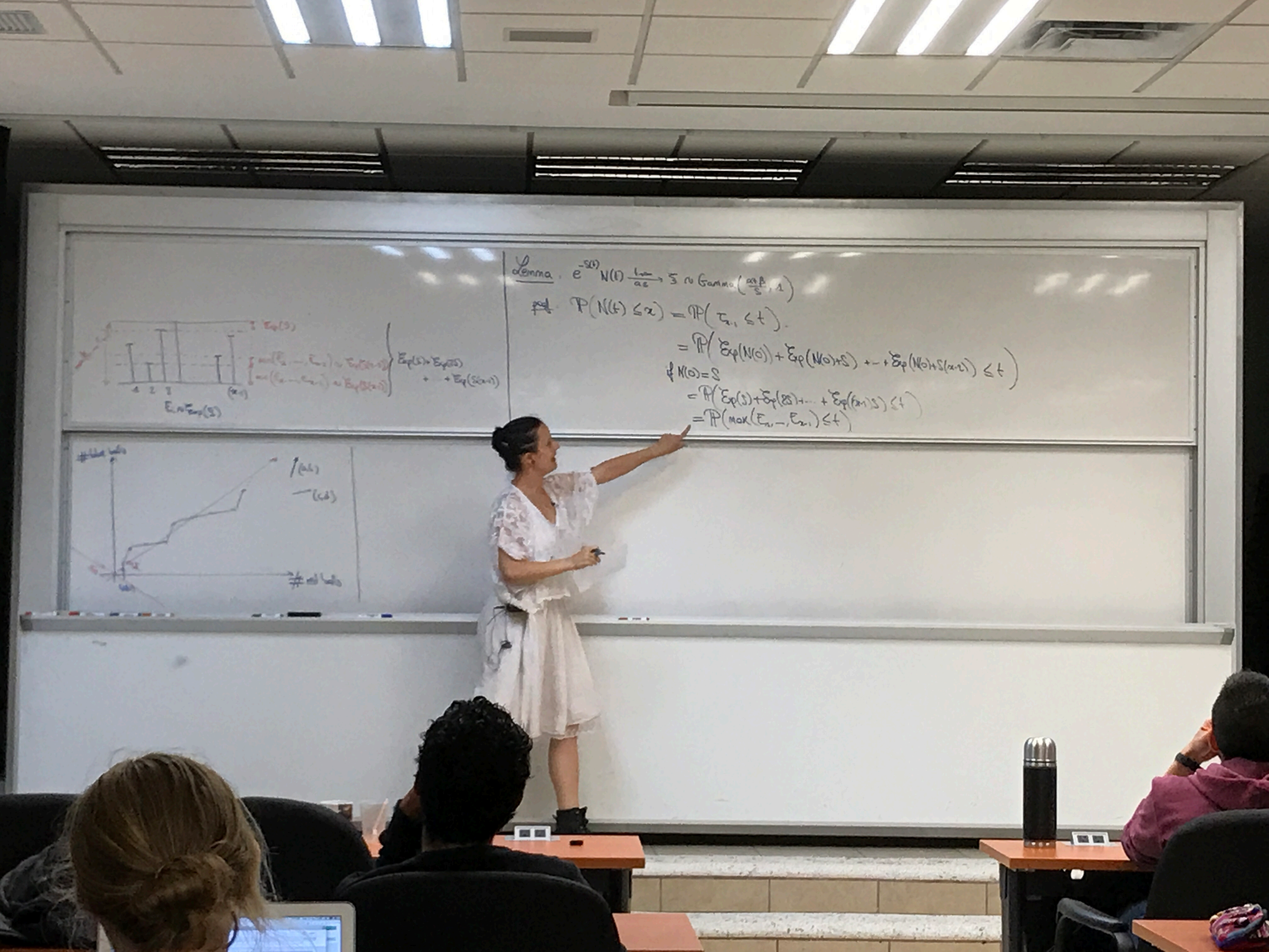
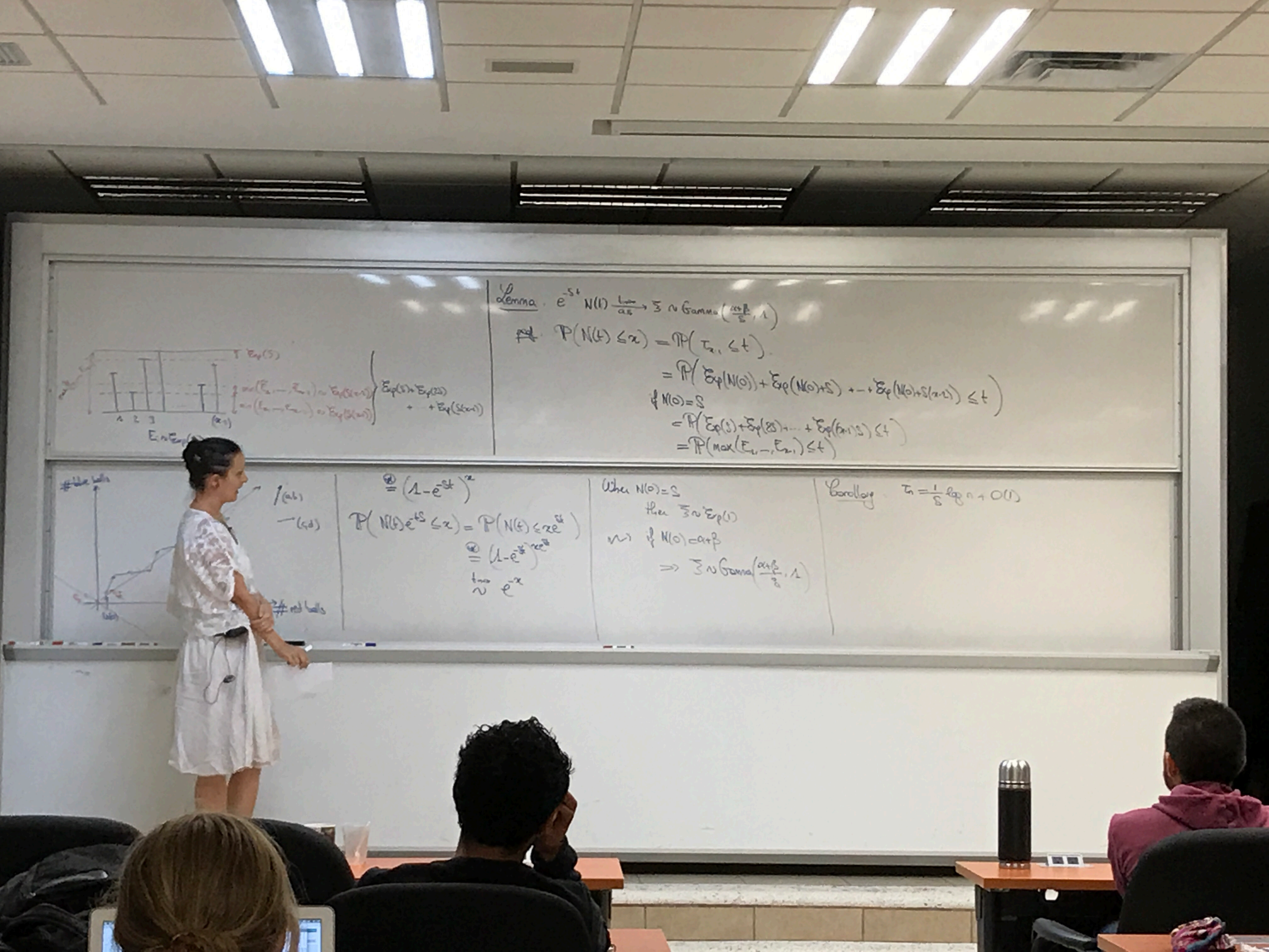
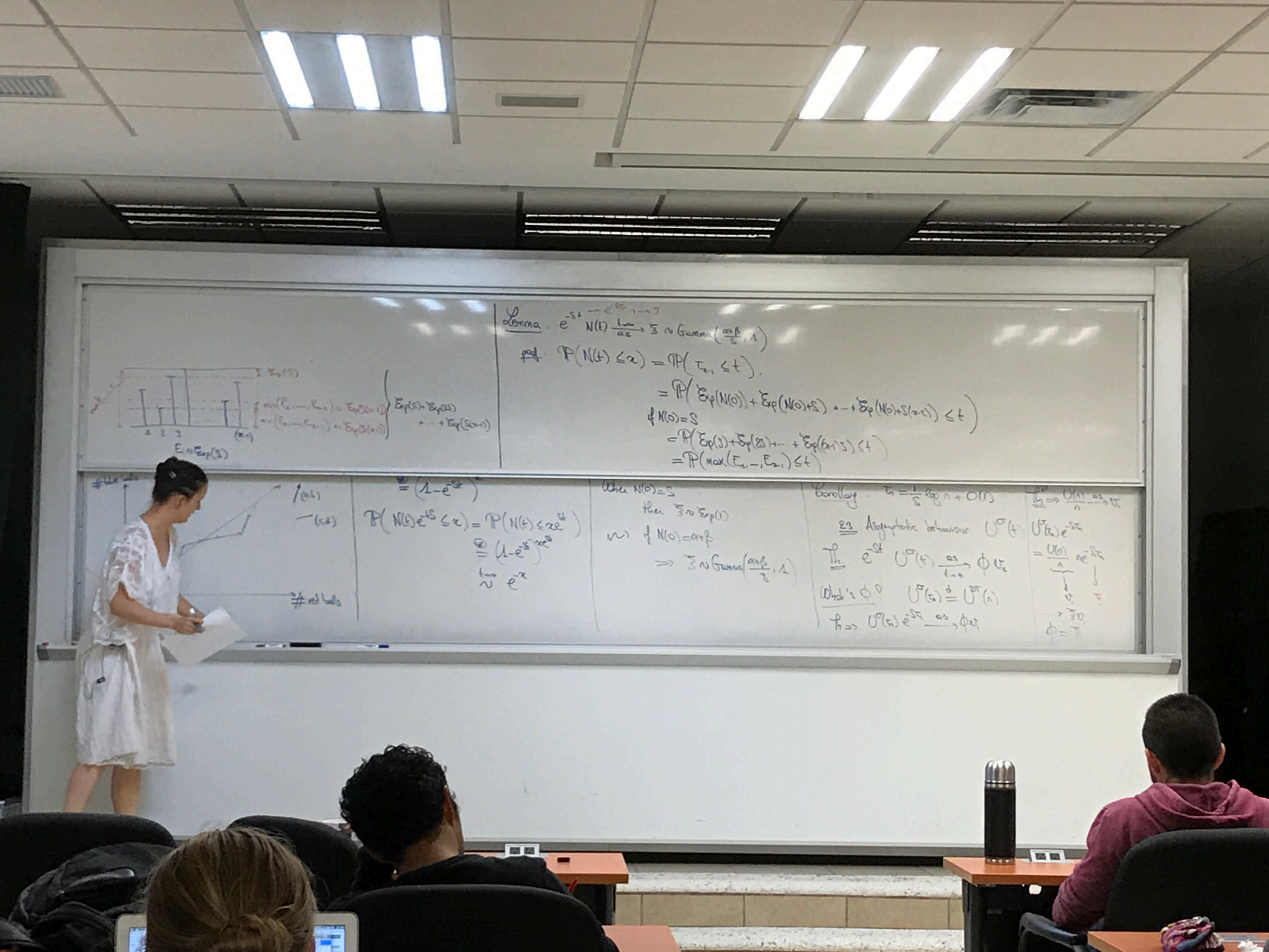
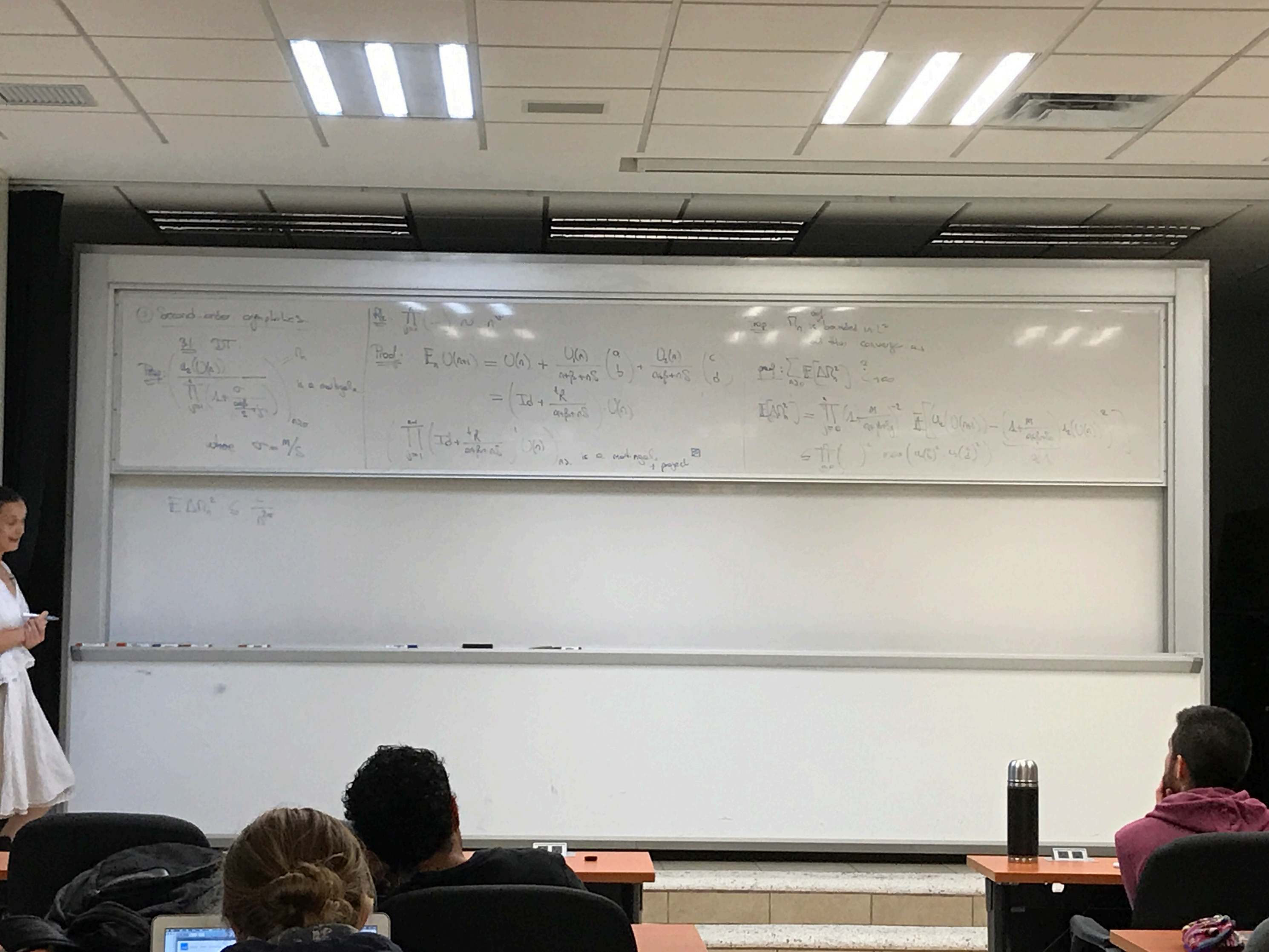

Lemma
Gamma
P(N(t) ≤ x) = P(τ ≤ t)
= P(Exp(N(0)) + Exp(N(0)+S) + ... + Exp(N(0)+S(x-2)) ≤ t)
if N(0) = S
= P(max(E, ..., E) ≤ t)


Lemma. e^{-St} N(t) → ξ ~ Gamma((α+β)/S, 1)
pf. P(N(t) ≤ x) = P(τ_{x-1} ≤ t)
= P(Exp(N(0)) + Exp(N(0)+S) + ... + Exp(N(0)+S(x-2)) ≤ t)
if N(0) = S
= P(Exp(S) + Exp(2S) + ... + Exp((x-1)S) ≤ t)
= P(max(E_1, ..., E_{x-1}) ≤ t)
Exp(S) + Exp(2S) + ... + Exp(S(x-1))
= (1 - e^{-St})^x
P(N(t)e^{-tS} ≤ x) = P(N(t) ≤ xe^{St})
= (1 - e^{-St})^{xe^{St}}
~ e^{-x}
When N(0) = S
then ξ ~ Exp(1)
if N(0) = α+β
⟹ ξ ~ Gamma((α+β)/S, 1)
Corollary τ_n = (1/S) log n + O(1)
blue balls
red balls


Lemma
Gamma
E_i ~ Exp(S)
P(N(t) ≤ x) = P(τ_{x+1} ≤ t)
= P(Exp(N(0)) + Exp(N(0)+S) + ... + Exp(N(0)+S(x-1)) ≤ t)
if N(0)=S
= P(max(E_1, ..., E_{x+1}) ≤ t)
#red balls
P(N(t)e^{-tS} ≤ x) = P(N(t) ≤ xe^{tS})
When N(0)=S
then ξ ~ Exp(1)
if N(0)=α+β
Corollary
2.3 Asymptotic behaviour
What's φ?


Second-order asymptotics
is a martingale
Proof
is a martingale + project
bounded in L^2
and thus converges a.s.

② Second order asymptotics

31. DT

Prop: $\left(\dfrac{u_2(U(n))}{\prod_{j=0}^{n-1} \left(1 + \frac{\sigma}{\frac{\alpha+\beta}{S} + j}\right)} \right)_{n \geq 0} = \Pi_n$ is a martingale

where $\sigma = M/S$

Rk: $\prod_{j=1}^{n} (\cdots) \sim n^{\sigma}$

Proof: $\mathbb{E}_n U(n+1) = U(n) + \dfrac{U_1(n)}{\alpha+\beta+nS} \begin{pmatrix} a \\ b \end{pmatrix} + \dfrac{U_2(n)}{\alpha+\beta+nS} \begin{pmatrix} c \\ d \end{pmatrix}$

$= \left(Id + \dfrac{{}^tR}{\alpha+\beta+nS} \right) U(n)$

$\left(\prod_{j=1}^{n-1} \left(Id + \dfrac{{}^tR}{\alpha+\beta+jS} \right)^{-1} U(n) \right)_{n \geq}$ is a martingale, project ☐

Prop: Π_n bounded in L^2 and thus converges a.s. if $\sigma > 1/2$!

proof: $\sum_{n \geq 0} \mathbb{E}[\Delta \Pi_n^2] \overset{?}{<} +\infty$

$\mathbb{E}[\Delta \Pi_n^2] = \prod_{j=0}^{n} \left(1 + \dfrac{M}{\alpha+\beta+Sj}\right)^{-2} \mathbb{E}\left[\left(u_2(U(n+1)) - \left(1 + \dfrac{M}{\alpha+\beta+nS}\right) u_2(U(n)) \right)^2 \right]$

$\leq \prod_{j=0} (\quad)^{-2} \max \left(u_2\binom{a}{b}^2, u_2\binom{c}{d}^2 \right)$ [illegible]

$\mathbb{E} \Delta \Pi_n^2 \leq \dfrac{c}{n^{2\sigma}} < +\infty$ if $\sigma > 1/2$ ■

[illegible] $\sigma \leq 1/2$??

$\sigma_n^2 = \mathbb{E} \Delta \Pi_n^2 \sim n^{-2\sigma}$ if $\sigma < 1/2$

$T_x = \min \{ n : \sum_{i}^{n} \sigma_i^2 \geq x \}$ $\Rightarrow$ $\dfrac{\Pi_{T_x}}{\sqrt{x}} \overset{d}{\Longrightarrow} N(0,*)$ $\rightsquigarrow$ $\dfrac{\Pi_{c x^{1/(1-2\sigma)}}}{\sqrt{x}} \Longrightarrow N(0,*)$

and $n^{1-2\sigma} \geq x$

$\Leftrightarrow n \geq c\, x^{\frac{1}{1-2\sigma}}$

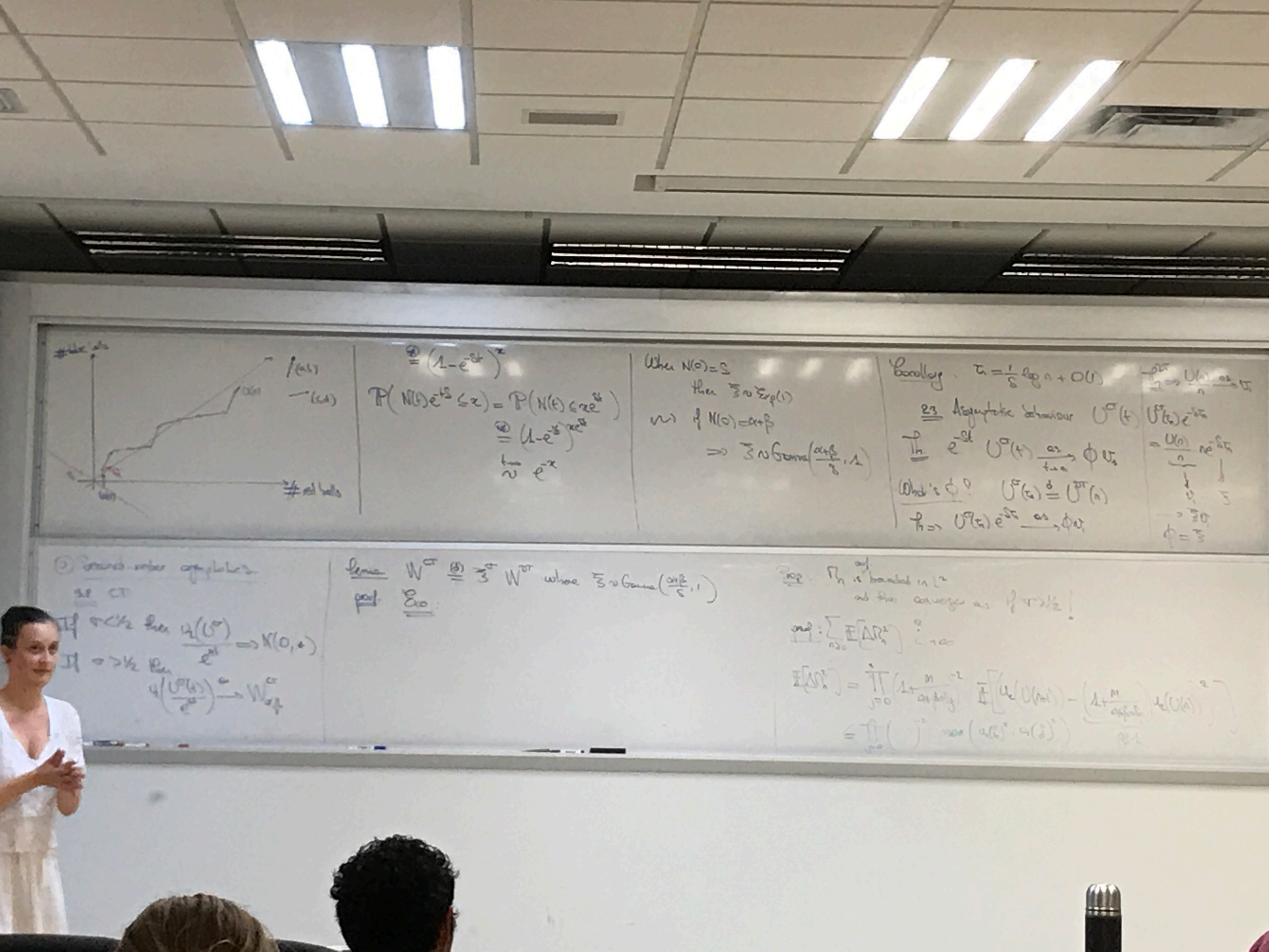
"$n = x^{1/(1-2\sigma)}$" $\rightsquigarrow$ $\dfrac{\Pi_n}{\sqrt{n^{1-2\sigma}}} \Longrightarrow N(0,*)$

Recall that $\Pi_n \sim \dfrac{u_2(U(n))}{n^{\sigma}}$ $\rightsquigarrow$ $\dfrac{u_2(U(n))}{\sqrt{n}} \Longrightarrow N(0,*)$.

explicit, does not depend on $U(0)$

If $\sigma = 1/2$ $\dfrac{u_2(U(n))}{\sqrt{n \log n}} \Longrightarrow N(0,*)$

If $\sigma > 1/2$ $\dfrac{u_2(U(n))}{n^{\sigma}} \xrightarrow{a.s.} W_{\alpha\beta}$



red balls
When N(0)=δ
then ξ ~ Exp(1)
~> if N(0)=α+β
⟹ ξ ~ Gamma((α+β)/δ, 1)
Corollary
2.3 Asymptotic behaviour
Th
What's φ ?
Th ⟹
Second-order asymptotics
3.1 CT
If σ<1/2 then
If σ>1/2 then
Lemma
where ξ ~ Gamma((α+β)/δ, 1)
proof. Exo
Prop.
is bounded in L²
and thus converge a.s. if σ>1/2 !
proof: