# The Pruning process on Trees

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1. Motivation

2. Definitions
   - Pointed Gromov-weak topology
   - Measure $\mathbb{R}$-trees
   - Subtree Gromov-weak topology
   - Bi-measure $\mathbb{R}$-trees and the LWV-topology

3. The pruning process

4. Applications
   - An approach via excursions
   - Galton-Watson trees
   - Brownian CRT and Stable Lévy tree
   - General Pruning

# Motivation

# Galton-Watson trees

Let $\eta$ be a probability measure on $\{0, 1, 2, \dots\}$ such that

$$\sum_{i \geq 0} i\eta(\{i\}) \leq 1 \quad \text{and} \quad \eta(\{1\}) < 1.$$

A Galton-Watson tree with offspring distribution $\eta$ is a random plane tree $\mathcal{G}$ such that:

(i) The number of children of the root is distributed according to $\eta$.

(ii) Conditionally on the fact that the root has $j$ children, the number of children of the $j$ children of the root are independent of law $\eta$, and so on (Branching Property).

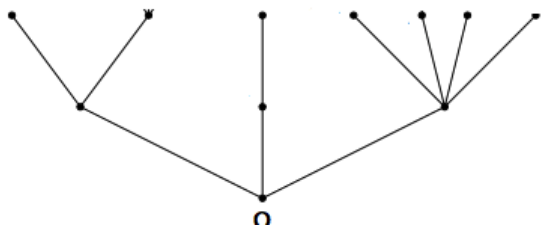

Figure: This happens with probability $\eta(0)^7\eta(1)^2\eta(2)\eta(3)\eta(4)$.

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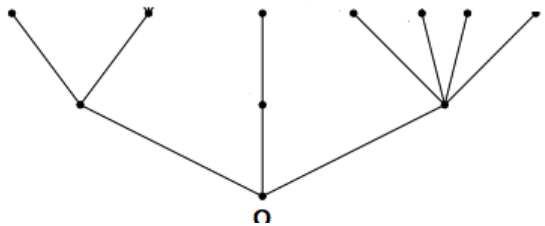

Figure: This happens with probability $\eta(0)^7\eta(1)^2\eta(2)\eta(3)\eta(4)$.

**Remark.**

- In this case, the tree is almost surely finite.

# Aldous and Pitman (1998)

Let $\mathcal{G}_1$ be a rooted G-W tree with offspring generating function

$$g(z) = \sum_{i \geq 0} z^i \eta(\{i\}), \quad z \in [0,1].$$

For $0 \leq u \leq 1$, let $\mathcal{G}_u$ be the subtree of $\mathcal{G}_1$ obtained by retaining each edge with probability $u$.

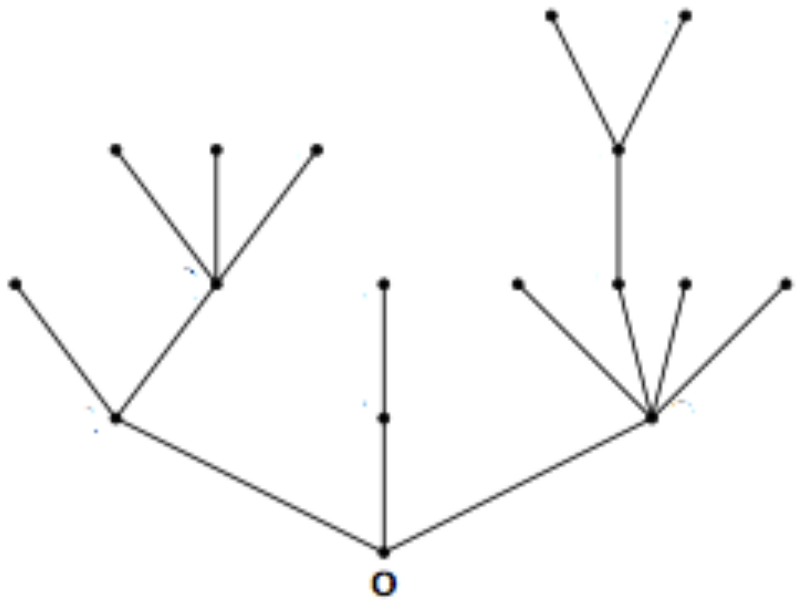


Figure: A G-W tree.

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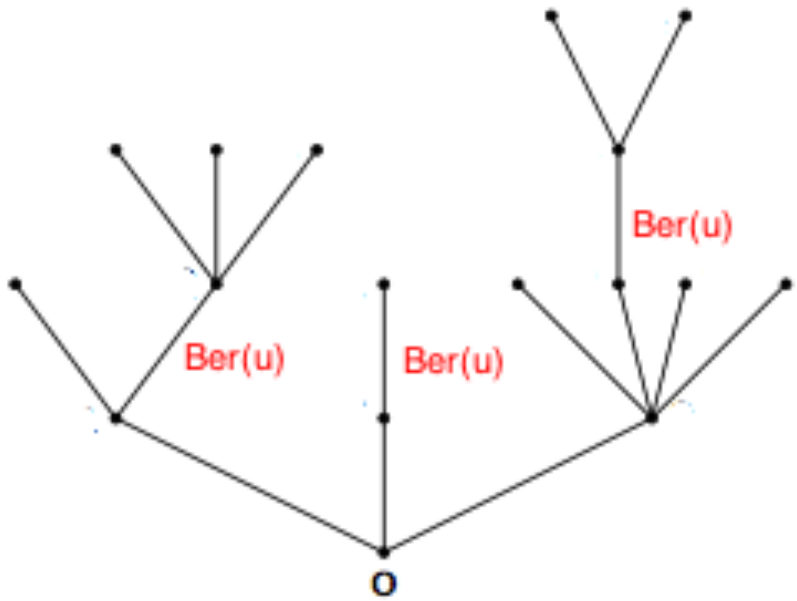


Figure: i.i.d. Bernoulli r.v. with parameter $u$.

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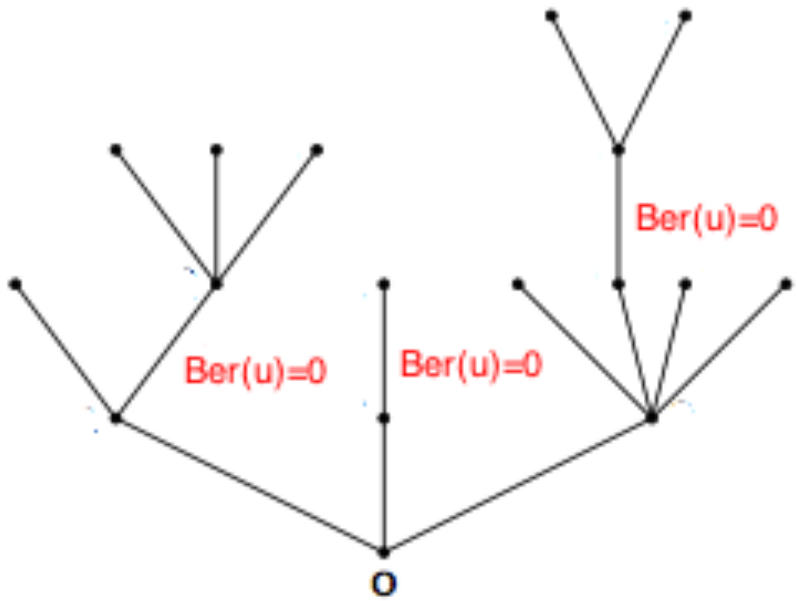


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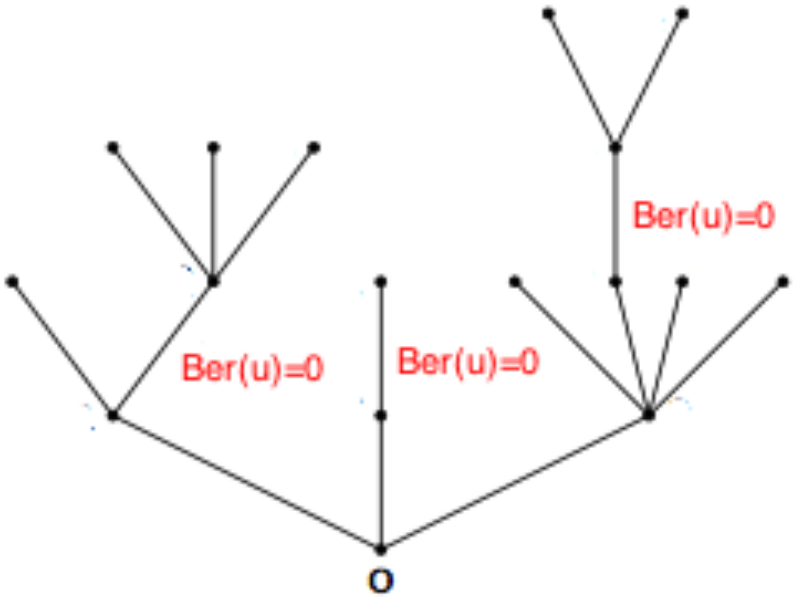


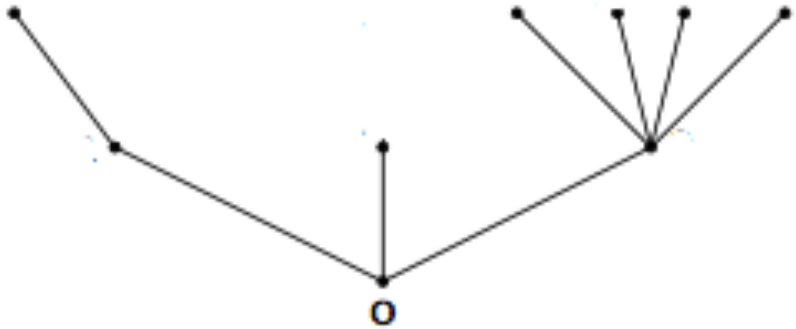

# Aldous and Pitman (1998)

- Lyons (1992) showed that $\mathcal{G}_u$ is again a G-W tree which corresponds to the offspring generating function

$$h(z) = g(1-u+uz), \quad 0 \leq u \leq 1.$$

- One can couple the pruning procedures for several $u \in [0,1]$ in such a way that $G_{u'} \subseteq G_u$ whenever $u' \leq u$.
- This gives rise to a non-decreasing tree-valued Markov process $(\mathcal{G}_u)_{u\in[0,1]}$.

# Related works

- Abraham, Delmas and He (2012) considered another pruning procedure on G-W trees where cut points fall on the branch points. Here each vertex of the initial G-W tree is cut independently with probability

$$1 - u^{n_v}$$

  where $n_v$ is the is the number of children of the vertex $v$.
- Aldous-Pitman (1998) and Abraham-Serlet (2002) for a pruning proportional to the length on the skeleton of a Brownian CRT.
- Miermont (2005) for a pruning on the infinite branch points of a stable Lévy tree.
- Aldous (2008) for a pruning on the infinite branch points of a Lévy tree without Brownian part.
- Abraham-Delmas-Voisin (2010, 2012) for a combined pruning proportional to the length and on the infinite branch points of a general Lévy tree.

# Motivation

In Abraham-Delmas (2012), it is conjecture that the pruning procedure on general Lévy trees is the limit of the suitably scaled pruning process of (discrete) G-W trees where the cut points fall on edges as well as on vertices (branch points).

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- Winkel and He (2014) answered this question. They got convergence in the Skorokhod topology on càdlàg functions with values in the space of (isometry classes of) locally compact $\mathbb{R}$-trees equipped with the Gromov-Hausdorff topology. (no publish).
- Löhr, Voissin and Winter (2015) answered this question in the case where the cut points fall only in the edges. They got convergence in a stronger topology, the Skorokhod topology on càdlàg functions with values in the space of (isometry classes of) bi-measure $\mathbb{R}$-trees equipped with the leaf-sampling weak vague topology (LWV-topology).

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# Definitions

# Pointed Gromov-weak topology

We follow Greven, Pfaffelhuber and Winter (2009):

- A metric measure space is a triple $(X, r, \mu)$, where $(X, r)$ is a metric space and $\mu$ is a finite measure on $\mathcal{B}(X)$ such that $(\text{supp}(\mu), r)$ is complete and separable.

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- We consider rooted (or pointed) metric measure spaces $(X, r, \rho, \mu)$, where $(X, r, \mu)$ is a metric measure space and $\rho \in X$ is a distinguished point which is referred as the root.

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**Notation.** To avoid heavy notations, we abbreviate

$$X = (X, r, \rho) \quad \text{and} \quad (X, \mu) = (X, r, \rho, \mu).$$

We want to extend these rooted metric measure spaces $(X, r, \rho, \mu)$ by fixing $n$ additional points $u_1, \ldots, u_n \in X$.

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- We call $(X, r, \rho, (u_1, \ldots, u_n), \mu)$ a (rooted) $n$-pointed metric measure space. Its support is defined by

$$\text{supp}((X, r, \rho, (u_1, \ldots, u_n), \mu)) := \text{supp}(\mu) \cup \{\rho, u_1, \ldots, u_n\}.$$

## Definition 1 (The space $\mathbb{M}_n$)

We call $\mathcal{X} = (X, r, \rho, (u_1, \ldots, u_n), \mu)$ and $\mathcal{X}' = (X', r', \rho', (u'_1, \ldots, u'_n), \mu')$ equivalent iff there exists an isometry $\phi : \text{supp}(\mu) \to \text{supp}(\mu')$ such that

(i) $\phi(\rho) = \rho'$,

(ii) $\phi(u_k) = u'_k$ for all $1 \leq k \leq n$, and

(iii) $\phi_* \mu = \mu'$, where

$$\phi_* \mu(A) = \mu(\phi^{-1}(A)), \quad A \subseteq X \text{ (measurable)}.$$

(the push forward of $\mu$ under the map $\phi$).

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**Notation.**

- We denote by $\mathbb{M}_n$ the set of equivalence classes of $n$-pointed metric measure spaces.
- To simplify notations, we do not distinguish between an $n$-pointed metric measure spaces and its equivalence class. That is, we write

$$\mathcal{X} = (X, r, \rho, (u_1, \dots, u_n), \mu) \in \mathbb{M}_n.$$

# Pointed Gromov-weak topology

**Remark.**

- The space $\mathbb{M}_0$ is the usual space of rooted metric measure spaces and its equivalence class.
- Metric measure spaces, or short $mm$-spaces, are discussed in Gromov's book (1999) in detail.

# Pointed Gromov-weak topology

For a (rooted) metric space $X$, we define a map $R^X$ that associates to a sequence of points the matrix of their distances to the root and to each other, i.e.,

$$R^X : \begin{cases} X^{\mathbb{N}} & \to & \mathbb{R}_+^{\binom{\mathbb{N}_0}{2}} \\ (x_i)_{i\geq 1} & \mapsto & (r(x_i, x_j))_{0\leq i<j} \text{ with } x_0 := \rho. \end{cases}$$

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The distance matrix distribution of an $n$-pointed metric measure spaces $\mathcal{X} = (X, r, \rho, (u_1, \ldots, u_n), \mu)$ is given by

$$\nu^{\mathcal{X}} := \| \mu \| \cdot R_*^X \left( \bigotimes_{k=0}^{n} \delta_{u_k} \otimes (\mu^\circ)^{\otimes \mathbb{N}} \right) \in \mathcal{M}_f(\mathbb{R}_+^{\binom{\mathbb{N}_0}{2}}), \quad \text{with} \quad u_0 = \rho,$$

where $\| \mu \| = \mu(X)$ (total mass) and $\mu^\circ = \mu / \| \mu \| \in \mathcal{M}_1(X)$ (the normalization).

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## Definition 2 (pGw-topology)

A sequence of n-pointed metric measure spaces $\mathcal{X}_N \in \mathbb{M}_n$ converges $n$-pointed Gromov-weakly (pGw) to $\mathcal{X} \in \mathbb{M}_n$ if

$$\nu^{\mathcal{X}_N} \underset{N\to\infty}{\Longrightarrow} \nu^{\mathcal{X}}$$

in the weak topology on $\mathcal{M}_1(\mathbb{R}_+^{\binom{\mathbb{N}_0}{2}})$.

**Remark.** Consider

$$\mathcal{X}_N = (X_N, r_N, \rho_N, (u_{N,1}, \dots, u_{N,n}), \mu_N) \in \mathbb{M}_n.$$

Let $U_{N,0} = \rho_N$, $U_{N,1} = u_{N,1}, \dots U_{N,n} = u_{N,n}$ and $(U_{N,n+k})_{k\geq 1}$ be an i.i.d. sequence of $\mu_N^\circ$-distributed r.v. The definition of pGw-convergence requires:

- The convergence of $\| \mu_N \|$.
- The convergence

$$(r_N(U_{N,i}, U_{N,j}) : i, j \geq 0) \xrightarrow[N\to\infty]{\text{f.d.}} (r(U_i, U_j) : i, j \geq 0),$$

  where $U_0 = \rho$, $U_1 = u_1, \dots U_n = u_n$ and $(U_{n+k})_{k\geq 1}$ be an i.i.d. sequence of $\mu^\circ$-distributed r.v.

# Pointed Gromov-weak topology

**Remark.** We see directly from the definition that functions of the form $\Phi : \mathbb{M}_n \to \mathbb{R}$, $\mathcal{X} \mapsto \langle \nu^{\mathcal{X}}, f \rangle$ with $f \in C_b(\mathbb{R}_+^{\binom{\mathbb{N}_0}{2}})$ are continuous.

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**Notation.**

- For $f$ that depends only on finitely many coordinates, $\Phi$ is called a polynomial.

  There exists $m \in \mathbb{N}$, $\varphi \in C_b(\mathbb{R}_+^{\binom{n+m+1}{2}})$ such that for $\mathcal{X} \in \mathbb{M}_n$,

$$\Phi(\mathcal{X}) = \Phi^{m,\varphi}(\mathcal{X}) := \int_{\mathcal{X}^m} \mu^{\otimes m}(\mathrm{d}\underline{v}) \varphi\left(R^X(\underline{u}, \underline{v})\right),$$

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- We define the class $\Pi_n \subseteq C_b(\mathbb{M}_n)$ of bounded test functions by
  $$\Pi_n := \left\{ \Phi^{\gamma,m,\varphi}(\mathcal{X}) := \gamma(\| \mu \|) \cdot \Phi^{m,\varphi}(\mathcal{X}) : \begin{array}{c} \Phi^{m,\varphi}(\mathcal{X}) \text{ is a polynomial}, \gamma \in C_b(\mathbb{R}_+), \\ \lim_{x\to\infty} x^k \gamma(x) = 0, \forall k \in \mathbb{N} \end{array} \right\}.$$

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**Remark.** (Vershik's reconstruction theorem (Gromov's book Chapter $3\frac{1}{2}.7$)) If $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{M}_n$ such that
$$\Phi(\mathcal{X}_1) = \Phi(\mathcal{X}_2) \quad \text{for all polynomials} \quad \Leftrightarrow \mathcal{X}_1 = \mathcal{X}_2.$$
Therefore, $\mathcal{X} \in \mathbb{M}_n$ is uniquely determined by its distance matrix distribution $\nu^{\mathcal{X}}$.

# Pointed Gromov-weak topology

Recall the Prokhorov distance $d_{\mathsf{Pr}}$ between two finite measures $\mu$, $\nu$ on a metric space $(Z, d)$,

$$d_{\mathsf{Pr}}^{(Z,d)}(\mu,\nu) := \inf\{\varepsilon > 0 : \mu(A^\varepsilon) + \varepsilon \geq \nu(A),\, \nu(A^\varepsilon) + \varepsilon \geq \mu(A),\ \forall A \subseteq Z \text{ closed}\},$$

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## Definition 3 ($n$-pointed Gromov-Prokhorov distance)

We define the $n$-pointed Gromov-Prokhorov distance between $\mathcal{X} = (X, r, \rho, (u_1, \ldots, u_n), \mu)$ and $\mathcal{X}' = (X', r', \rho', (u'_1, \ldots, u'_n), \mu')$ in $\mathbb{M}_n$ by

$$d_{\mathrm{pGP}}(\mathcal{X},\mathcal{X}') := \inf_{(\phi_{\mathcal{X}},\phi'_{\mathcal{X}'},Z)} \left\{ d_{\mathrm{Pr}}^{(Z,r_Z)}((\phi_{\mathcal{X}})_*\mu, (\phi'_{\mathcal{X}'})_*\mu') \vee \max_{0\leq i\leq n} r_Z(\phi_{\mathcal{X}}(u_i), \phi'_{\mathcal{X}'}(u'_i)) \right\},$$

where $u_0 = \rho$, $u'_0 = \rho'$, and the infimum is taken over all isometric embeddings $\phi_{\mathcal{X}}$ and $\phi'_{\mathcal{X}'}$ from $\mathcal{X}$ and $\mathcal{X}'$, respectively, into some common metric space $(Z, r_Z)$.

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## Definition 3 ($n$-pointed Gromov-Prokhorov distance)

We define the $n$-pointed Gromov-Prokhorov distance between $\mathcal{X} = (X, r, \rho, (u_1, \ldots, u_n), \mu)$ and $\mathcal{X}' = (X', r', \rho', (u'_1, \ldots, u'_n), \mu')$ in $\mathbb{M}_n$ by

$$d_{\mathrm{pGP}}(\mathcal{X},\mathcal{X}') := \inf_{(\phi_{\mathcal{X}},\phi'_{\mathcal{X}'},Z)} \left\{ d_{\mathrm{Pr}}^{(Z,r_Z)}((\phi_{\mathcal{X}})_*\mu, (\phi'_{\mathcal{X}'})_*\mu') \vee \max_{0\leq i\leq n} r_Z(\phi_{\mathcal{X}}(u_i), \phi'_{\mathcal{X}'}(u'_i)) \right\},$$

where $u_0 = \rho$, $u'_0 = \rho'$, and the infimum is taken over all isometric embeddings $\phi_{\mathcal{X}}$ and $\phi'_{\mathcal{X}'}$ from $\mathcal{X}$ and $\mathcal{X}'$, respectively, into some common metric space $(Z, r_Z)$.

**Remark.**

- $d_{\mathrm{pGP}}$ defines a metric on $\mathbb{M}_n$.

# Pointed Gromov-weak topology

**Remark.** Recall that a set $\mathcal{F} \subseteq C_b(X)$ is convergence determining (on the topological space $X$) if, for two probability measures $\mu_N$, $\mu$ on $X$, the weak convergence

$$\mu_N \underset{N\to\infty}{\Longrightarrow} \mu \quad \text{is equivalent to} \quad \int f \mathrm{d}\mu_N \underset{N\to\infty}{\longrightarrow} \int f \mathrm{d}\mu$$

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## Proposition 1 ($\Pi_n$ is convergence determining)

*Let $\mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_2, \cdots \in \mathbb{M}_n$. The following statements are equivalent:*

(i) $\mathcal{X}_N \xrightarrow{pGw} \mathcal{X}$, *as* $N \to \infty$.

(ii) $\Phi(\mathcal{X}_N) \xrightarrow[N\to\infty]{} \Phi(\mathcal{X})$, *for all polynomials* $\Phi$.

(iii) $d_{pGP}(\mathcal{X}_N, \mathcal{X}) \xrightarrow[N\to\infty]{} 0$.

*Furthermore, $(\mathbb{M}_n, d_{pGP})$ is complete and separable, and the class $\Pi_n \subseteq C_b(\mathbb{M}_n)$ is convergence determining on $\mathbb{M}_n$.*

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**Remark.** (Depperschmit, Greven & Pfaffelhuber (2012), Löhr (2013)) For $(\mathcal{X}_N)_{N\geq 1}$ and $\mathcal{X}$ random elements in $\mathbb{M}_n$, we have that

$$\mathbb{E}[\phi(\mathcal{X}_N)] \xrightarrow[N\to\infty]{} \mathbb{E}[\phi(\mathcal{X})] \quad \text{for all polynomials} \quad \Leftrightarrow \quad \mathcal{X}_N \xrightarrow[N\to\infty]{\text{pGw}} \mathcal{X}.$$

**Definition 4 (Dress, et. al.: $T$-theory (1996))**

A $n$-pointed metric measure space $\mathcal{X}$ is an $\mathbb{R}$-tree iff

- $\mathcal{X}$ is connected.
- $\mathcal{X}$ satisfies the is 4-point condition, i.e., for all $x_1, x_2, x_3, x_4 \in \text{supp}(\mathcal{X})$,

$$r(x_1, x_2) + r(x_3, x_4) \leq \max\{r(x_1, x_3) + r(x_2, x_4), r(x_1, x_4) + r(x_2, x_3)\}.$$

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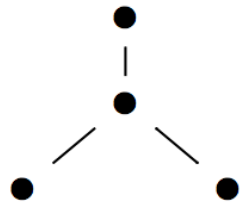
Figure: I am not a tree



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- $\mathbb{R}$-trees are understood as continuum trees.
- The G-W tree $\mathcal{G}$ can be seen as a rooted $\mathbb{R}$-tree with unit length edges (equipped with the graph distance).
- Ultra metric spaces, 0-hyperbolic spaces, graph-theoretical trees are $\mathbb{R}$-trees.

**Definition 6 (Dress, et. al.: $T$-theory (1996))**

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**Notation.** Denote by

$$\mathbb{H}_n := \{\mathcal{X} \in \mathbb{M}_n : \mathcal{X} \text{ is } \mathbb{R}\text{-tree}\} \subseteq \mathbb{M}_n, \quad \text{and} \quad \mathbb{H} := \mathbb{H}_0 \subseteq \mathbb{M}_0.$$

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**Remark.** We notice the following.

- Evans, Pitman and Winter (2006) showed that for each $n \in \mathbb{N}$, $(\mathbb{H}_n, d_{\text{pGP}})$ is complete and separable.

**Example.** (Via "excursion type" functions). Let

$$g : [0,1] \to \mathbb{R}_+$$

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- We consider the pseudo-distance on $[0,1]$ by setting,

$$r_g(x,y) := g(x) + g(y) - 2 \inf_{z \in [x \wedge y, x \vee y]} g(z), \quad \text{for} \quad x, y \in [0,1].$$

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- We define an equivalence relation on $[0,1]$ by setting $x \sim_g y$ iff $r_g(x,y) = 0$.

**Example.** (Via "excursion type" functions).

- The image of the projection $p_g : [0,1] \to [0,1]\setminus \sim_g$ endowed with the push forward of $r_g$ (again denoted $r_g$), i.e.

$$T_g = (T_g, r_g, \rho_g) := (p_g([0,1]), r_g, p_g(0)),$$

is a (rooted) $\mathbb{R}$-tree; see Lemma 3.1 Evans and Winter (2006).

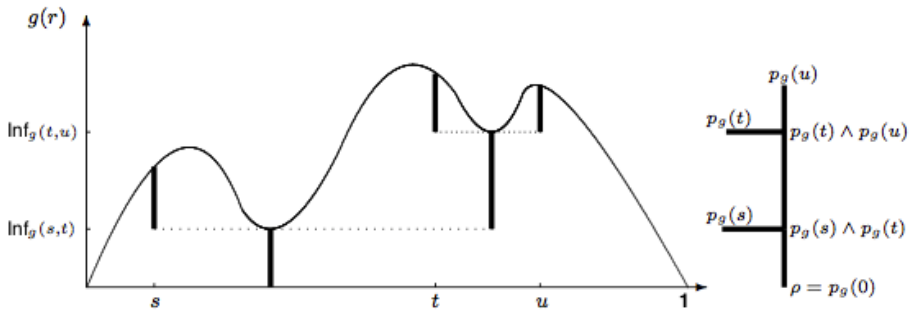


Figure: $T_g$

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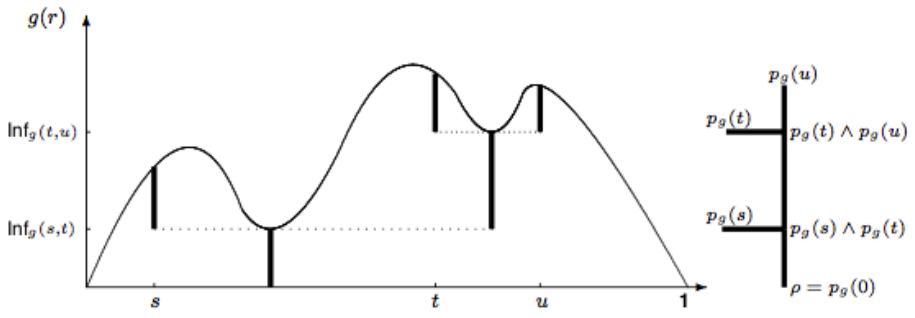



Figure: $T_g$

- We endow $T_g$ with the probability measure $\mu_g := (p_g)_* \lambda_{[0,1]}$, the push forward of the Lebesgue measure on $[0,1]$.

**Notation.** Given a (rooted) $\mathbb{R}$-tree $(T, r, \rho, \mu)$,

- We denote the unique path between two points $x, y \in T$ by $[x, y]$, i.e.

$$[x, y] := \{z \in T : r(x, z) + r(z, y) = r(x, y)\},$$

and write $[x, y[ := [x, y] \setminus \{y\}$.

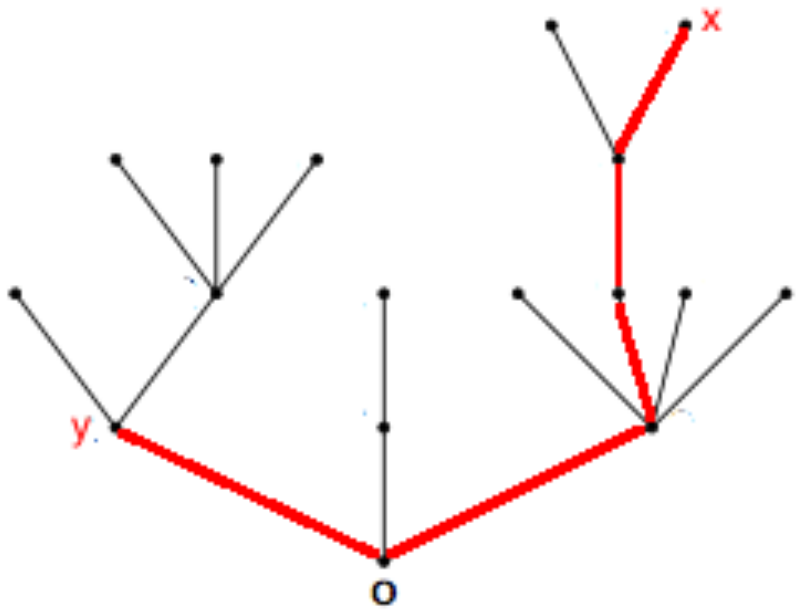



Figure: $[x, y]$

**Notation.** Given a (rooted) $\mathbb{R}$-tree $(T, r, \rho, \mu)$,

- For $x, y, z \in T$ there exists a unique branch point $c(x, y, z) \in T$ with

$$[x, y] \cap [y, z] \cap [x, z] = \{c(x, y, z)\}.$$

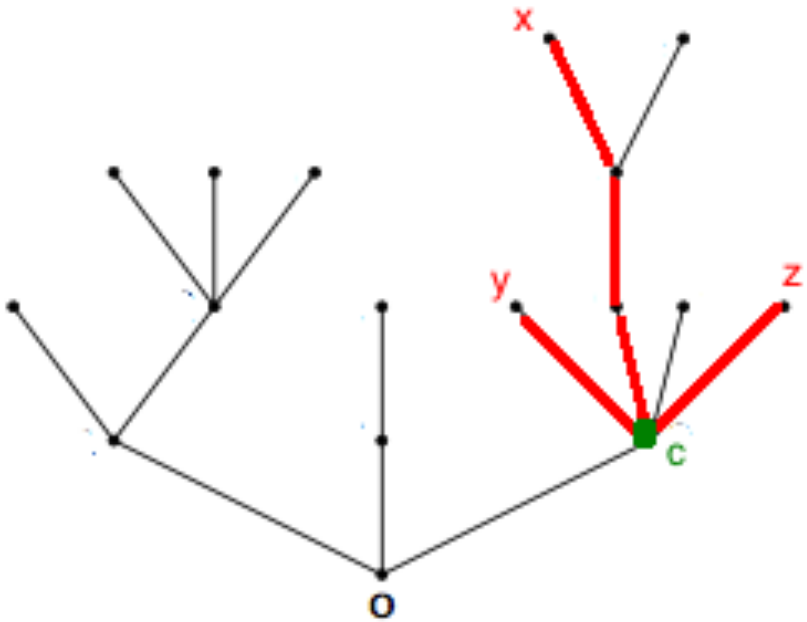



Figure: branch point

**Notation.** Given a (rooted) $\mathbb{R}$-tree $(T, r, \rho, \mu)$,

- The set of leaves of the tree is

$$\mathsf{Lf}(T) := T \setminus \bigcup_{x \in T} [\rho, x[.$$

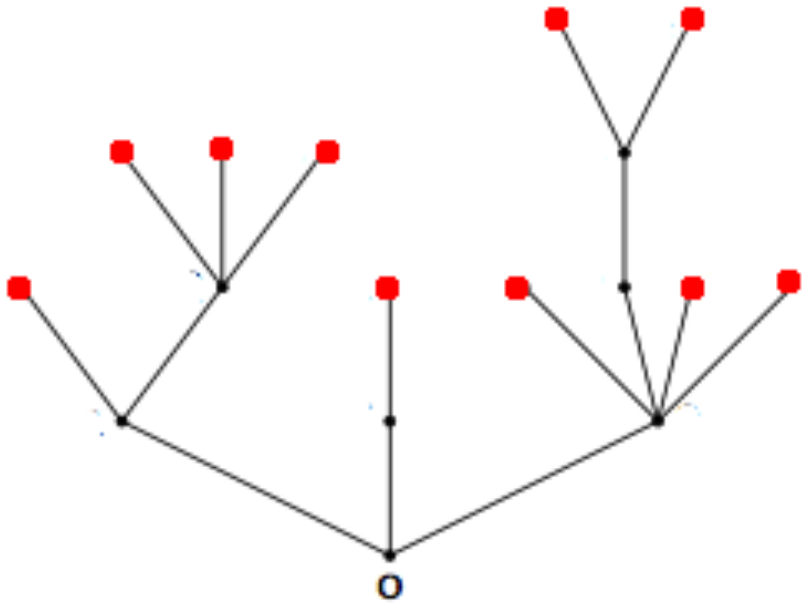



Figure: $\mathsf{Lf}(T)$

**Notation.** Given a (rooted) $\mathbb{R}$-tree $(T, r, \rho, \mu)$,

- We write $[\![\underline{v}]\!]$ for the tree spanned by the root $\rho$ and the vector $\underline{v} \in T^n$, i.e.,

$$[\![\underline{v}]\!] := \bigcup_{i=1}^{n} [\rho, v_i].$$

(We refer to any $\mathbb{R}$-tree of the previous form as a finite tree).

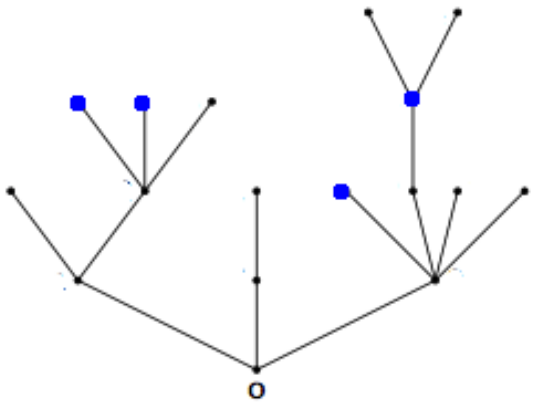
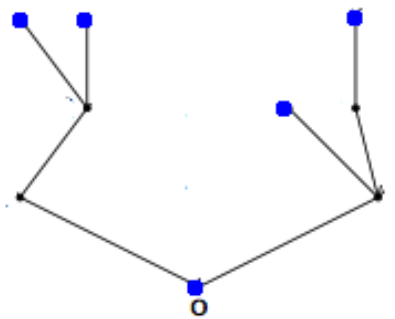



Figure: $\underline{v}$

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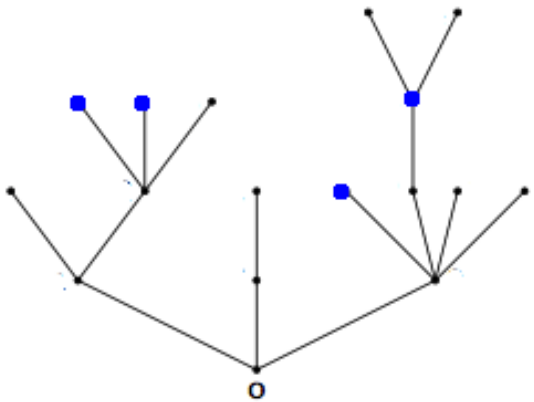


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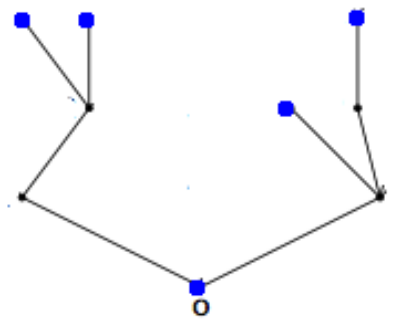


Figure: $[\![\underline{v}]\!]$

- We sometimes abbreviate

$$T = (T, r, \rho) \quad \text{and} \quad (T, \mu) = (T, r, \rho, \mu).$$

# Subtree Gromov-weak topology

We now define a topology on $\mathbb{H}_n$.

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We define a projection map which sends a list $\underline{u} \in T^n$ to the sublist indexed by $I = \{i_1, \ldots, i_k\}$ for a given $1 \leq i_1 < \cdots < i_k \leq n$. That is,

$$\pi_I^n : \begin{cases} T^n & \to & T^k \\ \underline{u} & \mapsto & (u_{i_1}, \ldots, u_{i_k}). \end{cases}$$

for $n \in \mathbb{N}$ and $1 \leq k \leq n$.

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**Notation.**

- The sublist $(u_1, \ldots, u_k)$ of $\underline{u} \in T^n$ is simply denoted by $\underline{u}^k$.
- We write

$$\pi_I^n(T, \underline{u}, \mu) := (\llbracket \pi_I^n(\underline{u}) \rrbracket, \pi_I^n(\underline{u}), \mu) := (\llbracket \pi_I^n(\underline{u}) \rrbracket, \pi_I^n(\underline{u}), \mu \restriction_{\llbracket \pi_I^n(\underline{u}) \rrbracket}),$$

  where the measure $\mu$ in the middle expression is understood to be restricted to the appropriate space, $\llbracket \pi_I^n(\underline{u}) \rrbracket$.

## Definition 7 (sGw-topology)

Consider a sequence of $n$-pointed measure $\mathbb{R}$-trees $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \cdots \in \mathbb{H}_N$. We say that $(\mathcal{X}_N)_{N\in\mathbb{N}}$ converges subtree Gromov-weakly (sGw) to $\mathcal{X}$ iff

$$\mathcal{X}_N \xrightarrow[N\to\infty]{\text{pGw}} \mathcal{X},$$

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**Notation.** Put

$$\tilde{\mathbb{H}}_n := \{(T, \underline{u}, \mu) \in \mathbb{H}_n : \text{supp}(\mu) \subseteq [\![\underline{u}]\!]\} \subseteq \mathbb{H}_n,$$

the set of finite trees with at most $n$ leaves.

# Subtree Gromov-weak topology

**Remark.**

- The sGw-topology is strictly stronger than the pGw-topology.
- On $\mathbb{H}_n$, sGw-convergence implies measured Gromov-Hausdorff convergence (Fukaya, 1987), also known as weighted Gromov-Hausdorff convergence (Evans and Winter (2006), Miermont (2009)).

## Lemma 8 (Sufficient condition for sGw-convergence)

*Consider n-pointed measure $\mathbb{R}$-trees $\mathcal{X} = (T, \underline{u}, \mu), \mathcal{X}_N = (T_N, \underline{u}_N, \mu_N) \in \tilde{H}_n$, $N \in \mathbb{N}$.*

- *Assume that $(\mathcal{X}_N)_{N\in\mathbb{N}}$ converges a.s. to $\mathcal{X}$ in the n-pointed Gromov-weak topology, as $N \to \infty$.*
- *Assume that there there is a strictly increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*
  - *$\psi(\| \mu \|)$ is integrable,*
  - *and*

$$\mathbb{E}\left[\psi(\mu_N([\![\pi_I^n(\underline{u}_N)]\!]))\right] \xrightarrow[N\to\infty]{} \mathbb{E}\left[\psi(\mu([\![\pi_I^n(\underline{u})]\!]))\right], \qquad \forall\, I \subseteq \{1,\dots,n\}.$$

*Then $(\mathcal{X}_N)_{N\in\mathbb{N}}$ converges subtree Gromov-weakly to $\mathcal{X}$, a.s., as $N \to \infty$.*

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# Bi-measure $\mathbb{R}$-trees

In this subsection,

- We give the definition of bi-measure R-trees.
- We equip the space of equivalence classes of bi-measure $\mathbb{R}$-trees with the leaf-sampling weak vague topology (LWV-topology).

**Notation.** For a rooted measure $\mathbb{R}$-trees,

- The $\mu$-skeleton

$$\mathsf{Sk}_\mu(T) := \bigcup_{v \in \mathsf{supp}(\mu)} [\rho, v[ \cup \{v \in T : \mu(\{v\}) > 0\}.$$

- The set of $\mu$-leaves

$$\mathsf{Lf}_\mu(T) := [\![\mathsf{supp}(\mu)]\!] \setminus \mathsf{Sk}_\mu(T).$$

**Definition 9**

We call $(T, \mu, \nu)$ a (rooted) bi-measure $\mathbb{R}$-trees if $(T, \mu)$ is a (rooted) measure $\mathbb{R}$-tree and $\nu$ is a ($\sigma$-finite) measure on $T$ which satisfies the following two conditions:

(i) $\nu([\rho, u])$ is $\mu$-a.s. finite for $u \in T$.

(ii) $\nu$ vanishes on the set of $\mu$-leaves, i.e., $\nu(\mathrm{Lf}_\mu(T)) = 0$.

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**Remark.** Note that (i) implies that

- $\nu$ is finite on subtrees of $T$ with finite number of leaves sampled with $\mu$, a.s.
- $\nu \restriction_{\mathrm{Sk}_\mu(T)}$ is $\sigma$-finite (separability of $\mathrm{supp}(\mu)$).

**Definition 10 (The space $\mathbb{H}^{f,\sigma}$)**

We call $(T,\mu,\nu)$ and $(T',\mu',\nu')$ equivalent if there exists an isometry $\phi : [\![\mathrm{supp}(\mu)]\!] \to T'$ such that

(i) $\phi(\rho) = \rho'$,

(ii) $\phi_*\mu = \mu'$,

(iii) and $\phi_*(\nu \restriction_{\mathsf{Sk}_\mu(T)}) = \nu' \restriction_{\mathsf{Sk}_{\mu'}(T')}$.

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**Remark.** In particular, $(T,\mu,\nu)$ is equivalent to $(T,\mu,\nu \restriction_{\mathrm{Sk}_\mu(T)})$.

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**Remark.** In particular, $(T,\mu,\nu)$ is equivalent to $(T,\mu,\nu \restriction_{\mathrm{Sk}_\mu(T)})$.

**Notation.** We denote by $\mathbb{H}^{f,\sigma}$ the space of equivalence classes of (rooted) bi-measure $\mathbb{R}$-trees.

# The subtree-vector-distribution

For a (rooted) bi-measure $\mathbb{R}$-trees $(T, \mu, \nu)$, we consider the function

$$\tau_{(T,\mu,\nu)} : \left\{ \begin{array}{lcl} \bigcup_{n\in\mathbb{N}} T^n & \to & \bigcup_{n\in\mathbb{N}} \mathbb{H}_n \\ (u_1, u_2, \ldots, u_n) & \mapsto & (\llbracket u_1, \ldots, u_n \rrbracket, (u_1, \ldots, u_n), \nu). \end{array} \right.$$

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We also define the function

$$\varsigma_{(T,\mu,\nu)} : \begin{cases} T^{\mathbb{N}} & \to & \prod_{n\in\mathbb{N}} \mathbb{H}_n \\ (u_1, u_2, \dots) & \mapsto & (\tau_{(T,\mu,\nu)}(u_1), \tau_{(T,\mu,\nu)}(u_1, u_2), \dots). \end{cases}$$

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We finally define the sub-tree vector distribution of a (rooted) bi-measure $\mathbb{R}$-tree $\mathcal{X} = (T, \mu, \nu)$ as

$$\varpi^{\mathcal{X}} := \| \mu \| \cdot (\varsigma_{\mathcal{X}})_* \left( (\mu^\circ)^{\otimes\mathbb{N}} \right) \in \mathcal{M}_f \left( \prod_{n\in\mathbb{N}} \mathbb{H}_n \right)$$

Leaf-sampling weak vague topology (LWV-topology); Löhr, Voisin and Winter (2015).

**Definition 11 (LWV-topology)**

A sequence of (rooted) bi-measure $\mathbb{R}$-tree $(\mathcal{X}_N)_{N\in\mathbb{N}}$ converges to $\mathcal{X} \in \mathbb{H}^{f,\sigma}$ in the LWV-topology if the corresponding sub-tree vector distribution converge, i.e.,

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**Remark.** Let $(U_{N,k})_{k\in\mathbb{N}}$ be an i.i.d. sequence of $\mu_N^\circ$-distributed r.v., and $\underline{U}_N^n := (U_{N,1}, \ldots, U_{N,n})$. The definition of LWV-convergence requires:

- The convergence of $\| \mu_N \|$.
- The joint convergence in law with respect to the pGw-topology of $\tau_{\mathcal{X}_N}(\underline{U}_N^n)$, $n \in \mathbb{N}$.

## Proposition 2

*Consider a sequence of bi-measure $\mathbb{R}$-trees $\mathcal{X}_N = (T_N, \mu_N, \nu_N) \in \mathbb{H}^{f,\sigma}$ and $\mathcal{X} = (T, \mu, \nu) \in \mathbb{H}^{f,\sigma}$ such that $\| \mu_N \| \to \| \mu \|$, as $N \to \infty$. The following statements are equivalent:*

(i) $\mathcal{X}_N \xrightarrow{LWV} \mathcal{X}$, *as* $N \to \infty$.

(ii) *For all* $n \in \mathbb{N}$,

$$(\tau_{\mathcal{X}_N})_*(\mu_N^\circ)^{\otimes n} \underset{N\to\infty}{\overset{pGw}{\Longrightarrow}} (\tau_{\mathcal{X}})_*(\mu^\circ)^{\otimes n}.$$

(iii) *Equipping* $\prod_{n\in\mathbb{N}} \mathbb{H}_n$ *with the product topology* $\prod(sGw)$,

$$(\varsigma_{\mathcal{X}_N})_*(\mu_N^\circ)^{\otimes \mathbb{N}} \underset{N\to\infty}{\overset{\prod(sGw)}{\Longrightarrow}} (\varsigma_{\mathcal{X}})_*(\mu^\circ)^{\otimes \mathbb{N}}.$$

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By the Skorohod'd representation theorem there exists:

- a list $\underline{U}^n := (U_1, \ldots, U_n)$ of $n$ i.i.d. r.v. with common distribution $\mu^\circ$ and
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such that

$$\tau_{\mathcal{X}_N}(\underline{U}_N^n) \xrightarrow[N\to\infty]{\text{pGw}} \tau_{\mathcal{X}_N}(\underline{U}^n), \qquad \text{a.s.} \tag{1}$$

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In order to obtain sGw-convergence, by Lemma 8, it sufficient to prove

$$\forall I \subseteq \{1, \ldots, n\} \quad \nu_N([\![\pi_I^n(\underline{U}_N^n)]\!]) \xrightarrow[N\to\infty]{\text{weakly}} \nu([\![\pi_I^n(\underline{U}^n)]\!]), \quad \text{(as } \mathbb{R}_+\text{-valued r.v.)}.$$

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- In order to obtain convergence of laws on products space, we have to show convergence of f.d. marginals (it follows from definition of sGw-convergence).

**Remark.** (Relation with Gromov-weak topology)

- LWV-convergence of $(T_N, \mu_N, \nu_N)_{N\in\mathbb{N}}$ implies Gromov-weak convergence of $(T_N, \mu_N)_{N\in\mathbb{N}}$.
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**Remark.** (Reconstruction theorem for $\mathbb{H}^{f,\sigma}$). If $\mathcal{X}, \mathcal{X}' \in \mathbb{H}^{f,\sigma}$ such that $\varpi^{\mathcal{X}} = \varpi^{\mathcal{X}'}$, then $\mathcal{X} = \mathcal{X}'$.

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**Remark.** (Separability & Metrizability). The space $\mathbb{H}^{f,\sigma}$ equip with the LWV-topology is separable and metrizable.

**Example.** We define the sequence of $\mathbb{R}$-trees,

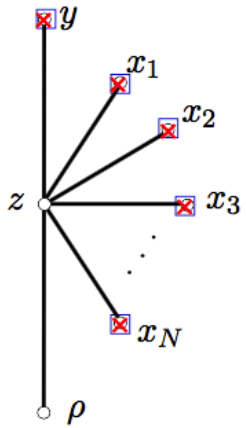


Figure: $T_N := \{\rho, z, x_1, x_2, \ldots, x_N\}$.

where $r_N(\rho, z) = r_N(z, y) = 1$ and $r_N(z, x_i) = \frac{1}{N}$, for all $i = 1, \ldots, N$. We define a probability measure

$$\mu_N = \lambda\delta_y + (1-\lambda)\sum_{i=1}^{N} \frac{1}{N}\delta_{x_i}$$

on the set of leaves of $T_N$.

Then

$$(T_N, \mu_N) \xrightarrow[N\to\infty]{\text{pGw}} (\{\rho, z, y\}, \lambda\delta_y + (1-\lambda)\delta_z).$$

If we endow these measure $\mathbb{R}$-trees with the measure $\nu_N = \mu_N$, then

$$(T_N, \mu_N, \nu_N) \xrightarrow[N\to\infty]{\text{LWV}} (\{\rho, z, y\}, \lambda\delta_y + (1-\lambda)\delta_z, \lambda\delta_y).$$

**Example.** The length measure, $\lambda_T$, on a $\mathbb{R}$-tree, generalizes the Lebesgue measure on $\mathbb{R}$. The length measure is defined by the following two requirements:

$$\forall\, x, y \in T : \lambda_T([x,y]) = r(x,y) \quad \text{and} \quad \lambda_T(\mathsf{Lf}(T)) = 0.$$

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We observe that the family of length measures $(\lambda_T)_{T\in\{\mathbb{R}-\text{trees}\}}$ depends continuously on the distances. More precisely,

## Definition 12

For a family $(T_i, \mu_i)_{i\in I}$ of measure $\mathbb{R}$-trees, we say that $(\nu_i)_{i\in I}$ (family of measures on $T_i$) depends continuously on the distances if for all $n \in \mathbb{N}$ there is a continuous mapping

$$F_n : \begin{cases} \mathbb{R}^{\binom{n+1}{2}} & \to & (\mathbb{H}_n, \mathrm{pGw}) \\ R^{T_i}(\underline{u}) & \mapsto & (\llbracket \underline{u} \rrbracket, \underline{u}, \nu_i). \end{cases}$$

for all $\underline{u} \in T_i^n$, for all $i \in I$.

## Proposition 3 (LWV-convergence from Gromov-weak convergence)

*Consider a sequence of* $(\mathcal{X}_N)_{N\in\mathbb{N}} := (T_N, \mu_N, \nu_N)_{N\in\mathbb{N}}$ *and* $\mathcal{X}_\infty := (T_\infty, \mu_\infty, \nu_\infty)$ *in* $\mathbb{H}^{f,\sigma}$ *such that the measures* $\nu_\infty$ *and* $(\nu_N)_{N\in\mathbb{N}}$ *depend continuously on the distances. If*

$$(T_N, \mu_N) \xrightarrow[N\to\infty]{pGw} (T_\infty, \mu_\infty) \quad \textit{then} \quad (T_N, \mu_N, \nu_N) \xrightarrow[N\to\infty]{LWV} (T_\infty, \mu_\infty, \nu_\infty).$$

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$$(T_N, \mu_N) \xrightarrow[N\to\infty]{pGw} (T_\infty, \mu_\infty) \quad \textit{then} \quad (T_N, \mu_N, \nu_N) \xrightarrow[N\to\infty]{LWV} (T_\infty, \mu_\infty, \nu_\infty).$$

**Proof.** For $N \in \mathbb{N}\cup\{\infty\}$, let $\underline{U}_N$ be a random vector in $T_N^n$ with distribution $(\mu_N^\circ)^{\otimes n}$. Then Gromov-weak convergence implies that $\| \mu_N \| \to \| \mu_\infty \|$ and

$$R^{T_N}(\underline{U}_N) \xRightarrow[N\to\infty]{\text{(law)}} R^{T_\infty}(\underline{U}_\infty)$$

By the continuous mapping theorem, we obtain

$$([\![\underline{U}_N]\!], \underline{U}_N, \nu_N) = F_n(R^{T_N}(\underline{U}_N)) \xRightarrow[N\to\infty]{\text{(law)}} F_n(R^{T_\infty}(\underline{U}_\infty)) = ([\![\underline{U}_\infty]\!], \underline{U}_\infty, \nu_\infty).$$

Since $([\![\underline{U}_N]\!], \underline{U}_N, \nu_N) \overset{(law)}{=} (\tau_{\mathcal{X}_N})_*((\mu_N^\circ)^{\otimes n})$ for all $N \in \mathbb{N}\cup\{\infty\}$, the claimed LWV-convergence now follows from Proposition 2.

## Lemma 13 (Sum of pruning measures)

*Let* $\mathcal{X}_N^i = (T_N, \mu_N, \nu_N^i) \in \mathbb{H}^{f,\sigma}$ *with*

$$\mathcal{X}_N^i \xrightarrow[N\to\infty]{LWV} \mathcal{X}^i = (T, \mu, \nu^i) \in \mathbb{H}^{f,\sigma} \quad \text{for} \quad i = 1, 2.$$

*If* $(\nu_N^1)_{N\in\mathbb{N}}$ *depends continuously on the distances, we obtain*

$$(T_N, \mu_N, \nu_N^1 + \nu_N^2) \xrightarrow[N\to\infty]{LWV} (T, \mu, \nu^1 + \nu^2).$$

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*Ann. Inst. Henri Poincaré Probab. Stat.*, 51(4):1342–1368, 2015.

# The pruning process

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- The set of cut points up to time $t \geq 0$ is the projection of $\pi \cap ([0, t] \times T)$ onto the tree, i.e.

$$\pi_t := \{v \in T : \exists\ s \leq t \text{ such that } (s, v) \in \pi\},$$

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- For every $\nu \in T$, the tree pruned at $v$ is defined by

$$T^v := \{w \in T : v \notin [\rho, w]\}.$$

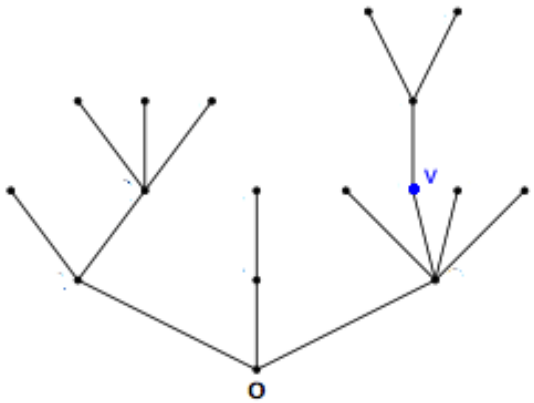


Figure: $T$

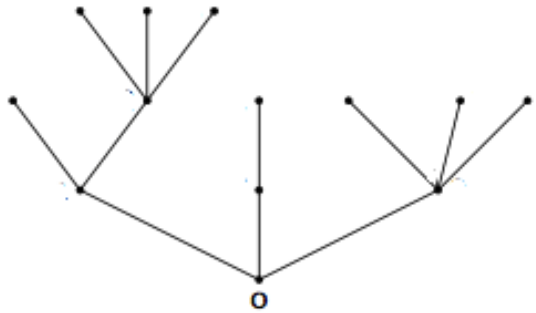


Figure: $T^v$

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- The pruned tree at the set $\pi_t \subseteq T$, it is the intersection of the trees $T^v$ pruned at $v \in \pi_t$, i.e.,

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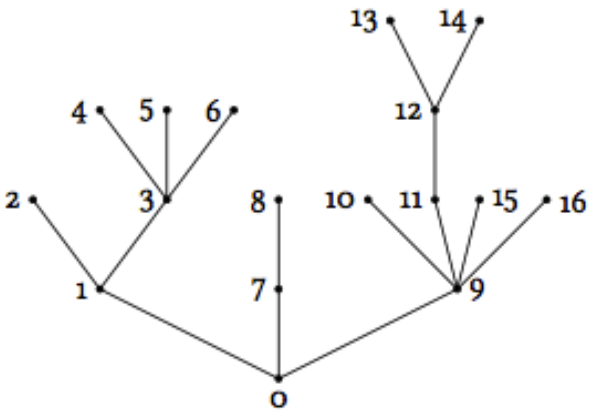


Figure: $T = \{0, 1, 2, \ldots, 16\}$

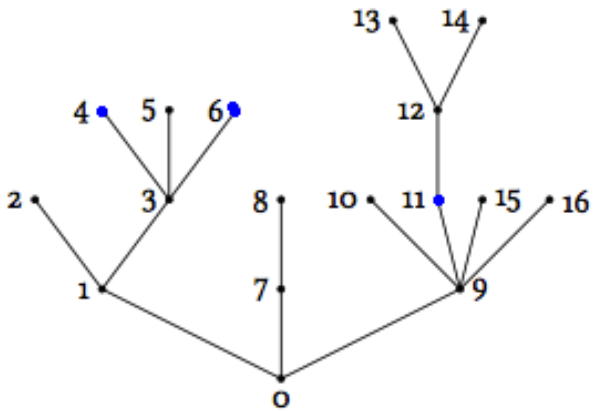


Figure: $\pi_t = \{4, 6, 11\}$

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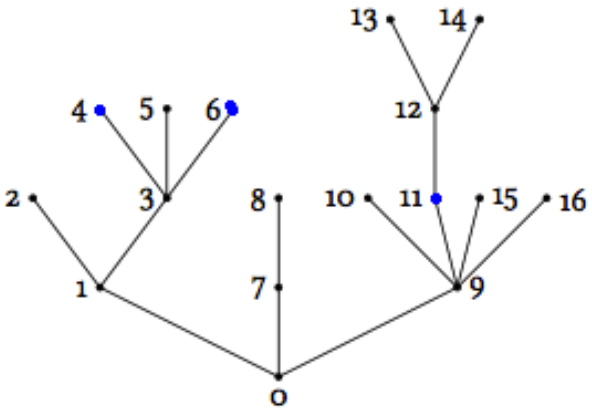


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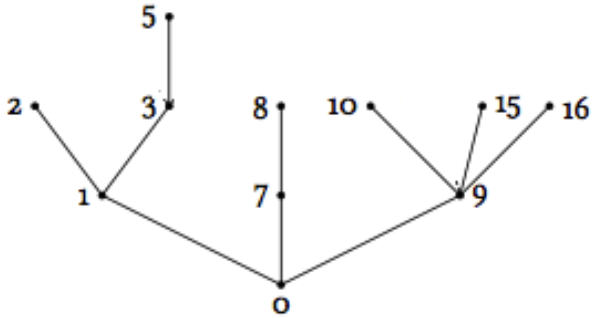


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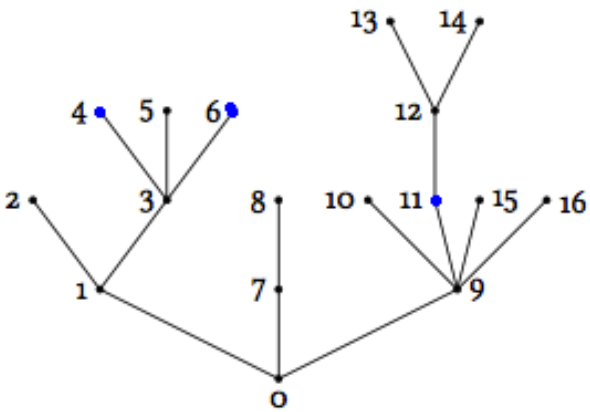


Figure: $\pi_t = \{4, 6, 11\}$

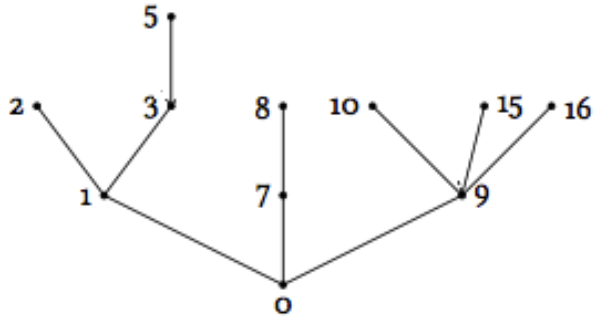


Figure: $T^{\pi_t}$

We equip the pruned tree $T^{\pi_t}$ with the restrictions of the measures $\mu$ and $\nu$, i.e.,

$$(T^{\pi_t}, \mu, \nu) \overset{\text{“”}}{=} (T^{\pi_t}, \mu \restriction_{T^{\pi_t}}, \nu \restriction_{T^{\pi_t}}) \in \mathbb{H}^{f,\sigma}.$$

**Lemma 14 (Càdlàg paths)**

*Fix $\mathcal{X} = (T, \mu, \nu) \in \mathbb{H}^{f,\sigma}$ and a set $\pi \in \mathbb{R}_+ \times T$. The map $t \mapsto \mathcal{X}_t := (T^{\pi_t}, \mu, \nu)$ is càdlàg with respect to the LWV-topology.*

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$$\Psi(\mathcal{X}) = \Psi^{\gamma,n,\Phi}(\mathcal{X}) := \gamma(\| \mu \|) \cdot \int_{T^n} \mu^{\otimes n}(\mathrm{d}\underline{u})\Phi(\tau_{\mathcal{X}}(\underline{u}))$$

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**Remark.** These kind of test functions are helpful characterizing LWV-convergence. In fact, they induce the LWV-topology.

**Proof.**

**Right continuity.** Let $0 < s < t$. As $T^{\pi_t} \subseteq T^{\pi_s}$, we obtain for $\Psi = \Psi^{1,n,\Phi}$,

$$\begin{aligned} |\Psi(\mathcal{X}_s) - \Psi(\mathcal{X}_t)| &= \left| \int_{(T^{\pi_s})^n \setminus (T^{\pi_t})^n} \mu^{\otimes n}(\mathrm{d}\underline{u}) \Phi(\tau_{\mathcal{X}}(\underline{u})) \right| \\ &\leq \| \Phi \|_\infty \cdot \mu^{\otimes n} \left( (T^{\pi_s})^n \setminus (T^{\pi_t})^n \right) \\ &\leq \| \Phi \|_\infty \cdot n \cdot \| \mu \|^{n-1} \mu \left( T^{\pi_s} \setminus T^{\pi_t} \right) \\ &\underset{t \downarrow s}{\longrightarrow} 0, \end{aligned}$$

since $\bigcap_{t>s} T^{\pi_s} \setminus T^{\pi_t} = \emptyset$ for fixed $s$.

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since $\bigcap_{t>s} T^{\pi_s} \setminus T^{\pi_t} = \emptyset$ for fixed $s$.

**Left limit.** Define $T_{t-} := \cap_{0 \leq s < t} T^{\pi_s} \supseteq T^{\pi_t}$ for each $t > 0$ and consider $\mathcal{Y}_t := (T_{t-}, \mu, \nu) \in \mathbb{H}^{f,\sigma}$. Similarly, as before, for all $0 < s < t$ there exists a constant $C^\Psi$ such that

$$\begin{aligned} |\Psi(\mathcal{X}_s) - \Psi(\mathcal{Y}_t)| &= C^\Psi \mu \left( T^{\pi_s} \setminus T^{t-} \right) \\ &\underset{t \uparrow s}{\longrightarrow} 0, \end{aligned}$$

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# The pruning process

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## Definition 15 (The pruning process)

We consider that $\lambda$ is the Lebesgue measure on $\mathbb{R}_+$. We define the pruning process, $X := (X_t)_{t \geq 0}$, as the bi-measure $\mathbb{R}$-tree-valued process obtained by pruning

$$X_0 := \mathcal{X} = (X, r, \rho, \mu, \nu)$$

at the points of the Poisson point process $\pi_t(\cdot) = \pi_t(\cdot)^{\mathcal{X}} := \pi^{\mathcal{X}}([0,t] \times \cdot)$, i.e.,

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**Notation.** We denote by $\mathbb{P}^{\mathcal{X}}$ (or $\mathbb{P}$) the distribution of the process $\mathcal{X}$ starting from $X_0 = \mathcal{X}$. We write $\mathbb{E}^{\mathcal{X}}$ (or $\mathbb{E}$) the expectation with respect $\mathbb{P}^{\mathcal{X}}$.

# The pruning process

## Lemma 16 (Strong Markov property)

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- $X$ is adapted to this filtration.
- The strong Markov property of the Poisson process: for every $t \geq 0$, stopping time $\sigma$, and $\underline{u} \in \mathbb{T}^n$, $n \in \mathbb{N}$,

$$\mathbb{P}\left(\pi_{\sigma+t}([\![\underline{u}]\!]) = 0 | \mathcal{A}_\sigma\right) = \mathbb{1}_{\{\pi_\sigma([\![\underline{u}]\!])=0\}} \mathbb{P}\left(\pi_t([\![\underline{u}]\!]) = 0\right).$$

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For every $\Psi = \Psi^{1,n,\Phi}$,

$$\begin{aligned}
\mathbb{E}\left[\Psi(X_{\sigma+t})|\mathcal{A}_\sigma\right] &= \int_{T^n} \mu^{\otimes n}(\mathrm{d}\underline{u}) \mathbb{P}\left(\pi_{\sigma+t}([\![\underline{u}]\!])) = 0 | \mathcal{A}_\sigma\right) \cdot \Phi(\tau_{\mathcal{X}}(\underline{u})) \\
&= \int_{(T^{\pi_\sigma})^n} \mu^{\otimes n}(\mathrm{d}\underline{u}) e^{-t\nu([\![\underline{u}]\!])} \cdot \Phi(\tau_{\mathcal{X}}(\underline{u})) \\
&= \mathbb{E}^{X_\sigma}\left[\Psi(X_t)\right].
\end{aligned}$$

Because $X_t \in \mathbb{H}^{\|\mu\|,\sigma}$, for all $t \geq 0$, the class of Polynomials is a separating class on this space, we obtain the strong Markov property.

# The pruning process

**Some crucial properties:** (Löhr, Voisin, and Winter (2015))

- The pruning process is Feller continuous.
- The law of pruning process $X = (X_t)_{t \geq 0}$ does depend continuously on the initial value $\mathcal{X}$, i.e., the map

$$\begin{aligned} \mathcal{M}_1(\mathbb{H}^{f,\sigma}) &\to \mathcal{M}_1(D_{\mathbb{H}^{f,\sigma}}(\mathbb{R}_+)), \\ \eta &\mapsto \mathcal{L}(X^\eta) \end{aligned}$$

  is continuous, where $D_{\mathbb{H}^{f,\sigma}}(\mathbb{R}_+)$ the space càdlàg paths with Skorokhod topology, $\mathcal{L}$ is the law of the process, and $X^\eta$ is the process with initial condition $\mathcal{L}(X_0^\eta) = \eta$.

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**Remark.** One could also compute the infinitesimal generator of $X$.

# Applications

# An approach via excursions

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- We consider the pseudo-distance on $[0,1]$ by setting,

$$r_g(x,y) := g(x) + g(y) - 2 \inf_{z \in [x \wedge y, x \vee y]} g(z), \quad \text{for} \quad x, y \in [0,1].$$

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- The image of the projection $p_g : [0,1] \to [0,1] \setminus \sim_g$ endowed with the push forward of $r_g$ (again denoted $r_g$), i.e.

$$T_g = (T_g, r_g, \rho_g) := (p_g([0,1]), r_g, p_g(0)),$$

is a (rooted) $\mathbb{R}$-tree; see Lemma 3.1 Evans and Winter (2006).

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- We endow $T_g$ with the probability measure $\mu_g := (p_g)_* \lambda_{[0,1]}$, the push forward of the Lebesgue measure on $[0,1]$.

# An approach via excursions

We denote by $G : \mathcal{E} \to \mathbb{H}_0$ the resulting "glue function",

$$G(g) := (T_g, \mu_g),$$

which sends an excursion to a rooted measure $\mathbb{R}$-tree.

**Remark.** The map $G$ is continuous:

- if $\mathbb{H}_0$ is endowed with the Gromov-weak topology, and
- $\mathcal{E}$ with the uniform topology (see Proposition 2.9 in Abraham et. al. (2014) for the case of continuous excursions) or, more generally, with the weaker excursion topology introduced in Theorem 4.8 in Löhr (2013).

# An approach via excursions

Consider a sequence of random excursions $g_N = (g_N(s), s \in [0,1]) \in \mathcal{E}$, $N \in \mathbb{N}$, such that

$$g_N \xrightarrow[N\to\infty]{(\mathrm{d})} g \in \mathcal{E}$$

(with respect to the uniform, respectively the excursion topology).

- We denote by $(X_t^N)_{t\geq 0}$ the pruning process started in the bi-measure tree $\mathcal{X}_{g_N} := (T_{g_N}, \mu_{g_N}, \lambda_{T_{g_N}})$, where $\lambda_{T_{g_N}}$ is the length measure on $T_{g_N}$.
- Similarly $(X_t)_{t\geq 0}$ for $\mathcal{X}_g$.

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We notice that

- By the continuity of $G$, $G(g_N) \xrightarrow[N\to\infty]{\mathsf{pGw}} G(g)$.
- By the Proposition 3, $\mathcal{X}_{g_N} \xrightarrow[N\to\infty]{\mathsf{LWV}} \mathcal{X}_g$.
- By the continuity of the initial distribution, we get the Skorokhod convergence

$$(X_t^N)_{t\geq 0} \xRightarrow[N\to\infty]{\mathsf{Sk}} (X_t)_{t\geq 0}$$

as $\mathbb{H}^{f,\sigma}$-valued process with LWV-topology.

# Galton-Watson trees

Let $\eta$ be a probability measure on $\{0, 1, 2, \dots\}$ such that

$$\sum_{i \geq 0} i\eta(\{i\}) \leq 1 \quad \text{and} \quad \eta(\{1\}) < 1.$$

A Galton-Watson tree with offspring distribution $\eta$ is a random plane tree $\mathcal{G}$ such that:

(i) The number of children of the root is distributed according to $\eta$.

(ii) Conditionally on the fact that the root has $j$ children, the number of children of the $j$ children of the root are independent of law $\eta$, and so on (Branching Property).

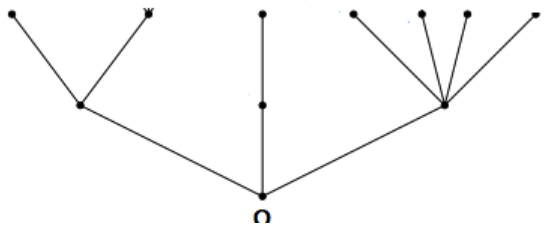

Figure: This happens with probability $\eta(0)^7\eta(1)^2\eta(2)\eta(3)\eta(4)$.

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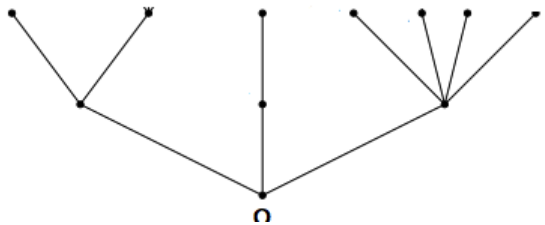

Figure: This happens with probability $\eta(0)^7\eta(1)^2\eta(2)\eta(3)\eta(4)$.

**Remark.**

- In this case, the tree is almost surely finite.
- We encode $\mathcal{G}$ as a rooted $\mathbb{R}$-tree with unit length edges.

# Galton-Wason trees.

For each $N \in \mathbb{N}$, let $\mathcal{G}_N$ be the G-W tree $\mathcal{G}$ conditioned to have $N$ vertices. We consider two sampling measures $\mu$ on $\mathcal{G}_N$:

- The normalized length measure

$$\mu_N^{\text{ske}} := \frac{1}{N}\lambda_{G_N}.$$

- The uniform measure on the vertices,

$$\mu_N^{\text{nod}} := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i},$$

where $\{x_1, \ldots, x_N\}$ are the vertices of $\mathcal{G}_N$.

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Notice that,

$$\mu_N^{\text{nod}}(A) = \sum_{x \in \text{nod}(A)} \mu_N^{\text{ske}}([x_-, x]) \leq \mu_N^{\text{ske}}(\{v \in \mathcal{G}_N : r_{G_N}(v, A) < 1\})$$

where $\text{nod}(A)$ is the set of vertices in $A$ and $x_-$ is the parent of $x$.

# Galton-Wason trees

In order to obtain convergence, we rescale the G-W tree $\mathcal{G}_N$ to have edge lengths $a_N/N > 0$, i.e., we leave the set unchanged and multiply the metric $r_{\mathcal{G}_N}$ by $a_N/N$.

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We observe that

$$d_{\mathsf{Pr}}^{\frac{a_N}{N}G_N}(\mu_N^{\mathsf{ske}}, \mu_N^{\mathsf{nod}}) \leq \frac{a_N}{N}. \tag{2}$$

Then $\mu_N^{\mathsf{ske}}$ and $\mu_N^{\mathsf{nod}}$ become arbitrary close whenever $\frac{a_N}{N} \to 0$, as $N \to \infty$.

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We consider two different pruning measures $\nu$:

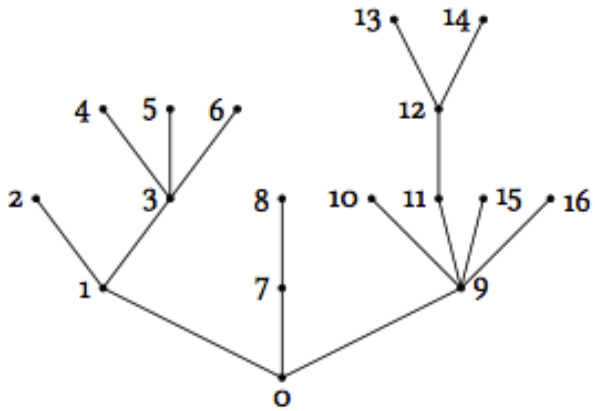
- The length measure on the rescaled tree,

$$\nu_N^{\mathsf{ske}} := \lambda_{\frac{a_N}{N}\mathcal{G}_N} = a_N \cdot \mu_N^{\mathsf{ske}}.$$

- The rescaled uniform measure on the vertices,

$$\nu_N^{\mathsf{nod}} := a_N \cdot \mu_N^{\mathsf{nod}}.$$

We associated the conditioned and rescaled bi-measure G-W tree with an excursion.

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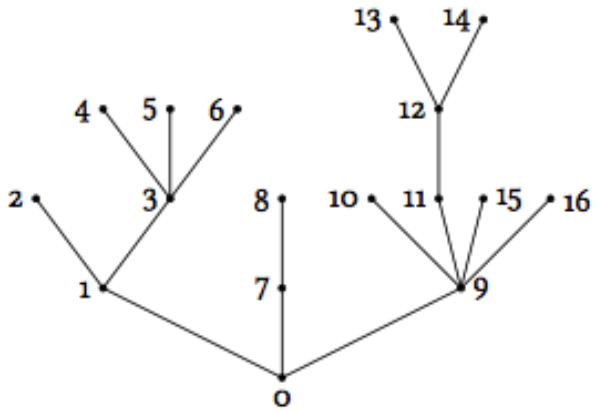


Figure: A G-W tree.

- We consider the depth-first search algorithm and obtain the contour sequence $\rho = y_0, y_1, \ldots, y_{2N-1}, y_{2N} = \rho$ in the discrete tree.

- The contour process $(C_N(t) : 0 \leq t \leq 1)$ of $\mathcal{G}_N$ is the linear interpolation of

$$C_N(k) := \text{height}(y_k) := r_{\mathcal{G}_N}(\rho, y_K), \quad k = 0, \ldots, 2N.$$

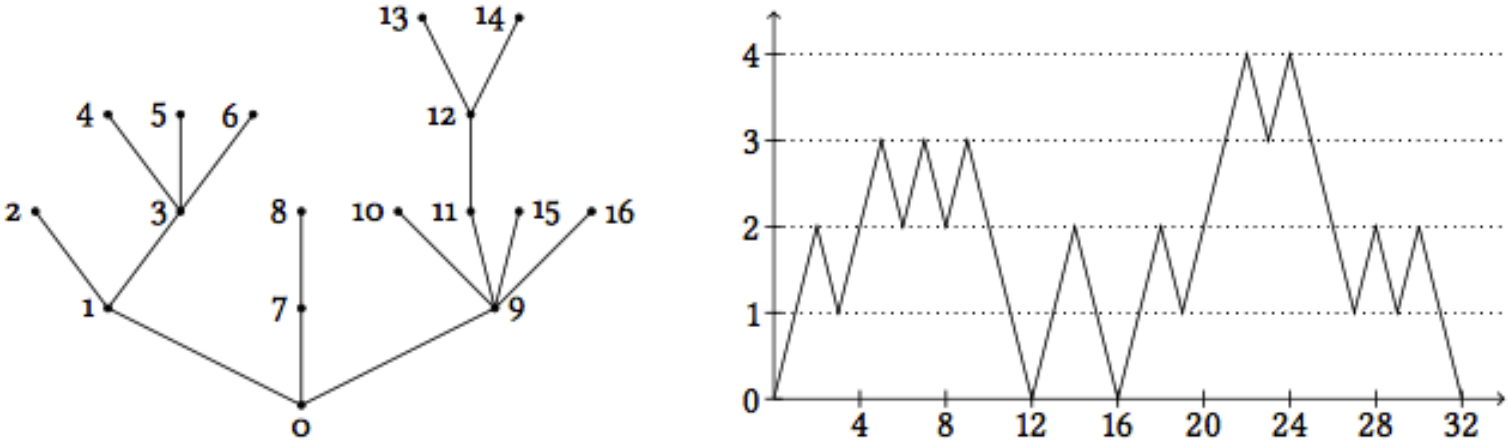


Figure: A G-W tree and its contour process.

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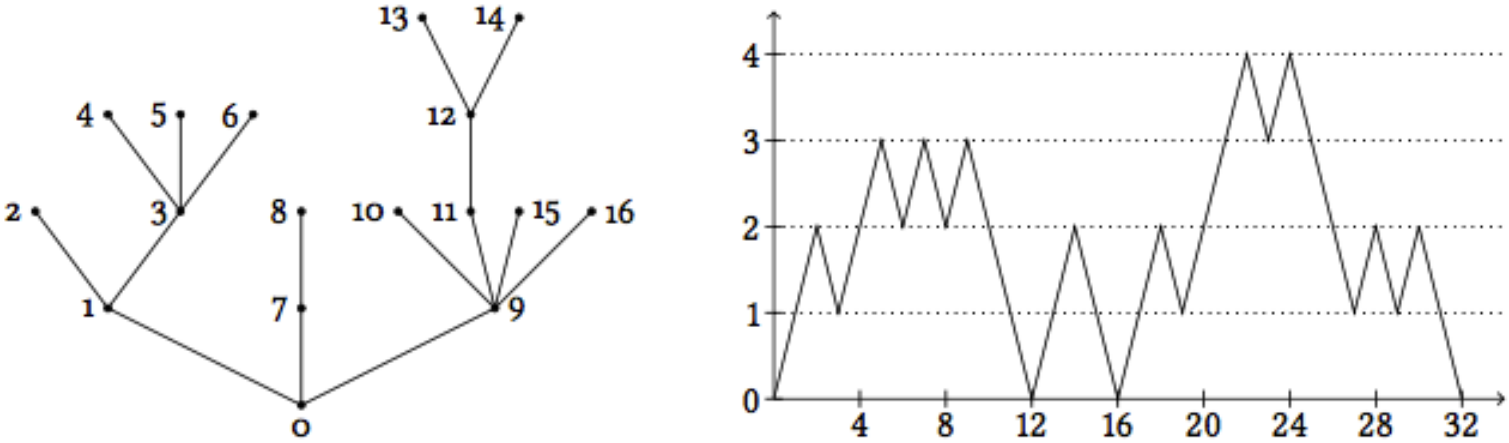


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- In fact, we consider $C_N\left(\frac{k}{2N}\right) := \text{height}(y_k)$ such that its domain is normalized to $[0,1]$. Then, we obtain that

$$g(C_N) = (\mathcal{G}_N, \mu_N^{\text{ske}}).$$

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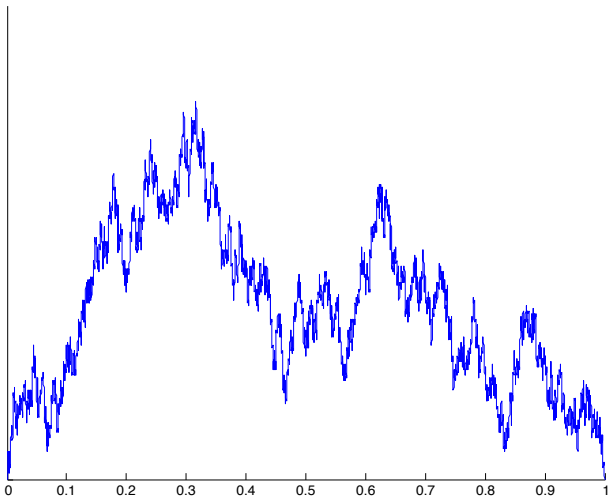
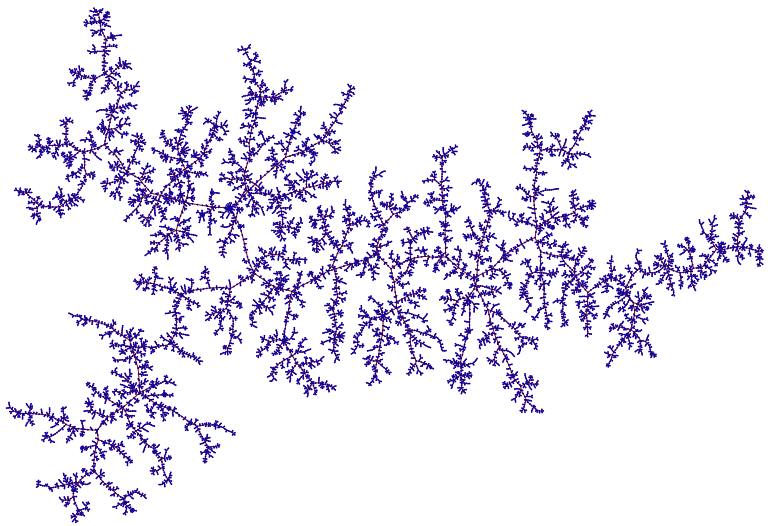
where $B$ is the standard Brownian excursion. By the approach via excursions:

$$\left(\frac{\sigma}{\sqrt{N}} \mathcal{G}_N, \mu_N^{\mathsf{ske}}, \nu_N^{\mathsf{ske}}\right) \xrightarrow[N\to\infty]{\mathsf{LWV}} (CRT, \mu, \lambda_{CRT}),$$

where $(CRT, \mu) = G(2B)$ is the $\mathbb{R}$-tree called the Brownian continuum random tree, and $\lambda_{CRT}$ is the length measure on the Brownian CRT.



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where $B$ is the standard Brownian excursion. By the approach via excursions:

$$\left(\frac{\sigma}{\sqrt{N}} \mathcal{G}_N, \mu_N^{\mathsf{ske}}, \nu_N^{\mathsf{ske}}\right) \xrightarrow[N\to\infty]{\mathsf{LWD}} (CRT, \mu, \lambda_{CRT}),$$

where $(CRT, \mu) = G(2B)$ is the $\mathbb{R}$-tree called the Brownian continuum random tree, and $\lambda_{CRT}$ is the length measure on the Brownian CRT. By equation (2),

$$\left(\frac{\sigma}{\sqrt{N}} \mathcal{G}_N, \mu_N, \nu_N\right) \xrightarrow[N\to\infty]{\mathsf{LWV}} (CRT, \mu, \lambda_{CRT}),$$

for all choices $\mu_N \in \{\mu_N^{\mathsf{ske}}, \mu_N^{\mathsf{nod}}\}$ and $\nu_N \in \{\nu_N^{\mathsf{ske}}, \mu_N^{\mathsf{nod}}\}$.

Finally, we have

$$\left(\frac{\sigma}{\sqrt{N}}\mathcal{G}_N^{\pi_t}, \mu_N, \nu_N\right)_{t\geq 0} \xrightarrow[N\to\infty]{\text{Sk}} (CRT^{\pi_t}, \mu, \lambda_{CRT})_{t\geq 0}.$$

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with the LWV-topology.

**Remark.**

- For $\nu_N = \nu_N^{\text{ske}}$, the pruning process $(\mathcal{G}_N^{\pi_t})_{t\geq 0}$ is, up to the time transformation $u = e^{-t/\sqrt{N}}$, the same as the pruning process $(\mathcal{G}_u^{\text{AP}})_{u\in[0,1]}$ uniformly on the edges of Aldous and Pitman (1998).
- The process $(CRT^{\pi_t}, \mu, \lambda_{CRT})_{t\geq 0}$ is the one considered by Aldous and Pitman (1998) and Abraham and Serlet (2002).

# The $\alpha$-stable Lévy tree

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$$\left(\frac{a_N}{N} C_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(\mathrm{d})} (H(t) : 0 \leq t \leq 1),$$

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**Remark.** Let $Y = (Y(t) : 0 \leq t \leq 1)$ be the normalized excursion of a stable spectrally positive Lévy process of index $\alpha$ above its infimum straddling time 1 (see Chaumont 1997).

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- For $\alpha = 2$, $H = 2X = 2B$.
- For $\alpha \in (1,2]$, we set

$$I(s,t) = \inf_{u\in[s,t]} Y(u), \qquad 0 \leq s \leq t \leq 1.$$

Then

$$H(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \mathrm{d}s \mathbb{1}_{\{Y(s) < I(s,t)+\varepsilon\}}, \qquad t \in [0,1].$$

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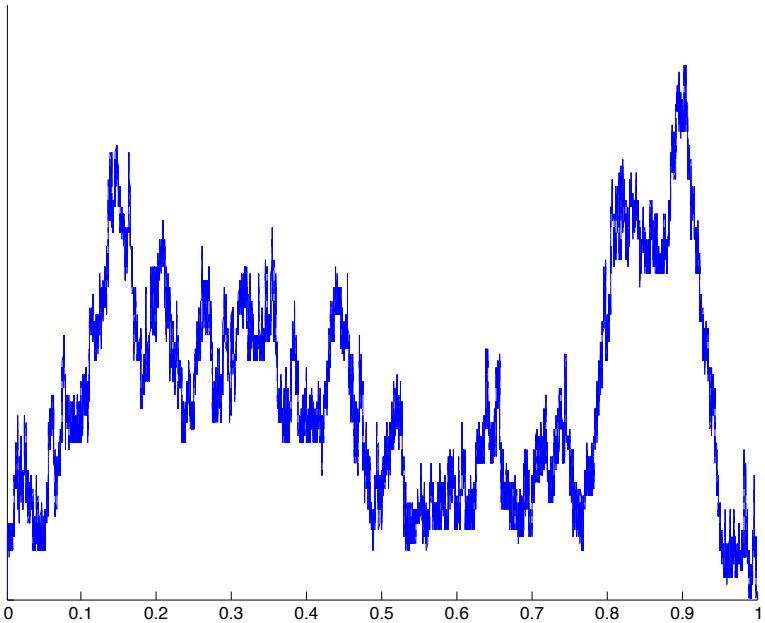


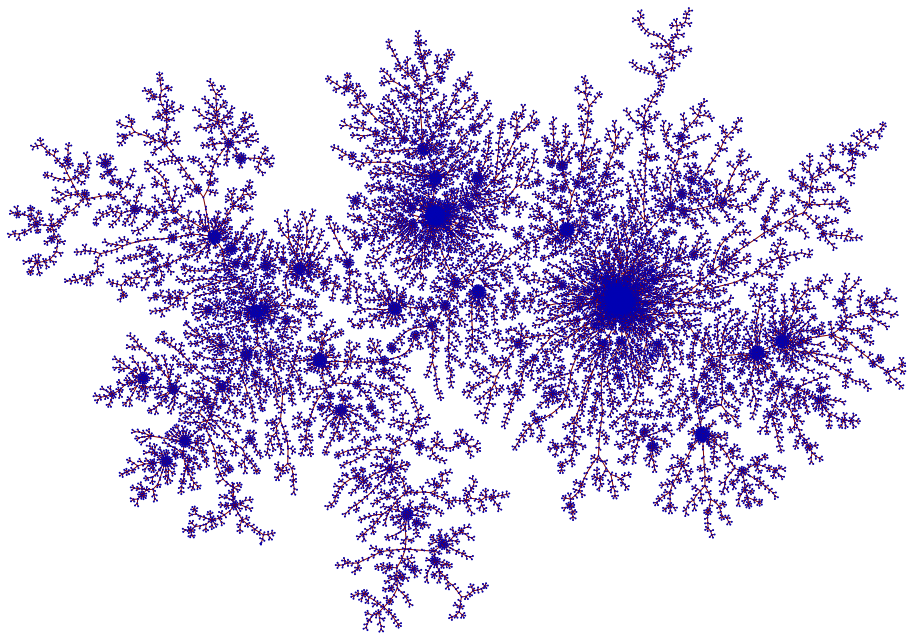

Figure: $\alpha = 1.1$-stable Lévy tree

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**Remark.** For instance:

$$\eta(\{k\}) \sim_{k\to\infty} Ck^{-1-\alpha}, \quad \text{we have } \frac{a_N}{N} = N^{-\overline{\alpha}} \left(\frac{\alpha(\alpha-1)}{C\Gamma(2-\alpha)}\right)^{-1/\alpha},$$

with $\overline{\alpha} = 1 - 1/\alpha$.

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By the approach via excursions:

$$\left(\frac{a_N}{N} \mathcal{G}_N^{\pi_t}, \mu_N, \nu_N\right)_{t\geq 0} \xRightarrow[N\to\infty]{\mathrm{Sk}} (LT_\alpha^{\pi_t}, \mu, \lambda_{LT_\alpha})_{t\geq 0},$$

for all choices $\mu_N \in \{\mu_N^{\mathrm{ske}}, \mu_N^{\mathrm{nod}}\}$ and $\nu_N \in \{\nu_N^{\mathrm{ske}}, \mu_N^{\mathrm{nod}}\}$.

# A non-uniform pruning process on the branch points

Abraham, Delmas and He (2012) consider the following pruning procedure:

> *They cut a branch point $v$ and its subtree above independently and with probability $1 - u^{c(v)-1}$,*

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Abraham, Delmas (2012) consider the following pruning procedure in the infinite branch points of a $\alpha$-stable Lévy tree:

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# A non-uniform pruning process on the branch points

## Conjecture of Abraham-Delmas, Löhr-Winter-Voisin

There exists a sequence $b_N$ such that

$$\left(\frac{a_N}{N}\mathcal{G}_N, \mu_N^{\text{nod}}, \nu_N^{\text{ske}} + b_N \nu^{\text{ADH}}\right) \xrightarrow[N\to\infty]{\text{LWV}} (LT_\alpha, \mu^{\text{nod}}, \nu^{\text{ske}} + \nu^{\text{AD}}),$$

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**Remark.**

- We expect a similar result for general Lévy trees.
- By Lemma 13, it is enough to show that

$$\left(\frac{a_N}{N}\mathcal{G}_N, \mu_N^{\text{nod}}, b_N \nu^{\text{ADH}}\right) \xrightarrow[N\to\infty]{\text{LWV}} (LT_\alpha, \mu^{\text{nod}}, \nu^{\text{AD}}),$$

(Recall that $\nu_N^{\text{ske}}$ depends continuously on the distances.)

# Idea of the proof

We consider

- Let $\underline{U}_N^n := (U_{N,1}, \ldots, U_{N,n})$ be i.d.d. random variables with distribution $\mu_N^{\text{nod}}$.
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In order to obtain LWV-convergence, it is sufficient to prove that

$$\forall I \subseteq \{1, \dots, n\} \quad \text{and} \quad b_N \nu_N^{ADH}([\![\pi_I^n(\underline{U}_N^n)]\!]) \xrightarrow[N\to\infty]{\text{weakly}} \nu^{AD}([\![\pi_I^n(\underline{U}^n)]\!]),$$

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**Remark.** We notice

$$\nu_N^{ADH}([\![\pi_1^n(\underline{U}_N^n)]\!]) = \nu_N^{ADH}([\![\rho, U_{N,i_1}]\!]) = \sum_{v \in [\![\rho, U_{N,i_1}]\!]} (c(v) - 1), \quad i_1 \in \{1, \dots, n\},$$

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**Remark.** We recognize the Łukasiewicz path walk.

# The Łukasiewicz path

Let $\mathcal{G}_N$ be a G-W tree.

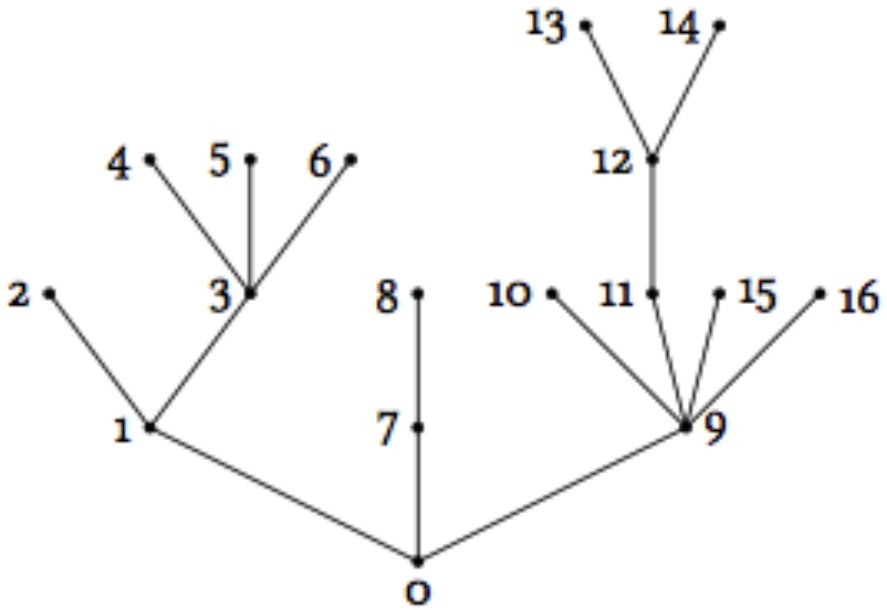


Figure: A G-W tree.

We let $\rho \preceq u_0 \preceq \cdots \preceq u_N$ be its vertices, listed in depth-first search order.

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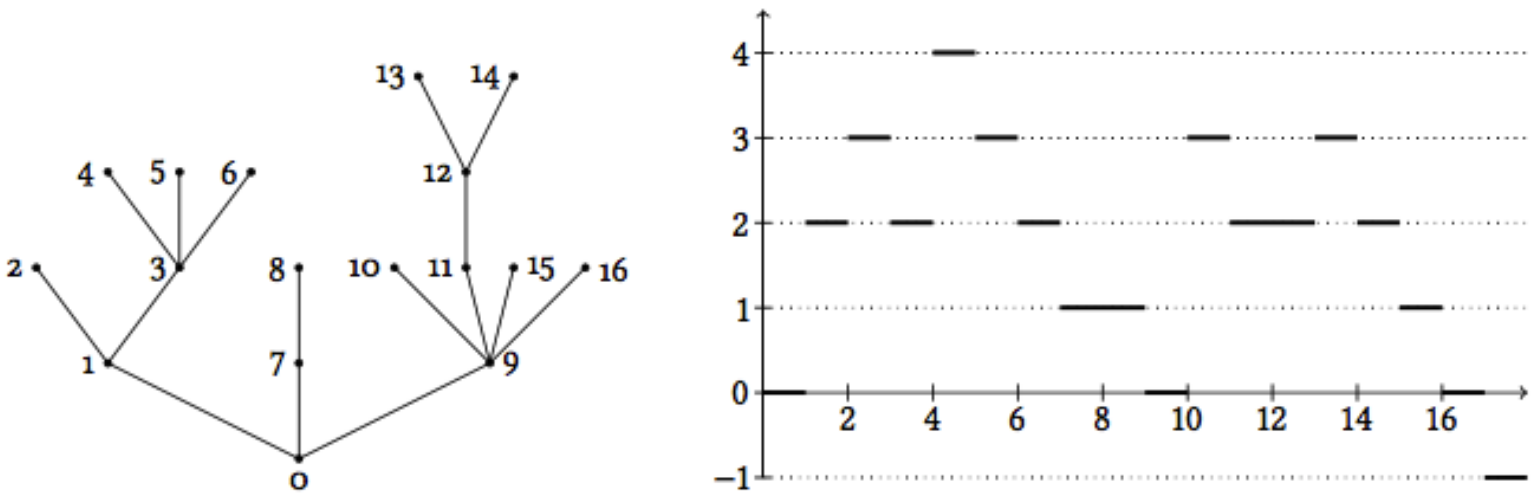


Figure: A G-W tree and its Łukasiewicz path.

Define $W_N = (W_N(k) : 0 \leq k \leq N)$ defined by $W_N(0) = 0$ and

$$W_N(k+1) = W_N(k) + c(u_k) - 1 = \sum_{i=1}^{k+1} (c(u_{i-1}) - 1), \quad \text{for} \quad 1 \leq k \leq N-1.$$

# The Łukasiewicz path

The walk $W_N = (W_N(k) : 0 \leq k \leq N)$,

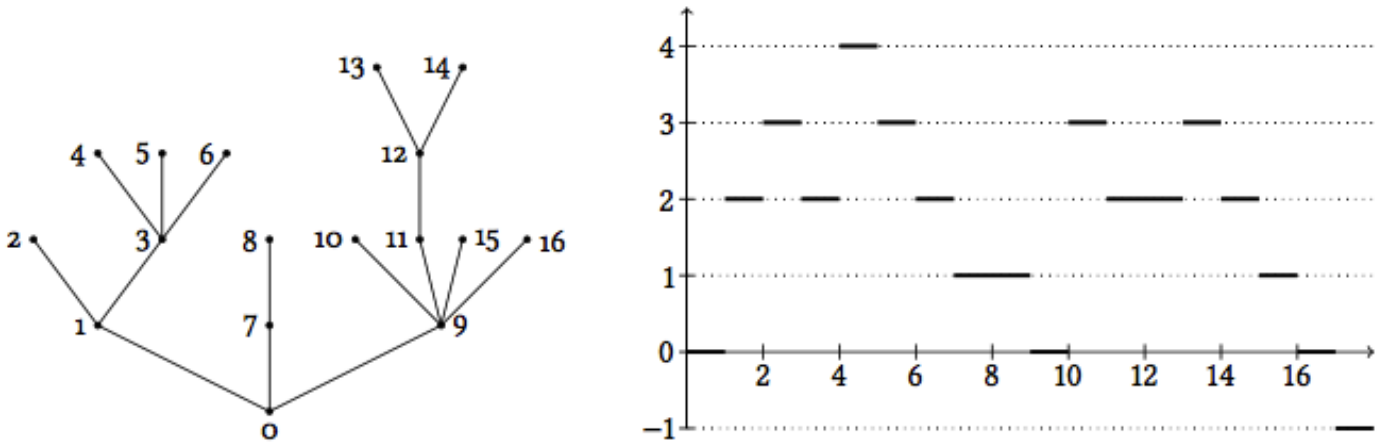


Figure: A G-W tree and its Łukasiewicz path.

**Remark.**

- Necessarily $W_N(N) = -1$.
- We observe that $\Delta W_N(k+1) = W_N(k+1) - W_N(k) = c(u_k) - 1$, for $0 \leq k \leq N-1$.
- The path $W_N$ associated with a G-W tree with offspring distribution $\eta$ is distributed as random walk on $\mathbb{Z}$ starting from $0$ with step distribution $\eta(\{\cdot + 1\})$ and stopped at the first hitting time of $-1$ (see Le Gall-Le Jan (1998)).

# The key observation

## The discrete genealogical order

The discrete genealogical order of $\mathcal{G}_N$ can be recovered from $W_N$. More precisely,

$$u(i) \preccurlyeq u(j) \quad \text{if and only if} \quad i \leq j \quad \text{and} \quad W_N(i) = \inf_{i \leq m \leq j} W_N(m).$$

In particular, $u(i) \prec u(j)$ when $u(i) \preccurlyeq u(j)$ and $i \neq j$.

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We let

$$\sigma_N(t) = \sum_{u_0 \preccurlyeq u_i \prec u_{\lfloor nt \rfloor}} \Delta W_N(i) \qquad \text{for} \qquad t \in [0,1].$$

**Remark.**

- The sum is over all the indexes $i$ such that $u_0 \preccurlyeq u_i \prec u_{\lfloor nt \rfloor}$.
- The quantity $\sigma_N(t)$ is the sum of the quantities $c(v) - 1$ over all strict ancestors $v \in \mathcal{G}_n$ of $u_{\lfloor nt \rfloor}$ in depth-first search order.

Theorem 3.1 of Duquesne 2003 implies that there exists a sequence $a_N$ such that

$$\left(\frac{a_N}{N}C_N(t), \frac{1}{a_N}W_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(\mathsf{d})} (H(t), Y(t) : 0 \leq t \leq 1),$$

where $H$ is a continuous excursion that codes an $\alpha$-stable Lévy tree and $Y$ is the normalized excursion of a strictly stable spectrally positive $\alpha$-stable Lévy process.

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**Notation.** For $0 < t \leq 1$, we write

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**Remark.** The genealogical order of $LT_\alpha$ can be easily recovered from $Y$ as follows. For every $s, t \in [0,1]$,

$$s \preccurlyeq t \quad \text{if} \quad s \leq t \quad \text{and} \quad Y(s-) \leq I(s,t),$$

In particular $s \prec t$ when $s \preccurlyeq t$ and $s \neq t$.

## Theorem 17

*We have that*

- *For* $\alpha \in (1,2)$,

$$\left(\frac{1}{a_N}\sigma_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(d)} (\sigma(t) : 0 \leq t \leq 1)$$

*in distribution for the Skorokhod topology on* $\mathbb{D}([0,1],\mathbb{R})$, *where*

$$\sigma(t) = \sum_{0 \preccurlyeq s \prec t} \Delta_s.$$

- *For* $\alpha = 2$,

$$\left(\frac{1}{a_N}\sigma_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(d)} (H_t : 0 \leq t \leq 1)$$

*in distribution for the Skorokhod topology on* $\mathbb{D}([0,1],\mathbb{R})$.

## Theorem 17

*We have that*

- *For* $\alpha \in (1,2)$,

$$\left(\frac{1}{a_N}\sigma_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(d)} (\sigma(t) : 0 \leq t \leq 1)$$

*in distribution for the Skorokhod topology on* $\mathbb{D}([0,1],\mathbb{R})$, *where*

$$\sigma(t) = \sum_{0 \preccurlyeq s \prec t} \Delta_s.$$

- *For* $\alpha = 2$,

$$\left(\frac{1}{a_N}\sigma_N(t) : 0 \leq t \leq 1\right) \xrightarrow[N\to\infty]{(d)} (H_t : 0 \leq t \leq 1)$$

*in distribution for the Skorokhod topology on* $\mathbb{D}([0,1],\mathbb{R})$.

**Remark.** In the conjecture, $b_N = 1/a_N$.

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# Thank You!