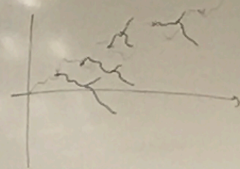


$$\beta \quad M_t = e^{-t(1 + \frac{r}{T})} \sum_{u \in Z_t} e^{\beta X_u(t)}$$

$$M_t \xrightarrow[t \rightarrow \infty]{ac} M_\infty$$

$$\frac{dP^+}{dP} \Big|_{T/T_0} = M_t$$

Power under P^+ :



Bei P^+
solche
Lösung

$\mu =$

Proposition. (i) $\int_0^1 |\beta| > \int_0^1 \beta$, $M_\infty = 0$ has

(ii) $\int_0^1 |\beta| < \int_0^1 \beta$, $M_\infty > 0$ P-as

(M_t) uniformly integrable

Proof. Use the following result

(M_t) is positive (\mathcal{F}_t)-martingale

Define $M_\infty = \limsup_{t \rightarrow \infty} M_t$

Define $\frac{dP^*}{dP} \Big|_{\mathcal{F}_t} = M_t$

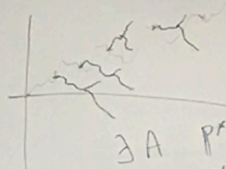
(i) $M_\infty = 0$ P-as $\Leftrightarrow M_\infty = \infty$ P*-as

$\Leftrightarrow P^*$ and P are singular

(ii) (M_t) u.i. (under P) $\Leftrightarrow M_\infty > 0$ P*-as $\Rightarrow P^* \ll P$

$$\frac{dP^*}{dP} \Big|_{\mathcal{F}_t} = M_t$$

Process under P^* :



stochastic solution

$\exists A$ $P^*(A) = 1$
 $P(A) = 0$

(i) (M_t) u.L. (under P) $\Leftrightarrow M_\infty < \infty$ P -as $\Leftrightarrow P^* \ll P$

(ii) $|\beta| > \sqrt{2}$ We want to show that $M_\infty = \infty$ P -as

$$M_t = e^{-t(1+\frac{\beta^2}{2})} \sum_{n \leq t} \beta^n X_n \rightarrow \limsup_{t \rightarrow \infty} M_t \rightarrow e^{-t(1+\frac{\beta^2}{2})} \beta^n X_t$$

$$\beta X_t - t(1+\frac{\beta^2}{2}) = \beta(X_t - \beta t) + t(\frac{\beta^2}{2} - 1)$$

Standard BM (under P^*) > 0

lim sup $\rightarrow \beta X_t - t(1+\frac{\beta^2}{2}) = \infty$ P^* -as

position of the spine at time t



(i) If $|\beta| > \sqrt{2}$, $M_\infty = 0$ P -as

(ii) If $|\beta| < \sqrt{2}$, $M_\infty > 0$ P -as (M_t uniformly integrable)

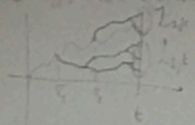
Proof: Use the following result.

$(M_t)_{t \geq 0}$ positive (F_t) -martingale
 Define $M_\infty = \limsup_{t \rightarrow \infty} M_t$

$$\frac{dP^*}{dP} \Big|_{F_t} = M_t$$

(i) $M_\infty = 0$ P -as $\Leftrightarrow M_\infty = \infty$ P^* -as
 $\Leftrightarrow P^*$ and P are singular.

(1) Let $\beta < \frac{1}{2}$. We want to show that $M_{\infty} = P=0$ s



(τ_1, τ_2, \dots) : birth times along the spine
 $\mathcal{L}_{k,t}$: { particles alive at time t which descend from the spine at time τ_k }

$$M_t = e^{-t(1+\frac{\beta t}{2})} \sum_{u \in \mathcal{L}_{k,t}} e^{\beta X_u}$$

$$\underbrace{\sum_{u \in \mathcal{L}_{k,t}} e^{\beta X_u}}_{\beta X_{\tau_k}} = e^{\beta X_{\tau_k}} + \sum_{k: \tau_k < t} \sum_{u \in \mathcal{L}_{k,t}} e^{\beta X_u}$$

$$E \left[\sum_{u \in \mathcal{L}_{k,t}} e^{\beta X_u} \right]$$

position of the spine at τ_k is X_{τ_k}

$$\beta X_{\tau_k} - t(1+\frac{\beta t}{2}) = \beta(X_{\tau_k} - \beta t) + t(\frac{\beta^2}{2} - 1)$$

standard BM (under \mathbb{P}) ≥ 0

now $\limsup_{t \rightarrow \infty} \beta X_t - t(1+\frac{\beta t}{2}) = -\infty$ $P=0$ s

$$\int \prod_{k \in \mathbb{Z}} e^{\beta X_k(t)} \Big| (X_{t=0}, \{v_k\}_{k \in \mathbb{Z}}) \Big|$$

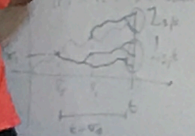
$$= e^{-t \sum_{k \in \mathbb{Z}} (1 + \frac{\beta^2}{2}) \beta X_k} e^{\beta X_{t=0}}$$

$$\langle M_t | (X_{t=0}, \{v_k\}_{k \in \mathbb{Z}}) \rangle = e^{-t(1 + \frac{\beta^2}{2}) \beta X_{t=0}} + \sum_{k \in \mathbb{Z}} e^{-v_k(1 + \frac{\beta^2}{2})} e^{\beta X_{v_k}}$$

$\Rightarrow \mathbb{E} [\lim_{t \rightarrow \infty} M_t | \dots] \leq \lim_{t \rightarrow \infty} M_t$
 $\Rightarrow \lim_{t \rightarrow \infty} M_t < \infty$ P-a.s.

$$\beta X_{t=0} - v(1 + \frac{\beta^2}{2}) = \beta(X_{t=0} - v) - v(1 - \frac{\beta^2}{2})$$

$|\beta| < 1$ We want to show that $M_t < \infty$ P-a.s.



(v_k, v_l, \dots) - birth times along the spine
 $\{z_k, t\}$ - particles alive at time t which originated from the spine at time v_k

$$M_t = e^{-t(1 + \frac{\beta^2}{2})} \sum_{k \in \mathbb{Z}} e^{\beta X_k(t)}$$

$$= e^{-t(1 + \frac{\beta^2}{2})} + \sum_{k \in \mathbb{Z}} \sum_{v_k < t} e^{\beta X_{v_k}}$$

$$\begin{aligned} & \mathbb{E}^* \left[\sum_{u \in \mathcal{Z}_t^+} e^{\beta X_u(t)} \mid (X_s)_{s \geq 0}, (\mathbb{Q}_t)_{t \geq 0} \right] \\ &= e^{-t - \alpha(1 + \frac{\beta^2}{2})t} e^{\beta X_t} \end{aligned}$$

$$\begin{aligned} & \mathbb{E}^* [M_t \mid (X_s)_{s \geq 0}, (\mathbb{Q}_t)_{t \geq 0}] = e^{-t(1 + \frac{\beta^2}{2})} e^{\beta X_t} + \sum_{h: \mathbb{Q}_h < t} e^{-\alpha h(1 + \frac{\beta^2}{2})} e^{\beta X_h} \\ & \Rightarrow \mathbb{E}^* \left[\liminf_{t \rightarrow \infty} M_t \mid \dots \right] \leq \liminf_{t \rightarrow \infty} M_t < \infty \text{ P-a.s.} \\ & \Rightarrow \liminf_{t \rightarrow \infty} M_t < \infty \text{ P-a.s.} \end{aligned}$$

$\beta X_s - \alpha(1 + \frac{\beta^2}{2})s = \beta(X_s - \beta s) - \alpha(1 - \frac{\beta^2}{2})s$

Under P^+ , $\frac{1}{\Pi_t}$ is ^{positive} a martingale (actually a martingale)
 $\Rightarrow \frac{1}{\Pi_t}$ (V P-a.s) $\Rightarrow \liminf_{t \rightarrow \infty} M_t = M_\infty$ P-a.s. \square

$$M_t = e^{-t(1 + \frac{\beta^2}{2})} \sum_{u \in \mathcal{Z}_t^+} e^{\beta X_u(t)} = e^{-t(1 + \frac{\beta^2}{2})} \left(e^{\beta X_t} + \sum_{h: \mathbb{Q}_h < t} \sum_{u \in \mathcal{Z}_h^+} e^{\beta X_u} \right)$$

Under \tilde{P} , $\frac{1}{\mu_t}$ is ^{positive} α_t -periodic (actually a martingale)
 $\Rightarrow \frac{1}{\mu_t}$ is P-a.s. $\Rightarrow \lim_{t \rightarrow \infty} \mu_t = \mu_\infty$ P-a.s. \square

Prop. (i) $\forall |\beta| < \sqrt{2}$, \exists a row P such that $\frac{X_R(t)}{t} \xrightarrow[t \rightarrow \infty]{} \beta$
 (ii) P-a.s, $\max_{x \in Z_t} X_x(t) - \sqrt{2}t \xrightarrow[t \rightarrow \infty]{} -\infty$

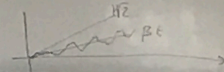
$(1 + \frac{\beta^2}{2}) e^{\beta X_{\sqrt{2}t}}$
 $S(\beta_t) = (1 + \frac{\beta^2}{2}) e^{\beta X_{\sqrt{2}t}}$
 $\sqrt{2}xt > 0$

Man in striped shirt talking on phone.

Students in the foreground.

Under P^* , $(\frac{1}{P_t})$ is a \mathbb{R}^n -martingale (actually a martingale)
 $\Rightarrow \frac{1}{P_t}$ is P^* -a.s. $\Rightarrow \lim_{t \rightarrow \infty} M_t = M_\infty$ P^* -a.s.

Prop. (i) $\forall |\beta| < \sqrt{2}$, \exists a law \tilde{P} such that $\frac{X_t(t)}{t} \xrightarrow[t \rightarrow \infty]{} \beta$
 (ii) P -a.s, $\max_{u \leq t} X_u(t) - \sqrt{2}t \xrightarrow[t \rightarrow \infty]{} -\infty$



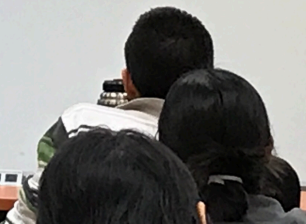
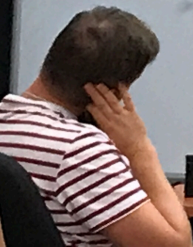
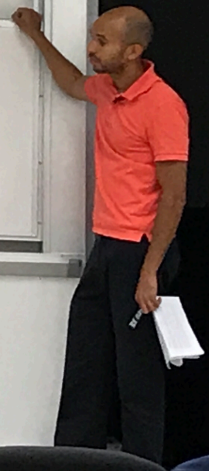
Proof: (i) $\exists \beta$ $|\beta| < \sqrt{2}$, P^* and \tilde{P} are equivalent.
 $P^*(\exists R: \frac{X_t(t)}{t} \rightarrow \beta) = 1$ (given)
 \Rightarrow under \tilde{P} as well.

(ii) $\beta = \sqrt{2}$

$$M_t = e^{-t(1 + \frac{\beta^2}{2})} \sum_{u \leq t} e^{\beta X_u(t)}$$

$$\sum_{u \leq t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \leq t} X_u(t) - t\sqrt{2})}$$

P -a.s. $0 \xrightarrow[t \rightarrow \infty]$



Def. (1) If $\|B\| < \sqrt{2}$, P^* and P are equivalent

$$P^*(\exists R: \frac{X_R(t)}{t} \rightarrow B) = 1 \text{ (ipm)}$$

$$\frac{dP^*}{dP} = M_\infty$$

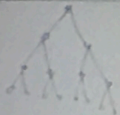
(ii) $B = \sqrt{2}$

$$M_t = e^{-t(1 + \frac{B^2}{2})} \sum_{u \in \mathcal{U}_t} e^{B X_u(t)}$$

$$\sum_{u \in \mathcal{U}_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \in \mathcal{U}_t} X_u(t) - t\sqrt{2})}$$

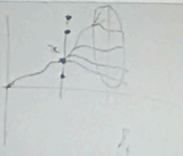
Pas. 0 \leftarrow even

$$M_t(x) = e^{-t(1-x)} \sum_{u \in \mathcal{U}_t} e^{B X_u(t)} \xrightarrow{t \rightarrow \infty} M_\infty(x)$$



$\partial T = \{ \text{set of roots} \}$

$$d(R, R') = e^{-r(R, R')}$$



particle x . we want to define

$$M_\infty(\{x\}) = \text{mass that go through } x$$

$$= M_\infty(x)$$

Prop:

$$\mathbb{E} \left[\int_0^T f(\omega, R) M_{\infty}(dR) \right] = \mathbb{E} \left[f(\omega, R) \right]$$

(1) $B = \sqrt{2}$

$$M_t = e^{-t(1 + \frac{B^2}{2})} \sum_{u \in \mathcal{U}_t} e^{B^2 X_u(t)}$$

$$\sum_{u \in \mathcal{U}_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \in \mathcal{U}_t} X_u(t) - t\sqrt{2})}$$

pas.

$0 \leftarrow t \rightarrow \infty$

$$M_t(z) = e^{-t(1 + \frac{B^2}{2})} \sum_{\substack{u \in \mathcal{U}_t \\ u \geq x}} e^{B^2 X_u(t)} \xrightarrow{t \rightarrow \infty} M_{\infty}(z)$$

$|B| < \sqrt{2}$



$\partial T = \{ \text{set of rays} \}$

$$d(R, R') = e^{-\sqrt{2}R(R')}$$



partir de x . we want to define

$$M_{\infty}(\text{set of rays that go through } x)$$

" $M_{\infty}(x)$

Prop:

$$E \left[\int_0^T f(\omega, R) M_{\infty}(dR) \right] = E^* \left[f(\omega, R) \right]$$

$$(0) B = \sqrt{2}$$

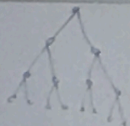
$$M_t = e^{-t(1 + \frac{\sigma^2}{2})} \sum_{u \in \mathcal{L}_t} \beta X_u(t)$$

$$\sum_{u \in \mathcal{L}_t} e^{\sqrt{2}(X_u(t) - t\sqrt{2})} \geq e^{\sqrt{2}(\max_{u \in \mathcal{L}_t} X_u(t) - t\sqrt{2})}$$

P.s. $0 \leftarrow t \rightarrow \infty$

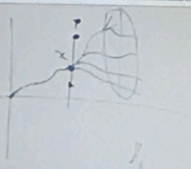
$$M_t(x) = e^{-t(\cdot)} \sum_{\substack{u \in \mathcal{L}_t \\ u \succ x}} \beta X_u(t) \xrightarrow{t \rightarrow \infty} M_{\infty}(x)$$

$\sqrt{2}$



$\partial T = \{x \text{ of rays}\}$

$$d(R, R') = e^{-\sqrt{2}R(R')}$$



possible z : we want to define

$$M_{\infty}(\{x\} \text{ of rays that go through } x) \\ \parallel M_{\infty}(x)$$

