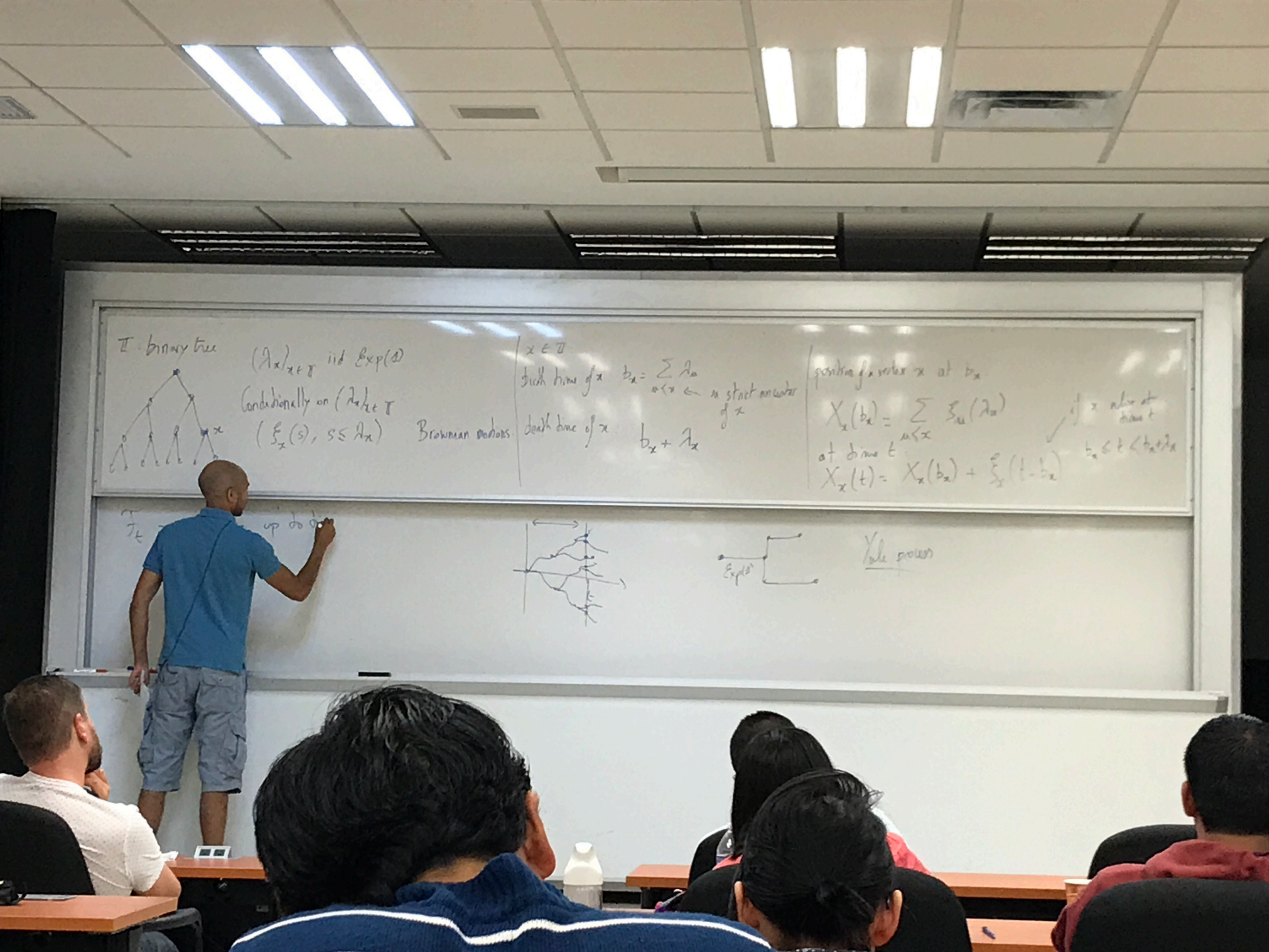


$\mathbb{T}$ binary tree
$(\lambda_x)_{x \in \mathbb{T}}$ iid $\mathcal{E}xp(1)$
Conditionally on $(\lambda_x)_{x \in \mathbb{T}}$
$(\xi_x(s), s \leq \lambda_x)$ Brownian motions
$x \in \mathbb{T}$
birth time of x $b_x = \sum_{u < x} \lambda_u$
u strict ancestor of x
death time of x $b_x + \lambda_x$
position of vertex x at b_x
$X_x(b_x) = \sum_{u < x} \xi_u(\lambda_u)$
at time t
$X_x(t) = X_x(b_x) + \xi_x(t - b_x)$
if x alive at time t
$b_x \leq t < b_x + \lambda_x$
up to
$\mathcal{E}xp(1)$
Yule process

For x alive at time t $(x \in \mathcal{Z}_t)$

population at time t

ω

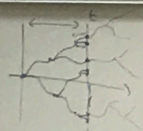
algebra up to time t

$\sigma(U_t)$

$U_t(\omega)$

Killing operator

tree: $\{x : b_x \leq t\}$



$x \in \mathbb{U}$:

birth time of x $\quad b_x = \sum_{u < x} \lambda_u \leftarrow$ a strict ancestor of x

death time of x $\quad b_x + \lambda_x$

position of a vertex x at b_x

$$X_x(b_x) = \sum_{u < x} \xi_u(\lambda_u)$$

at time t

$$X_x(t) = X_x(b_x) + \xi_x(t - b_x)$$

ω: marked tree

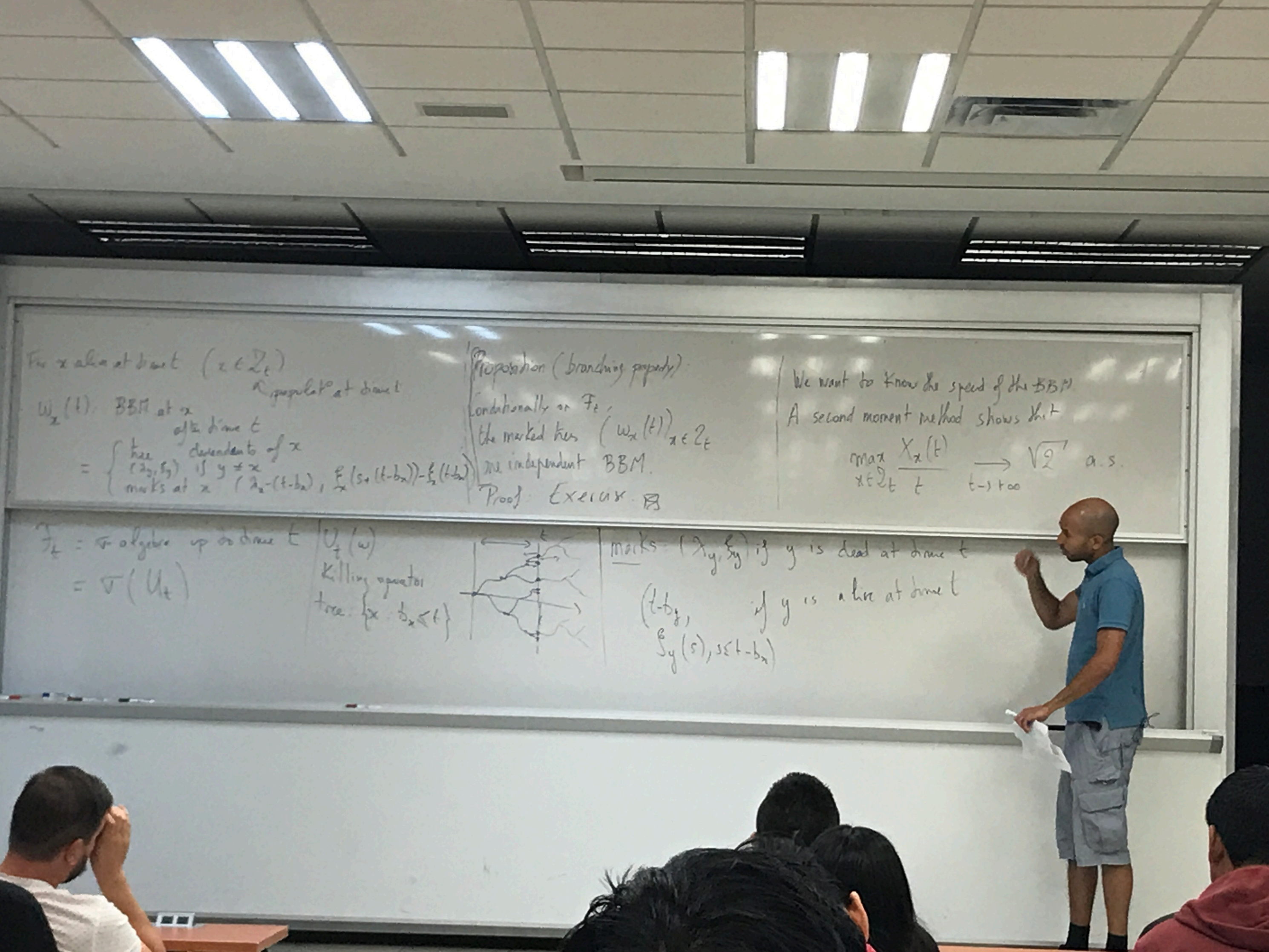
if x alive at time t

$b_x \leq t < b_x + \lambda_x$

marks: (λ_y, ξ_y) if y is dead at time t

$(t - b_y, \quad$ if y is alive at time t

$\xi_y(s), s \leq t - b_x)$


For x alive at time t (x ∈ Z_t)
population at time t
W_x(t): BBM at x of the time t
= { tree descendants of x; (λ_y, ξ_y) if y ≠ x; marks at x (λ_x − (t − b_x), ξ_x(s + (t − b_x)) − ξ_x(t − b_x)) }
Proposition (branching property)
Conditionally on F_t, the marked trees (W_x(t))_{x ∈ Z_t} are independent BBM.
Proof: Exercise
We want to know the speed of the BBM
A second moment method shows that
max_{x ∈ Z_t} X_x(t)/t → √2 a.s. (t → +∞)
F_t = σ-algebra up to time t = σ(U_t)
U_t(ω)
Killing operator
tree: {x : b_x ≤ t}
marks: (λ_y, ξ_y) if y is dead at time t
(t − b_y, ξ_y(s), s ≤ t − b_x) if y is alive at time t

For x alive at time t ($x \in Z_t$)

population at time t

$W_x(t)$: BBM at x after time t

$$= \begin{cases} \text{tree} & \text{descendants of } x \\ (\lambda_y, \xi_y) & \text{if } y \neq x \\ \text{marks at } x & (\lambda_x - (t - b_x),\ \xi_x(s + (t - b_x)) - \xi_x(t - b_x)) \end{cases}$$

Proposition (branching property)

Conditionally on $\mathcal{F}_t$, the marked trees $(W_x(t))_{x \in Z_t}$ are independent BBM.

Proof: Exercise $\blacksquare$

We want to know the speed of the BBM.

A second moment method shows that

$$\max_{x \in Z_t} \frac{X_x(t)}{t} \xrightarrow[t \to +\infty]{} \sqrt{2} \quad \text{a.s.}$$

<u>Exercise</u>: (i) Let $\beta > \sqrt{2}$. Show with first moment that there is no $x \in Z_t$ such that $X_x(t) \geq \beta t$ for t large enough.

(ii) Let $|\beta| < \sqrt{2}$. Show that $\forall \varepsilon > 0$, there exists a ray R such that $X_R(t) \in [(\beta - \varepsilon)t, (\beta + \varepsilon)t]$ for t large enough.

($\forall Z$ integer-valued

$$P(Z \geq 1) \geq \frac{E[Z]^2}{E[Z^2]}$$)

$\forall$ real β, introduce

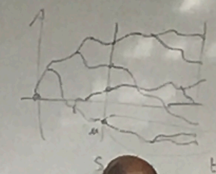
$$M_t(\beta) := e^{-t\left(1+\frac{\beta^2}{2}\right)} \sum_{x \in Z_t} e^{\beta X_x(t)}$$

(often: we write M_t instead $M_t(\beta)$)

Proposition: $(M_t(\beta))_{t \geq 0}$ is a positive martingale / $(\mathcal{F}_t)$.

Proof. $s < t$

$$M_t = e^{-s\left(1+\frac{\beta^2}{2}\right)} \sum_{u \in Z_s} e^{\beta X_u(s)} \Big\{ e^{-(t-s)\left(1+\frac{\beta^2}{2}\right)} \sum_{\substack{x \in Z_t \\ x > u}} e^{\beta\left(X_x(t) - X_u(s)\right)}$$



Exercise: (i) Let $\beta > \sqrt{2}$. Show with first moment that there is no $x \in Z_t$ such that $X_x(t) \geq \beta t$ for t large enough

(ii) Let $|\beta| < \sqrt{2}$. Show that $\forall \varepsilon > 0$, there exists a ray R such that $X_R(t) \in [(\beta-\varepsilon)t, (\beta+\varepsilon)t]$ for t large enough.

($\forall x$: Z integer-valued

$$\mathbb{P}(Z \geq 1) \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}$$

$\forall$ real β, introduce

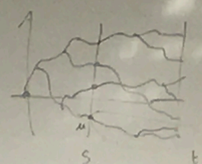
$$M_t(\beta) = e^{-t(1+\frac{\beta^2}{2})} \sum_{x \in Z_t} e^{\beta X_x(t)}$$

(often we write M_t instead $M_t(\beta)$)

Proposition: $(M_t(\beta))_{t \geq 0}$ is a positive martingale / $(\mathcal{F}_t)$.

Proof: $s < t$

$$M_t = e^{-s(1+\frac{\beta^2}{2})} \sum_{u \in Z_s} e^{\beta X_u(s)} \left\{ e^{-(t-s)(1+\frac{\beta^2}{2})} \sum_{\substack{x \in Z_t \\ x > u}} e^{\beta(X_x(t) - X_u(s))} \right.$$

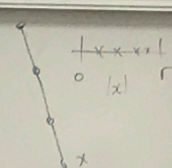


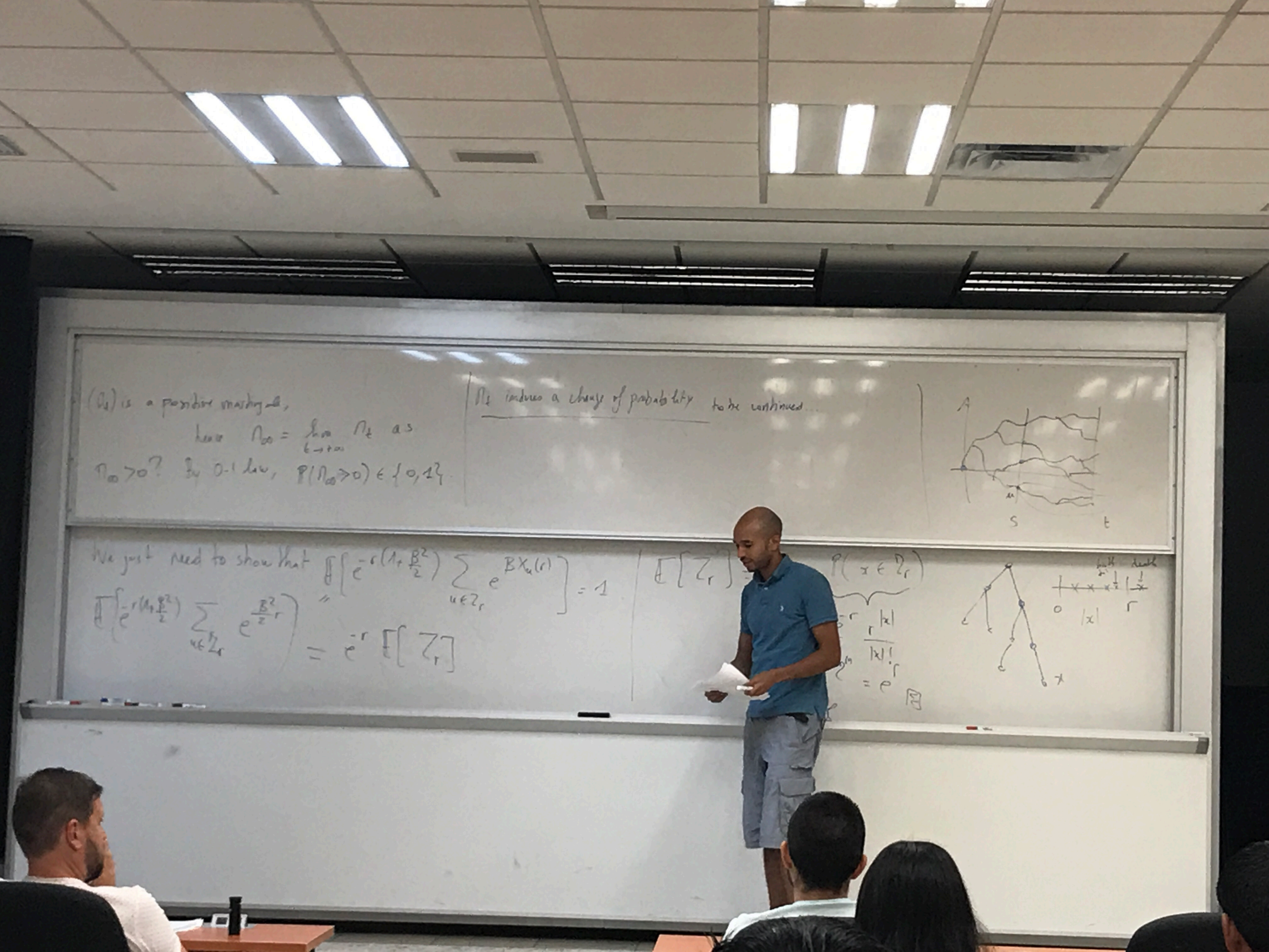
We just need to show that $E\left[e^{-r(1+\frac{\beta^2}{2})} \sum_{u \in Z_r} e^{\beta X_u(r)}\right] = 1$

$$E\left[e^{-r(1+\frac{\beta^2}{2})} \sum_{u \in Z_r} e^{\frac{\beta^2}{2} r}\right) = e^{-r} E[Z_r]$$

$$E[Z_r] = \sum_{x \in \mathbb{T}} \underbrace{P(x \in Z_r)}_{e^{-r} \frac{r^{|x|}}{|x|!}}$$

$$= \sum_n e^{-r} \frac{r^n}{n!} 2^n = e^r \quad \boxed{}$$





(Λ_t) is a positive martingale,
hence $\Lambda_\infty = \lim_{t \to +\infty} \Lambda_t$ a.s
$\Lambda_\infty > 0$? By 0-1 law, $\mathbb{P}(\Lambda_\infty > 0) \in \{0, 1\}$.
Λ_t induces a change of probability
to be continued...
s
t
We just need to show that $\mathbb{E}\left[e^{-r(1+\frac{\beta^2}{2})} \sum_{u \in Z_r} e^{\beta X_u(r)}\right] = 1$
$\mathbb{E}\left[e^{-r(1+\frac{\beta^2}{2})} \sum_{u \in Z_r} e^{\frac{\beta^2}{2} r}\right] = e^{-r}\, \mathbb{E}[Z_r]$
$\mathbb{E}[Z_r]$
$\mathbb{P}(x \in Z_r)$
$\frac{r^{|x|}}{|x|!}$
$= e^r$
birth
death
0
$|x|$
r
x