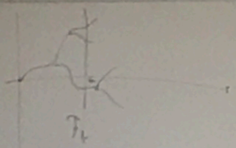


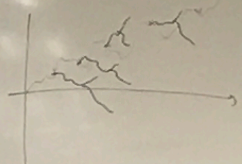

$\beta$

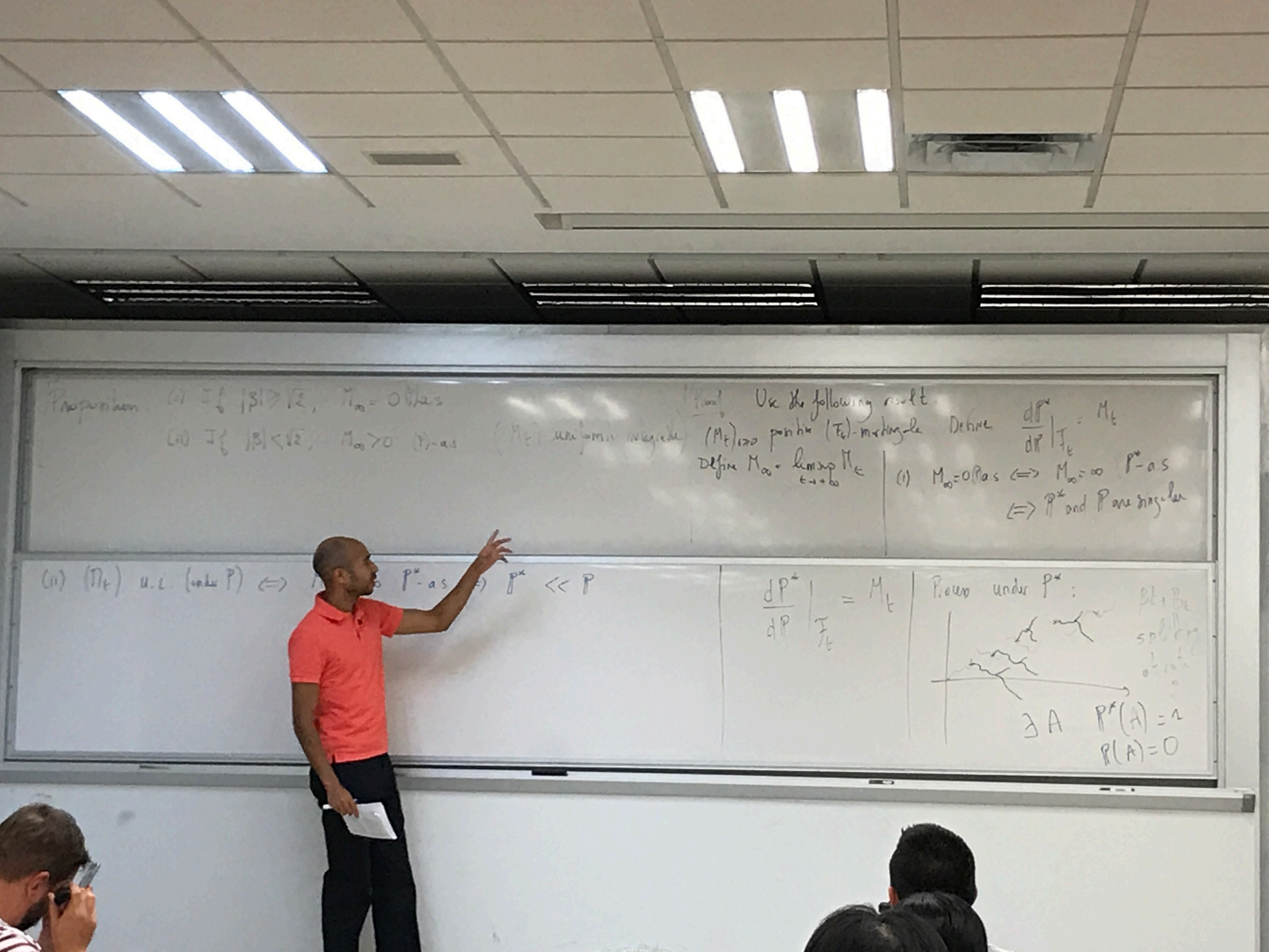
$$M_t = e^{-t\left(1+\frac{\beta^2}{2}\right)} \sum_{u \in Z_t} e^{\beta X_u(t)}$$

$$M_t \xrightarrow[t \to +\infty]{a.s.} M_\infty$$

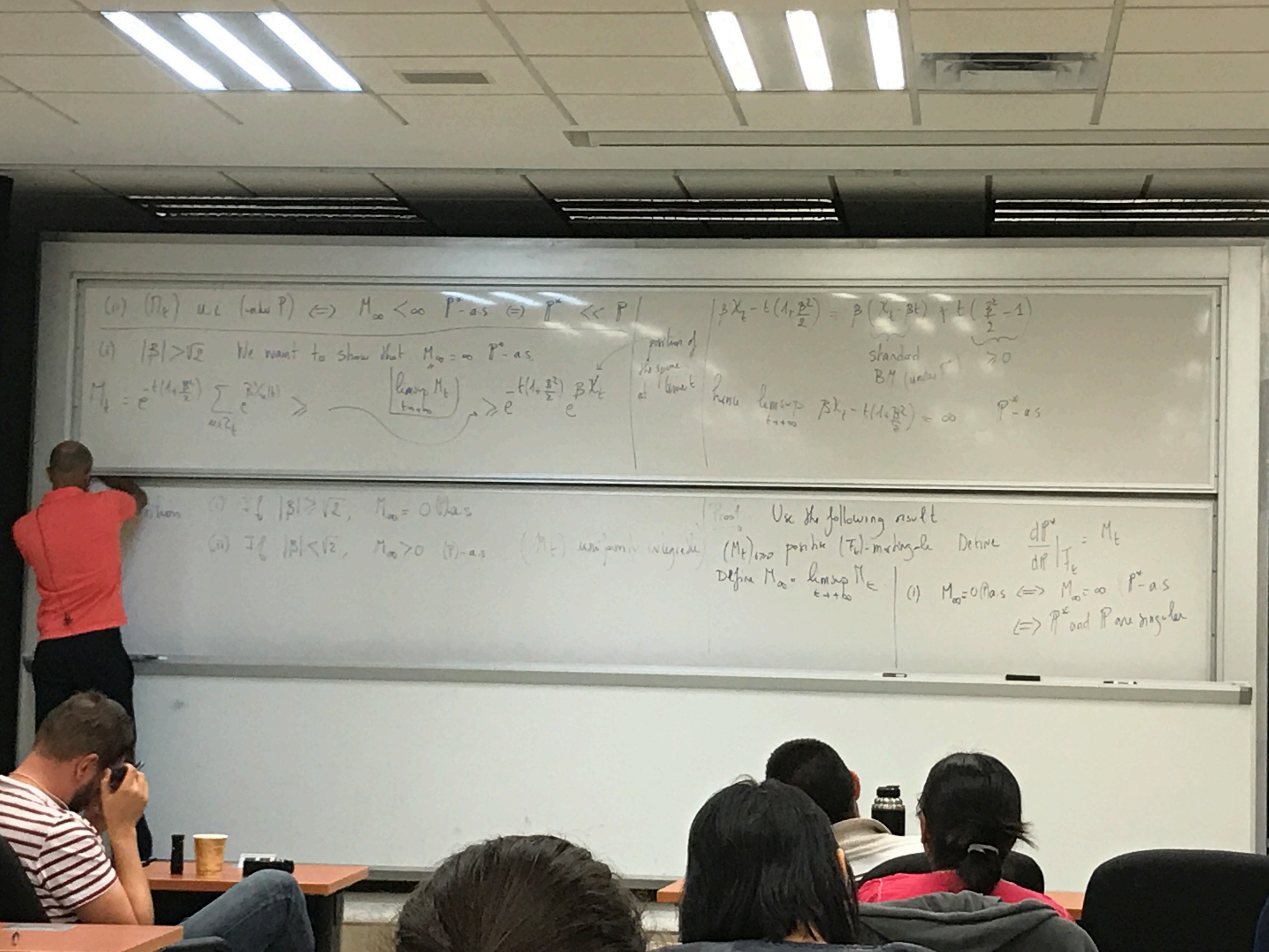
$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = M_t$$

Pows under $P^*$:

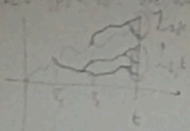




Proposition (i) If |β| ≥ √2, M∞ = 0 P-a.s
(ii) If |β| < √2, M∞ > 0 P-a.s
(Mt) uniformly integrable
Proof: Use the following result
(Mt)t≥0 positive (Ft)-martingale
Define M∞ = limsup Mt
t→+∞
Define dP*/dP |Ft = Mt
(i) M∞ = 0 P-a.s ⟺ M∞ = ∞ P*-a.s
⟺ P* and P are singular
(ii) (Mt) u.i. (under P) ⟺
P*-a.s
P* << P
dP*/dP |Ft = Mt
Proof under P*:
∃ A P*(A) = 1
P(A) = 0


(ii) $(M_t)$ u.i. (under $\mathbb{P}$) $\Leftrightarrow$ $M_\infty < \infty$ $\mathbb{P}^*$-a.s $\Leftrightarrow$ $\mathbb{P}^* \ll \mathbb{P}$
(i) $|\beta| > \sqrt{2}$ We want to show that $M_\infty = \infty$ $\mathbb{P}^*$-a.s
$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)} \geq$ $\limsup_{t\to+\infty} M_t$ $\geq e^{-t(1+\frac{\beta^2}{2})} e^{\beta X_t}$
position of the spine at time t
$\beta X_t - t(1+\frac{\beta^2}{2}) = \beta(X_t - \beta t) + t(\frac{\beta^2}{2} - 1)$
standard BM (under $\mathbb{P}^*$)
$\geq 0$
hence $\limsup_{t\to+\infty} \beta X_t - t(1+\frac{\beta^2}{2}) = \infty$ $\mathbb{P}^*$-a.s
(i) If $|\beta| \geq \sqrt{2}$, $M_\infty = 0$ $\mathbb{P}$-a.s
(ii) If $|\beta| < \sqrt{2}$, $M_\infty > 0$ ($\mathbb{P}$)-a.s $(M_t)$ uniformly integrable
Proof: Use the following result
$(M_t)_{t\geq 0}$ positive $(\mathcal{F}_t)$-martingale Define $\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M_t$
Define $M_\infty = \limsup_{t\to+\infty} M_t$
(i) $M_\infty = 0$ $\mathbb{P}$-a.s $\Leftrightarrow$ $M_\infty = \infty$ $\mathbb{P}^*$-a.s
$\Leftrightarrow$ $\mathbb{P}^*$ and $\mathbb{P}$ are singular

(ii) Let $|\beta| < \sqrt{2}$. We want to show that $M_\infty < \infty$ $\tilde{\mathbb{P}}$-a.s.




$(\tau_1, \tau_2, \dots)$ : birth times along the spine

$Z_{k,t}$ = { particles alive at time $t$ which separated from the spine at time $\tau_k$ }

$$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)}$$

$$\sum_{u \in Z_t} e^{\beta X_u(t)} \longrightarrow e^{\beta X_t} + \sum_{k : \tau_k < t} \sum_{u \in Z_{k,t}} e^{\beta X_u}$$

$\tilde{\mathbb{E}}\left[ \sum_{u \in Z_{k,t}} e^{\beta X_u(t)} \right|$

position of the spine at time $t$

$$\beta X_t - t\left(1 + \frac{\beta^2}{2}\right) = \beta \underbrace{(X_t - \beta t)}_{\text{standard BM (under } \tilde{\mathbb{P}})} + t \underbrace{\left(\frac{\beta^2}{2} - 1\right)}_{\geq 0}$$

Hence $\limsup_{t \to +\infty} \beta X_t - t\left(1 + \frac{\beta^2}{2}\right) = \infty$ $\tilde{\mathbb{P}}$-a.s.

$$\mathbb{E}^*\Big[ \sum_{u \in Z_{k,t}} e^{\beta X_u(t)} \,\Big|\, (X_s)_{s \geq 0}, (\sigma_k)_{k \geq 1} \Big]$$

$$= e^{(t-\sigma_k)(1+\frac{\beta^2}{2})} e^{\beta X_{\sigma_k}}$$

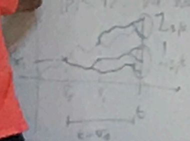
$$\mathbb{E}^*\big[ M_t \mid (X_s)_{s \geq 0}, (\sigma_k)_{k \geq 1} \big] = e^{-t(1+\frac{\beta^2}{2})} e^{\beta X_t} + \sum_{k : \sigma_k < t} e^{-\sigma_k(1+\frac{\beta^2}{2})} e^{\beta X_{\sigma_k}}$$

$$\Rightarrow \mathbb{E}^*\big[ \liminf_{t \to +\infty} M_t \mid \dots \big] \leq \liminf \quad < \infty \quad \mathbb{P}^*\text{-a.s}$$

$$\boxed{\beta X_s - s(1+\tfrac{\beta^2}{2}) = \beta(X_s - \beta s) - s\underbrace{(1 - \tfrac{\beta^2}{2})}_{>0}}$$

$$\Rightarrow \liminf_{t \to +\infty} M_t < \infty \quad \mathbb{P}^*\text{-a.s}$$

$|\beta| < \sqrt{2}$ We want to show that $M_\infty < \infty$ $\mathbb{P}^*$-a.s




$(\sigma_1, \sigma_2, \dots)$ birth times along the spine

$Z_{k,t}$ = { particles alive at time $t$ which separated from the spine at time $\sigma_k$ }

$$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)}$$

$$\sum_{u \in Z_t} \longrightarrow e^{\beta X_t} + \sum_{k : \sigma_k < t} \sum_{u \in Z_{k,t}} e^{\beta X_u}$$

$$\mathbb{E}^*\Big[ \sum_{u \in Z_{k,t}} e^{\beta X_u(t)} \,\Big|\, (X_s)_{s\geq 0}, (\sigma_k)_{k\geq 1} \Big]$$

$$= e^{(t-\sigma_k)(1+\frac{\beta^2}{2})} e^{\beta X_{\sigma_k}}$$

$$\mathbb{E}^*\big[ M_t \,\big|\, (X_s)_{s\geq 0}, (\sigma_k)_{k\geq 1} \big] = e^{-t(1+\frac{\beta^2}{2})} e^{\beta X_t} + \sum_{k : \sigma_k < t} e^{-\sigma_k(1+\frac{\beta^2}{2})} e^{\beta X_{\sigma_k}}$$

$$\Rightarrow \mathbb{E}^*\big[ \liminf_{t\to+\infty} M_t \,\big|\, \dots \big] \leq \liminf \qquad < \infty \quad \mathbb{P}^*\text{-a.s}$$

$$\boxed{\beta X_s - s(1+\tfrac{\beta^2}{2}) = \beta(X_s - \beta s) - s\underbrace{(1-\tfrac{\beta^2}{2})}_{>0}}$$

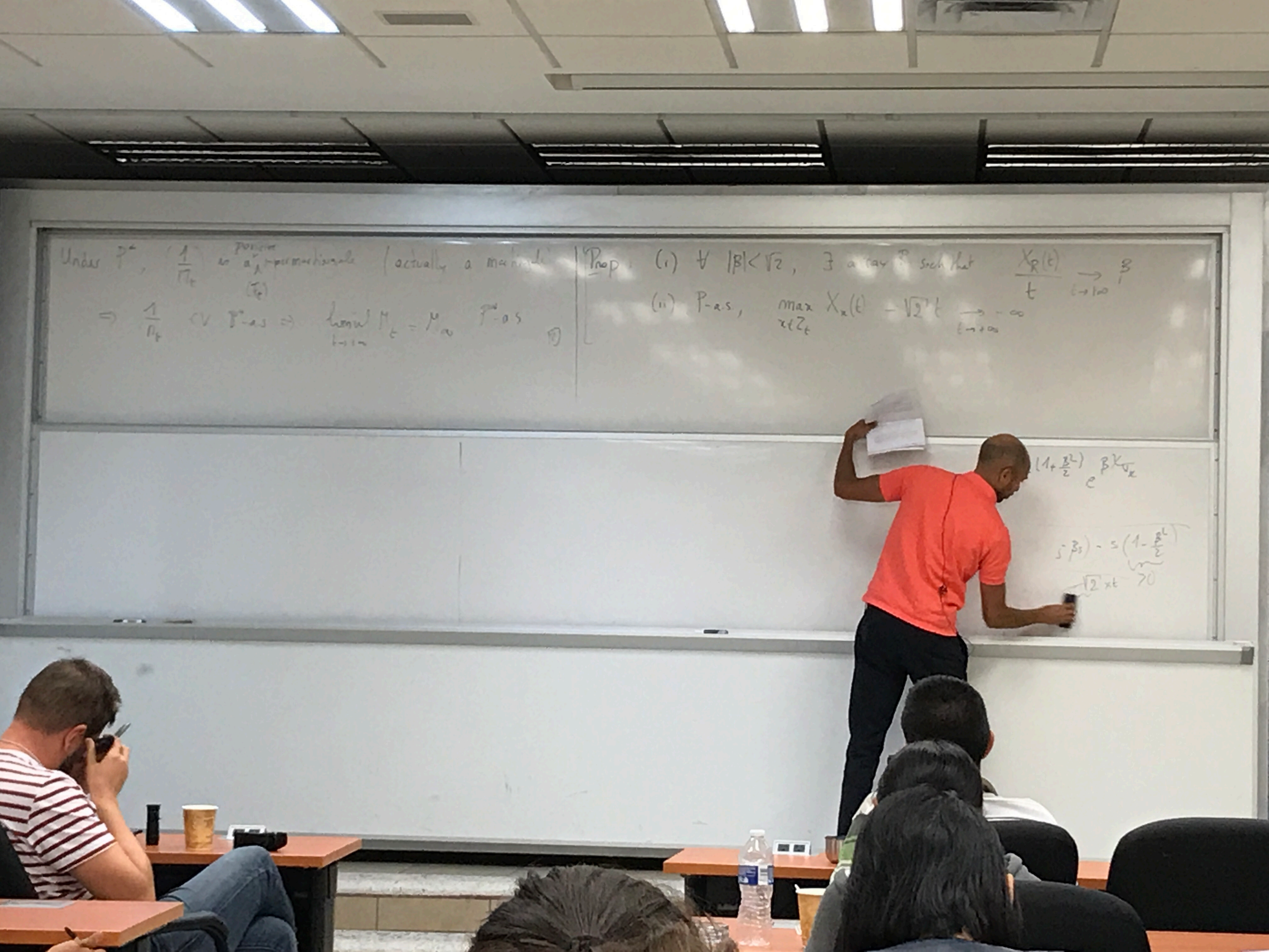
$$\Rightarrow \liminf_{t\to+\infty} M_t < \infty \quad \mathbb{P}^*\text{-a.s}$$

Under $\mathbb{P}^*$, $\left(\frac{1}{\Pi_t}\right)$ is a positive supermartingale $(\mathcal{F}_t)$ (actually a martingale)

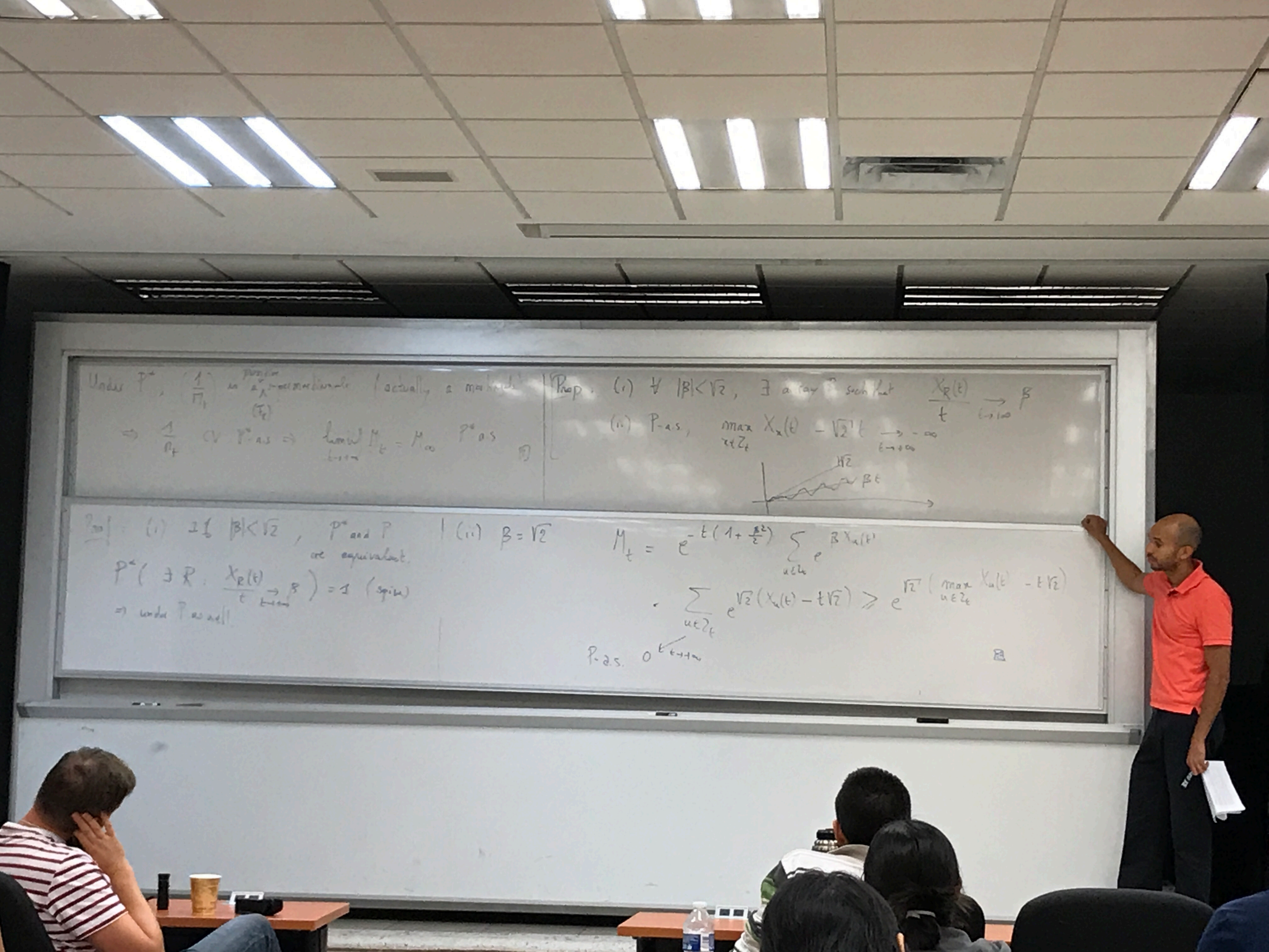
$$\Rightarrow \frac{1}{\Pi_t} \text{ CV } \mathbb{P}^*\text{-a.s} \Rightarrow \liminf_{t\to+\infty} M_t = M_\infty \quad \mathbb{P}^*\text{-a.s} \qquad \square$$

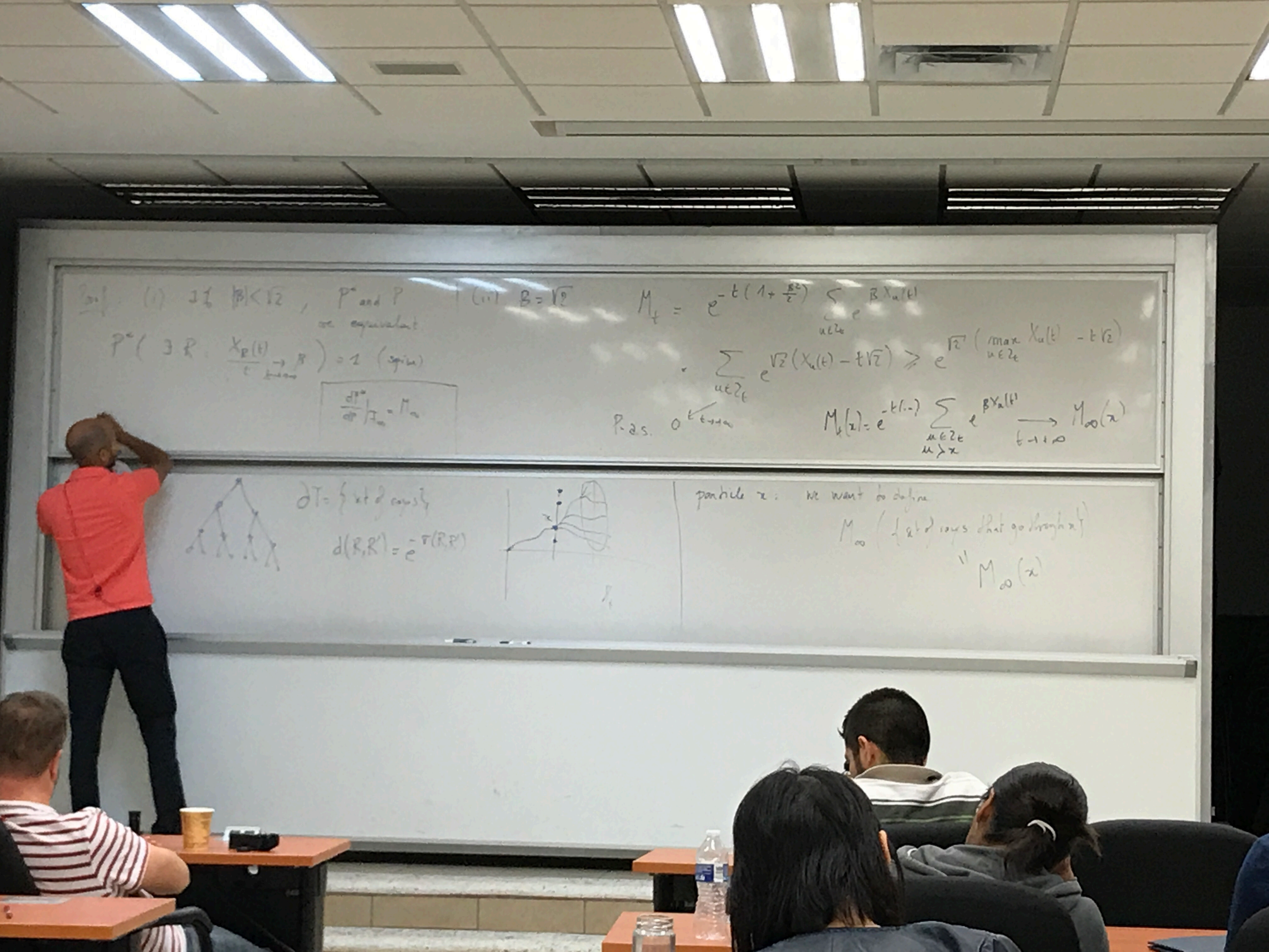
$$M_t = e^{-t(1+\frac{\beta^2}{2})} \underbrace{\sum_{u \in Z_t} e^{\beta X_u(t)}}$$

$$\longrightarrow e^{\beta X_t} + \sum_{k : \sigma_k < t} \sum_{u \in Z_{k,t}} e^{\beta X_u}$$

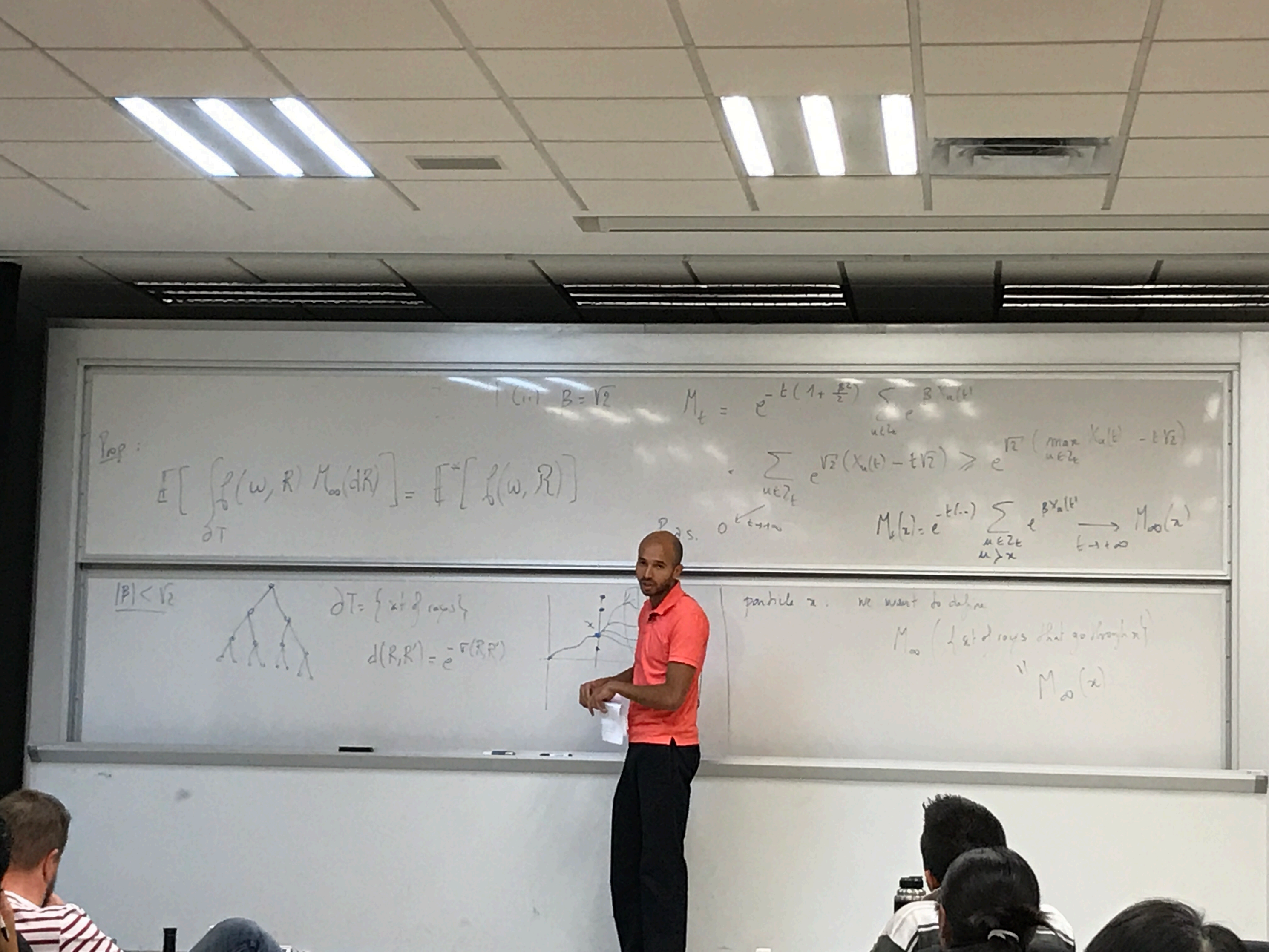


Under $\mathbb{P}^{\omega}$, $\left(\frac{1}{M_t}\right)$ is a positive supermartingale ($\mathcal{F}_t$) (actually a martingale)
$\Rightarrow \frac{1}{M_t}$ CV $\mathbb{P}^{\omega}$-a.s $\Rightarrow \liminf_{t\to+\infty} M_t = M_\infty$ $\mathbb{P}^{\omega}$-a.s
Prop: (i) $\forall |\beta| < \sqrt{2}$, $\exists$ a ray $\xi$ such that $\frac{X_\xi(t)}{t} \xrightarrow[t\to+\infty]{} \beta$
(ii) $\mathbb{P}$-a.s, $\max_{x \in Z_t} X_x(t) - \sqrt{2}\,t \xrightarrow[t\to+\infty]{} -\infty$


Under $P^*$, $\left(\frac{1}{M_t}\right)$ is a positive supermartingale (actually a martingale) $(\mathcal{F}_t)$
$\Rightarrow \frac{1}{M_t}$ CV $P^*$-a.s. $\Rightarrow \liminf_{t\to+\infty} M_t = M_\infty$ $P^*$ a.s.
Prop: (i) $\forall |\beta|<\sqrt{2}$, $\exists$ a ray $R$ such that $\frac{X_R(t)}{t} \xrightarrow[t\to+\infty]{} \beta$
(ii) P-a.s., $\max_{x\in Z_t} X_x(t) - \sqrt{2}\,t \xrightarrow[t\to+\infty]{} -\infty$
$\sqrt{2}$
$\beta t$
Proof: (i) If $|\beta|<\sqrt{2}$, $P^*$ and $P$ are equivalent.
$P^*\left(\exists R : \frac{X_R(t)}{t} \xrightarrow[t\to+\infty]{} \beta\right) = 1$ (spine)
$\Rightarrow$ under P as well
(ii) $\beta = \sqrt{2}$
$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u\in Z_t} e^{\beta X_u(t)}$
$\sum_{u\in Z_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u\in Z_t} X_u(t) - t\sqrt{2})}$
P-a.s. $0 \xleftarrow[t\to+\infty]{}$



(ii) $B = \sqrt{2}$
$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)}$
$\sum_{u \in Z_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \in Z_t} X_u(t) - t\sqrt{2})}$
P-a.s.
$M_t(x) = e^{-t(\ldots)} \sum_{u \in Z_t, u > x} e^{\beta X_u(t)} \xrightarrow[t \to +\infty]{} M_\infty(x)$
$d(R,R') = e^{-\tau(R,R')}$
particle x: we want to define
$M_\infty(x)$



(1.1) $\beta = \sqrt{2}$
$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)}$
Prop:
$\mathbb{E}\left[ \int_{\partial T} f(\omega, R)\, M_\infty(dR) \right] = \mathbb{E}^*\left[ f(\omega, R) \right]$
$\sum_{u \in Z_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \in Z_t} X_u(t) - t\sqrt{2})}$
P a.s. $0 \xleftarrow[t \to +\infty]{}$
$M_t(x) = e^{-t(\dots)} \sum_{u \in Z_t, u > x} e^{\beta X_u(t)} \xrightarrow[t \to +\infty]{} M_\infty(x)$
$|\beta| < \sqrt{2}$
$\partial T = \{\text{set of rays}\}$
$d(R, R') = e^{-\tau(R, R')}$
$x$
particle $x$. we want to define
$M_\infty(\{\text{set of rays that go through } x\})$
$\shortparallel$
$M_\infty(x)$

(ii) $\beta = \sqrt{2}$

$$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{u \in Z_t} e^{\beta X_u(t)}$$

Prop :

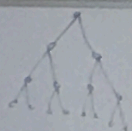
$$\mathbb{E}\left[ \int_{\partial T} f(\omega, R)\, M_\infty(dR) \right] = \mathbb{E}^*\left[ f(\omega, R) \right]$$

$$\sum_{u \in Z_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}\left(\max_{u \in Z_t} X_u(t) - t\sqrt{2}\right)}$$

P-a.s. $0 \xleftarrow{t \to +\infty}$

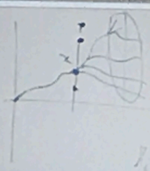
$$M_t(x) = e^{-t(\dots)} \sum_{\substack{u \in Z_t \\ u \succ x}} e^{\beta X_u(t)} \xrightarrow[t \to +\infty]{} M_\infty(x)$$

$\sqrt{2}$



$\partial T = \{$set of rays$\}$

$d(R, R') = e^{-\tau(R,R')}$



particle $x$ : we want to define

$M_\infty(\{$set of rays that go through $x\})$

" $M_\infty(x)$