In the plane A domain D is an open connected subset of the plane

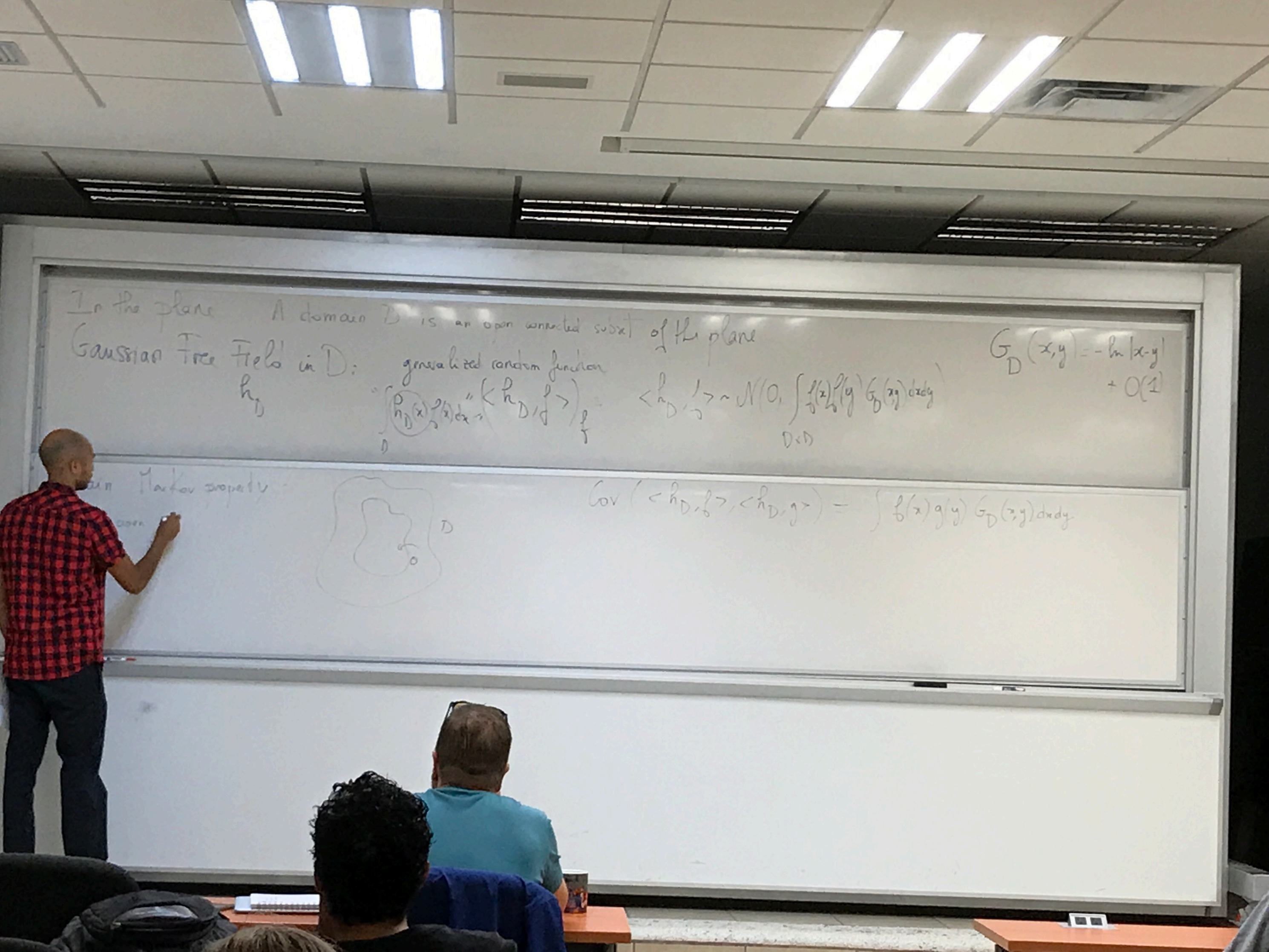
Gaussian Free Field in D: generalized random function

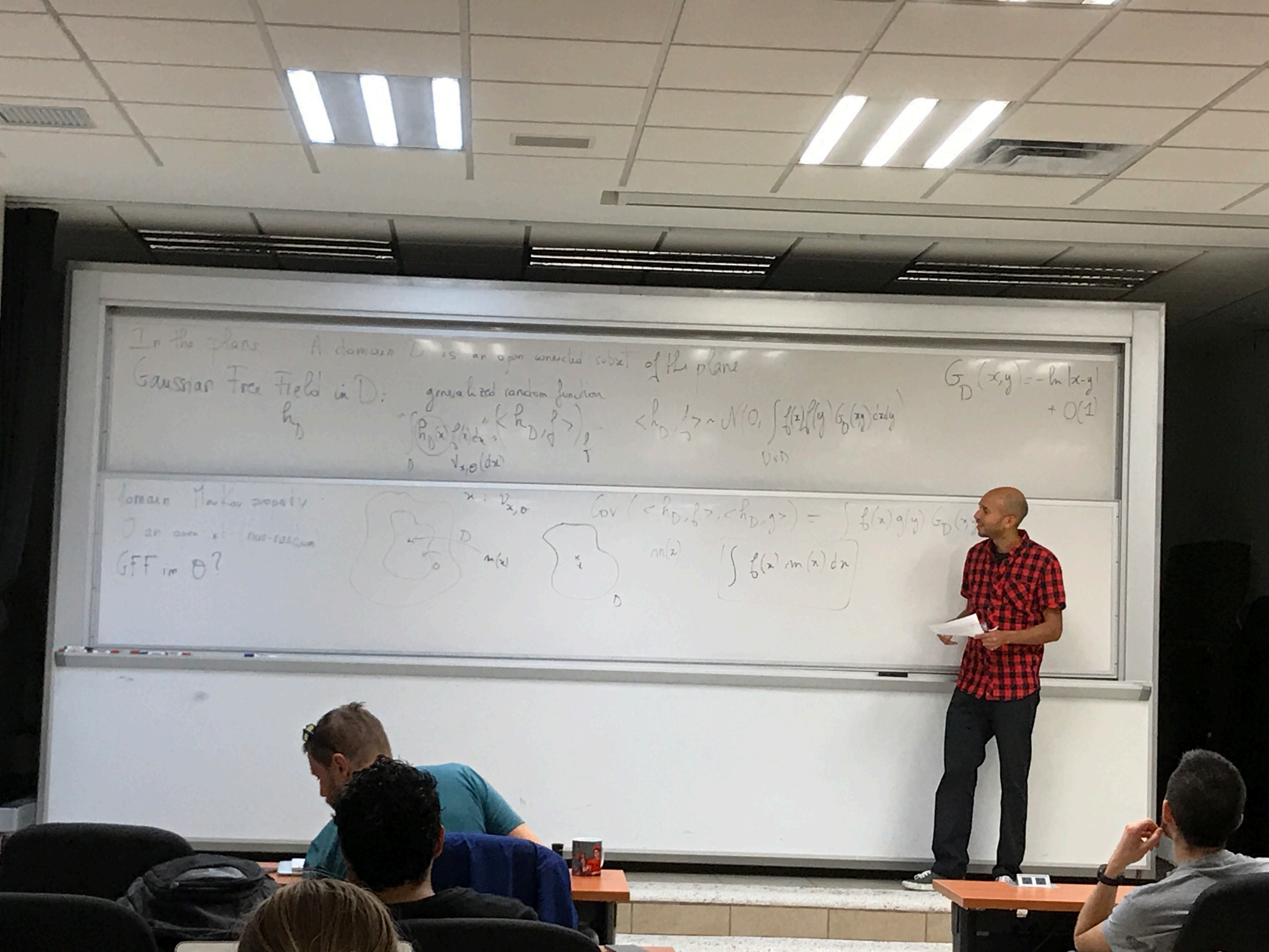
h_D

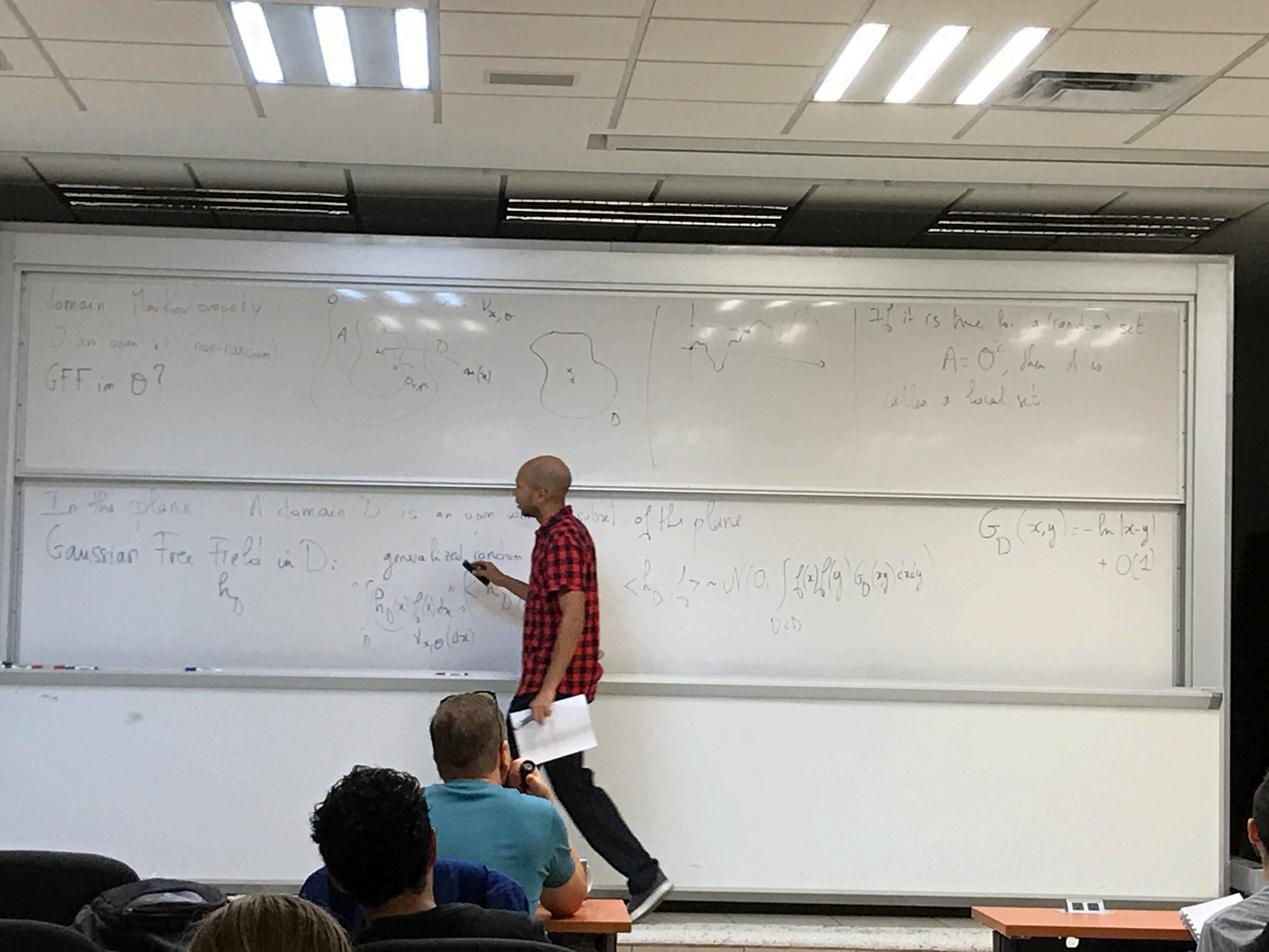
$$\text{"}\int_D h_D(x) f(x)\,dx\text{"} = \left(\langle h_D, f \rangle \right)_f$$

$$\langle h_D, f \rangle \sim \mathcal{N}\left(0, \int_{D \times D} f(x) f(y)\, G_D(x,y)\, dx\, dy\right)$$

$$G_D(x,y) = -\ln|x-y| + O(1)$$


In the plane
A domain D is an open connected subset of the plane
Gaussian Free Field in D: generalized random function
h_D
"∫_D h_D(x) f(x) dx" (<h_D, f>)_f
<h_D, f> ~ N(0, ∫_{D×D} f(x) f(y) G_D(x,y) dx dy)
G_D(x,y) = - ln|x-y| + O(1)
Markov property
Cov(<h_D, f>, <h_D, g>) = ∫ f(x) g(y) G_D(x,y) dx dy
D

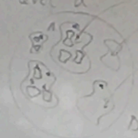

In the plane
A domain D is an open connected subset of the plane
Gaussian Free Field in D:
generalized random function
h_D
$G_D(x,y) = -\ln|x-y| + O(1)$
domain Markov property
GFF in O?
$\mathrm{Cov}(\langle h_D, f\rangle, \langle h_D, g\rangle) = \int f(x)g(y)G_D(x,y)$
$\int f(x)\, m(x)\, dx$


Domain Markov property
GFF in O?
If it is true for a 'random' set
A = O^c, then A is
called a local set
In the plane
A domain D is an open subset of the plane
Gaussian Free Field in D:
generalized random
G_D(x,y) = - ln |x-y| + O(1)
D×D

Aru, Lupu, Sepulveda   First passage set at level a

it is a local set A

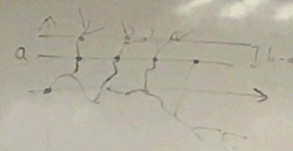
$a - h_A$ positive measure

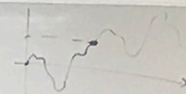


independent GFF

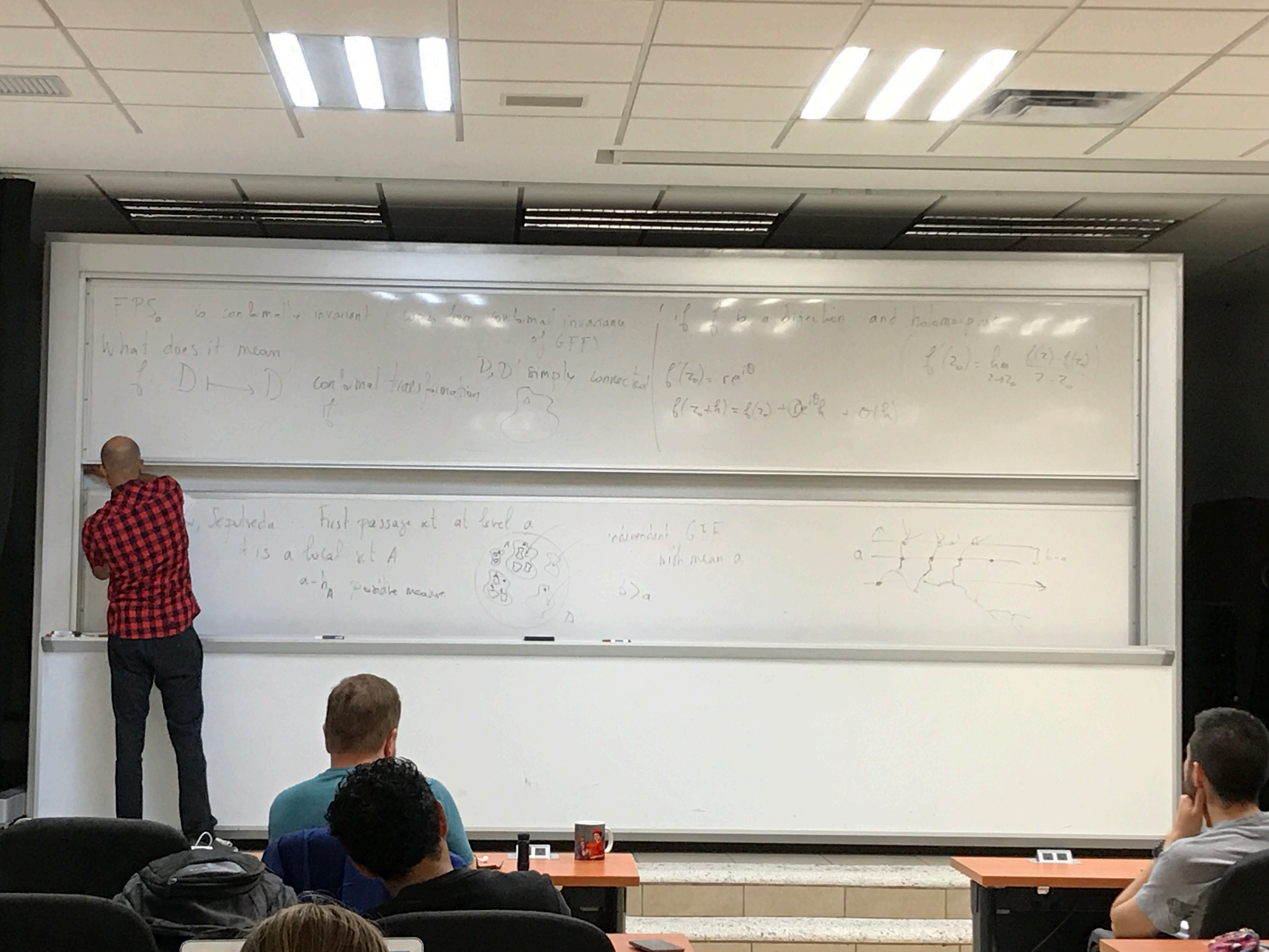
with mean a

$b > a$

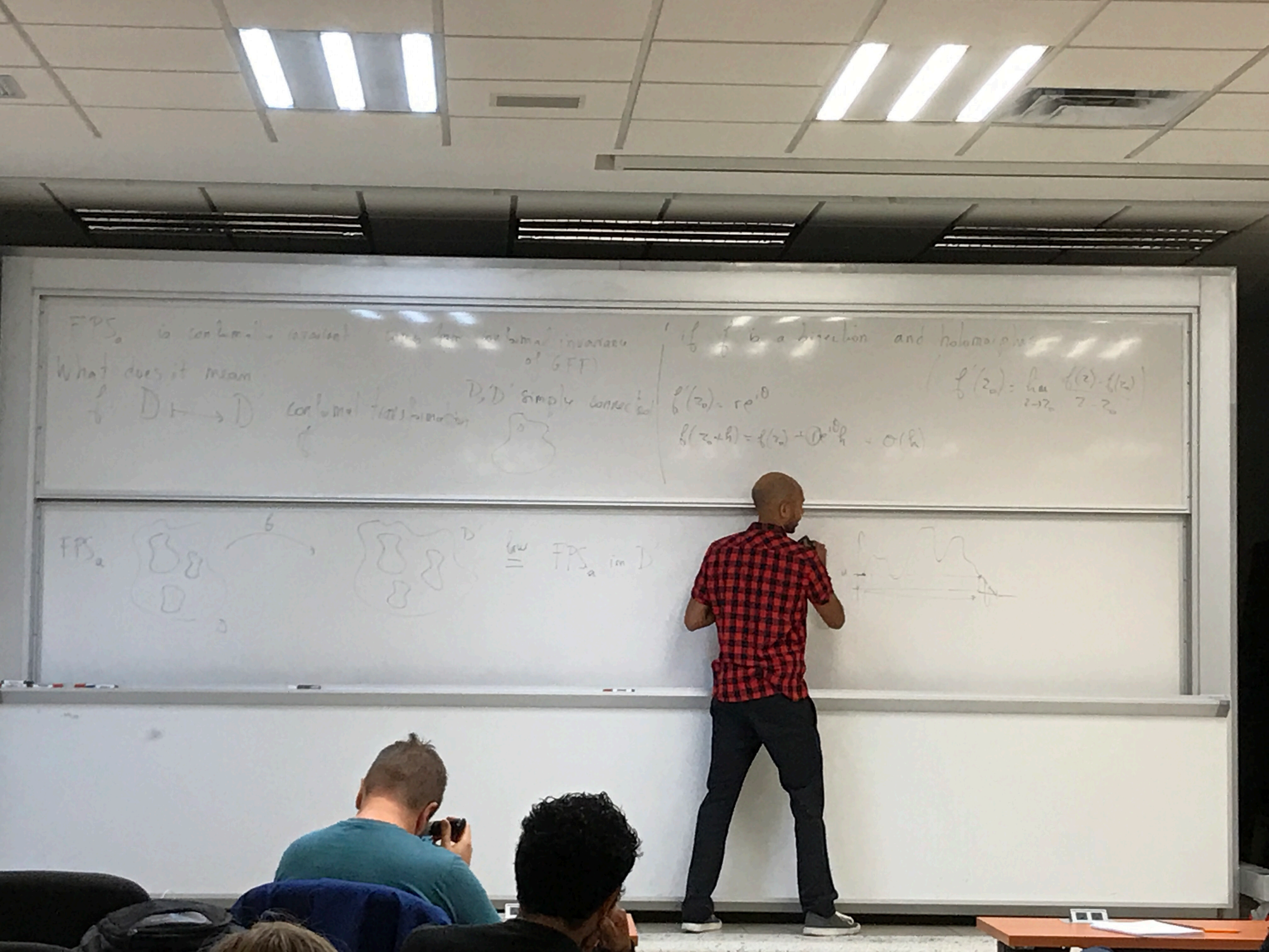




If it is true for a (random) set $A = O^c$, then A is called a local set.



FPS$_a$ is conformally invariant (comes from conformal invariance of GFF)
What does it mean
$f: D \longmapsto D'$ conformal transformation
D, D' simply connected
if f is a bijection and holomorphic
$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$
$f'(z_0) = re^{i\theta}$
$f(z_0 + h) = f(z_0) + re^{i\theta}h + o(h)$
Sepulveda First passage set at level a
it is a local set A
$a - h_A$ positive measure
D
independent GFF
with mean a
$b > a$
a



FPS_a is conformally invariant
of GFF)
What does it mean
D, D' simply connected
f : D ⟼ D' conformal transformation
if f is a bijection and holomorphic
f'(z_0) = re^{iθ}
f'(z_0) = lim_{z→z_0} (f(z) - f(z_0))/(z - z_0)
f(z_0 + h) = f(z_0) + re^{iθ}h + o(h)
FPS_a
f
D
law
=
FPS_a in D'


FPS_a
law =
FPS_a in D
For Lebesgue a.e. x ∈ D = unit disc = 𝔻
L_t(x) domain at level t which contains x
(L_0(x) = unit disc)
X_t(x) = - log CR(L_t(x), x)
conformal radius
Definition: D simply connected bounded
x ∈ D
CR(D, x)?
Let Ψ: unit disc → D
conformal transformation
Ψ(0) = x
Ψ'(0) > 0
and holomorphic
f'(z_0) = lim_{z→z_0} (f(z) - f(z_0))/(z - z_0)
o(h)

$X_t(x) = -\log \underbrace{CR(L_t(x), x)}_{\text{conformal radius}}$
D simply connected bounded
$x \in D$
$CR(D,x)$?
Let Ψ unit disc $\to D$
conformal transformation
$\Psi(0) = x$
$\Psi'(0) \geq 0$
Then $CR(D,x) = \Psi'(0)$
$CR(D,x) \leq CR(D',x)$
Exercise:
1) Let Ψ_t $U \to L_0(t)$ such that
$\Psi_t(0) = 0$
$\Psi_t'(0) > 0$
$CR(L_0(t), 0)$
Show that $\Psi_t^{-1}(L_0(t+s)) \perp L_0(t)$
$(L_0(r), r \leq t)$
FPS_a
$\overset{law}{=}$ FPS
$L_t(x)$ domain at level t which contains x
$L_0(x)$ = unit disc)

$X_t(x) = -\log \underbrace{CR(L_t(x), x)}$
conformal radius
D simply connected
bounded
$x \in D$
$CR(D,x)$?
Let ψ unit disc $\to D$
conformal transformation,
$\psi(0) = x$
$\psi'(0) > 0$
Then $CR(D,x) = \psi'(0)$
$CR(D,x) \leq CR(D',x)$
Exercise
1) Let ψ_t $U \to L_0(t)$ such that
$\psi_t(0) = 0$
$\psi_t'(0) > 0$
$CR(L_0(t), 0)$
Show that $\psi_t^{-1}(L_0(t+s)) \perp\!\!\!\perp L_0(t)$
$(L_0(r), r \leq t)$
2) Show that $\psi_t^{-1} \circ \psi_{t+s} \perp\!\!\!\perp (L_0(r), r \leq t)$
and $\psi_t^{-1} \circ \psi_{t+s}$ law $= \psi_s$
3) Show that $(X_0(t))_{t \geq 0}$ is a subordinator. Use $\psi'_{t+s}(0) = \psi'_t(0)(\psi_t^{-1} \circ \psi_{t+s})'(0)$
For Lebesgue a.e. $x \in D$ = unit disc = U
$L_t(x)$ domain at level t which contains x
$(L_0(x)$ = unit disc$)$