

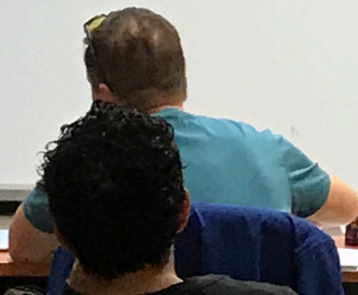
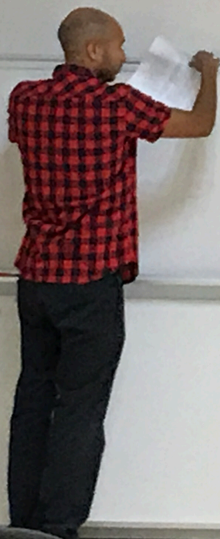
In the plane A domain D is an open connected subset of the plane

Gaussian Free Fields in D :
 h_D

generalized random function
 $\langle h_D(x), h_D(y) \rangle_f$

$$\langle h_D(x), h_D(y) \rangle \sim \mathcal{N}(0, \int_D f(x)f(y) G_D(x,y) dx dy)$$

$$G_D(x,y) = -\ln|x-y| + O(1)$$



In the plane A domain D is an open connected subset of the plane

Gaussian Free Fields in D : h_D

generalized random function $\langle h_D, f \rangle = \int_D f(x) h_D(x) dx$

$$\langle h_D, 1 \rangle \sim \mathcal{N}(0, \int_D f(x) g(y) G_D(z, y) dx dy)$$

$$G_D(x, y) = -\ln|x-y| + O(1)$$

Markov property



$$\text{Cov}(\langle h_D, f \rangle, \langle h_D, g \rangle) = \int_D f(x) g(y) G_D(z, y) dx dy$$

In the plane A domain D is an open connected subset of the plane

Gaussian Free Field in D :

h_D

generalized random function
 $\left(\int_D \varphi(x) \Delta h_D(x) \right)_{\varphi \in C_c^\infty(D)}$

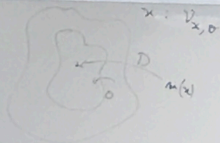
$$\langle h_D, \varphi \rangle \sim \mathcal{N}(0, \int_D \varphi(x) \Delta \varphi(x) dx)$$

$$G_D(x, y) = -\ln|x-y| + O(1)$$

domain Markov property

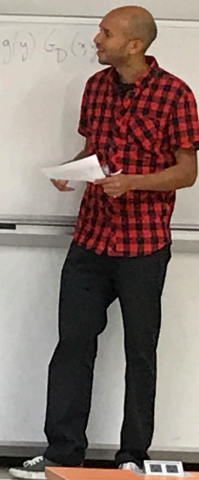
\mathcal{D} an open set non-random

GFF in \mathcal{D} ?

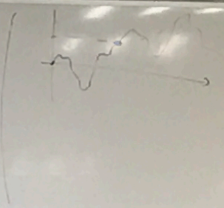
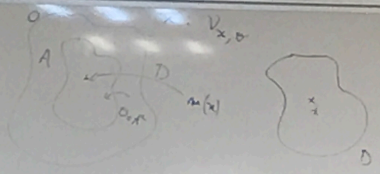


$$\text{Cov}(\langle h_D, \varphi \rangle, \langle h_D, \psi \rangle) = \int_D \varphi(x) \psi(x) \Delta G_D(x, x) dx$$

$$\int_D \varphi(x) \Delta \varphi(x) dx$$



domain Markov property
 Is an open set non-random
 GFF in \mathbb{D} ?



If it is true for a (random) set
 $A = \mathbb{O}^c$, then D is
 called a local set

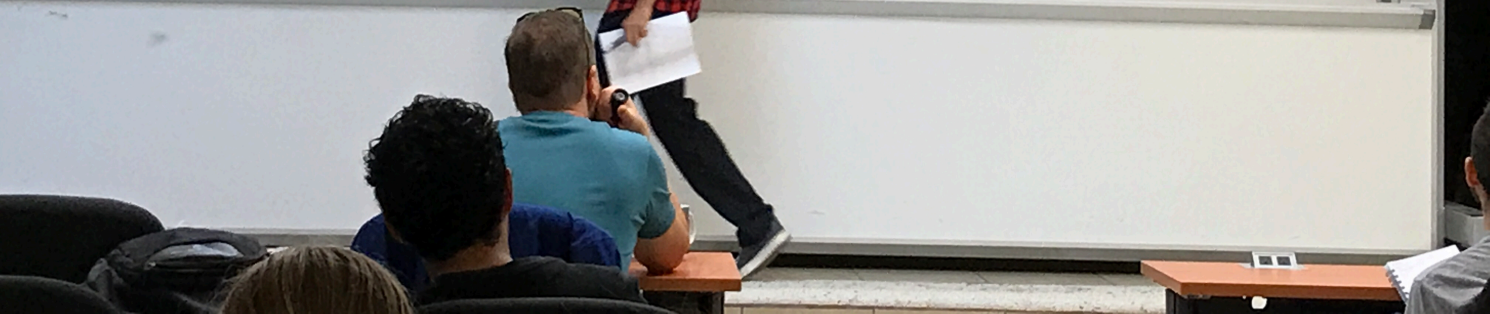
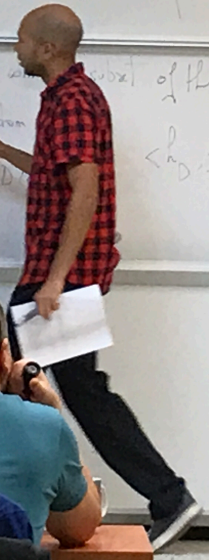
In the plane A domain D is an open subset of the plane

Gaussian Free Field in D :
 h_D

generalized random
 $\int_D \phi(x) \phi(y) dx dy$
 $\int_{\partial D} \phi(x) dx$

$$\langle h_D, \mathbf{1}_D \rangle \sim \mathcal{N}(0, \int_D \int_D \phi(x) \phi(y) G_D(x,y) dx dy)$$

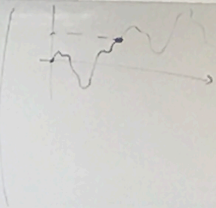
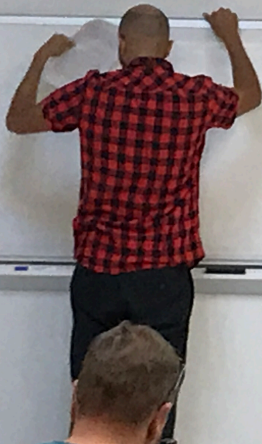
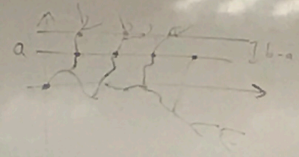
$$G_D(x,y) = -\ln|x-y| + O(1)$$



Aru, Lupa, Szukada First passage xt at level a
 It is a local set A
 $a - h_A$ width measure

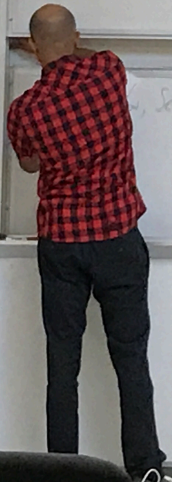


independent GFE
 with mean a
 $b > a$



If it is true for a (random) set
 $A = O^c$, then it is
 called a local set.

FPS_0 is conformally invariant (under conformal invariance of GFF)
 What does it mean
 $f: D \rightarrow D'$ conformal transformation
 D, D' simply connected
 if f is a diffeomorphism and holomorphic
 $f(z) = \frac{a(z-z_0)}{z-z_0}$
 $f(z+h) = f(z) + Oe^{i\theta}h + o(h)$



Sphereda First passage at level a
 t is a local set A
 $a-h$ purple measure




independent GFF
 with mean a
 $b > a$



FPS_a is conformal isomorphism between maximal domains of GFT

What does it mean
 $f: D_1 \rightarrow D_2$ conformal transformation

D_1, D_2 simply connected



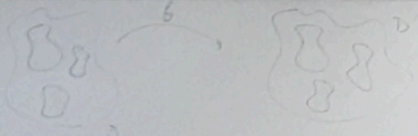
If f is a direction and holomorphic

$f'(z_0) \neq 0$

$f(z+h) = f(z) + f'(z)h + o(h)$

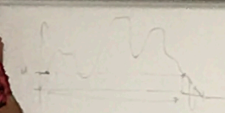
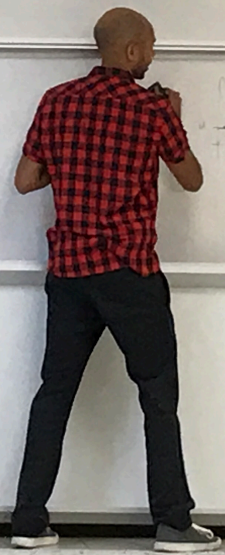
$f'(z) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$

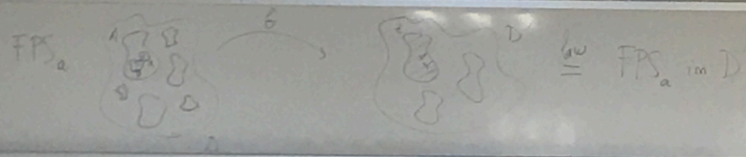
FPS_a



$\xrightarrow{f} D_2$

$\text{loc} = \text{FPS}_a \text{ in } D_2$





For Lebesgue a.e. $z \in D = \text{unit disc} = \mathbb{D}$
 $L_t(x)$ domain at level t which contains x
 $(L_0(x) = \text{unit disc})$

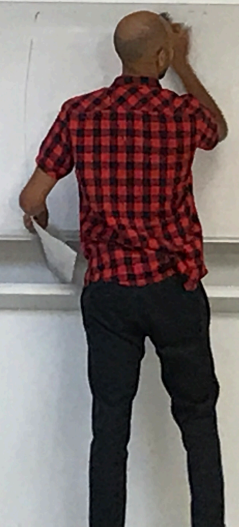
$X_t(x) = -\log CR(L_t(z), x)$
 radial index

Lemma: D simply connected bounded
 $CR(D, x)^2 \leq CR(D, x)$
 Let ψ : unit disc $\rightarrow D$
 conformal transformation
 $\psi(0) = x$
 $\psi'(0) > 0$

ψ is and holomorphic

$$\psi'(z_0) = \lim_{z \rightarrow z_0} \frac{\psi(z) - \psi(z_0)}{z - z_0}$$

$\sigma(\mathbb{R})$



$$\chi_g(x) = -\log \frac{CR(L_g(x), x)}{\text{infimal radius}}$$



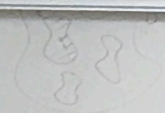
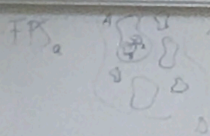
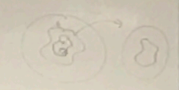
Invariant. To simplify consider bounded
 $x \in D$
 $CR(D, x)$
 Let ψ unit disc $\rightarrow D$
 conformal fixed-point
 $\psi(0) = x$
 $\psi'(0) > 0$

$$\chi_{\psi \circ f \circ \psi^{-1}}(x) = \chi_f(\psi^{-1}(x))$$

$$CR(D, x) = CR(\psi(D), \psi(x))$$

Exerc.
 1) Let $\psi: U \rightarrow L_0(t)$ such that
 Show that $U = \bigcup_{s \in \mathbb{R}} L_0(t+s) \perp L_0(t)$
 $(L_0(t), \text{rot})$

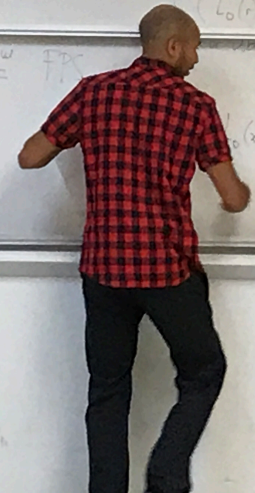
where $t \in D \subset D$
 $CR(D, x) \leq CR(D, x)$
 $\psi(0) = 0$
 $\psi'(0) > 0$
 $CR(L_0(t), 0)$



low = FR

$L_0(x)$ domain: a line t which contains x

$L_0(x) = \text{unit disc}$



$$X_2(x) = -\log CR(L_1(x), x)$$



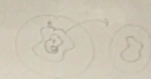
$CR(D, x)$ is simply connected bounded
 $x \in D$
 Let ψ unit disc $\rightarrow D$
 conformal transformation,
 $\psi(0) = x$
 $\psi'(0) > 0$

$$\psi^{-1}(CR(D, x)) = \psi^{-1}(0)$$

$$CR(D, x) \in CR(D, 0)$$

$\psi^{-1}(0) \in CR(D, x)$
 Let $\psi_t: U \rightarrow L_0(t)$ such that
 Show that $\psi_t^{-1}(L_0(t)) \subset L_0(t)$
 $(L_0(t), x)$

$CR(D, 0) \in CR(D, x)$
 $CR(D, x) \subset CR(D, 0)$
 $\psi(0) = 0$
 $\psi'(0) > 0$
 $CR(L_0(t), 0)$



- 2) Show that $\psi_t^{-1}(0) \in L_0(t) \iff (L_0(t), x)$
- 3) Show that $(L_0(t), x)$ is a subdomain. Use $\psi_t^{-1}(0) = \psi_t^{-1}(\psi_t^{-1}(0))$

For arbitrary $a \in D =$ unit disc $= U$
 $L_0(x)$ domain at level t which contains x
 $L_0(x) =$ unit disc

