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The mass of super-Brownian motion upon exiting balls and Sheu's compact support condition

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Abstract

We study the mass of a d-dimensional super-Brownian motion as it first exits an increasing sequence of balls. The mass process is a time-inhomogeneous continuous-state branching process, where the increasing radii of the balls are taken as the time-parameter. We characterise its time-dependent branching mechanism and show that it converges, as time goes to infinity, towards the branching mechanism of the mass of a one-dimensional super-Brownian motion as it first crosses above an increasing sequence of levels.

Our results identify the compact support criterion in Sheu (1994) as Grey's condition (1974) for the aforementioned limiting branching mechanism.

MSC: 60J68; 60J80

Keywords: Super-Brownian motion; Exit measures; Time-dependent continuous state branching processes; Compact support condition

1. Introduction and main results

Suppose that $X = (X_t, t \geq 0)$ is a super-Brownian motion in $\mathbb{R}^d, d \geq 1$, with general branching mechanism ψ of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x), \quad \lambda \geq 0, \tag{1}$$

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where $\alpha = -\psi'(0+) \in (-\infty, \infty)$, $\beta \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (x \wedge x^2) \Pi(\mathrm{d}x) < \infty$. Assume $\psi(\infty) = \infty$. Denote by $\mathbf{P}_\mu$ the law of X with initial configuration according to $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, the space of finite measures on $\mathbb{R}^d$ with compact support. We write $\mathcal{M}_F(D)$ for the space of finite measures supported on $D \subset \mathbb{R}^d$.

A construction of superprocesses with a general branching mechanism ψ as in (1) can be found in Fitzsimmons [8], see also Section 2.3 in Li [14] which provides a comprehensive account on the theory of superprocesses.

We call X (sub)critical if $\psi'(0+) \geq 0$ and supercritical if $\psi'(0+) < 0$. Denote the root of ψ by $\lambda^* := \inf\{\lambda \geq 0 : \psi(\lambda) > 0\}$. In the (sub)critical case, we have $\lambda^* = 0$. In the supercritical case, convexity of ψ and the condition $\psi(\infty) = \infty$ ensure that there is a unique and finite $\lambda^* > 0$. In both cases,

$$\mathbf{P}_\mu(\lim_{t\to\infty} \|X_t\| = 0) = e^{-\lambda^* \|\mu\|},$$

where $\|\mu\|$ denotes the total mass of the measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$.

We want to study the mass of the super-Brownian motion X upon its first exit from an increasing sequence of balls. Fix an initial radius $r > 0$ and let $D_s := \{x \in \mathbb{R}^d : \|x\| < s\}$ be the open ball of radius $s \geq r$ around the origin. According to Dynkin's theory of exit measures [4], we can describe the mass of X as it first exits the growing sequence of balls $(D_s, s \geq r)$ as a sequence of random measures on $\mathbb{R}^d$, known as branching Markov exit measures. We denote this sequence of branching Markov exit measures by $\{X_{D_s}, s \geq r\}$. Informally, the measure X_{D_s} is supported on the boundary ∂D_s and it is obtained by 'freezing' mass of the super-Brownian motion when it first hits ∂D_s.

Formally, $\{X_{D_s}, s \geq r\}$ is characterised by the following branching Markov property, see for instance Section 1.1 in Dynkin and Kuznetsov [6]. Let $\mu \in \mathcal{M}_F(D_r)$ and, for $z \geq r$, define $\mathcal{F}_{D_z} := \sigma(X_{D_{z'}}, r \leq z' \leq z)$. For any positive, bounded, continuous function f on ∂D_s,

$$\mathbf{E}_\mu[e^{-\langle f, X_{D_s}\rangle} | \mathcal{F}_{D_z}] = e^{-\langle v_f(\cdot,s), X_{D_z}\rangle}, \quad 0 < r \leq z \leq s, \tag{2}$$

where the Laplace functional v_f is the unique non-negative solution to

$$v_f(x, s) = \mathbb{E}_x[f(\xi_{T_s})] - \mathbb{E}_x\Big[\int_0^{T_s} \psi(v_f(\xi_z, s))\, \mathrm{d}z\Big], \tag{3}$$

and $((\xi_z, z \geq 0), \mathbb{P}_x)$ is an $\mathbb{R}^d$-Brownian motion with $\xi_0 = x$ and with $T_s := \inf\{z > 0 : \xi_z \notin D_s\}$ denoting its first exit time from D_s. In (2), we have used the inner product notation $\langle f, \mu\rangle = \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x)$.

For $s \geq r$, let $Z_s := \|X_{D_s}\|$ denote the mass that is 'frozen' when it first hits the boundary of the ball D_s. We can then define the mass process $(Z_s, s \geq r)$ which uses the radius s as its time-parameter. Let us write P_r, for the law of the process $(Z_s, s \geq r)$ starting at time $r > 0$ with unit initial mass. In case we start with non-unit initial mass $a > 0$ we shall use the notation $P_{a,r}$ for its law.

It is not difficult to see that Z is a time-inhomogeneous continuous-state branching process and we can characterise it as follows.

Theorem 1. (i) *Let $r > 0$. The process $Z = (Z_s, s \geq r)$ is a time-inhomogeneous continuous-state branching process. This is to say it is a $[0, \infty]$-valued strong Markov process with* *càdlàg paths satisfying the branching property*

$$E_{(a+a'),r}[e^{-\theta Z_s}] = E_{a,r}[e^{-\theta Z_s}]E_{a',r}[e^{-\theta Z_s}],$$

for all $a, a' > 0, \theta \geq 0$ *and* $s \geq r$.

(ii) *Let* $r > 0$ *and* $\mu \in \mathcal{M}_F(\partial D_r)$ *with* $\|\mu\| = a$. *Then, for* $s \geq r$, *we have*

$$E_{a,r}[e^{-\theta Z_s}] = e^{-u(r,s,\theta)a}, \quad \theta \geq 0, \tag{4}$$

where the Laplace functional $u(r, s, \theta)$ *satisfies*

$$u(r, s, \theta) = \theta - \int_r^s \Psi(z, u(z, s, \theta))\, \mathrm{d}z, \tag{5}$$

for a family of branching mechanisms $(\Psi(r, \cdot), r > 0)$ *of the form*

$$\Psi(r, \theta) = -q_r + a_r\theta + b_r\theta^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x \mathbf{1}_{(x<1)})\Lambda_r(\mathrm{d}x), \tag{6}$$

for $\theta \geq 0$, *and for each* $r > 0$ *we have* $q_r \geq 0, a_r \in \mathbb{R}, b_r \geq 0$ *and* Λ_r *is a measure concentrated on* $(0, \infty)$ *satisfying* $\int_{(0,\infty)}(1 \wedge x^2)\Lambda_r(\mathrm{d}x) < \infty$.

(iii) *The branching mechanism* Ψ *satisfies the PDE*

$$\begin{aligned} &\frac{\partial}{\partial r}\Psi(r, \theta) + \frac{1}{2}\frac{\partial}{\partial \theta}\Psi^2(r, \theta) + \frac{d-1}{r}\Psi(r, \theta) = 2\psi(\theta) \quad r > 0,\ \theta \in (0, \infty) \\ &\Psi(r, \lambda^*) = 0, \quad r > 0. \end{aligned} \tag{7}$$

The authors are not aware of a result in the literature which states that the definition of the time-dependent CSBP in (i) implies the characterisation in (ii). It is therefore outlined in the proof of Theorem 1(ii) in Section 2.1 how this implication can be derived as a generalisation of the equivalent result for standard CSBPs in Silverstein [18].

As part of Theorem 1, we later prove that the root λ^* of ψ is also the root for each $\Psi(r, \cdot), r > 0$, cf. Lemma 6. This will be a key property for the forthcoming analysis of the family of branching mechanisms $(\Psi(r, \cdot), r > 0)$.

Let us now describe how Ψ changes as r increases. We observe the following change in the shape of the branching mechanism, see Fig. 1.

Proposition 2. (i) *For (sub)critical* ψ, *we have, for* $0 < r \leq s$,

$$\Psi(r, \theta) \leq \Psi(s, \theta) \quad \text{for all } \theta \geq 0.$$

(ii) *For supercritical* ψ, *we have, for* $0 < r \leq s$,

$$\begin{aligned} &\Psi(r, \theta) \geq \Psi(s, \theta) \quad \text{for all } \theta \leq \lambda^* \\ &\Psi(r, \theta) \leq \Psi(s, \theta) \quad \text{for all } \theta \geq \lambda^*. \end{aligned}$$

This result suggests that there is a limiting branching mechanism $\Psi_\infty(\cdot) := \lim_{r\to\infty} \Psi(r, \cdot)$. Intuitively speaking, in the case where the initial mass is supported on a large ball, the local behaviour of the super-Brownian motion when exiting increasingly larger balls should look like a one-dimensional super-Brownian upon crossing levels. This idea is supported by the following result.

Theorem 3. *For each* $\theta \geq 0$, *the limit* $\lim_{r\uparrow\infty} \Psi(r, \theta) = \Psi_\infty(\theta)$ *is finite and the convergence holds uniformly in* θ *on any bounded, closed subset of* $\mathbb{R}_+$.

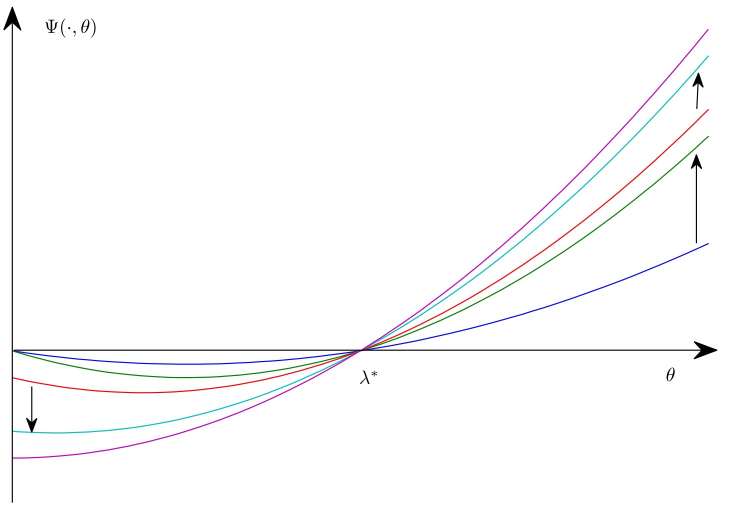

Fig. 1. Shape of the branching mechanism $\Psi(r,\cdot)$ as $r \to \infty$ in the supercritical case.

(i) *For any* $\theta \geq 0$, *we have*

$$\Psi_\infty(\theta) = 2\mathrm{sgn}(\psi(\theta)) \sqrt{\int_{\lambda^*}^{\theta} \psi(\lambda)\,\mathrm{d}\lambda}, \tag{8}$$

with $\lambda^* = 0$ *in the (sub)critical case.*

(ii) *Denote by* $((Z^{\infty_s}, s \geq 0), P^\infty)$ *the standard CSBP associated with the limiting branching mechanism* Ψ_∞*, with unit initial mass at time* 0.

Then, $(Z^{\infty_s}, s \geq 0)$ *is the mass of the process of branching Markov exit measures of a one-dimensional super-Brownian motion with unit initial mass at the origin at time zero as it first exits the family of intervals* $((-\infty, s), s \geq 0)$.

Further, for any $s > 0, \theta \geq 0$,

$$\lim_{r\to\infty} E_r[e^{-\theta Z_{r+s}}] = E^\infty[e^{-\theta Z^{\infty_s}}]. \tag{9}$$

Let us remark that, in the supercritical case, the limiting branching mechanism Ψ_∞ is critical and possesses an explosion coefficient, that is $\Psi_\infty'(0+) = 0$ and $\Psi_\infty(0) < 0$. Thanks to the uniform continuity in θ, this implies that $\Psi(t, 0) < 0$ for all sufficiently large t.

The limiting process Z^∞ in Theorem 3 has already been studied in Theorem 3.1 in Kyprianou et al. [13]. Note that therein the underlying Brownian motion has a positive drift which is chosen such that the resulting branching mechanism is conservative. The characterisation can easily be adapted to the driftless case as in Theorem 3(ii). Kaj and Salminen [10,11] studied the analogous process in the setting of branching particle diffusions, that is the process of the number of particles of a one-dimensional branching Brownian motion stopped upon exiting the interval $((-\infty, s), s \geq 0)$. They discover in the supercritical case [10] that the resulting offspring distribution is degenerate, meaning that

$$\sum_{i\geq 0} p_i < 1, \tag{10}$$

where p_i is the probability of having i offspring, $i \geq 0$. In particular, the probability of a birth event with an infinite number of offsprings is strictly positive. In this view, (10) is the analogue of $\Psi_\infty(0) < 0$.

In Sheu [16,17], asymptotics of the process Z are studied in order to obtain a compact support criterion for the super-Brownian motion X. It is found that the event of extinction of Z, i.e. $\{\exists s > 0 : Z_s = 0\}$, and the event $\{X \text{ has a compact support}\}$ agree $\mathbf{P}_\mu$-a.s., c.f. [17, Theorem 4.1].

The following result on the asymptotic behaviour of Z is given by Sheu [16].

Theorem (*Sheu [16, Theorems 1.1, 1.2, Corollary 1.1]*). *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. The event $\{\exists s > 0 : Z_s = 0\}$ agrees $\mathbf{P}_\mu$-a.s. with the event $\{\lim_{s\to\infty} Z_s = 0\}$ if ψ satisfies*

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\lambda \psi(\theta)\,\mathrm{d}\theta}}\,\mathrm{d}\lambda < \infty. \tag{11}$$

Otherwise, $\{\exists s > 0 : Z_s = 0\}$ has probability 0.

In short, the event of extinction of Z agrees with the event of extinguishing of Z, denoted by $\mathcal{E}(Z) := \{\lim_{s\to\infty} Z_s = 0\}$, if and only if (11) holds, and it has zero probability otherwise. We have stated the theorem slightly differently from its original version in which, in the supercritical case, condition (11) reads $\int_s^\infty \frac{1}{\sqrt{\int_0^\lambda \phi(\theta)\,\mathrm{d}\theta}}\,\mathrm{d}\lambda < \infty$, for $\phi(s) := \psi(s) - \alpha s$. The equivalence of these two conditions was already pointed out in [13].

The unusual condition (11) corresponds to Grey's condition in [9] for extinction vs. extinguishing in the following sense. Recall that Grey's condition says that, for a standard CSBP with branching mechanism F, the event of extinction agrees with the event of becoming extinguished if and only if $\int^\infty F(\theta)^{-1}\,\mathrm{d}\theta < \infty$, and has probability zero otherwise. The following interpretation of (11) is an immediate consequence of Theorem 3(i).

Corollary 4. *Sheu's compact support condition (11) is Grey's condition for the limiting standard CSBP Z^∞ with branching mechanism Ψ_∞ in (8).*

Sheu's compact support condition (11) plays an important role when studying the radial speed of the support of supercritical Super-Brownian motion. In the one-dimensional case, assuming (11), Kyprianou et al. [13, Corollary 3.2], show that, on the event of non-extinction of X,

$$\lim_{t\to\infty} \frac{\mathcal{R}_t}{t} = \sqrt{-2\psi'(0+)}, \quad \mathbf{P}_\mu\text{-a.s}, \ \mu \in \mathcal{M}_F(\mathbb{R}), \tag{12}$$

where $\mathcal{R}_t := \sup\{r > 0 : X_t(r, \infty) > 0\}$ is the right-most point of the support of X_t. A key step in the proof is to study the mass of the process of branching exit measures of a one-dimensional super-Brownian motion with drift $c := -\sqrt{-2\psi'(0+)}$ upon exiting the increasing sequence of intervals $((-\infty, s), s \geq 0)$, which we denote here by $Z^c = (Z_s^c, s \geq 0)$. It is proved in Theorem 3.1 in [13] that Z^c is a subcritical standard CSBP. Now condition (11) comes in. Corollary 4 interprets (11) as Grey's condition for the standard CSBP Z^∞. The CSBPs Z^∞ and Z^c only differ in that the underlying Brownian motion of the latter has drift c and it is not difficult to convince ourselves that the drift term is irrelevant when studying the extinction vs. extinguishing problem, see (29) in [13] for a rigorous argument. Therefore condition (11) is also equivalent to Grey's condition for the subcritical CSBP Z^c and hence ensures that Z^c becomes extinct $\mathbf{P}_\mu$-a.s. This now implies that the right-most point of the support cannot travel at a speed faster than

$\sqrt{-2\psi'(0+)}$. In order to make this last conclusion, extinguishing of Z^c is clearly not sufficient and it remains an open question whether a strong law for $(\mathcal{R}_t, t \geq 0)$ can exist when (11) fails.

In the d-dimensional case, $d \geq 1$, and with a quadratic branching mechanism of the form $\psi(\lambda) = -\alpha\lambda + \beta\lambda^2$, for $\alpha, \beta \geq 0$, Kyprianou [12] shows that (12) holds, where $\mathcal{R}_t$ is now replaced by $\tilde{\mathcal{R}}_t := \sup\{r > 0 : X_t(\mathbb{R}^d \setminus D_r) > 0\}$, the radius of the support of X_t. It can be checked that condition (11) is satisfied for this choice of ψ. It is possible to adapt the higher-dimensional result in [12] to hold for general branching mechanisms provided (11) holds.

The remainder of the paper is organised as follows. In Section 2 we prove Theorem 1 which is followed by the proof of Proposition 2 and Theorem 3 in Section 3.

2. Characterising the process Z—Proof of Theorem 1

2.1. *Proof of Theorem 1(i) and (ii)*

Proof of Theorem 1(i). Take a look at Eq. (2) which characterises the sequence of branching exit measures $(X_{D_s}, s \geq r)$. For any measure $\mu \in \mathcal{M}_F(\partial D_r)$ and $\|\mu\| = a$, we can write

$$E_{a,r}[e^{-\theta Z_s}] = \mathbf{E}_\mu[e^{-\theta\|X_{D_s}\|}] = e^{-\langle v_\theta(\cdot,s),\mu\rangle} = e^{-v_\theta(x,s)a},$$

for any $x \in \partial D_r$, by radial symmetry. The branching property of Z now follows easily from the branching property of $(X_{D_s}, s > r)$ in (2) since, for $a, a' > 0$, $0 < r \leq s$,

$$\begin{aligned} E_{(a+a'),r}[e^{-\theta Z_s}] &= \mathbf{E}_{\mu+\mu'}[e^{-\theta\|X_{D_s}\|}] \\ &= e^{-v_\theta(x,s)(a+a')} \\ &= \mathbf{E}_\mu[e^{-\theta\|X_{D_s}\|}]\mathbf{E}_{\mu'}[e^{-\theta\|X_{D_s}\|}] = E_{a,r}[e^{-\theta Z_s}]E_{a',r}[e^{-\theta Z_s}], \end{aligned}$$

for measures $\mu, \mu' \in \mathcal{M}_F(\partial D_r)$ with $\|\mu\| = a$, $\|\mu'\| = a'$. The Markov property is also an immediate consequence of (2). □

Proof of Theorem 1(ii). First note that, by radial symmetry as seen in the proof of Theorem 1(i), (4) holds with $u(r, s, \theta) = v_\theta(x, s)$ for $x \in \partial D_r$ where $r = \|x\|$. It remains to show that (5) and (6) are satisfied.

For any $0 < r \leq z \leq s, \theta \geq 0$,

$$E_r[e^{-\theta Z_s}] = E_r[E_{Z_z,z}[e^{-\theta Z_s}]] = E_r[e^{-u(z,s,\theta)Z_z}] = e^{-u(r,z,u(z,s,\theta))},$$

which shows that the Laplace functional satisfies the composition property

$$u(r, s, \theta) = u(r, z, u(z, s, \theta)) \quad \text{for } 0 < r \leq z \leq s,\ \theta \geq 0. \tag{13}$$

The branching property of Z implies that, for any fixed $0 < r \leq s$, the law of (Z_s, P_r) is an infinitely divisible distribution on $[0, \infty]$. It follows from the Lévy–Khintchin formula that, for fixed r and s, $u(r, s, \theta)$ is a non-negative, completely concave function as considered in Section 4 in Silverstein [18]. The process Z thus has the properties of the time-dependent version of the CSBP considered in Definition 4 in [18]. We can then adapt the proof of Theorem 4 in [18] to show that there exists a branching mechanism Ψ of the form (6) such that

$$\left.\frac{\partial}{\partial r}u(r, s, \theta)\right|_{r=s} = \Psi(s, \theta), \quad \text{for } s > 0,\ \theta \geq 0.$$

With the composition property (13), we then get

$$\frac{\partial}{\partial r}u(r, s, \theta) = \Psi(r, u(r, s, \theta)), \quad \text{for } 0 < r \leq s,\ \theta \geq 0.$$

Indeed it was already discussed at the end of Section 4 in [18] that it is possible to allow time-dependence in Theorem 4 in [18].

Together with the initial condition $u(r, r, \theta) = \theta$, we obtain Eq. (5). $\square$

From (5), we get an alternative characterisation of the relation between the Laplace functional u and the branching mechanism Ψ as

$$\frac{\partial}{\partial s}u(r, s, \theta) = -\Psi(s, \theta)\frac{\partial}{\partial \theta}u(r, s, \theta) \tag{14}$$

$$\frac{\partial}{\partial r}u(r, s, \theta) = \Psi(r, u(r, s, \theta)) \tag{15}$$

$$u(r, r, \theta) = \theta,$$

for any $s > r > 0$ and $\theta \geq 0$. To see where Eq. (14) comes from, compare the derivatives of (5) in s and θ, that is

$$\frac{\partial}{\partial s}u(r, s, \theta) = -\Psi(s, \theta) - \int_r^s \frac{\partial}{\partial u}\Psi(z, u(z, s, \theta))\frac{\partial}{\partial s}u(z, s, \theta)\,\mathrm{d}z$$

$$\frac{\partial}{\partial \theta}u(r, s, \theta) = 1 - \int_r^s \frac{\partial}{\partial u}\Psi(z, u(z, s, \theta))\frac{\partial}{\partial \theta}u(z, s, \theta)\,\mathrm{d}z,$$

where $\partial\Psi(\cdot, \cdot)/\partial u$ denotes the derivative in the second component of Ψ. We see that $\frac{\partial}{\partial s}u(r, s, \theta)$ and $-\Psi(s, \theta)\frac{\partial}{\partial \theta}u(r, s, \theta)$ are solutions to the same integral equation. With an application of Gronwall's inequality it can be shown that this integral equation has a unique solution.

2.2. *Proof of Theorem 1(iii)*

We have already seen in the previous section that, for any measure $\mu \in \mathcal{M}_F(\partial D_r)$ with $\|\mu\| = a$, we can write

$$E_{a,r}[e^{-\theta Z_s}] = \mathbf{E}_\mu[e^{-\theta\|X_{D_s}\|}] = e^{-\langle v_\theta(\cdot,s),\mu\rangle} = e^{-v_\theta(x,s)a},$$

for any $x \in \partial D_r$, by radial symmetry. In particular, we saw that $u(r, s, \theta) = v_\theta(x, s)$ for any $x \in \partial D_r$. From the semi-group equation for v in (3), we thus get a semi-group representation of u, alternative to the representation in (5), as the unique non-negative solution to

$$u(r, s, \theta) = \theta - \mathbb{E}_r^{\mathrm{R}}\Big[\int_0^{\tau_s} \psi(u(R_z, s, \theta))\,\mathrm{d}z\Big], \tag{16}$$

where $(R, \mathbb{P}_r^{\mathrm{R}})$ is a d-dimensional Bessel process and $\tau_s := \inf\{z > 0 : R_z > s\}$ its first passage time above level s.

Eq. (16) tells us that the process Z can be viewed as the mass process of the branching exit measures of a d-dimensional super-Bessel process with branching mechanism ψ as it first exits the intervals $(0, s)$, $s \geq r$.

Equivalently to the characterisation of $u(r, s, \theta)$ as the unique non-negative solution to the integral equation (16), we can characterise it as the unique non-negative solution to the differential equation

$$\frac{1}{2}\frac{\partial^2}{\partial r^2}u(r, s, \theta) + \frac{d-1}{2r}\frac{\partial}{\partial r}u(r, s, \theta) = \psi(u(r, s, \theta)), \quad 0 < r < s, \theta \geq 0,$$

$$u(r, r, \theta) = \theta. \tag{17}$$

We will show this equivalence in Appendix A. In the following section, we will use the differential equation (17) to prove the PDE characterisation of the branching mechanism Ψ in Theorem 1(iii).

We prove Theorem 1(iii) in two parts. In Lemma 5 we show that Ψ satisfies the PDE in (7) before we prove that $\Psi(r, \lambda^*) = 0$, for all $r > 0$, in Lemma 6 below.

Lemma 5. *The branching mechanism Ψ satisfies the PDE (7), i.e.*

$$\frac{\partial}{\partial r}\Psi(r,\theta) + \frac{1}{2}\frac{\partial}{\partial \theta}\Psi^2(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) = 2\psi(\theta) \quad r > 0,\ \theta \in (0,\infty).$$

Proof of Lemma 5. Using (15), the left-hand side of (17) becomes

$$\begin{aligned}
\frac{\partial^2}{\partial r^2}u(r,s,\theta) + \frac{d-1}{r}\frac{\partial}{\partial r}u(r,s,\theta) &= \frac{\partial}{\partial r}\Psi(r,u(r,s,\theta)) + \frac{d-1}{r}\Psi(r,u(r,s,\theta)) \\
&= \frac{\partial}{\partial y}\Psi(y,u(r,s,\theta))\Big|_{y=r} \\
&\quad + \frac{\partial}{\partial u}\Psi(r,u(r,s,\theta))\ \Psi(r,u(r,s,\theta)) \\
&\quad + \frac{d-1}{r}\Psi(r,u(r,s,\theta)) \\
&= \frac{\partial}{\partial y}\Psi(y,u(r,s,\theta))\Big|_{y=r} \\
&\quad + \frac{1}{2}\frac{\partial}{\partial u}\Psi^2(r,u(r,s,\theta)) + \frac{d-1}{r}\Psi(r,u(r,s,\theta)),
\end{aligned}$$

where $\partial\Psi(\cdot,\cdot)/\partial u$ denotes the derivative with respect to the second argument. Note that this equation holds for all $s > r$ and $\theta \geq 0$. Since $u(r,s,\theta) \to \theta$ as $s \downarrow r$, we see that, for fixed r, the range of $u(r,s,\theta)$ is $(0,\infty)$ as we vary $s \in (r,\infty)$ and $\theta \in [0,\infty)$. Hence, we can replace $u(r,s,\theta)$ above by an arbitrary $\theta \in (0,\infty)$ and conclude that the PDE (7) holds true. □

Recall that $\lambda^* = \inf\{\lambda \geq 0 : \psi(\lambda) > 0\}$ denotes the root of ψ and define $\lambda^*(r) := \inf\{\lambda \geq 0 : \Psi(r,\lambda) > 0\}$, for $r > 0$.

Lemma 6. (i) *In the (sub)critical case, for all $r > 0$, we have $\lambda^*(r) = 0$. In particular, $\Psi(r,\theta) \geq 0$ for all $\theta \geq 0$.*

(ii) *In the supercritical case, for all $r > 0$, we have $\lambda^*(r) = \lambda^*$. In particular, $\Psi(r,\theta) \leq 0$ for $\theta \leq \lambda^*$, while $\Psi(r,\theta) \geq 0$ for $\theta \geq \lambda^*$.*

Proof of Lemma 6(i). As we are in the (sub)critical case we have $\psi(\theta) \geq 0$ for all $\theta \geq 0$. For $r < z < s$, (16) yields

$$\begin{aligned}
u(r,s,\theta) &= \theta - \mathbb{E}_r^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\,\mathrm{d}v \\
&= \theta - \mathbb{E}_r^{\mathrm{R}}\int_0^{\tau_z}\psi(u(R_v,s,\theta))\,\mathrm{d}v - \mathbb{E}_z^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\,\mathrm{d}v \\
&\leq \theta - \mathbb{E}_z^{\mathrm{R}}\int_0^{\tau_s}\psi(u(R_v,s,\theta))\,\mathrm{d}v \\
&= u(z,s,\theta).
\end{aligned}$$

Hence, $u(r, s, \theta)$ is non-decreasing in r. With (15) we thus see that, for all $0 < r < s, \theta \geq 0$,

$$\Psi(r, u(r, s, \theta)) = \frac{\partial}{\partial r} u(r, s, \theta) \geq 0. \tag{18}$$

As we take $s \downarrow r$, we get $u(r, s, \theta) \to \theta$ and hence $\Psi(r, \theta) \geq 0$ for all $\theta > 0, r > 0$. Continuity of Ψ ensures $\Psi(r, 0) = 0$ and, in particular, $\lambda^*(r) = 0$ for all $r > 0$. $\square$

The key to the proof of part (ii) of Lemma 6 is the following lemma.

Lemma 7. *Fix $r > 0$.*

(i) *For any $\lambda > 0$, the process*

$$M_s^\lambda = e^{-\lambda Z_s} - \int_r^s \Psi(v, \lambda) Z_v e^{-\lambda Z_v} \mathbf{1}_{\{Z_v < \infty\}} \mathrm{d}v, \quad s \geq r, \tag{19}$$

is a P_r-martingale.

(ii) *The process $(e^{-\lambda^* Z_s}, s \geq r)$ is a P_r-martingale.*

Here we use the convention $e^{-\lambda Z_s} \mathbf{1}_{\{Z_s = \infty\}} = 0$, for any $\lambda > 0$.

Proof of part (i). Taking expectations in (19) and interchanging expectation and integral gives

$$E_r[M_s^\lambda] = e^{-u(r,s,\lambda)} - \int_r^s \Psi(v, \lambda) \frac{\partial}{\partial \lambda} u(r, v, \lambda)\, e^{-u(r,v,\lambda)} \mathrm{d}v.$$

Differentiating in s, together with (14), gives

$$\frac{\partial}{\partial s} E_r[M_s^\lambda] = \Big(-\frac{\partial}{\partial s} u(r, s, \lambda) - \Psi(s, \lambda) \frac{\partial}{\partial \lambda} u(r, s, \lambda) \Big) e^{-u(r,s,\lambda)} = 0.$$

Hence, $E_r[M_s^\lambda]$ is constant for all $s \geq r$ and in particular, taking $s = r$, equal to $e^{-\lambda}$. Note that the same computation gives that $E_{a,v}[M_s^\lambda] = e^{-\lambda a}$, for $a > 0$ and $0 < r \leq v \leq s$. An application of the Markov property then shows that $(M^{\lambda_s}, s \geq r)$ is a martingale for any $\lambda > 0$. $\square$

The proof of Lemma 7(ii) relies on the following idea. Since $(\|X_t\|, t \geq 0)$ is a CSBP with branching mechanism ψ it is well-known that the process $(e^{-\lambda^* \|X_t\|}, t \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ where $\mathcal{F}_t = \sigma(\|X_u\|, u \leq t)$. The martingale property follows on account of the fact that

$$\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\| \to 0\}} | \mathcal{F}_t] = e^{-\lambda^* \|X_t\|}, \quad t \geq 0,$$

by a simple application of the tower property. Now, fix $r > 0$, and consider the filtration $(\mathcal{G}_s, s \geq r)$ where $\mathcal{G}_s = \sigma(\|X_{D_v}\|, r \leq v \leq s) = \sigma(Z_v, r \leq v \leq s)$ instead. If we can show that, for $\mu \in \mathcal{M}_F(\partial D_r)$,

$$\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\| \to 0\}} | \mathcal{G}_s] = e^{-\lambda^* \|X_{D_s}\|} = e^{-\lambda^* Z_s},$$

holds, then we can deduce in the same way that the process $(e^{-\lambda^* \|X_{D_s}\|}, s \geq r)$ is a martingale with respect to the filtration $(\mathcal{G}_s, s \geq r)$. The proof is slightly cumbersome and therefore postponed to the end of this section.

The proof of Lemma 6(ii) is now a simple consequence of Lemma 7.

Proof of Lemma 6(ii). By Lemma 7, the process

$$e^{-\lambda^* Z_s} - M_s^{\lambda^*} = \int_r^s \Psi(v, \lambda^*) Z_v e^{-\lambda^* Z_v} \mathbf{1}_{\{Z_v < \infty\}}\, \mathrm{d}v, \quad s \geq r,$$

must be a P_r-martingale. However this is only possible if the expectation of the Lebesgue-integral above is constant in s which requires $\Psi(s, \lambda^*) = 0$ on $\{0 < Z_s < \infty\}$ for all $s \geq r$. Since the event $\{0 < Z_s < \infty\}$ has a positive probability under P_r, we reason that $\Psi(s, \lambda^*) = 0$ for all $s \geq r$. Choosing $r > 0$ arbitrarily small yields $\Psi(s, \lambda^*) = 0$ for all $s > 0$. Convexity of $\Psi(s, \theta)$ immediately implies that $\Psi(s, \theta) \geq 0$ for $\theta \geq \lambda^*$ and, further noting that $\Psi(s, 0) \leq 0$, that $\Psi(s, \theta) \leq 0$ for $\theta \leq \lambda^*$. $\square$

Proof of Theorem 1(iii). Combine Lemmas 5 and 6. $\square$

Let us now come to the proof of Lemma 7(ii). For $r > 0, t \geq 0$, define the space–time domain D_r^t as

$$D_r^t = \{(x, u) : \|x\| < r, u < t\} \subset \mathbb{R}^d \times [0, \infty).$$

Let $(X_{D_r^t}, t \geq 0, r > 0)$ be the system of branching Markov exit measures describing the mass of X as it first exits the space–time domains D_r^t, see again Dynkin [4].

For the proof of Lemma 7(ii), we will need the following result which seems rather obvious but nevertheless needs a careful proof.

Lemma 8. *Let $r > 0$. For any $\mu \in \mathcal{M}_F(D_r)$, we have $\mathbf{P}_\mu$-a.s.,*

$$\lim_{t\to\infty} \|X_{D_r^t}\| = \|X_{D_r}\| = Z_r.$$

Proof. For $r > 0, t \geq 0$, denote by ∂D_r^t the boundary of the set D_r^t, i.e.

$$\begin{aligned}\partial D_r^t &= (\{x : \|x\| = r\} \times [0, t)) \cup (\{x : \|x\| < r\} \times \{t\}) \\ &=: \partial D_r^{t-} \cup \partial D_{r-}^t.\end{aligned}$$

By monotonicity, we have $\lim_{t\to\infty} \|X_{D_r^t}|_{\partial D_r^{t-}}\| = \|X_{D_r}\| = Z_r$, $\mathbf{P}_\mu$-a.s. Next, define the event that X becomes extinguished within D_r, i.e.

$$\mathcal{E}(X, D_r) := \{\lim_{t\to\infty} \|X_{D_r^t}|_{\partial D_{r-}^t}\| = 0\}.$$

On the complement of $\mathcal{E}(X, D_r)$, we have

$$\lim_{t\to\infty} \|X_{D_r^t}|_{\partial D_{r-}^t}\| = \infty, \quad \mathbf{P}_\mu\text{-a.s.}$$

This is to say that, on $\mathcal{E}(X, D_r)^c$, the total mass within the open ball D_r at time t tends to infinity as t tends to infinity. This follows from Proposition 7 in [7] which says that $\limsup_{t\to\infty} \|X_{D_r^t}|_{B\times\{t\}}\| \in \{0, \infty\}$, $\mathbf{P}_\mu$-a.s. for any nonempty open set $B \subset D_r$ (noting that Proposition 7 in [7] indeed holds for the general branching mechanism we are considering here). Hence, we have shown so far that

$$\lim_{t\to\infty} \|X_{D_r^t}\| = Z_r + \infty \mathbf{1}_{\mathcal{E}(X,D_r)^c}.$$

Thus it remains to prove that, on $\mathcal{E}(X, D_r)^c$, Z_r is also infinite. Fix a $K > 0$. Thanks to Proposition 7 of [7], on $\mathcal{E}(X, D_r)^c$, we can define an infinite sequence of stopping times

$$\begin{aligned}T_0 &= \inf\{t > 0 : \|X_{D_r^t}|_{\partial D_{r-}^t}\| \geq K\} \\ T_{i+1} &= \inf\{t > T_i + 1 : \|X_{D_r^t}|_{\partial D_{r-}^t}\| \geq K\}, \quad i = 1, 2, \ldots .\end{aligned}$$

At times $T_i, i \geq 0$, the total mass within the open ball D_r is greater than or equal to K. Fix an $M > 0$ and define the event

$$A_i = \{\|X_{D_r^{T_i}}|_{[T_{i-1},T_i)\times\partial D_r}\| > M\}, \quad i = 1, 2, \ldots$$

which is the event that the mass that exits D_r during the time interval $[T_{i-1}, T_i)$ exceeds M. Note that there exists a strictly positive constant $\epsilon(M, K)$, such that

$$\begin{aligned}\mathbf{P}_{X_{D_r^{T_i}}}(A_{i+1}) &\geq \mathbf{P}_{K\delta_0}(A_1)\\ &\geq \mathbf{P}_{K\delta_0}(\|X_{D_r^1}|_{[0,1)\times\partial D_r}\| > M) > \epsilon(M, K).\end{aligned} \tag{20}$$

Thus, we can partition time into infinitely many intervals $[T_i, T_{i+1}), i \geq 0$, of length at least 1. During each time interval the mass that exits D_r, and thus contributes to Z_r, exceeds M with positive probability. These probabilities are uniformly bounded from below by $\epsilon(M, K) > 0$ in (20). Therefore $\|X_{D_r}\| = Z_r = \infty, \mathbf{P}_\mu$-a.s on the event $\mathcal{E}(X, D_r)^c$. This completes the proof. □

Proof of Lemma 7(ii). For $s > 0, t \geq 0$, define $\mathcal{F}_{D_s^t} = \sigma(X_{D_{s'}^{t'}}, s' \leq s, t' \leq t)$. Fix $r > 0$. The characterising branching Markov property for exit measures, see for instance Section 1.1 in [6], yields that, for $\mu \in \mathcal{M}_F(D_r)$, $s \geq r$ and $u \geq t \geq 0$, we have

$$\mathbf{E}_\mu[e^{-\theta\|X_u\|}|\mathcal{F}_{D_s^t}] = \exp\{-\langle w_\theta(u-\cdot), X_{D_s^t}\rangle\} \tag{21}$$

where w_θ is the Laplace functional of the standard CSBP $(\|X_u\|, u \geq 0)$ with branching mechanism ψ. Taking $\theta = \lambda^*$, it is well known that $w_{\lambda^*}(t) = \lambda^*$ for all $t \geq 0$. Therefore (21), with θ replaced by λ^*, turns into

$$\mathbf{E}_\mu[e^{-\lambda^*\|X_u\|}|\mathcal{F}_{D_s^t}] = \exp\left\{-\int w_{\lambda^*}(u-t')\,\mathrm{d}X_{D_s^t}(x,t')\right\} = e^{-\lambda^*\|X_{D_s^t}\|}.$$

Taking $u \to \infty$, we conclude

$$\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\mathcal{F}_{D_s^t}] = \lim_{u\to\infty}\mathbf{E}_\mu[e^{-\lambda^*\|X_u\|}|\mathcal{F}_{D_s^t}] = e^{-\lambda^*\|X_{D_s^t}\|}. \tag{22}$$

Now, we want to take the limit in t. By Lemma 8, we have $\|X_{D_s^t}\| \to Z_s$ as $t \to \infty$ and thus the right-hand side of (22) tends to $\exp\{-\lambda^* Z_s\}$, $\mathbf{P}_\mu$-a.s. For the left-hand side, by the strong Markov property, we can replace $\mathcal{F}_{D_s^t}$ by $\sigma(X_{D_s^t})$. Further, note that $\mathbf{P}_\mu(\|X_u\| \to 0) = e^{-\lambda^*\|\mu\|}$ for any $\mu \in \mathcal{M}_F(D_s)$, with $\mathbf{P}_\mu(\|X_u\| \to 0) = 0$ if μ has infinite mass. Thus, the event $\{\|X_u\| \to 0\}$ only depends on the total mass of μ. Therefore we can replace $\sigma(X_{D_s^t})$ by $\sigma(\|X_{D_s^t}\|)$ on the left-hand side in (22). To sum up, we get

$$\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\mathcal{F}_{D_s^t}] = \mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(X_{D_s^t})] = \mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(\|X_{D_s^t}\|)].$$

By Lemma 8, we have $\lim_{t\to\infty}\|X_{D_s^t}\| = Z_s$, with the possibility of the limit being infinite. Hence,

$$\lim_{t\to\infty}\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(\|X_{D_s^t}\|)] = \mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(Z_s)].$$

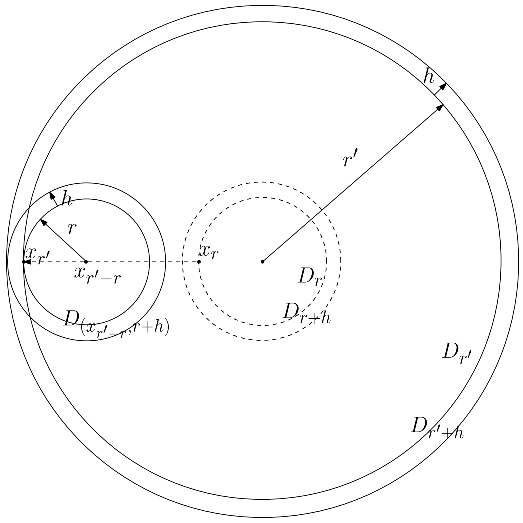

Fig. 2. Shifting the balls D_r and D_{r+h} by a distance $r' - r$.

Putting the pieces together, we get

$$\mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(Z_s)] = \lim_{t\to\infty} \mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\mathcal{F}_{D_s^t}] = \lim_{t\to\infty} e^{-\lambda^*\|X_{D_s^t}\|} = e^{-\lambda^* Z_s}.$$

Finally take $\mu \in \mathcal{M}_F(\partial D_r)$ and let $r \le s' \le s$. Then conditioning on $\sigma(Z_s)$ and using the tower property, gives

$$\begin{aligned} e^{-\lambda^* Z_{s'}} &= \mathbf{E}_\mu[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(Z_{s'})] \\ &= \mathbf{E}_\mu[\mathbf{E}[\mathbf{1}_{\{\|X_u\|\to 0\}}|\sigma(Z_s)]|\sigma(Z_{s'})] = E_r[e^{-\lambda^* Z_s}|\sigma(Z_{s'})], \end{aligned}$$

from which we conclude that $(e^{-\lambda^* Z_s}, s \ge r)$ is a P_r-martingale. □

3. The limiting branching mechanism—Proof of Proposition 2 and Theorem 3

3.1. Changing shape—Proof of Proposition 2

Proof of Proposition 2. (i) Fix $0 < r \le r', h > 0$ and $\theta \ge 0$. The first step is to show that $u(r, r+h, \theta) \ge u(r', r'+h, \theta)$. Said another way, we want to show that

$$E_{r'}[e^{-\theta Z_{r'+h}}] \ge E_r[e^{-\theta Z_{r+h}}]. \tag{23}$$

Recall that (Z_{r+h}, P_r) is the mass of X as it first exists the ball D_{r+h}, when X is initiated from one unit of mass distributed on ∂D_r. By radial symmetry of X, we may assume that the initial mass is concentrated in a point $x_r \in \partial D_r$, i.e. $E_r[e^{-\theta Z_{r+h}}] = \mathbf{E}_{\delta_{x_r}}[e^{-\theta\|X_{D_{r+h}}\|}]$.

Now we shift the point x_r to the point $x_{r'} \in \partial D_{r'}$ where $\|x_{r'} - x_r\| = r' - r$. We also shift the ball D_{r+h} in the same direction and by the same distance $r' - r$ and denote its new centre by $x_{r'-r}$, see Fig. 2. By translation invariance of X we then have

$$E_r[e^{-\theta Z_{r+h}}] = \mathbf{E}_{\delta_{x_r}}\left[e^{-\theta\|X_{D_{r+h}}\|}\right] = \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D(x_{r'-r}, r+h)}\|}\right],$$

where $D(x_{r'-r}, r+h)$ is the open ball centred at $x_{r'-r}$ with radius $r+h$. We can then write (23) as

$$\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right] \geq \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D(x_{r'-r},r+h)}\|}\right]. \tag{24}$$

Recall that Eq. (2) shows that the process of branching exit measure X_{D_s} indexed by the increasing sequence of balls $(D_s, s \geq r)$ has the strong Markov property. By Dynkin [4], the strong Markov property holds more generally for any increasing sequence of open Borel subsets of $\mathbb{R}^d$. In particular,

$$\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\,\middle|\,\mathcal{F}_{D(x_{r'-r},r+h)}\right] = \mathbf{E}_{X_{D(x_{r'-r},r+h)}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right], \tag{25}$$

where $\mathcal{F}_{D(x_{r'-r},r+h)} = \sigma(X_{D(x_{r'-r},s)}, s \leq r+h)$. Hence, assuming that

$$\mathbf{E}_{X_{D(x_{r'-r},r+h)}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right] \geq e^{-\theta\|X_{D(x_{r'-r},r+h)}\|} \tag{26}$$

holds true, we get, together with (25), that

$$\begin{aligned}
\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right] &= \mathbf{E}_{\delta_{x_{r'}}}\left[\mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\,\middle|\,\sigma(X_{D(x_{r'-r},r+h)})\right]\right] \\
&= \mathbf{E}_{\delta_{x_{r'}}}\left[\mathbf{E}_{X_{D(x_{r'-r},r+h)}}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right]\right] \\
&\geq \mathbf{E}_{\delta_{x_{r'}}}\left[e^{-\theta\|X_{D(x_{r'-r},r+h)}\|}\right],
\end{aligned}$$

which is the desired inequality (24). Thanks to the branching Markov property for exit measures, for (26) to hold it suffices to show that

$$\mathbf{E}_{\delta_x}\left[e^{-\theta\|X_{D_{r'+h}}\|}\right] \geq e^{-\theta}, \quad \text{for any } x \in \partial D(x_{r'-r}, r+h). \tag{27}$$

For fixed $x \in \partial D(x_{r'-r}, r+h)$, set $s = \|x\|$ and note that $s \leq r'+h$. By (18), $u(s, r'+h, \theta)$ is increasing in s and bounded from above by $u(r'+h, r'+h, \theta) = \theta$. Hence we obtain

$$\mathbf{E}_{\delta_x}[e^{-\theta\|X_{D_{r'+h}}\|}] = E_s[e^{-\theta Z_{r'+h}}] = e^{-u(s,r'+h,\theta)} \geq e^{-\theta},$$

which is (27). This means we have proved (23) and thus $u(r, r+h, \theta) \geq u(r', r'+h, \theta)$. The latter yields that, for all $\theta \geq 0$,

$$\begin{aligned}
\frac{\partial}{\partial s}u(r,s,\theta)\bigg|_{s=r} &= \lim_{h\downarrow 0}\frac{u(r,r+h,\theta)-u(r,r,\theta)}{h} \\
&\geq \lim_{h\downarrow 0}\frac{u(r',r'+h,\theta)-u(r',r',\theta)}{h} = \frac{\partial}{\partial s}u(r',s,\theta)\bigg|_{s=r'}.
\end{aligned} \tag{28}$$

Now we apply (14) to get

$$\frac{\partial}{\partial s}u(r,s,\theta)\,|_{s=r} = \left(-\Psi(s,\theta)\frac{\partial}{\partial\theta}u(r,s,\theta)\right)|_{s=r} = -\Psi(r,\theta)\cdot 1, \tag{29}$$

where we used that $\lim_{s\downarrow r}\frac{\partial}{\partial\theta}u(r,s,\theta) = 1$ which can be seen as follows. By dominated convergence, we have

$$\lim_{s\downarrow r}\frac{\partial}{\partial\theta}e^{-u(r,s,\theta)} = \lim_{s\downarrow r}\frac{\partial}{\partial\theta}E_r[e^{-\theta Z_s}\mathbf{1}_{\{Z_s<\infty\}}] = \lim_{s\downarrow r}E_r[-Z_s e^{-\theta Z_s}\mathbf{1}_{\{Z_s<\infty\}}] = -e^{-\theta}.$$

On the other hand,

$$\lim_{s\downarrow r}\frac{\partial}{\partial\theta}e^{-u(r,s,\theta)} = -\lim_{s\downarrow r}\frac{\partial}{\partial\theta}u(r,s,\theta)\, e^{-u(r,s,\theta)} = -\lim_{s\downarrow r}\frac{\partial}{\partial\theta}u(r,s,\theta)\, e^{-\theta}$$

and we may conclude that $\lim_{s\downarrow r}\frac{\partial}{\partial\theta}u(r,s,\theta) = 1$ as claimed.

Combining (28) with (29) gives $\Psi(r,\theta) \le \Psi(r',\theta)$ for $\theta \ge 0$ and $r \le r'$, which completes the proof.

(ii) Define $\Psi^*(r,\theta) := \Psi(r,\lambda^*+\theta)$ for $\theta \ge 0$. Then $(\Psi^*(r,\cdot), r > 0)$ is a family of subcritical branching mechanisms which, by part (i), has the property that $\Psi^*(r,\theta) \le \Psi^*(r',\theta)$ for $r \le r'$ and all $\theta \ge 0$. Clearly this gives $\Psi(r,\theta) \le \Psi(r',\theta)$ for $r \le r'$ and $\theta \ge \lambda^*$.

Let $\theta \le \lambda^*$. First, note that $u(r,s,\lambda^*) = -\log E_r[e^{-\lambda^* Z_s}] = \lambda^*$, which is a consequence of Lemma 7(ii). Thus, $u(r,s,\theta) \le u(r,s,\lambda^*) = \lambda^*$ for all $\theta \le \lambda^*, 0 < r \le s$, and in particular $\psi(u(r,s,\theta)) \le 0$. We therefore get

$$\begin{aligned} u(r,s,\theta) &= \theta - \mathbb{E}^{\mathrm{R}}_r\int_0^{\tau_z}\psi(u(R_v,s,\theta))\,\mathrm{d}v - \mathbb{E}^{\mathrm{R}}_z\int_0^{\tau_s}\psi(u(R_v,s,\theta))\,\mathrm{d}v \\ &\ge \theta - \mathbb{E}^{\mathrm{R}}_z\int_0^{\tau_s}\psi(u(R_v,s,\theta))\,\mathrm{d}v \\ &= u(z,s,\theta) \end{aligned}$$

for any $0 < r \le z \le s, \theta \le \lambda^*$. We can then use $\frac{\partial}{\partial r}u(r,s,\theta) \le 0$ in place of the inequality (18) in the proof of part (i). Thus, following the same arguments as in the proof of part (i) with all inequalities reversed, we see that $\Psi(r,\theta) \ge \Psi(r',\theta)$ for $r \le r'$ and all $\theta \le \lambda^*$. □

3.2. Limiting branching mechanism—Proof of Theorem 3

To begin with, we show the existence and finiteness of the limiting branching mechanism Ψ_∞ and derive a PDE characterisation.

Proposition 9. *For each $\theta \ge 0$, the limit $\lim_{r\uparrow\infty}\Psi(r,\theta) = \Psi_\infty(\theta)$ is finite and the convergence holds uniformly in θ on any bounded, closed subset of $\mathbb{R}_+$.*

(i) *In the (sub)critical case, Ψ_∞ satisfies the equation*

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial\theta}\Psi^2_\infty(\theta) = 2\psi(\theta), \quad \theta \ge 0, \\ &\Psi_\infty(0) = 0. \end{aligned} \tag{30}$$

(ii) *In the supercritical case, Ψ_∞ satisfies (30) with the initial condition at 0 replaced by*

$$\Psi_\infty(0) = -2\sqrt{\int_0^{\lambda^*}|\psi(\theta)|\,\mathrm{d}\theta}$$

and $\Psi_\infty(\lambda^*) = 0$.

Proof. From the monotonicity in Proposition 2, we conclude that the pointwise limit $\Psi_\infty(\theta) := \lim_{r\uparrow\infty}\Psi(r,\theta)$ exists. We will have to show that $|\Psi_\infty(\theta)|$ is finite for each $\theta \ge 0$. Uniform convergence on any bounded, closed subset of $\mathbb{R}$ will then follow by convexity, see for example Theorem 10.8 in [15]. We consider the (sub)critical case and the supercritical case separately.

(i) Suppose we are in the (sub)critical case. We have $\Psi(r, 0) = 0$ for all $r > 0$ and hence $\Psi_\infty(0) = 0$. For $\theta > 0$, recall the PDE (7), which can be written slightly differently as

$$\frac{\partial}{\partial r}\Psi(r,\theta) + \Psi(r,\theta)\frac{\partial}{\partial \theta}\Psi(r,\theta) + \frac{d-1}{r}\Psi(r,\theta) = 2\psi(\theta), \quad r > 0, \theta > 0. \tag{31}$$

By Proposition 2(i), $\frac{\partial}{\partial r}\Psi(r,\theta) \geq 0$ and, by Lemma 6(i), $\Psi(r,\theta) \geq 0$. Thus,

$$\Psi(r,\theta)\frac{\partial}{\partial \theta}\Psi(r,\theta) \leq 2\psi(\theta), \quad \text{for all } r > 0 \text{ and } \theta \geq 0. \tag{32}$$

Fix a $\theta_0 > 0$. Suppose for contradiction that $\Psi(r, \theta_0) \uparrow \infty$ as $r \to \infty$. For any $K > 0$, we can find an r_0 large enough such that

$$\Psi(r_0, \theta_0) > 2K\psi(\theta_0). \tag{33}$$

By (32), this implies that $\frac{\partial}{\partial\theta}\Psi(r_0,\theta_0) < \frac{1}{K}$. As Ψ is convex in θ with $\Psi(r_0, 0) = 0$, we get that

$$\Psi(r_0,\theta_0) \leq \frac{\theta_0}{K}.$$

Now we can choose K large enough such that $\theta_0/K < 2K\psi(\theta_0)$, which then contradicts (33). Hence, $\lim_{r\to\infty}\Psi(r,\theta) = \Psi_\infty(\theta) < \infty$ for all $\theta \geq 0$.

Note that $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta)$ is also finite for each $\theta \geq 0$. Indeed, if we supposed the contrary for some $\theta > 0$, that is, $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta) = \infty$, then (32) would imply that $\liminf_{r\to\infty}\Psi(r,\theta) = 0$, which contradicts Lemma 6(i). By convexity, we can pick any $\theta > 0$ to get $\limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r, 0+) \leq \limsup_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta) < \infty$.

Next, we want to take $r \to \infty$ in (31) and we know that the limit of the left-hand side exists since the right-hand side does not depend on r. We keep $\theta_0 > 0$ fixed and consider each term on the left-hand side of (31) separately.

We have just seen that $\lim_{r\to\infty}\Psi(r,\theta_0) < \infty$ which implies that the third term on the left-hand side of (31), namely $\frac{d-1}{r}\Psi(r,\theta_0)$, vanishes as $r \to \infty$.

Consider the term $\Psi(r,\theta_0)\frac{\partial}{\partial\theta}\Psi(r,\theta_0)$ next. Since $\Psi(r,\cdot)$ is a sequence of continuous, convex functions, the pointwise limit Ψ_∞ is also continuous and convex in θ, cf. Theorem 10.8 in Rockafellar [15]. The convexity ensures that the set of points at which Ψ_∞ is not differentiable is at most countable. If Ψ_∞ is differentiable at θ_0, then by Theorem 25.7 in [15], it follows that $\lim_{r\to\infty}\frac{\partial}{\partial\theta}\Psi(r,\theta_0) = \frac{\partial}{\partial\theta}\Psi_\infty(\theta_0)$ and hence

$$\lim_{r\to\infty}\Psi(r,\theta_0)\frac{\partial}{\partial\theta}\Psi(r,\theta_0) = \Psi_\infty(\theta_0)\frac{\partial}{\partial\theta}\Psi_\infty(\theta_0). \tag{34}$$

So far we have seen that, for all $\theta \geq 0$ at which Ψ_∞ is differentiable, the second and third terms on the left-hand side of (31) converge to a finite limit as $r \to \infty$ which implies that the limit of the first term, that is $\lim_{r\to\infty}\frac{\partial}{\partial r}\Psi(r,\theta)$, also exists and is finite. With $\lim_{r\to\infty}\Psi(r,\theta) < \infty$ it thus follows that $\frac{\partial}{\partial r}\Psi(r,\theta)$ tends to 0 as $r \to \infty$, for all $\theta \geq 0$ at which Ψ_∞ is differentiable.

In conclusion, for any θ at which Ψ_∞ is differentiable, the first and third term on the left-hand side of (31) vanish as $r \to \infty$ and with (34) we get

$$\Psi_\infty(\theta)\frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\psi(\theta). \tag{35}$$

For $\theta > 0$, we have $\Psi_\infty(\theta) > 0$ and we can write (35) as

$$\frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\frac{\psi(\theta)}{\Psi_\infty(\theta)}, \tag{36}$$

which again holds for all $\theta > 0$ at which Ψ_∞ is differentiable. By convexity, Ψ_∞ admits left and right derivatives for every $\theta > 0$. Since the right-hand side of (36) is continuous and (36) holds true for all but countably many $\theta > 0$, we conclude that the left and the right derivative of $\Psi_\infty(\theta)$ agree for every $\theta > 0$. Thus (36), and equivalently (30), holds in fact for every $\theta > 0$. By convexity, for any $\theta > 0$, we get

$$\frac{\partial}{\partial\theta}\Psi_\infty(0+) \leq \frac{\partial}{\partial\theta}\Psi_\infty(\theta) = 2\frac{\psi(\theta)}{\Psi_\infty(\theta)} < \infty,$$

which shows that (30) holds true for $\theta = 0$ with both sides being equal to 0.

(ii) We consider the supercritical case now. Again we first have to show that $\Psi_\infty(\theta)$ is finite for each $\theta \geq 0$.

Let us begin with the case $\theta \in [\lambda^*, \infty)$. We can consider the (sub)critical branching mechanisms $\Psi^*(r, \lambda) := \Psi(r, \lambda + \lambda^*)$ for $\lambda \geq 0$. Then part (i) applies to the (sub)critical Ψ^* and we conclude that, for any $\theta \geq \lambda^*$,

$$\Psi_\infty(\theta) = \lim_{r\to\infty}\Psi(r, \theta) = \lim_{r\to\infty}\Psi^*(r, \theta - \lambda^*) = \Psi^*_\infty(\theta - \lambda^*) < \infty.$$

In particular, Eq. (30) holds for all $\theta \geq \lambda^*$ and $\Psi_\infty(\lambda^*) = \Psi^*_\infty(0) = 0$.

Further, it follows from the monotonicity in Proposition 2 that $\frac{\partial}{\partial\theta}\Psi^*(r, 0+) \leq \frac{\partial}{\partial\theta}\Psi^*_\infty(0+)$. The latter derivative was shown to be finite in the proof of part (i). Thus, for any $r > 0$,

$$\left.\frac{\partial}{\partial\theta}\Psi(r, \theta)\right|_{\theta=\lambda^*} = \frac{\partial}{\partial\theta}\Psi^*(r, 0+) \leq \frac{\partial}{\partial\theta}\Psi^*_\infty(0+) < \infty.$$

Hence, we have a uniform upper bound for the θ-derivative of $\Psi(r, \cdot)$ at λ^*. Recalling that $\Psi(r, \lambda^*) = 0$, convexity ensures that $\Psi(r, \cdot)$ is uniformly bounded from below by the function $\frac{\partial}{\partial\theta}\Psi^*_\infty(0+)(\cdot - \lambda^*)$. This implies already that $\lim_{r\to\infty}|\Psi(r, \theta)| < \infty$ for all $\theta \in [0, \lambda^*]$.

To show that Eq. (30) holds for all $\theta \leq \lambda^*$ we can now simply repeat the argument given in the proof of part (i). Finally, with $\Psi_\infty(\lambda^*) = 0$, we can derive the initial condition for $\Psi_\infty(0)$ by integrating (30) from 0 to λ^*. □

Proof of Theorem 3. Proposition 9 guarantees the existence and finiteness of Ψ_∞. If we integrate (30) from λ^* to θ, and note that $\Psi_\infty(\theta)$ and $\psi(\theta)$ are negative if and only if $\theta \leq \lambda^*$, we obtain the expression in (8). It thus remains to show (ii).

It follows from an obvious adaptation of the proof of Theorem 3.1 in Kyprianou et al. [13] that Z^∞ is the process of the mass of the branching Markov exit measures of a one-dimensional super-Brownian as it first exits the family of intervals $((-\infty, s), s \geq 0)$ as claimed.

Concerning the convergence in (9), we will show that, for $s \geq 0$ and $\theta \geq 0$, $u^\infty(s, \theta) := \lim_{r\to\infty} u(r, s + r, \theta)$ exists and solves

$$u^\infty(s, \theta) = \theta - \int_0^s \Psi_\infty(u^\infty(s - v, \theta))\,\mathrm{d}v, \tag{37}$$

which is the characterising equation for the Laplace functional of Z^∞.

This is trivially satisfied for $s = 0$. Henceforth, let $s > 0$ and $\theta \geq 0$ be fixed. Recall that $u(r, s + r, \theta)$ solves equation (5), which can be written as

$$u(r, s + r, \theta) = \theta - \int_0^s \Psi(v + r, u(v + r, s + r, \theta))\,\mathrm{d}v, \qquad r > 0.$$

Note that the convergence of the convex functions $\Psi(r, \cdot)$ to $\Psi_\infty(\cdot)$ in Theorem 3 holds uniformly in θ on each bounded closed subset of $\mathbb{R}_+$. Therefore, for fixed $\epsilon > 0$, we can choose r large enough such that $|\Psi(s+r,\lambda)-\Psi_\infty(\lambda)|<\epsilon$ for all $\lambda\in\{u(v+r,s+r,\theta),0\le v\le s\}$. Thus, for large r,

$$\begin{aligned}&\Big|u(r,s+r,\theta)-\big(\theta-\int_0^s\Psi_\infty(u(v+r,s+r,\theta))\,\mathrm{d}v\big)\Big|\\&\quad=\Big|\int_0^s\Psi(v+r,u(v+r,s+r,\theta))\,\mathrm{d}v-\int_0^s\Psi_\infty(u(v+r,s+r,\theta))\,\mathrm{d}v\Big|\\&\quad\le\epsilon\, s.\end{aligned}\tag{38}$$

Now assume for a contradiction that $\limsup_{r\to\infty}u(r,s+r,\theta)=+\infty$. Since Ψ_∞ is convex and $\Psi'_\infty(0+)\ge 0$ (with $\Psi'_\infty(0+)=0$ in the supercritical case), the integrand in the first line of (38) is bounded from below by $\Psi_\infty(0)$. Therefore, the expression in the first line of (38) tends to ∞ along a subsequence of r which is an obvious contradiction.

Hence, $u(r,s+r,\theta)$ is bounded as a sequence in r. It therefore contains a convergent subsequence, say $u(r_n,s+r_n,\theta)$ where $(r_n,n\ge 1)$ is a strictly monotone sequence which tends to ∞.

Let us show that every subsequence converges to the same limit. Let $(r'_n,n\ge 1)$ be another strictly monotone sequence which tends to ∞. To begin with we note that $\sup_{n\in\mathbb{N}}\{u(v+r_n,v+r_n,\theta)\}<\infty$ by boundedness. Then we set

$$\bar u=\sup_{v\in(0,s)}\sup_{n\in\mathbb{N}}\{u(v+r_n,v+r_n,\theta)\}<\infty$$

and define $\bar u'$ accordingly using the sequence $(r'_n,n\ge 1)$ in place of $(r_n,n\ge 1)$.

By (38), for any $\epsilon>0$, we can find an $N\in\mathbb{N}$ large enough such that for all $n\ge N$

$$\begin{aligned}&|u(r_n,s+r_n,\theta)-u(r'_n,s+r'_n,\theta)|\\&\quad\le 2\epsilon s+\int_0^s\Big|\Psi_\infty(u(v+r_n,s+r_n,\theta))-\Psi_\infty(u(v+r'_n,s+r'_n,\theta))\Big|\,\mathrm{d}v\\&\quad\le 2\epsilon s+\int_0^s M|u(v+r_n,s+r_n,\theta)-u(v+r'_n,s+r'_n,\theta)|\,\mathrm{d}v\end{aligned}\tag{39}$$

where $M:=\sup\{\Psi'_\infty(w):w\in(0,\max\{\bar u,\bar u'\})\}<\infty$. Set

$$F_n(s')=M\int_0^{s'}|u(v+r_n,s+r_n,\theta)-u(v+r'_n,s+r'_n,\theta)|\,\mathrm{d}v,\quad\text{for }0\le s'\le s,$$

and note that $\partial F_n(s')/\partial s'=M|u(s'+r_n,s+r_n,\theta)-u(s'+r'_n,s+r'_n,\theta)|$. By (39),

$$\frac{\partial}{\partial s'}F_n(s')-2\epsilon M(s-s')-M(F_n(s)-F_n(s'))\le 0.$$

Multiplying by $e^{Ms'}$, we derive $\partial[\big(F_n(s)-F_n(s')+2\epsilon(s-s')+\frac{2\epsilon}{M}\big)e^{Ms'}]/\partial s'\ge 0$. Therefore,

$$\big(F_n(s)-F_n(s')+2\epsilon(s-s')+\frac{2\epsilon}{M}\big)e^{Ms'}\le\frac{2\epsilon}{M}e^{Ms},\quad\text{for any }\ 0\le s'\le s.$$

Hence, $F_n(s)-F_n(s')\le 2\epsilon\big(\frac{1}{M}(e^{M(s-s')}-1)-(s-s')\big)$, for $0\le s'\le s$. Since $\epsilon>0$ can be chosen arbitrarily small, we conclude from the definition of $F_n(s')$ that $u(r'_n,s'+r'_n,\theta)$ converges to the same limit as $u(r_n,s'+r_n,\theta)$ as $n\to\infty$. We have thus shown that, considered as a sequence in r, all subsequences of $u(r,s+r,\theta)$ converge to the same limit. Therefore

$u^\infty(s,\theta) = \lim_{r\to\infty} u(r, s+r, \theta)$ exists and, with (38), it satisfies (37). By uniqueness of solutions to (37), $u^\infty(s,\theta)$ agrees with the Laplace functional associated with Z^∞ which in turn implies the desired convergence. □

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Appendix. Derivation of the differential equation (17) corresponding to the semi-group equation (16)

The reader familiar with the superprocess literature will readily believe that any solution to the differential equation (17) also solves the semi-group equation (16) and conversely that solutions to (16) also solve (17). Results of this fashion can be found for instance in the work of Dynkin, see [2], Section 3 in [3] or Section 5.2 in [5]. However, in these references only (sub)critical branching mechanisms are allowed and the authors are unaware of a rigorous proof in the literature for the case of a supercritical branching mechanism. Although it seems possible to adapt Dynkin's arguments to the supercritical case, we will offer a self-contained proof here instead.

Recall that the Laplace functional u of Z, defined in (4), is the unique non-negative solution to the equation

$$u(r,s,\theta) = \theta - \mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_l,s,\theta))\,\mathrm{d}l, \quad 0 < r \le s,\ \theta \ge 0, \tag{A.1}$$

where $(R, \mathbb{P}^{\mathrm{R}})$ is a d-dimensional Bessel process and $\tau_s := \inf\{l > 0 : R_l > s\}$ its first passage time above level s, see (16).

Fix $0 < r \le s$ and $\theta \ge 0$ from now on. Let us apply a Lamperti transform to the d-Bessel process R in the integral on the right-hand side of (A.1). Define $\varphi(s) = \int_0^{r^2 s} R_l^{-2}\mathrm{d}l$, $s \ge 0$, then

$$B_s = \log(r^{-1} R_{r^2\varphi^{-1}(s)}), \quad s \ge 0,$$

is a one-dimensional Brownian motion with drift $\frac{d}{2} - 1$ starting from 0. Let us denote the law of B by $\mathbb{P}_0$. Thus we get

$$\begin{aligned}\mathbb{E}_r^{\mathrm{R}} \int_0^{\tau_s} \psi(u(R_l,s,\theta))\,\mathrm{d}l &= \mathbb{E}_r^{\mathrm{R}} \int_0^{\varphi(r^{-2}\tau_s)} \psi(u(R_{r^2\varphi^{-1}(l)},s,\theta)) R^2_{r^2\varphi^{-1}(l)}\,\mathrm{d}l \\ &= \mathbb{E}_0 \int_0^{T_{\log(s/r)}} \psi(u(e^{B_l+\log r},s,\theta)) e^{2(B_l+\log r)}\,\mathrm{d}l \\ &= \mathbb{E}_{\log r} \int_0^{T_{\log s}} \psi(u(e^{B_l},s,\theta)) e^{2B_l}\,\mathrm{d}l,\end{aligned}$$

where $T_{\log s}$ is the first time B crosses level $\log s$. Eq. (A.1) becomes

$$u(r,s,\theta) = \theta - \mathbb{E}_{\log r} \int_0^{T_{\log s}} \psi(u(e^{B_l},s,\theta)) e^{2B_l}\,\mathrm{d}l. \tag{A.2}$$

We split the integral on the right hand side into its excursions away from the maximum. This gives

$$\begin{aligned}
&\mathbb{E}_{\log r} \int_0^{T_{\log s}} \psi(u(e^{B_l}, s, \theta))e^{2B_l}\, \mathrm{d}l \\
&\quad = \mathbb{E}_{\log r} \sum_{\log r \le u \le \log s} \int_0^{\zeta^{(u)}} \psi(u(e^{u-e_u(l)}, s, \theta))e^{2(u-e_u(l))}\, \mathrm{d}l,
\end{aligned}$$

where e_u is an excursion away from the maximum with lifetime $\zeta^{(u)}$ and the sum is taken over all left end-points u of the excursion intervals in $(T_{\log r}, T_{\log s})$. It follows from the Compensation formula for excursions (Bertoin [1, Corollary 11, p. 110]) that

$$\begin{aligned}
&\mathbb{E}_{\log r} \sum_{\log r \le u \le \log s} \int_0^{\zeta^{(u)}} \psi(u(e^{u-e_u(l)}, s, \theta))e^{2(u-e_u(l))}\, \mathrm{d}l \\
&\quad = \int_{\log r}^{\log s} \eta\left(\int_0^{\zeta} \psi(u(e^{u-e(l)}, s, \theta))e^{2(u-e(l))}\, \mathrm{d}l\right) \mathrm{d}u,
\end{aligned}$$

where η denotes the excursion measure and e is a generic excursion with length ζ. Then we apply Exercise 5, chapter VI, [1], to get

$$\begin{aligned}
&\int_{\log r}^{\log s} \eta\left(\int_0^{\zeta} \psi(u(e^{u-e(s)}, s, \theta))e^{2(u-e(l))}\, \mathrm{d}l\right) \mathrm{d}u \\
&\quad = \int_{\log r}^{\log s} \int_0^{\infty} \psi(u(e^{u-y}, s, \theta))e^{2(u-y)}\, \hat{V}(\mathrm{d}y)\, \mathrm{d}u,
\end{aligned}$$

where $\hat{V}$ is the renewal function of the dual ladder height process (the dual process is here simply Brownian motion with drift $-(\frac{d}{2}-1)$). We see from Eq. (4), p. 196 in [1] that $\hat{V}(\mathrm{d}y) = 2e^{-2(\frac{d}{2}-1)y}\mathrm{d}y$ and obtain

$$\begin{aligned}
&\int_{\log r}^{\log s} \int_0^{\infty} \psi(u(e^{u-y}, s, \theta))e^{2(u-y)}\, \hat{V}(\mathrm{d}y)\, \mathrm{d}u \\
&\quad = 2\int_{\log r}^{\log s} e^{2u} \int_0^{\infty} \psi(u(e^{u-y}, s, \theta))\, e^{-dy}\, \mathrm{d}y\, \mathrm{d}u \\
&\quad \overset{z=e^{u-y}}{=} -2\int_{\log r}^{\log s} e^{2u} \int_{e^u}^{0} \psi(u(z, s, \theta))z^d e^{-du}\, z^{-1}\, \mathrm{d}z\, \mathrm{d}u \\
&\quad \overset{v=e^u}{=} -2\int_r^s v^2 \int_v^0 \psi(u(z, s, \theta))\, z^{d-1}v^{-d}\, \mathrm{d}z\, v^{-1}\, \mathrm{d}v \\
&\quad = 2\int_r^s v^{1-d} \int_0^v \psi(u(z, s, \theta))\, z^{d-1}\, \mathrm{d}z\, \mathrm{d}v.
\end{aligned}$$

Thus the characterising semi-group equation (A.1) resp. (A.2) becomes

$$u(r, s, \theta) = \theta - 2\int_r^s v^{1-d} \int_0^v \psi(u(z, s, \theta))\, z^{d-1}\, \mathrm{d}z\, \mathrm{d}v.$$

Differentiation in r gives

$$\frac{\partial}{\partial r}u(r,s,\theta) = 2r^{1-d}\int_0^r \psi(u(z,s,\theta))z^{d-1}\,\mathrm{d}z,$$
$$\frac{\partial^2}{\partial r^2}u(r,s,\theta) = 2(1-d)r^{-d}\int_0^r \psi(u(z,s,\theta))z^{d-1}\,\mathrm{d}z + 2\psi(u(r,s,\theta)).$$

Hence, we obtain the differential equation in (17), i.e. for $\theta \geq 0$,

$$\frac{1}{2}\frac{\partial^2}{\partial r^2}u(r,s,\theta) + \frac{d-1}{2r}\frac{\partial}{\partial r}u(r,s,\theta) = \psi(u(r,s,\theta)) \quad 0 < r \leq s,$$
$$u(r,r,\theta) = \theta.$$

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