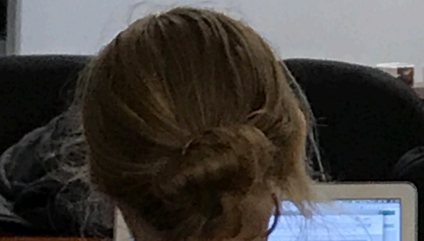
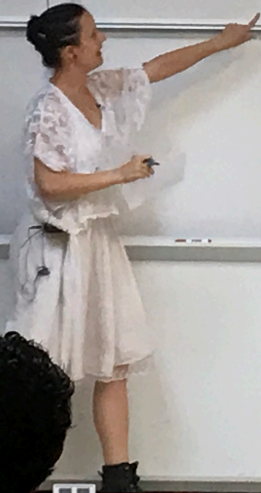
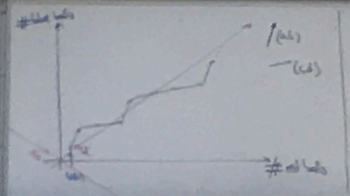


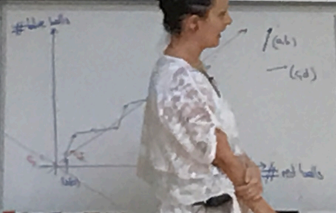
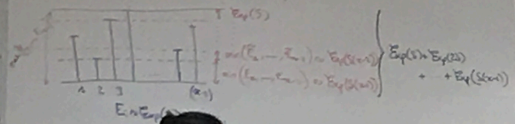
Lemma. $e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \rightarrow \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \sim \text{Gamma}(\frac{\lambda t}{\lambda}, 1)$

$$\begin{aligned}
 \mathbb{P}(N(t) \leq x) &= \mathbb{P}(\tau_1 \leq t) \\
 &= \mathbb{P}(\text{Exp}(\lambda) + \text{Exp}(\lambda) + \dots + \text{Exp}(\lambda) \leq t) \\
 \text{if } N(t) = S & \\
 &= \mathbb{P}(\text{Exp}(S) + \text{Exp}(S) + \dots + \text{Exp}(S) \leq t) \\
 &= \mathbb{P}(\max(E_1, E_2) \leq t)
 \end{aligned}$$



Lemma: $e^{-s} N(t) \xrightarrow{t \rightarrow \infty} \exists \sim \text{Gamma}(\frac{c\lambda}{s}, 1)$

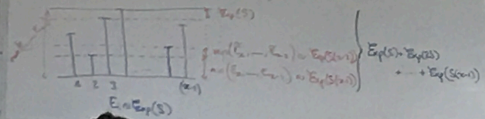
Pr. $P(N(t) \leq x) = P(\tau_x \leq t)$
 $= P(\sum_{i=1}^{N(t)} E_i \leq t)$
 $\uparrow N(t) = S$
 $= P(E_1 + E_2 + \dots + E_S \leq t)$
 $= P(\max(E_1, \dots, E_S) \leq t)$



$\stackrel{D}{=} (1 - e^{-st})^{-c}$
 $P(N(t)e^{-st} \leq x) = P(N(t) \leq xe^{st})$
 $\stackrel{D}{=} (1 - e^{-sxe^{st}})^c$
 $\xrightarrow{t \rightarrow \infty} e^{-x}$

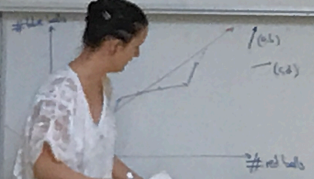
When $N(0) = 0$
then $\exists \sim \text{Exp}(\lambda)$
 $\Rightarrow \uparrow N(0) = \text{arr}$
 $\Rightarrow \exists \sim \text{Gamma}(\frac{c\lambda}{s}, 1)$

Corollary: $\tau_x = \frac{1}{s} \log \frac{c}{s} + O(1)$



Lemma $e^{-st} - \frac{e^{-st} - 1}{-s} \xrightarrow{t \rightarrow \infty} \frac{1}{s} \sim \text{Gamma}\left(\frac{\alpha s}{s}, 1\right)$

$$\begin{aligned} \mathbb{P}(N(t) \leq x) &= \mathbb{P}(T_n \leq t) \\ &= \mathbb{P}(Exp(N(t)) + Exp(N(t) \cdot s) + \dots + Exp(N(t) \cdot s) \leq t) \\ \text{if } N(t) = S \\ &= \mathbb{P}(Exp(S) + Exp(2S) + \dots + Exp(nS) \leq t) \\ &= \mathbb{P}(\max\{E_1, E_2, \dots\} \leq t) \end{aligned}$$



$$\begin{aligned} &= (1 - e^{-st}) \\ \mathbb{P}(N(t) e^{st} \leq x) &= \mathbb{P}(N(t) \leq x e^{-st}) \\ &= (1 - e^{-s x e^{-st}}) \\ &\xrightarrow{t \rightarrow \infty} e^{-x} \end{aligned}$$

Other $N(t) = S$
then $\exists n \sim Exp(S)$
so $\mathbb{P}(N(t) = \alpha t) \Rightarrow \exists n \sim \text{Gamma}\left(\frac{\alpha t}{s}, 1\right)$

Paralog. $t_n = \frac{1}{s} \log n + O(1)$
eg Asymptotic behaviour $\mathcal{U}^\sigma(t)$
 $\lim_{t \rightarrow \infty} e^{-st} \mathcal{U}^\sigma(t) \xrightarrow{t \rightarrow \infty} \phi \psi$
 What is ϕ ? $\mathcal{U}^\sigma(t) \stackrel{d}{=} \mathcal{U}^{\sigma^*}(n)$
 $\lim_{t \rightarrow \infty} \mathcal{U}^\sigma(t) e^{st} \xrightarrow{t \rightarrow \infty} \phi \psi$

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{U(t)}{n} \xrightarrow{t \rightarrow \infty} \frac{U(t)}{n} \xrightarrow{t \rightarrow \infty} \frac{U(t)}{n} e^{-st} \\ &= \frac{U(t)}{n} e^{-st} \\ &\quad \downarrow \quad \downarrow \\ &\quad \phi \quad \psi \\ &\quad \rightarrow \exists \phi \end{aligned}$$

① Second order systems

$$\frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{b_0}{1} = \frac{b_2}{s^2} + \frac{b_1}{s} + b_0$$

where $\omega = \frac{1}{s}$

Prod: $E_n U(n) = U(n) + \frac{U(n)}{a_1 + a_2 n} + \frac{U(n)}{b_1} + \frac{U(n)}{a_2 n^2} + \frac{c}{d}$

$$= \left(\text{Id} + \frac{1}{a_1 + a_2 n} \right) U(n)$$

$\prod_{n=1}^{\infty} \left(\text{Id} + \frac{1}{a_1 + a_2 n} \right) U(n)$ is a constant, \square

if $\prod_{n=1}^{\infty} (1 + \frac{1}{a_1 + a_2 n})$ is convergent as

$$\sum_{n=1}^{\infty} \frac{1}{a_1 + a_2 n} < \infty$$

$$E_n U(n) = \prod_{j=0}^{n-1} \left(1 + \frac{1}{a_1 + a_2 j} \right) U(n) = \frac{1}{a_1} U(n)$$

$$= \prod_{j=0}^{n-1} \left(1 + \frac{1}{a_1 + a_2 j} \right) U(n)$$

$$E_n U(n) \in \frac{1}{a_1}$$

① second order approximation

$$\frac{\sum_{i=1}^n \Delta t_i}{\prod_{i=1}^n (1 + \frac{\sigma^2 \Delta t_i}{2})} \approx e^{-\frac{\sigma^2}{2} \sum_{i=1}^n \Delta t_i} = e^{-\frac{\sigma^2}{2} t}$$
 is a martingale.

done $\sigma = \sigma/2$

$$\mathbb{E} \left[\prod_{i=1}^n (1 + \frac{\sigma^2 \Delta t_i}{2}) \right] \approx e^{\frac{\sigma^2}{2} t}$$

$$\text{Prod: } \mathbb{E}_t(U(n)) = U(t) + \frac{U(t)}{\sigma^2 + n^2} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{U(t)}{\sigma^2 + n^2} \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \left(\text{Id} + \frac{t}{\sigma^2 + n^2} \right) U(t)$$

$$\left(\prod_{i=1}^n \left(\text{Id} + \frac{t}{\sigma^2 + n^2} \right) \right) U(t) \quad n \text{ is a mathematically perfect } \square$$

sup \mathbb{P}_n bounded in L^2
 and the convergence as if $\sigma > 1/2$!

$$\text{prod: } \sum_{i=1}^n \mathbb{E}[\Delta R_i^2] \xrightarrow{t \rightarrow \infty}$$

$$\mathbb{E}[\Delta R_i^2] = \prod_{j=1}^i \left(1 + \frac{\sigma^2 \Delta t_j}{2} \right)^{-2} \mathbb{E} \left[(U(t_{i+1}) - \left(1 + \frac{\sigma^2 \Delta t_{i+1}}{2} \right) U(t_i))^2 \right]$$

$$\leq \prod_{j=1}^i \left(\frac{\sigma^2 \Delta t_j}{2} \right) \approx \sigma^2 (U(t_i))^2 \Delta t_{i+1}$$

$\mathbb{E}[\Delta R_i^2] \leq \frac{\sigma^2}{2} \Delta t_i < \infty$ if $\sigma > 1/2$

done $\sigma > 1/2$!!

$$V_n^2 = \mathbb{E}[\Delta R_i^2] \xrightarrow{t \rightarrow \infty}$$

$$V_n^2 = \text{var} \left[\sum_{i=1}^n \sigma^2 \Delta t_i \right] \approx \sigma^2 t$$

$\Rightarrow \frac{\mathbb{P}_n}{\sqrt{t}} \xrightarrow{d} N(0, \sigma^2)$

$\text{or } n^{1/2} \geq \alpha$

$\Leftrightarrow n \geq 2 \alpha^{1/2}$

$\frac{\mathbb{P}_n}{\sqrt{t}} \xrightarrow{d} N(0, \sigma^2)$

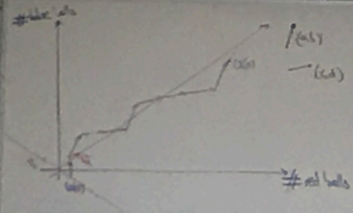
$n = \alpha^{1/2} \approx \frac{\mathbb{P}_n}{\sqrt{t}} \Rightarrow N(0, \sigma^2)$

Recall that $\mathbb{P}_n \approx \frac{U(t)}{\sqrt{t}} \Rightarrow \frac{U(t)}{\sqrt{t}} \Rightarrow N(0, \sigma^2)$

If $\sigma = 1/2$ $\frac{U(t)}{\sqrt{t}} \Rightarrow N(0, \sigma^2)$

If $\sigma > 1/2$ $\frac{U(t)}{\sqrt{t}} \Rightarrow W_{\sigma^2}$

done not equal to $U(t)$



$$\begin{aligned}
 & \mathbb{P} = (1 - e^{-\lambda t})^x \\
 & \mathbb{P}(N(t) e^{-\lambda t} \leq x) = \mathbb{P}(N(t) \leq x e^{\lambda t}) \\
 & \stackrel{(*)}{=} (1 - e^{-\lambda t})^{x e^{\lambda t}} \\
 & \lim_{t \rightarrow \infty} \frac{1}{t} e^{\lambda t} = e^{\lambda}
 \end{aligned}$$

When $N(0) = 0$
 then $\Xi \sim \text{Exp}(\lambda)$
 \Rightarrow if $N(0) = \text{const}$
 $\Rightarrow \Xi \sim \text{Gamma}(\frac{\text{const}}{\lambda}, 1)$

Corollary: $\tau = \frac{1}{\lambda} \log n + O(1)$
 as Asymptotic behaviour $U^\sigma(t)$
 $\frac{1}{t} \int_0^t e^{-\lambda t} U^\sigma(t) dt \xrightarrow{a.s.} \phi$
 What's ϕ ? $U^\sigma(t) \stackrel{d}{=} U^{\text{P}^\sigma}(n)$
 $\frac{1}{n} \int_0^n U^\sigma(t) e^{-\lambda t} dt \xrightarrow{a.s.} \phi$
 $\phi = \frac{1}{\lambda}$

2) Grand order completion
 at CT
 If $\nu < 1/2$ then $U(U^\sigma) \rightarrow K(0, +)$
 If $\nu > 1/2$ then $U(U^\sigma) \rightarrow W_{\text{off}}$

Lemma $W^\sigma \stackrel{d}{=} \Xi^\sigma W^{\text{P}^\sigma}$ where $\Xi \sim \text{Gamma}(\frac{\nu}{\lambda}, 1)$
 proof Ξ_σ

Proof: \mathbb{P}_n is bounded in L^2
 and then converge as $\nu > 1/2$!

$$\text{and } \sum_{n=0}^{\infty} \mathbb{E}[R_n^2] < \infty$$

$$\begin{aligned}
 \mathbb{E}[R_n^2] &= \mathbb{E} \left[\sum_{j=0}^n \left(\Delta t + \frac{n}{\text{order}} \right)^2 \mathbb{E} \left[\left(U(U^{\sigma(n)}) - \left(\Delta t + \frac{n}{\text{order}} \right) U(U^{\sigma(n)}) \right)^2 \right] \right] \\
 &\leq \sum_{j=0}^n \left(\right)^2 \text{var}(U(U^{\sigma(n)}))
 \end{aligned}$$

